Abstract

In this paper, we give a new formulation of a geometric lemma for the metaplectic groups over $p$–adic field $F$, analogous to the existing one for classical groups. This enables us to give a Zelevinsky type classification of irreducible admissible genuine representations of metaplectic groups. As an application of our Jacquet module technique, we explicitly calculate Jacquet modules of even and odd Weil representations.

1 Introduction

This paper is the first in a series where we systematically study the representation theory of a double cover $\tilde{Sp}(n)$ of a symplectic group $Sp(n)$ of rank $n$ over a local non–archimedean field $F$ of characteristic different from two. In this paper we establish fundamental results on parabolic induction and Jacquet modules relying on general results of Bernstein–Zelevinsky on the $l$–group theory ([3], [21], [2]), and general principles established in [9]. In the sequel to this paper ([8]) we give the description of rank–one cuspidal reducibility of $\tilde{Sp}(n)$ via theta correspondence. We expect applications in the theory of automorphic forms where metaplectic groups play a prominent role.

Now, we explain the content of the paper as well as the results that we establish. One of the nice properties of classical groups such as $Sp(n)$ is
that all Levi subgroups have a particularly nice form: every proper parabolic subgroup $P$ of $Sp(n)$ has a Levi subgroup $M$ of the form:

$$GL(n_1, F) \times \cdots \times GL(n_k, F) \times Sp(n'), \quad n' = n - (n_1 + \cdots + n_k).$$  \hfill (1)

This is used extensively in studying its representation theory (see for example [19], [9]). The group $\tilde{Sp}(n)$ is no longer a linear algebraic group, and the notion of parabolic subgroups needs to be introduced. This is done in [11] (see also [13]) as follows. A parabolic subgroup $\tilde{P}$ is the preimage of $P$ under the canonical (surjective) map $\tilde{Sp}(n) \to Sp(n)$. If we write $P = MN$ for a Levi decomposition, where $N$ is the unipotent radical, then $\tilde{Sp}(n)$ splits over $N$ and we have the following Levi decomposition $\tilde{P} = \tilde{M}N$. The only problem is that the analogue of (1) is no longer true for $\tilde{M}$. But this does not cause serious problems since we have the following epimorphism (see [11]):

$$GL(n_1, F) \times \cdots \times GL(n_k, F) \times \tilde{Sp}(n') \to \tilde{M},$$  \hfill (2)

whose kernel is easy to describe. Here $\tilde{GL}(n, F)$ is a very simple double cover of $GL(n, F)$ (see Section 1) whose representation theory is easy to describe in terms of that of $GL(n, F)$. A Zelevinsky type classification for $\tilde{GL}(n, F)$ is described in Section 4.1 in Propositions 4.1, 4.2, and 4.3.

Also, although $\tilde{M}$ is not exactly the product $GL(n_1, F) \times \cdots \times GL(n_k, F) \times \tilde{Sp}(n')$, it differs from it by a finite subgroup that enables to write every irreducible representation $\pi$ of $\tilde{M}$ in the form $\pi_1 \otimes \cdots \otimes \pi_k \otimes \pi'$ where all representations $\pi_1, \ldots, \pi_k, \pi'$ are genuine or not at the same time. This simple property enables us to set–up Tadić’s machinery ([19], [9]) of parabolic induction and Jacquet functors. This is obtained in Section 4.2 in Propositions 4.4 and 4.5. The results of Proposition 4.4 and 4.5 and later results in the same section depend on a precise form of the composition of parabolic induction functor and Jacquet functor (i.e., the form of a geometric lemma (see Theorem 3.3), also Section 2.1 of [3]). This ultimately depends on a non–trivial computation of the action of the action of the non–trivial element $w_0$ in $N_{Sp(n)}(M)/M$ ($P$ a maximal parabolic subgroup) on $\tilde{M}$. This computation covers almost all Section 3. Surprisingly this was not done before except using global methods but for the special case of the minimal parabolic subgroup recently in a paper by Savin and Loke ([16]).
As we said above, the reward for these tedious computations is a rather precise description of irreducible representations and intertwining operators for $\tilde{Sp}(n)$. The Zelevinsky type classification is obtained in Theorem 4.7 using methods of our previous work [9]. The construction and analytic continuation can be obtained directly from [14] since this approach uses a geometric lemma and general principles established by Bernstein (see the Remark at the end of 4.2).

Finally, in Section 5 we give a sample of the possible applications of our results. In Theorem 5.1 of Section 5 we explicitly calculate Jacquet modules for even and odd Weil representations.

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2 Preliminaries

2.1 Central extensions

The groups we study are two–sheeted central extensions of linear, orthogonal and symplectic $p$–adic groups. More precisely, we study the groups of $F$–points of $F$–split symplectic, general linear and $F$–(quasi)- split orthogonal groups and their two–sheeted central extensions. In each case we have the following central extension (with $\mu_2 = \{1, -1\}$)

$$1 \longrightarrow \mu_2 \xrightarrow{i} \tilde{G} \xrightarrow{p} G \longrightarrow 1,$$

where $G$ is any of the groups mentioned above. These extensions will be topological, i.e., the topology on $\tilde{G}$ is such that $i$ and $p$ are continuous, $p$ is open and $i$ is closed. We will recall of the topology on $\tilde{G}$ in the section dealing with the geometric lemma. To conclude, $\tilde{G}$ will be a Hausdorff, $l$–group. From the construction of the topology on $\tilde{G}$, it will be obvious that there exists an open compact subgroup $K'$ of $G$ such that (3) splits on $K'$.

We note that Schur’s lemma holds in the setting of $l$–groups (which are countable at infinity) by [4], Chapter 1, Section 4.2. So, for an irreducible
smooth representation $\pi$ of $\widehat{G}$, there exists a character $\chi_\pi$ of $\mu_2$ (which is a restriction of the central character of $\pi$ to the central subgroup $\mu_2$).

### 2.2 The symplectic group

We use the same realization of the symplectic group as in [10]. So, for every $n \in \mathbb{Z}_{\geq 0}$, let $(W_n, \langle \cdot, \cdot \rangle)$ be a non-degenerate symplectic vector space of dimension $2n$ over $F$. We have a complete polarization $W_n = W'_n + W''_n$, where \{${e_1, \ldots, e_n}$\} is a basis for $W'_n$ and \{${e'_1, \ldots, e'_n}$\} for $W''_n$ with $\langle e_i, e'_j \rangle = \delta_{ij}$. Let $\widetilde{Sp}(W_n)$ be the unique two-fold central extension of $Sp(W_n)$, i.e., the following holds

$$1 \to \mu_2 \to \widetilde{Sp}(W_n) \to Sp(W_n) \to 1.$$ 

The multiplication in $\widetilde{Sp}(W_n)$ is given by Rao’s cocycle $c_{Rao}$ ([15],[10]):

$$[g_1, \epsilon_1][g_2, \epsilon_2] = [g_1 g_2, \epsilon_1 \epsilon_2 c_{Rao}(g_1, g_2)],$$

where $g_i \in Sp(W_n)$, $\epsilon_i \in \mu_2$, $i = 1, 2$. If $G$ is a subgroup of $Sp(W_n)$, we denote by $\widetilde{G}$ its inverse image in $\widetilde{Sp}(W_n)$. Now, we follow ([11], Section 1.4) very closely. We fix a diagonal maximal split torus in $Sp(W_n)$ (having in mind the basis introduced above) and we fix a Borel subgroup stabilizing a maximal isotropic flag consisting of vectors \{${e'_1, \ldots, e'_n}$\}. Every standard Levi subgroup $M_s$ of $Sp(W_n)$ is isomorphic to $GL(n_1, F) \times \cdots \times GL(n_k, F) \times Sp(W_{n-|s|})$, where $s = (n_1, n_2, \ldots, n_k)$ is a sequence of positive integers with $n_1 + \cdots + n_k = |s| \leq n$. There is a natural splitting from the unipotent radical of $N_s$ of the corresponding standard parabolic subgroup $P_s$ to its cover ([13], Lemma 2.9 on p. 43); let $N'_s$ be the image of that homomorphism. We then have $\widetilde{P}_s \cong \widetilde{M}_s \rtimes N'_s$. The subgroups of $\widetilde{Sp}(W_n)$ of this form we call the parabolic subgroups of $Sp(W_n)$ and this decomposition is their Levi decomposition.

We can explicitly describe $\widetilde{M}_s$ as follows: There is a natural epimorphism

$$\phi : \widetilde{GL}(n_1, F) \times \cdots \times \widetilde{GL}(n_k, F) \times \widetilde{Sp}(W_{n-|s|}) \to \widetilde{M}_s$$

given by

$$([g_1, \epsilon_1], \ldots, [g_k, \epsilon_k], [h, \epsilon]) \mapsto ([g_1, g_2, \ldots, g_k, h], \epsilon_1 \ldots \epsilon_k \epsilon \alpha),$$

with $\alpha = \prod_{i < j} (\det g_i, \det g_j)_F \left(\prod_{i=1}^k (\det g_i, x(h))_F \right)$, where $x(h)$ is defined in [15], Lemma 5.1, and $(\cdot, \cdot)_F$ denotes the Hilbert symbol of the field $F$. We
may think of $\tilde{GL}(n_i, F)$ as the two–fold cover of $GL(n_i, F)$ with multiplication $[g_1, \epsilon_1][g_2, \epsilon_2] = [g_1 g_2, \epsilon_1 \epsilon_2 (\det g_1, \det g_2)_F]$. Of course, this agrees with Rao’s cocycle under an identification of $GL(n_i, F)$ and its diagonal embedding in the Siegel Levi subgroup of $Sp(W_n)$ (we will explain (4) more thoroughly in the subsection dealing with the action of the Weyl group on maximal, non–Siegel parabolic subgroups).

We have to calculate Rao’s cocycle several times, so we write it down explicitly. For $g_1, g_2 \in Sp(W_n)$ the following holds:

$$c_{Rao}(g_1, g_2) = (x(g_1), x(g_2))_F (-x(g_1)x(g_2), x(g_1 g_2))_F (-1, \det(2q))_F$$

$$= (-1, -1)_F \varepsilon (2q).$$

(5)

All the quantities in this expression are explained in the first chapter of [10]; we will further describe $q$ in Section 3.1.

Having in mind the definition of the parabolic subgroups in the two–sheeted metaplectic cover $\tilde{Sp}(W_n)$ of the symplectic group (Section 1.3 of [11]), we can choose essentially the same Weyl group there as in the symplectic group ([13], Section 4 on p. 59).

3 The action of the Weyl group

This section gives some technical results which enable us, in many instances, to give the same proof of some representation–theoretic statement for the metaplectic group as for the symplectic group.

3.1 The Siegel case

Let $\tilde{M}$ be the (standard) Levi subgroup of the Siegel parabolic of $\tilde{Sp}(W_n)$. We want to calculate the action of the longest element of the Weyl group $w_0$ on the representation $\rho$ of $\tilde{M}$. Having in mind the matrix representation of the symplectic group we have mentioned, we may take $w_0 = \left[\begin{array}{cc} 0 & I_n \\ -I_n & 0 \end{array}\right]$. Let $[m, \epsilon] = \left[\begin{array}{cc} A & 0 \\ 0 & A^{-t} \end{array}\right], \epsilon \in \tilde{M}$. Here $A^{-t}$ is the transpose of the inverse of $A$. Then we calculate $[w_0, 1][m, \epsilon][w_0, 1]^{-1}$.

Lemma 3.1. Let $\rho$ be an irreducible representation of $GL(n, F)$ thought of as a representation of $\tilde{M}$, the (standard) Levi subgroup of the Siegel parabolic
subgroup of $\widetilde{Sp}(W_n)$. Then, with an element $[w_0, 1]$ of the Weyl group of $\widetilde{Sp}(W_n)$ just described, we have the following:

- If $\rho$ is not a genuine representation of $GL(n, F)$, i.e., $\rho$ factors through $\mu_2$, then $[w_0, 1] \rho \cong \rho$;

- If $\rho$ is a genuine representation of $GL(n, F)$, then $[w_0, 1] \rho \cong \alpha \tilde{\rho}$, where $\alpha$ is a character of $GL(n, F)$ which factors through $GL(n, F)$, given by $\alpha(g) = (\det g, \det g)_F = (\det g, -1)_F$.

Proof. First, we assume that the residual characteristic of $F$ is different from two. Then, with maximal compact subgroup $K = \widetilde{Sp}(W_n, O_F)$, $K$ splits in $\widetilde{Sp}(W_n)$ ([13], p. 43) so that $[w_0, 1]^{-1} = [w_0^{-1}, 1]$. Also, in this case, $(-1, -1)_F = 1$ ([17], Chapter 3, Section 1.2 Theorem 1, noting that this theorem refers to $F = \mathbb{Q}_p$ with but this is easily extended to a finite extension of $\mathbb{Q}_p$). This simplifies the expression for Rao’s cocycle in (5). We use [10], p. 19 to see that $x(w_0) = (-1)^n \tilde{A}^{*, 2}$, $x(m) = \det A \tilde{A}^{*, 2}$, and $x(w_0 m) = (-1)^n \det A \tilde{A}^{*, 2}$. We have

$$c_{\text{Rao}}(w_0, m) = ((-1)^n, \det A)_F (\det A, (-1)^n \det A)_F (\det A, (\det(2q))_F \epsilon(2q)).$$

The second factor equals one by a basic property of the Hilbert symbol that $(c, c)_F = 1$ for all $c \in F$. $q$ is attached to the Leray invariant of the triple of isotropic spaces; in our case $L(W''_n, W''_n m^{-1}, W''_n w_0)$, where elements of the symplectic group act from the right on the isotropic subspaces. Here we abuse the notation, because, following Kudla, the Leray invariant is actually the pair $(W_R, q)$ (Section I.1.3, p. 11-12 of [10]), but we go on with $q = L(W''_n, W''_n m^{-1}, W''_n w_0)$. Because of the form of $m$ and $w_0$, we get $L(W''_n, W''_n m^{-1}, W''_n w_0)$, and when we have a repetition of the maximal isotropic subspaces in the triple, as we have here, the attached quadratic form is trivial, i.e., $q$ is defined on the null–space. Because of that, $l = 0$ and $t = 0$ since $j(w_0) + j(m) - j(w_0 m) = 0$ ([10], p. 19). We get $c_{\text{Rao}}(w_0, m) = ((-1)^n, \det A)_F$. We now calculate $c_{\text{Rao}}(w_0 m, w_0^{-1})$. We get, in this case, $q = L(W''_n, W''_n, W''_n) = 0$, and we may take $\epsilon(2q) = 1$. Also, $2t = n+n-l$ and $t = n$. Now, we know $\det(2q) = x(w_0 m w_0^{-1}) x(w_0 m x(w_0^{-1}) (-1)^t$, so $\det(2q) = 1$. We finally get $c_{\text{Rao}}(w_0 m, w_0^{-1}) = ((-1)^n \det A, \det A)_F$. This means that

$$[w_0, 1][m, \epsilon][w_0, 1]^{-1} = [w_0 m w_0^{-1}, \epsilon((-1)^n, \det A)_F (\det A, \det A)_F]$$

$$= [w_0 m m w_0^{-1}, \epsilon],$$
by the well–known properties of the Hilbert symbol.

When the residual characteristic of \( F \) is two, we need a slight modification: in this case \( K \) does not necessarily split in \( \tilde{Sp}(W_n) \), and we calculate \([w_0, 1]^{-1} = [w_0^{-1}, (-1, -1)_F^{n(n-1)/2}] \). If \( F \) is such that \((-1, -1)_F = -1 \) and if \( n(n-1)/2 \) is odd, we have \([w_0, 1]^{-1} = [w_0^{-1}, -1] \). When we calculate \( c_{\text{Rao}}(w_0, m) \) using formula (5) we get the same result as in odd residual case, since \( \psi \) is not a genuine representation of \( \rho \). Now, let \( \tilde{\psi} : (\rho, \epsilon) \) be an irreducible representation of \( \tilde{GL}(n, F) \) (we consider it as a representation of \( \tilde{M} \)). We fix a non–trivial additive character \( \psi \) of \( F \). Then, there is a genuine (i.e., which does not factor through \( GL(n, F) \)) character of \( \tilde{GL}(n, F) \) given as \( \chi(g, \epsilon) = \chi(\psi)(g, \epsilon) = c_R(\psi(\omega_1), -t)^{-1} \). Here \( \gamma \) denotes the Weil invariant, and, in general, \( \psi_a \) denotes the character \( \psi_a(x) = \psi(ax) \). From the previous calculation, it follows

\[
([w_0, 1]^{(w_0, 1)} \rho) \left[ \begin{array}{cc} A & 0 \\ 0 & A^{-t} \end{array} \right], \epsilon \right] = \rho \left[ \begin{array}{cc} A^{-t} & 0 \\ 0 & A \end{array} \right], \epsilon \right].
\]

If \( \rho \) is not a genuine representation of \( \tilde{GL}(n, F) \), we have \(([w_0, 1]^{(w_0, 1)} \rho) \cong \tilde{\rho} \); this is a basic fact about representations of \( GL(n, F) \) ([21], proof of Theorem 1.9). If \( \rho \) is a genuine representation of \( \tilde{GL}(n, F) \), \( \chi(\psi) \rho \) is not, and we immediately get \([w_0, 1]^{(w_0, 1)} \rho \cong ([w_0, 1]^{(w_0, 1)}(\chi(\psi)))^{-1} \tilde{\psi} \). The character \( \alpha = ([w_0, 1]^{(w_0, 1)}(\chi(\psi)))^{-1} \tilde{\psi} = \chi_\psi^2 = \chi_\psi^2 \) (since \( \chi_\psi^4 = 1 \)) factors through \( GL(n, F) \) and \( \alpha(g) = (\det(g), \det(g)) F = (\det(g), -1) F \).

\( \square \)

### 3.2 Non–Siegel maximal parabolic

Assume that \( s_1 \) is positive integer smaller than \( n \). We define \( W_{s_1} = \text{span}\{e_1, \ldots, e_{s_1}\} \), and \( W_{s_1}^* = \text{span}\{e_1', \ldots, e_{s_1}'\} \). In accordance with our previous notation, \( W_{s_1} = W_{s_1}' + W_{s_1}^* \) is a complete polarization of a non–degenerate symplectic space. Also, with \( W_{s_1} = W_{s_1}' \) we have a decomposition \( W_n = W_{s_1} + W_{s_1}' \). Of course, \( W_{s_1}^* \) is a \( 2(n-s_1) \)– dimensional non–degenerate symplectic space. The stabilizer of an isotropic subspace \( W_{s_1}^* \) is a maximal standard parabolic subgroup \( P_{s_1} \) of \( Sp(W_n) \) with Levi decomposition \( P_{s_1} = M_{s_1} N_{s_1} \) (again, we follow the notation from [11]). We have an obvious
inclusion \( j : \text{Sp}(W_n) \times \text{Sp}(W_{s_1}^0) \to \text{Sp}(W_n) \) (cf. [10], p. 5) which leads to the following commutative diagram (all the arrows are explained below):

\[
\begin{array}{ccc}
\text{Sp}(W_n) \times \text{Sp}(W_{s_1}) & \xrightarrow{f} & \text{Sp}(W_n) \times \text{Sp}(W_{s_1}^0) \\
\uparrow & & \uparrow \\
\text{GL}(W_n) \times \text{Sp}(W_{s_1}) & \xrightarrow{\phi} & \tilde{M}_{s_1}
\end{array}
\]

We need this diagram to describe the action of the Weyl group on the irreducible representations of \( \tilde{M}_{s_1} \).

Let \( w_0 \) be a representative of the longest element in the Weyl group of \( \text{Sp}(W_n) \) modulo the longest one in the Weyl group of \( M_{s_1} \). We know that \( w_0 \) belongs to \( j(\text{Sp}(W_{s_1}) \times \text{Sp}(W_{s_1}^0)) \); moreover, \( w_0 = j(\tilde{w}_0, 1) \), where \( \tilde{w}_0 \) denotes the non–trivial element in the Weyl group (this follows from the explicit description of \( w_0 \), according to p. 7 of [5]). We know describe all the mappings appearing in this commutative diagram. The lower horizontal epimorphism \( \phi \) is described by (4). Here, \( \tilde{Sp}(W_{s_1}) \) and \( \tilde{Sp}(W_{s_1}^0) \) are considered as two–fold covers of the appropriate symplectic groups in their own right, and \( \tilde{Sp}(W_{s_1}) \times \tilde{Sp}(W_{s_1}^0) \) is, as before, an inverse image of \( j(\text{Sp}(W_{s_1}) \times \text{Sp}(W_{s_1}^0)) \) in \( \tilde{Sp}(W_n) \). From now on, we will simply denote \( j(g_1, g_2) \) by \( (g_1, g_2) \in \text{Sp}(W_n) \).

Let \( f \) be a mapping defined by

\[
f([g_1, \epsilon_1], [g_2, \epsilon_2]) = [(g_1, g_2), \epsilon_1 \epsilon_2 c_{Rao}((g_1, 1), (1, g_2))].
\]

For an irreducible representation \( \rho \) of \( \text{GL}(n, F) \) or \( \tilde{\text{Sp}}(n, F) \), we denote by \( \chi_\rho \) the restriction of the central character of \( \rho \) to \( \{[1, \epsilon] ; \epsilon \in \mu_2 \} \). The same notation for that character will be used if we study an irreducible representation of an inverse image of the Levi subgroup of these groups.

**Lemma 3.2.** Let \( [w_0, 1] \) be the element in the Weyl group of \( \tilde{\text{Sp}}(W_n) \) described above, and let \( \pi \) be an irreducible representation of \( \tilde{M}_{s_1} \). Then, there exist irreducible representations \( \rho \) of \( \text{GL}(W_{s_1}^n) \) and \( \sigma \) of \( \tilde{\text{Sp}}(W_{s_1}^0) \) satisfying \( \chi_\rho = \chi_\sigma \) such that \( \pi \) pulls back via \( \phi \) from the diagram above to \( \rho \otimes \sigma \), and then we have the following:

- If \( \chi_\rho = \chi_\sigma = 1 \), then \( [w_0, 1] \phi^*(\pi) = [w_0, 1](\rho \otimes \sigma) = \tilde{\rho} \otimes \sigma \);
• if $\chi_\rho = \chi_\sigma \neq 1$, then $[w_0, 1] \phi^*(\pi) = [w_0, 1](\rho \otimes \sigma) = \alpha \tilde{\rho} \otimes \sigma$, where $\alpha$ is a character of $GL(W''_s)$, analogous to that of Lemma 3.1. Observe that, in this case, $\rho \cong \alpha \tilde{\rho}$ is equivalent to $\chi^{-1}_\psi \rho \cong \chi^{-1}_\psi \rho$.

Proof. First, we describe the homomorphisms of this diagram and comment its commutativity.

Now, if our metaplectic groups were realized not with the aid of the cocycle, but as groups of unitary operators (as in [10], I.1) the existence of the mapping of the upper horizontal line in the diagram would be obvious. Since we realize our groups through Rao’s cocycles, we just have to check that the restriction of the Rao’s cocycle of $\tilde{Sp}(W'_n)$ to $\tilde{Sp}(W'_s)$ coincides with the original Rao’s cocycle for $\tilde{Sp}(W'_s)$ (here we have to check explicit realizations of Rao’s cocycle since it depends on the choice of the symplectic basis of the symplectic space). The same thing must be checked for $\tilde{Sp}(W'_0)$.

Let $f_1 : \tilde{Sp}(W'_s) \to \tilde{Sp}(W'_n)$ be a mapping denoted by $f_1([g_1, \epsilon]) = [(g_1, 1), \epsilon]$. We prove that this is a homomorphism, i.e., that

$$c_{\text{Rao}}(g_1, g'_1) = c_{\text{Rao}}((g_1, 1), (g'_1, 1)),$$

where the first cocycle is with respect to the basis $\{e_1, \ldots, e_{s_1}, e'_1, \ldots, e'_{s_1}\}$. In the same way, let $f_2 : \tilde{Sp}(W'_0) \to \tilde{Sp}(W'_n)$ defined by $f_2([g, \epsilon]) = [(1, g), \epsilon]$. Analogously, we have to prove that $c_{\text{Rao}}(g_1, g_2) = c_{\text{Rao}}((1, g_1), (1, g_2))$. Now we return to $f_1$. In order to calculate both cocycles, we fix $g \in \tilde{Sp}(W'_s)$ and calculate $x(g)$, then compare it to $x(g, 1)$. Let $w'_j$ be an element of the Weyl group of $\tilde{Sp}(W'_s)$ such that $g = p_1 w'_j^{-1} p_2$, for some $p_1, p_2$ from the maximal parabolic subgroup of $\tilde{Sp}(W'_s)$ stabilizing $Y_1 = \text{span}\{e'_1, \ldots, e'_{s_1}\}$ (and $Y_2$ will be $\text{span}\{e'_{s_1+1}, \ldots, e'_n\}$). We have $x(g) = \det(p_1 p_2|Y_1)$. In this symplectic basis, $w'_j^{-1}$ is represented by the following matrix ([10], p. 19)

$$\begin{bmatrix} I_{s_1-j} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_j \\ 0 & 0 & I_{s_1-j} & 0 \\ 0 & I_j & 0 & 0 \end{bmatrix}.$$

In the symplectic basis $\{e_1, \ldots, e_n, e'_1, \ldots, e'_n\}$, an element $\left(w'_j^{-1}, 1\right)$ is repre-
sented by the matrix

\[
\begin{bmatrix}
I_{s_1-j} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -I_j & 0 \\
0 & 0 & I_{n-s_1} & 0 & 0 & 0 \\
0 & 0 & 0 & I_{s_1-j} & 0 & 0 \\
0 & I_j & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_{n-s_1} & 0
\end{bmatrix}.
\]

For \( g \in Sp(W_n) \), \( j(g) \) denotes the rank of the lower left \( n \times n \) block in the matrix realization of \( Sp(W_n) \). We see that \( j(g) = j(g, 1) \) and then

\[
(g, 1) = (p_1, 1)(w_j^{-1}, 1)(p_2, 1) = (p_1, 1)\overline{p}_1 w_j^{-1} \overline{p}_2 (p_2, 1)
\]

for some \( \overline{p}_1, \overline{p}_2 \in P \) where now \( w_j \) is of the same form as \( w_j' \), but now acting on the space \( W_n \). We immediately see that we can take \( \overline{p}_1 \) and \( \overline{p}_2 \) belonging to the Levi subgroup of the Siegel parabolic \( P \). Then, if \( \pi = \begin{bmatrix} a_i & 0 \\ 0 & d_i \end{bmatrix}, i = 1, 2 \) we can take \( d_1 \) to be the permutational matrix corresponding to the permutation

\[
\begin{pmatrix}
1 & 2 & \ldots & s_1 - j & s_1 - j + 1 & \ldots & s_1 & s_1 + 1 & \ldots & n \\
1 & 2 & \ldots & s_1 - j & n - j + 1 & \ldots & n & s_1 - j + 1 & \ldots & n - j
\end{pmatrix}
\]

and \( d_2 \) to be the permutational matrix corresponding to the permutation

\[
\begin{pmatrix}
1 & 2 & \ldots & s_1 - j & s_1 - j + 1 & \ldots & n - j & n - j + 1 & \ldots & n \\
1 & 2 & \ldots & s_1 - j & s_1 + 1 & \ldots & n & s_1 - j + 1 & \ldots & s_1
\end{pmatrix}.
\]

We get \( x(g, 1) = \det((p_1, 1)(p_2, 1)|_Y) \det(d_1 d_2) = x(g) \det(d_1 d_2) = x(g) \det(d_1 d_2) \). We see that \( \det(d_i) = (-1)^j(n-s_1) \), and \( x(g, 1) = x(g) \).

On the other hand, the situation with \( x(1, g) \) is simpler, i.e., the matrix of \( (1, w_j''^{-1}) \) in the full symplectic basis is of the form

\[
\begin{bmatrix}
I_{s_1} & 0 & 0 & 0 & 0 & 0 \\
0 & I_{n-s_1-j} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -I_j \\
0 & 0 & 0 & I_{s_1} & 0 & 0 \\
0 & 0 & 0 & 0 & I_{n-s_1-j} & 0 \\
0 & 0 & I_j & 0 & 0 & 0
\end{bmatrix},
\]

so \( (1, w_j''^{-1}) = w_j^{-1} \) and \( x(1, g) = x(g) \).
Now we calculate other features appearing in the expression (5). First, for the cocycle related to \( Sp(W^0_{s_1}) \) we calculate the quadratic form \( q' = L(Y, Y(1, g_2^{-1}), Y(1, g_1)) \), for \( g_1, g_2 \in Sp(W^0_{s_1}) \) in terms of \( q = L(Y_2, Y_2g_2^{-1}, Y_2g_1) \).

Following [10], p. 12 we calculate

\[
R_1 = Y \cap Y(1, g_2^{-1}) + Y \cap Y(1, g_1) + Y(1, g_2^{-1}) \cap Y(1, g_1)
\]

and

\[
R = Y_2 \cap Y_2g_2^{-1} + Y_2 \cap Y_2g_1 + Y_2g_2^{-1} \cap Y_2g_1.
\]

We write \( g_2^{-1} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \) and \( g_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \).

First, we have

\[
R = (\text{Ker } c_2)d_2 + (\text{Ker } c_1)d_1 + V,
\]

where \( V \subset X_2 + Y_2 = W^0_{s_1} \) came from the last part in the expression for \( R \).

It is straightforward that

\[
R_1 = Y_1 + (\text{Ker } c_2)d_2 + Y_1 + (\text{Ker } c_1)d_1 + Y_1 + V,
\]

so that \( R_1 = Y_1 + R \). If we denote by \( R^\perp \) the orthogonal complement of \( R \) in \( W^0_{s_1} \), we immediately get \( R_1^\perp = Y_1 + R^\perp \) and \( W_{R_1} = R_1^\perp / R_1 \approx R^\perp / R = W_R \).

Also, the intersections and projections of \( Y_1 + (\text{Ker } c_2)d_2, Y_1 + (\text{Ker } c_1)d_1 \) and \( Y_1 + V \) behave accordingly so that \( q \) is a quadratic form isomorphic to \( q' \), so they have the same dimension (\( l \) in the expression for the Rao’s cocycle), Hasse invariant \( \epsilon(2q) \) and quantity \( t \) ([10], Proposition 4.3). The same reasoning shows that \( q = L(Y, Y(g^{-1}, 1), Y(1, 1)) \) is a quadratic form isomorphic to \( q' = L(Y_1, Y_1g_2^{-1}, Y_1g_1) \) and the same conclusions follow. We can now conclude that \( c_{Rao}(g_1, 1, (g_2, 1)) = c_{Rao}(g_1, g_2), \) for \( g_i \in Sp(W_{s_1}), i = 1, 2 \) and \( c_{Rao}(1, g_1), (1, g_2)) = c_{Rao}(g_1, g_2), \) for \( g_i \in Sp(W^0_{s_1}), i = 1, 2 \), so the functions \( f_i, i = 1, 2 \) are homomorphisms.

The following fact is useful:

\[
c_{Rao}((g_1, 1), (1, g_2)) = c_{Rao}((1, g_2), (g_1, 1)). \tag{6}
\]

We use the following equation which relates Leray’s and Rao’s cocycles ([10], I, Theorem 4.5)

\[
c_{Y}(g'_1, g'_2) = \beta_{e, \psi}(g'_1g'_2)\beta_{e, \psi}(g'_1)^{-1}\beta_{e, \psi}(g'_2)^{-1}c_{Rao}(g'_1, g'_2),
\]

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for \( g'_1, g'_2 \in Sp(W_n) \) where \( e \) is our fixed symplectic basis, \( Y \) as before and \( c_Y(\cdot , \cdot ) \) Leray’s cocycle. Since \((g_1, 1)\) and \((1, g_2)\) commute, we can plug it into the expression for \( \beta_{\epsilon, \psi} \) and we only have to prove that \( c_Y((g_1, 1), (1, g_2)) = c_Y((1, g_2), (g_1, 1)) \). Since \( c_Y(g'_1, g'_2) = \gamma(\psi \circ q) = \gamma(\det(q), \psi)\gamma(\psi)^t \epsilon(q) \), where \( q \) is a Leary invariant as before, it is enough to show that

\[
L(Y, Y(1, g_2^{-1}), Y(g_1, 1)) = L(Y, Y(g_1^{-1}, 1), Y(1, g_2)).
\]

If we denote the space corresponding to the quadratic form on the left–hand side of the above relation by \( W_{R'_1} \) and on the right–hand side by \( W_{R'_2} \), we get that \( W_{R'_1} = Y / Y = \{0\} \); also \( W_{R'_2} = Y / Y = \{0\} \). In more words, \( Y_1 \subset Y \cap Y(1, g_2^{-1}) \) and \( Y_2 \subset Y \cap Y(g_1^{-1}, 1) \), so that \( Y = Y_1 + Y_2 \subset R'_1 \). Since \( R'_1 \) is isotropic, \( R'_1 = Y \) and \( W_{R'_2} = Y_1 / Y = Y / Y = \{0\} \).

We can now conclude, using the fact that \( f_1 \) and \( f_2 \) are homomorphisms and relation (6), that the mapping \( f \) (which satisfies \( f([g_1, \epsilon_1], [g_2, \epsilon_2]) = f_1([g_1, \epsilon_1]) f_2([g_2, \epsilon_2]) \)) is a homomorphism. To prove that this diagram is commutative, we have to see that

\[
f([h, \epsilon_1], [g, \epsilon]) = [(h, g), \epsilon_1 \epsilon(\det h, x(g)) F]
\]

for \( h \in GL(Y_1) \subset Sp(W_{s_1}) \), i.e., we have to see that \( c_{Rao}((h, 1), (1, g)) = (\det h, x(g)) F \). But, in the expression for the Rao’s cocycle (5), we have already calculated \( q, l \) and \( t \) in the calculation following (6). Also, \( x(h, 1) x(1, g) = x(h, g) \) since \((h, 1)\) belongs to Siegel parabolic of \( Sp(W_n) \). Hence,

\[
c_{Rao}((h, 1), (1, g)) = (x(h, 1), x(1, g)) F = (x(h), x(g)) F = (\det h, x(g)) F.
\]

In the diagram, \( Sp(W_{s_1}) \times Sp(W_{s_0}) \) is obviously an image of the homomorphism \( f \) in \( Sp(W_n) \).

We want to calculate \([w_0, 1][m, \epsilon][w_0, 1]^{-1}\), for \([m, \epsilon] \in M_{s_1}\) using the commutative diagram above, noting that this expression belongs to \( Sp(W_{s_1}) \times Sp(W_{s_0}) \).

Let \( m = f(h, g) \); then \([m, \epsilon] = \phi([h, x(g)] F, [g, \epsilon]) \). On the other hand, \([w_0, 1] = [\tilde{w}_0, 1, 1] = f([\tilde{w}_0, 1, 1]) \). We need to calculate

\[
f(([\tilde{w}_0, 1], [1, 1]) ([h, (\det h, x(g)) F], [g, \epsilon]) ([\tilde{w}_0, 1]^{-1}, [1, 1])) =
\]

\[
f(([\tilde{w}_0, 1][h, (\det h, x(g)) F][\tilde{w}_0, 1]^{-1}, [g, \epsilon]) .
\]

The calculation in the first component (\( Sp(W_{s_1}) \) part) is obtained in the Siegel case of the previous section. So, if we take \( h = \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} \), the last
expression is equal to
\[ \phi([\tilde{w}_0 h \tilde{w}_0^{-1}, (\det h, x(g))_F], [g, \epsilon]) = [(A^{-t}, g), \epsilon], \]
as expected.

The epimorphism \( \phi \) of the diagram above shows that we can view an irreducible representation \( \pi \) of \( \tilde{M}_s \) as an irreducible representation of \( \tilde{GL}(s_1, F) \times \tilde{Sp}(W_{s_1}) \), say \( \rho \otimes \sigma \), provided it is trivial on \( \text{Ker} \phi = \{(1, \epsilon), (1) : \epsilon \in \mu_2 \} \).

Now, if \( \chi_\rho = \chi_\sigma = 1 \), then we immediately get that \([w_0, 1](\rho \otimes \sigma) = \tilde{\rho} \otimes \sigma \), and if \( \chi_\rho = \chi_\sigma \neq 1 \), we have \([w_0, 1](\rho \otimes \sigma) = \alpha \tilde{\rho} \otimes \sigma \).

### 3.3 The geometric lemma

We briefly recollect the topology introduced on the covering groups \( \tilde{G} = GL(n, F) \) and \( \tilde{G} = Sp(W_n) \). Let \( \beta \) in both cases denotes the corresponding cocycle, i.e., Rao’s cocycle or the Hilbert symbol of the determinants. In both cases, \( \beta : G \times G \to \{1, -1\} \) is a continuous mapping with respect to the usual topology on \( G \) which makes it an \( l \)-group. We note that \( \beta(h, (\det h, x(g)) F), [g, \epsilon]) = [(h, g), \epsilon]. \)

To prescribe the topology on \( \tilde{G} \) it is enough to prescribe the system of neighbourhoods of the identity of the group \( \tilde{G} \), which is, as a set, given by \( G \times \mu_2 \). We prescribe that the system of neighbourhoods of the identity consists of the images of the neighborhoods of the identity in \( K \) under these splittings. These sets satisfy the usual conditions for the system of neighbourhoods, and they generate a unique topology on \( \tilde{G} \) (especially nice if residual characteristic is odd, cf. Lemme and Remarque II.10, p. 43 in [13], or [22]).

It is easily checked that \( \tilde{G} \) is now an Hausdorff, \( l \)-group; the mapping \( i \) from (3) is continuous and closed, and \( p \) is continuous and open.

Now we return to \( Sp(W_n) \). We defined the standard parabolic subgroups of \( \tilde{Sp}(W_n) \) as inverse images of standard parabolic subgroups of \( Sp(W_n) \); (we use the notation from the second section, the second subsection) and we have \( \tilde{P}_s = \tilde{M}_s \ltimes N'_s \). Here \( N'_s \) is the image of the homomorphism \( u \mapsto [u, 1] \). The subgroup \( N'_s \) is closed in \( Sp(W_n) \) and we identify Haar measures on \( N_s \) and \( N'_s \). The topological module \( \delta_{N'_s}(\tilde{m}) \) (for \( \tilde{m} \in \tilde{M}_s \)) is defined in the usual way.
where the case of the groups from the set we have \( m = p(\tilde{m}) \). In the future, we use \( \delta_{\tilde{P}} \) instead of \( \delta_{N_i} \). The groups \( Sp(W_n), \tilde{M}_s, N_i' \) and \( \tilde{P} \) satisfy the conditions of [3], Subsection 1.8 so that the functors of parabolic induction and Jacquet modules are defined. We denote them by \( Ind_{\tilde{P}}^{\tilde{Sp}(W_n)} \) (or \( i_{\tilde{Sp}(W_n),M_1} \)) and \( r_{\tilde{Sp}(W_n),P_1} \), respectively.

Now it is easy to that the additional conditions (1) to (4) and (*) for the general statement of the geometric lemma of the fifth section of [3] are satisfied. To be precise, let \( P_{s_1} \) and \( P_{s_2} \) be standard parabolic subgroups of \( Sp(W_n) \), where \( s_1 \) and \( s_2 \) are the sequences of positive integers, and \( |s_i| \leq n, i = 1, 2 \). Then \( N_{s_i}, i = 1, 2 \) is the limit of compact subgroups (since the induced topology there coincides with the original topology on \( N_{s_i}, i = 1, 2 \)).

Also, as topological spaces with the quotient topology, we have \( \tilde{P}_{s_1} \setminus \tilde{Sp}(W_n) \cong P_{s_1} \setminus Sp(W_n) \). The group \( \tilde{P}_{s_2} \) acts by right multiplication on \( \tilde{P}_{s_1} \setminus \tilde{Sp}(W_n) \) with a finite number of orbits, moreover

\[
\tilde{P}_{s_1} \setminus \tilde{Sp}(W_n)/\tilde{P}_{s_2} \cong P_{s_1} \setminus Sp(W_n)/P_{s_2} \quad (= W_{s_1} \setminus W/W_{s_2})
\]

this is easily verified. Here we use the notation for the Weyl groups from [3]. We choose a distinguished element in each class of \( W_{s_1} \setminus W/W_{s_2} \) from the set \( W_{M_1}^{s_1},M_{s_2}^{s_1},M_{s_2}^{s_1} \) ([3], p. 484). For \( w \) with this property, the groups \( [w,1]\tilde{P}_{s_1},[w,1]^{-1} \), \( [w,1]\tilde{M}_s, [w,1]^{-1} \) and \( [w,1](N_{s_1}')[w,1]^{-1} \) are decomposable with respect to the pair \( (M_{s_2}',N_{s_1}') \) so the property (4) is satisfied. In the case of the groups \( \tilde{P}_{s_1} \) and \( \tilde{M}_s \) this follows immediately, and for the group \( N_i \) it follows directly from the fact that \( c_{Ra_0}(x,u) = c_{Ra_0}(u,x) = 1 \) for every \( x \in Sp(W_n) \) and \( u \) in any unipotent radical of a standard parabolic subgroup. We conclude that the following geometric lemma holds:

**Theorem 3.3.** We keep the notation from above. The functor \( F = r_{Sp(W_n),M_{s_2}} \circ i_{Sp(W_n),M_{s_1}} : \text{Alg}_{\tilde{M}_1} \rightarrow \text{Alg}_{\tilde{M}_2} \) is glued from the functors \( F_w, w \in W_{M_1}^{s_1},M_{s_2}^{s_1} \), where \( F_w : \text{Alg}_{\tilde{M}_1} \rightarrow \text{Alg}_{\tilde{M}_2} \) is defined by \( F_w = i_{\tilde{M}_{s_2},L} \circ w \circ r_{\tilde{M}_{s_1},R} \). Here \( L = \tilde{M}_{s_2} \cap w(\tilde{M}_1) \) and \( R = \tilde{M}_1 \cap w^{-1}(\tilde{M}_{s_2}) \).
4 Classification of irreducible genuine representations of metaplectic groups; another form of the geometric lemma

We start with a description of irreducible representations of $\widetilde{GL(n, F)}$ and intertwining operators in that setting.

4.1 Representations of $\widetilde{GL(n, F)}$; intertwining operators for $GL(n, F)$

Let $H_{n_i} = \widetilde{GL(n_i, F)}$, $i = 1, 2$ be the two–sheeted coverings of general linear groups defined as before. By $\text{Irr}\, \widetilde{GL(n, F)}_{\text{gen}}$ we denote all the equivalence classes of irreducible smooth genuine representations of $\widetilde{GL(n, F)}$. This set is naturally equivalent to $\text{Irr}\, GL(n, F)$, the set of all the equivalence classes of irreducible smooth representations of $GL(n, F)$ through the multiplication with the character $\chi_{\psi}$. Analogously we define $\text{Irr}\, \widetilde{GL(n, F)}_{\text{non-gen}}$.

Let $\tilde{M}$ be the inverse image of a standard Levi subgroup $M \cong GL(n_1, F) \times GL(n_2, F)$ in $GL(n_1 + n_2, F)$ ($M = \{ \begin{bmatrix} g_1 & \epsilon_1 \\ g_2 & \epsilon_2 \end{bmatrix} : g_i \in GL(n_i, F), i = 1, 2 \}$). From the previous section, we know that there is an epimorphism $\phi : H_{n_1} \times H_{n_2} \rightarrow \tilde{M}$ given by

$$\phi([g_1, \epsilon_1], [g_2, \epsilon_2]) = [(g_1, g_2), \epsilon_1 \epsilon_2 (\det g_1, \det g_2)_F].$$

We obviously have $\text{Ker} \, \phi = \{ ([1, \epsilon], [1, \epsilon]) : \epsilon \in \mu_2 \}$. In this way, we can identify the irreducible representations of $\tilde{M}$ with the tensor product of irreducible representations of $H_{n_i}$, $i = 1, 2$, provided the restrictions of their central characters to $\{ (1, \epsilon) : \epsilon \in \mu_2 \}$ are the same.

We know that $w_0$, the longest element in the Weyl group modulo that of $M$ can be represented by $w_0 = \begin{bmatrix} 0 & I_{n_2} \\ I_{n_1} & 0 \end{bmatrix}$, and that $w_0 M w_0^{-1} = \{ \begin{bmatrix} g_2 \\ g_1 \end{bmatrix} : g_i \in GL(n_i, F), i = 1, 2 \}$. Now, let $\phi'$ be the analogously defined map $\phi' : H_{n_2} \times H_{n_1} \rightarrow M'$, where $M' = w_0 M w_0^{-1}$.

With $\tilde{w}_0 = [w_0, 1] \in H_{n_1 + n_2}$, we immediately get the following action of $\tilde{w}_0$ on $\tilde{M}$:

$$\tilde{w}_0 \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}, \epsilon \tilde{w}_0^{-1} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} w_0^{-1}, \epsilon.$$
We have just seen that the following diagram commutes

\[
\begin{array}{ccc}
H_{n_1} \times H_{n_2} & \xrightarrow{\phi} & \tilde{M} \\
\downarrow{j} & & \downarrow{\lambda} \\
H_{n_2} \times H_{n_1} & \xrightarrow{\phi'} & \tilde{M}',
\end{array}
\]

where \(j(x, y) = (y, x)\), and \(\lambda(\tilde{m}) = \tilde{w}_0 \tilde{m} \tilde{w}_0^{-1}\).

As usual, for an irreducible representation \(\pi\) of \(\tilde{M}\), we denote by \(\tilde{w}_0(\pi)\) the representation of \(\tilde{M}'\) defined by \(\tilde{w}_0(\pi)(\tilde{m}') = \pi(\tilde{w}_0^{-1}\tilde{m}'\tilde{w}_0)\). Since \(\phi'\) are epimorphisms, the pullback of an irreducible representation \(\pi\) via \(\phi\) is an irreducible representation of \(H_{n_1} \times H_{n_2}\), say \(\rho_1 \otimes \rho_2\), so \(\pi \circ \phi \simeq \rho_1 \otimes \rho_2\).

As before, \(\chi_{\rho_1} = \chi_{\rho_2}\), and the other way around; i.e., an irreducible representation \(\rho_1 \otimes \rho_2\) of \(H_{n_1} \times H_{n_2}\) defines an irreducible representation of \(\tilde{M}\) if and only if \(\chi_{\rho_1} = \chi_{\rho_2}\). From the diagram, we immediately have

\[
\tilde{w}_0(\pi) \circ \phi' \circ j \simeq \pi \circ \phi. \quad (7)
\]

We easily extend this situation to the non–maximal Levi subgroups of \(GL(n, F)\). Let \(M\) be a standard Levi subgroup of \(GL(n, F)\), corresponding to \((n_1, n_2, \ldots, n_k)\) (partition of \(n\)). Then, there is an epimorphism \(\phi : H_{n_1} \times \cdots \times H_{n_k} \to \tilde{M}\), given by

\[
\phi([h_1, \epsilon_1], \ldots, [h_k, \epsilon_k]) = \begin{bmatrix}
   h_1 & & \\
   & \ddots & \\
   & & h_k
\end{bmatrix}, \epsilon_1 \cdots \epsilon_k \prod_{i<j}(\det h_i, \det h_j)_F.
\]

Again, an irreducible representation \(\rho_1 \otimes \cdots \otimes \rho_k\) of \(H_{n_1} \times \cdots \times H_{n_k}\) defines an irreducible representation \(\pi\) of \(\tilde{M}\) provided \(\chi_{\rho_1} = \cdots = \chi_{\rho_k}\) and this is equal to \(\chi_{\pi}\).

Also, we immediately get that the restriction of the central character of \(\text{Ind}^{H_{n_v}}(\pi)\) to \([\{1, \epsilon \} : \epsilon \in \mu_2]\) is equal to \(\chi_{\pi}\).

We can pass from genuine to non–genuine representations of \(H_n\) using character \(\chi_\psi(g, \epsilon) = \epsilon \gamma(\det g, \psi(\frac{1}{2}))^{-1}\). We recall that \(\psi\) is a non–trivial additive character of \(F\). Let \(n = n_1 + n_2\) and \(M \cong GL(n_1, F) \times GL(n_2, F)\). Using the properties of the Weil index (Appendix to [15]), we obtain

\[
\chi_{\tilde{M}' \circ \phi} = \chi_\psi \otimes \chi_\psi,
\]

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where the characters $\chi_\psi$ on the right-hand side are viewed as characters of $H_{n_1}$ and $H_{n_2}$, respectively, and $\chi_\psi$ on the left-hand side denotes the character on $H_{n_1+n_2}$. Having this in mind, we can adopt Zelevinsky’s notation for parabolic induction in $H_n$. In more words, we denote $\rho_1 \times \rho_2 \times \cdots \times \rho_k = \text{Ind}_{P}^{H_n}(\pi)$, where $\pi \circ \phi = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k$. A direct consequence of this is the following corollary:

**Proposition 4.1.** Let $\rho_i$ be a representation of $H_{n_i}$, $i = 1, 2$ such that $\chi_{\rho_1} = \chi_{\rho_2}$. Then,

$$\chi_\psi(\rho_1 \times \rho_2) \cong \chi_{\psi \rho_1} \times \chi_{\psi \rho_2}.$$

**Proof.** Using the above mentioned properties of $\chi_\psi$, we obtain that the mapping $T : \chi_\psi(\rho_1 \times \rho_2) \to \chi_{\psi \rho_1} \times \chi_{\psi \rho_2}$ given by $(T f)(\tilde{g}) = \chi_\psi(\tilde{g}) f(\tilde{g})$ is an isomorphism. \qed

**Remark.** As we observed above, we can pass from genuine to non-genuine irreducible representations of $\widetilde{\text{GL}}(n, F)$ using a character $\chi_\psi$. The dependence of this transition on $\psi$ is given by the following commutative diagram

$$
\begin{array}{ccc}
\text{Irr}\widetilde{\text{GL}}(n, F)_{\text{gen}} & \xrightarrow{\text{id}} & \text{Irr}\widetilde{\text{GL}}(n, F)_{\text{gen}} \\
\downarrow T_\psi & & \downarrow T_{\psi'} \\
\text{IrrGL}(n, F) & \xrightarrow{T_{\psi, \psi'}} & \text{IrrGL}(n, F).
\end{array}
$$

If $\psi'$ is another non-trivial additive character of $F$, there exists $a \in F$ such that $\psi' = \psi_a$, where $\psi_a(x) = \psi(ax)$. $T_\psi$ is a mapping given by $T_\psi(\rho) = \chi_{\psi^{-1}} \rho$; $T_{\psi'}$ is defined analogously. Then, $T_{\psi, \psi'}$ is given by $[T_{\psi, \psi'}(\rho)](g) = (\det(g), a)_{F} \rho(g)$.

In this way, we can use known properties of the representations of general linear groups to obtain analogous properties for the covering groups. We just comment on the definitions of the discrete series representations, cuspidal representations and relation to segments as in [21], and Langlands’ classification.

We define an irreducible representation of $H_n$ to be square-integrable if it has a unitary central character and the matrix coefficients of that representation are square-integrable modulo (the full) center. The center of $H_n$ is (as a set) $Z(\text{GL}(n, F)) \times \mu_2$. If $\pi$ is an irreducible representation of $H_n$, and $\chi_{\pi} = 1$, then $\pi$ is square-integrable if and only if it is square-integrable as
representation of $GL(n, F)$ since $H_n/Z(GL(n, F)) \cong GL(n, F)/Z(GL(n, F))$ as topological groups, and if $\chi_\pi \neq 1$, then it is square–integrable if and only if $\chi_\psi \pi$ is square–integrable as a representation of $GL(n, F)$.

Cuspidal representations of $H_n$ are defined in the usual way, using the space of coinvariants ([4],[5]). It is straightforward that an irreducible representation $\rho$ of $H_n$ is cuspidal if and only if $\rho$ ($\chi_\psi \rho$, respectively) is cuspidal as a representation of $GL(n, F)$ if $\chi_\rho = 1$ (if $\chi_\rho \neq 1$, respectively).

We note that a similar characterization of non–genuine cuspidal representations of $\widetilde{Sp}(W_n)$ in terms of cuspidal representations of $Sp(W_n)$ also holds.

We recall that an irreducible essentially square–integrable representation $\pi$ of $GL(n, F)$ is attached to a segment of cuspidal representations ([21]), i.e., there exists an irreducible cuspidal representation $\rho$ of some $GL(k, F)$ such that $km = n$ and $\pi$ is a unique subrepresentation of

$$\rho \nu^{m-1} \times \rho \nu^{m-2} \times \cdots \times \rho \nu \times \rho,$$

(10)

where $\nu$ is a determinant character on $GL(k, F)$ composed with the absolute value on $F$.

From this discussion and Proposition 4.1, we have the following:

**Proposition 4.2.** 1. Let $\pi$ be an irreducible essentially square–integrable representation of $H_n$. Then, there exists a segment of cuspidal representations $\Delta = \{\rho, \rho \nu, \ldots, \rho \nu^{m-1}\}$ such that $\rho$ is an irreducible cuspidal representation of $H_k$, where $mk = n$, and $\pi$ is a unique subrepresentation of

$$\rho \nu^{m-1} \times \rho \nu^{m-2} \times \cdots \times \rho \nu \times \rho.$$  

We then denote $\pi = \delta(\Delta)$. Note that $\delta(\Delta)$ is genuine if only if $\rho$ is genuine.

2. The representation $\delta(\Delta_1) \times \delta(\Delta_2)$ of $H_{n_1+n_2}$, where $\delta(\Delta_i)$ is an essentially square–integrable representation of $H_{n_i}$, $i = 1, 2$, as defined above, is reducible if and only if the segments $\Delta_1$ and $\Delta_2$ are connected in the sense of Zelevinsky ([21])–the same definition applies.

3. **Langlands' classification**

For every irreducible essentially square–integrable representation $\delta$ of $H_n$, there exists a real number $e(\delta)$ and a (unitarizable) square–integrable
representation \( \delta_u \) of \( H_n \) such that \( \delta(g, \epsilon) = \delta_u(g, \epsilon)\nu(g) \epsilon(\delta) \) for all \((g, \epsilon) \in H_n\). For every irreducible representation \( \pi \) of \( H_n \), there exist essentially square–integrable representations \( \delta_i = \delta(\Delta_i) \), \( i = 1, \ldots, k \) of \( H_n \), \( i = 1, \ldots, k \) such that \( \epsilon(\delta_1) \geq \epsilon(\delta_2) \geq \cdots \geq \epsilon(\delta_k) \) and \( \pi \) is a unique quotient of \( \delta_1 \times \delta_2 \times \cdots \times \delta_k \). Given \( \pi \), the representations \( \delta_1, \ldots, \delta_k \) are unique (up to a permutation).

**Remark.** The discussion before the previous proposition shows that, for an irreducible genuine square–integrable \( \delta \) of \( H_n \), \( e(\delta) = e(\chi \psi \delta) \).

Let \( w_0 \) be an element of the Weyl group of \( GL(n, F) \) transferring Levi \( M \) to Levi \( M' \), as in the previous commutative diagram, and let \( \tilde{P} \) and \( \tilde{P}' \) be the corresponding standard parabolic subgroups. We have the following result on the existence of the intertwining operators:

**Proposition 4.3.** Let \( \pi \) be an irreducible representation of \( \tilde{M} \). Then, the following holds:

\[
\text{Hom}_{H_n}(\text{Ind}_{\tilde{P}}^{H_n}(\pi), \text{Ind}_{\tilde{P}'}^{H_n}(w_0(\pi))) \neq 0.
\]

**Proof.** We can rephrase (7), using the representations \( \rho_1 \) of \( H_{n_1} \) and \( \rho_2 \) of \( H_{n_2} \) such that \( \pi \circ \phi = \rho_1 \otimes \rho_2 \), as \( \tilde{w}_0(\rho_1 \otimes \rho_2) = \rho_2 \otimes \rho_1 \). If \( \chi_{\rho_1} = \chi_{\rho_2} = 1 \), the representation \( \pi \) factors through \( M \), and \( \text{Ind}_{\tilde{P}}^{H_n}(\pi) \) is a representation of \( GL(n, F) \); the same thing holds for \( \text{Ind}_{\tilde{P}'}^{H_n}(w_0(\pi)) \). In this situation, we can apply the Gelfand–Kazdhan arguments for general linear groups and irreducible representations \( \rho_i \) of \( GL(n_i, F) \), \( i = 1, 2 \):

\[
\text{Hom}_{GL(n_1+n_2,F)}(\rho_1 \times \rho_2, \rho_2 \times \rho_1) \neq 0.
\]

If \( \chi_{\rho_1} = \chi_{\rho_2} \neq 1 \), we just apply above statements to \( \chi_\psi \rho_1 \) and \( \chi_\psi \rho_2 \). We get

\[
0 \neq \text{Hom}_{GL(n,F)}(\chi_\psi \rho_1 \times \chi_\psi \rho_2, \chi_\psi \rho_2 \times \chi_\psi \rho_1) \cong \text{Hom}_{H_n}(\rho_1 \times \rho_2, \rho_2 \times \rho_1).
\]

The first isomorphism follows from Proposition 4.1.

In this situation (of \( H_n \), we can study the analytic properties of these intertwining operators using the results of [14].
We can, of course, extend the Zelevinsky notation for the parabolic induction in $\tilde{GL}(n, F)$ in the settings of non–maximal parabolic subgroups, as in the relation (8). Since parabolic induction preserves the genuine representations, we can prove that the transitivity of induction holds in the same way for $GL(n, F)$ as for $GL(n, F)$, using Proposition 4.1 and diagram (9).

4.2 Representations of $\tilde{Sp}(W_n)$; intertwining operators for $\tilde{Sp}(W_n)$

In this section, we prove a classification result for the representations of the metaplectic group $\tilde{Sp}(W_n)$. It is quite analogous to the purely algebraic, Zelevinsky type classification of the irreducible representations of the classical groups obtained in [9]. To obtain this classification, we need a structure formula for the Jacquet modules of the metaplectic groups as comodules for the representations of the double–cover of $GL(n, F)$, which we studied in the previous section. This is another expression for the geometric lemma, analogous in spirit to Tadić’s result on classical groups.

**Proposition 4.4.** For an irreducible representation $\sigma$ of $\tilde{Sp}(W_n)$ there exists an irreducible cuspidal representation $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k \otimes \rho$ of some $\tilde{M}_s$, where $s = (n_1, n_2, \ldots, n_k)$, $\rho_i$ is a representation of $H_{n_i}$, $i = 1, \ldots, k$ and $\rho$ of $Sp(W_{n\mid s})$ such that

$$\sigma \hookrightarrow \rho_1 \times \rho_2 \times \cdots \times \rho_k \times \rho.$$ 

**Proof.** The proof is the same as for the reductive algebraic groups (cf. [5], Theorem 5.1.2); it uses an induction over $n$, provided that the following fact is proved: if $\pi$ is an irreducible admissible representation of $Sp(W_n)$, every Jacquet module $r_{\tilde{Sp}(W_n), \tilde{M}_s}(\pi)$ is admissible (and finitely generated). This follows from Section 3.3 of [5], having in mind that there exists an open compact subgroup of $Sp(W_n)$ which splits in $\tilde{Sp}(W_n)$. 

Now we prove a very important formula (a version of a geometric lemma) which is a basis for our calculations with Jacquet modules of the metaplectic group. Tadić proved it in the context of classical groups (more precisely, for the symplectic and split odd-orthogonal groups ([19]); there are some other
generalizations ([1],[7],[12]). For a representation $\sigma$ of the group $\widetilde{Sp(W_n)}$, we denote by $r_k(\sigma)$ the normalized Jacquet module of $\sigma$ with respect to the maximal standard Levi subgroup $\overline{M(k)}$, i.e., $r_{\widetilde{Sp(W_n)},\overline{M(k)}}(\sigma)$. Let $R(\widetilde{Sp(W_n)})_{\text{gen}}$ denote the Grothendieck group of the category of smooth finite length genuine representations of $\widetilde{Sp(W_n)}$. Analogously we define

$$R(\widetilde{Sp(W_n)})_{\text{non-gen}}, R(\widetilde{GL(n,F)})_{\text{gen}}, R(\widetilde{GL(n,F)})_{\text{non-gen}},$$

and then

$$R^\text{gen}_1 = \oplus_n R(\widetilde{Sp(W_n)})_{\text{gen}}, R^\text{non-gen}_1 = \oplus_n R(\widetilde{Sp(W_n)})_{\text{non-gen}},$$

$$R^\text{gen}_n = \oplus_n R(\widetilde{GL(n,F)})_{\text{gen}}, R^\text{non-gen}_n = \oplus_n R(\widetilde{GL(n,F)})_{\text{non-gen}}.$$  

We denote by $m$ the linear extension to $R^\text{gen} \otimes R^\text{gen}$ ($R^\text{non-gen} \otimes R^\text{non-gen}$, respectively) of the parabolic induction (from a maximal parabolic subgroup). We can easily check that if $\sigma$ is an irreducible genuine representation of $\widetilde{Sp(W_n)}$, then $r_k(\sigma)$ is a genuine representation of $\overline{M(k)}$ and, as such, can be interpreted as a (genuine) representation of $\widetilde{GL(k,F)} \times \widetilde{Sp(W_{n-k})}$, i.e., as an element of $R^\text{gen} \otimes R^\text{gen}_1$. So for irreducible genuine $\sigma$ we can introduce $\mu^*(\sigma) \in R^\text{gen} \otimes R^\text{gen}_1$ by

$$\mu^*(\sigma) = \sum_{k=0}^{n} \text{s.s.}(r_k(\sigma)),$$

where s.s. stands for the semisimplification. We can extend $\mu^*$ linearly to the whole $R^\text{gen}_1$. We have the analogous construction for a non–genuine irreducible $\sigma$. Analogously, using Jacquet modules for the maximal parabolic subgroups of $\widetilde{GL(n,F)}$ we can define $m^*(\pi) = \sum_{k=0}^{n} \text{s.s.}(r_k(\pi)) \in R^\text{gen} \otimes R^\text{gen}$, for a genuine, irreducible representation $\pi$ of $\widetilde{GL(n,F)}$ and then extend $m^*$ linearly to the whole $R^\text{gen}$. We have the same procedure for a non–genuine $\pi$, but then we consider $m^*(\pi) \in R^\text{non-gen} \otimes R^\text{non-gen}$.

Let $\kappa : R^\text{gen} \otimes R^\text{gen} \rightarrow R^\text{gen} \otimes R^\text{gen}$ ($\kappa : R^\text{non-gen} \otimes R^\text{non-gen} \rightarrow R^\text{non-gen} \otimes R^\text{non-gen}$, respectively) be defined by $\kappa(x \otimes y) = y \otimes x$. We extend the contragredient $\sim$ to an automorphism of $R^\text{gen}$ ($R^\text{non-gen}$, respectively) in a natural way. Finally, we define:

$$M^* = (m \otimes \text{id}) \circ (\alpha \otimes m^*) \circ \kappa \circ m^*.$$
Here $\sim\alpha$ means taking contragredient of a representation, and then multiplying by the character $\alpha$, acting on the general linear group as $\alpha(g) = (\det g, -1)_F$.

We have the following

**Proposition 4.5.** For $\pi$ in $R^{\text{gen}}$ and $\sigma$ from $R_1^{\text{gen}}$, the following holds

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma).$$

**Proof.** The proof can be adapted from ([19]) in the following way: the representatives of the Weyl group can be taken of the form $[w, 1]$ where $w$ is Weyl group element from $Sp(W_n)$ and then we use the considerations of Section 3.2. and Section 3.3. Essentially, we have to check the action of the appropriate representatives of the elements in the Weyl group on the representations, to see if the structure formula from the claim of this proposition holds. We, essentially, have to check how the element $q_n(d, k)_{i_1, i_2}$ (defined just before Lemma 4.5 of [19]) acts on the appropriate representation (described before Lemma 5.1 of [19]). Now we use ([18]), Lemma 2.1.2 to see that the action of $q_n(d, k)_{i_1, i_2}$ can be decomposed in the generalized rank one cases of our Sections 3.2 and 4.1. We see that action of $q_n(d, k)_{i_1, i_2}$ is only on the symplectic group–part of the element of the appropriate Levi subgroup of the metaplectic group. If we are dealing with genuine representations, from p. 23 and Lemma 5.4 of ([19]), we see that we only have to change $\sim$ to $\sim\alpha$ in the definition of the function $M^*$, according to our results in Section 3.2. \hfill \Box

**Remark.** If $\pi \in R^{\text{non-gen}}$ and $R_1^{\text{non-gen}}$, then the proposition above reduces precisely to Theorem 5.4 in [19] about representations of symplectic groups. We just note that, in that case, there is no $\alpha$ in the expression for $M^*$.

The previous proposition enables us to prove the classification of irreducible genuine representations of $\widetilde{Sp}(W_n)$ analogous to the classification of the irreducible smooth representations of symplectic groups, as stated in ([9]).

The classification there is stated in terms of so-called negative (and strongly negative) representations. The notion of negative representation extends directly to genuine representations of $Sp(W_n)$.

Let $\Delta$ be a segment of genuine cuspidal representations, and let $\delta(\Delta)$ be as in Proposition 4.2. We define $e(\Delta) = e(\chi_{\psi}^{-1}\delta(\Delta))$ (as for the representations of general linear groups). For a segment $[\rho, \rho\nu, \ldots, \rho\nu^k] = \Delta$ of cuspidal
representations and $k \in \mathbb{Z}_{\geq 0}$, let $[[\rho, \rho_1, \ldots, \rho_k]] = \langle \Delta \rangle$ denote Zelevinsky’s segment representation ([21]) (i.e., $L(\rho_1, \ldots, \rho_k)$).

**Definition 4.1.** Let $\sigma \in \text{Irr}(\widetilde{Sp(W_n)})_{\text{gen}}$. Then $\sigma$ is a negative representation if and only if for every embedding $\sigma \hookrightarrow \rho_1 \times \rho_2 \times \cdots \times \rho_t \rtimes \sigma_{\text{sc}}$, where $\rho_i$, $i = 1, \ldots, t$ is irreducible genuine supercuspidal representation of $GL(m_{\rho_i}, F)$ (this defines $m_{\rho_i}$) and $\sigma_{\text{sc}}$ is irreducible supercuspidal genuine representation of some $\widetilde{Sp(W_n')}$, we have the following:

$$e(\rho_1)m_{\rho_1} \leq 0,$$

$$e(\rho_1)m_{\rho_1} + e(\rho_2)m_{\rho_2} \leq 0,$$

$$\vdots$$

$$e(\rho_1)m_{\rho_1} + e(\rho_2)m_{\rho_2} + \cdots + e(\rho_t)m_{\rho_t} \leq 0.$$

The following two theorems hold:

**Theorem 4.6.** Suppose that $\Delta_1, \ldots, \Delta_k$ is a sequence of segments satisfying $e(\Delta_1) \geq \cdots \geq e(\Delta_k) > 0$. (We allow the empty sequence here: in this case $k = 0$.) Let $\sigma_{\text{neg}}$ be a negative representation. Then we have the following:

(i) The induced representation $\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \rtimes \sigma_{\text{neg}}$ has a unique irreducible subrepresentation (we call it the Zelevinsky subrepresentation); we will denote it by $\langle \Delta_1, \ldots, \Delta_k; \sigma_{\text{neg}} \rangle$.

(ii) We have $\langle \Delta_1, \ldots, \Delta_k; \sigma_{\text{neg}} \rangle \hookrightarrow \langle \Delta_1 \rangle \rtimes \langle \Delta_2, \ldots, \Delta_k; \sigma_{\text{neg}} \rangle$.

(iii) The representation $\langle \Delta_1, \ldots, \Delta_k; \sigma_{\text{neg}} \rangle$ appears with the multiplicity one in the composition series of $\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_k \rangle \rtimes \sigma_{\text{neg}}$.

(iv) The representation $\langle \Delta_1, \ldots, \Delta_k; \sigma_{\text{neg}} \rangle$ is negative if and only if $k = 0$; in that case $\langle \emptyset; \sigma_{\text{neg}} \rangle \simeq \sigma_{\text{neg}}$.

(v) The induced representation $\langle \widetilde{\Delta_1} \rangle \times \langle \widetilde{\Delta_2} \rangle \times \cdots \times \langle \widetilde{\Delta_k} \rangle \rtimes \sigma_{\text{neg}}$ has a unique maximal proper subrepresentation; the corresponding quotient is $\langle \Delta_1, \ldots, \Delta_k; \sigma_{\text{neg}} \rangle$.

(vi) $\langle \Delta_1, \ldots, \Delta_k; \sigma_{\text{neg}} \rangle \simeq \langle \Delta_1', \ldots, \Delta_k'; \sigma'_{\text{neg}} \rangle$ if and only if $(\Delta_1, \ldots, \Delta_k)$ is a permutation of $(\Delta_1', \ldots, \Delta_k')$ and $\sigma_{\text{neg}} \simeq \sigma'_{\text{neg}}$. 

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Finally, we have the following theorem which ends the classification of irreducible genuine representations in terms of negative representations:

**Theorem 4.7.** If $\sigma \in \text{Irr}(\widetilde{\text{Sp}}(W_n))_{\text{gen}}$, then there exists a sequence of segments $\Delta_1, \ldots, \Delta_k$ satisfying $e(\Delta_1) \geq \ldots \geq e(\Delta_k) > 0$ and a negative representation $\sigma_{\text{neg}}$ such that $\sigma \simeq \langle \Delta_1, \ldots, \Delta_k; \sigma_{\text{neg}} \rangle$.

**Proof.** The proofs of the previous two classification theorems follow the same lines as in [9], as long as we have the following ingredients: structure formula of Tadić for the calculations with Jacquet modules (we have a slightly modified version of it in Proposition 4.5), calculation of $m^*(\delta(\Delta))$ (quite analogous to the case of general linear groups, due to Proposition 4.2). We can also prove that, in the Grothendieck group of the smooth, finite length representations of $\widetilde{\text{Sp}}(W_{n+k})$ the following holds: for $\pi \in \text{Irr}(\widetilde{\text{GL}}(k, F))$, and $\sigma \in \text{Irr}(\widetilde{\text{Sp}}(W_n))_{\text{gen}}$ we have

$$\pi \times \sigma = \overline{\pi} \alpha \times \sigma. \quad (11)$$

We can prove this relation using geometric construction of intertwining operators by Muić ([14]) relying on Bernstein’s continuation principle which carries over to metaplectic groups in a direct way. \qed

**Remark.** Having all the ingredients at hand, as shown above, we can extend Theorem 9.4 of ([9]) (about factorization of the long intertwining operator) to the metaplectic case.

## 5 Jacquet modules of even and odd Weil representations

As an application of our formula for the calculation of Jacquet modules (Proposition 4.5), we explicitly calculate Jacquet modules for even and odd Weil representations. For that, we will introduce some basic notation related to theta correspondence.

The pair $(\text{Sp}(W_n), O(V_r))$ constitutes a dual pair in $\text{Sp}(W_{n \cdot \dim V_r}) ([10],[11])$. Here $r$ denotes the Witt index of a quadratic space $V_r$. When $\dim(V_r)$ is odd, the group $\text{Sp}(W_n)$ does not split in $\widetilde{\text{Sp}}(W_{n \cdot \dim V_r})$, so the theta correspondence relates the representations of $\text{Sp}(W_n)$ and $O(V_r)$, or more generally,
the representations of the metaplectic groups (as two–fold coverings of symplectic groups attached to the symplectic tower) with the representations of the orthogonal groups attached to the orthogonal tower (Section 5 of [10]).

For a fixed additive, non–trivial character \( \psi \) of \( F \) related to \( \Theta \) corresponds to \( \chi_{\psi} \) on \( GL(n, F) \) by \( \chi_{\psi}(g, \epsilon) = (\chi_{\psi}(g \epsilon \psi^{-1})(\det g)\epsilon \gamma(\det g, \psi_2^{-1})^{-1} \) (Proposition 4.3, p. 37 of [10]). Here \( \chi_{\psi} \) is related to a quadratic form on \( V \).

Let \( \pi \) with respect to the unipotent radical of \( V \). We again denote \( \chi = \chi_{\psi}^2 \). \( \alpha \) is a quadratic character on \( GL(n, F) \).

Let \( (Sp(W_n), O(V_\ell)) \) be a reducible dual pair in \( Sp(W_{n, \dim V_\ell}) \); let \( n' = n \cdot \dim V_\ell \) (with \( \dim V_\ell \) odd). Let \( \omega_{\psi, \psi} \) be the Weil representation of \( Sp(W_{n'}) \) depending on the non–trivial additive character \( \psi \) (\( [10], [11] \), and let \( \omega_{\psi, \psi} = \omega_{\psi, \psi} \) be the pull–back of that representation to the pair \( (Sp(W_n), O(V_\ell)) \). For an irreducible genuine smooth representation \( \pi_1 \) of \( Sp(W_{n_1}) \), let \( \Theta(\pi_1, l) \) be a smooth representation of \( O(V_\ell) \), given as the full lift of \( \pi_1 \) to the \( l \)–level of the orthogonal tower. This is the biggest quotient of \( \omega_{\pi_1, l} \) on which \( Sp(W_{n_1}) \) acts as a multiple of \( \pi_1 \) and is given as \( \pi_1 \otimes \Theta(\pi_1, l) \), as a representation of \( Sp(W_{n_1}) \times O(V_\ell) \) (\([10] \), p. 33, \([13] \), p. 45).

As in the Preliminaries section, for \( s = (n_1, n_2, \ldots, n_k) \) with \( n_1 + n_2 + \cdots + n_k \leq n \) we denote by \( \tilde{M}_s \) the pre–image of the corresponding Levi subgroup of \( Sp(W_{n_k}) \) in \( Sp(W_n) \). As with a maximal parabolic subgroup we denote a normalized Jacquet module of a smooth finite length representation \( \pi \) with respect to the unipotent radical of \( P_\ell \) by \( r_\ell(\pi) \). So, an irreducible genuine representation of a minimal standard Levi subgroup of \( Sp(W_{n_k}) \) will be of the form \( \chi_{\psi, \psi} \chi_1 \otimes \cdots \otimes \chi_{\psi, \psi} \chi_n \otimes \omega_0 \), where \( \chi_s \) are characters of \( F^* \), and \( \omega_0 \) is a non-trivial character of \( \mu_2 \), viewed as a genuine representation of \( Sp(W_0) \cong \mu_2 \).

We follow ([10], p. 88) to introduce even and odd Weil representations. Let \( V_0 \) be the field \( F \) with the quadratic form given by \( x \mapsto x^2 \). Then \( \chi_V = 1 \). We continue to keep a non–degenerate additive character \( \psi \) of \( F \) fixed. Then, we observe the occurrence of the representation \( 1_{O(V_0)} \) in the associated metaplectic tower. Since we may consider \( 1_{O(V_0)} \) cuspidal, \( \Theta(1_{O(V_0)}, W_n) \) is an irreducible representation, which we denote by \( \omega_{\psi, \psi}^+ \) and call it the even Weil representation of \( Sp(W_n \otimes V_0) \cong Sp(W_n) \).

In the same way \( sgn_{O(V_0)} \) is a cuspidal representation of \( O(V_0) \). Observe that \( 1_{O(V_0)} \) appears at the “zeroth–level” in the metaplectic tower, i.e., we
The representation $\Theta(sgnO_{V_0}, W_n)$ is an irreducible representation of $\widetilde{Sp}(W_n)$, called the odd Weil representation and denoted by $\omega_{V,n}^-$. Observe that $sgnO_{V_0}$ appears for the first time at the first level of the metaplectic tower, and $\Theta(sgnO_{V_0}, W_1)$ is, accordingly, a cuspidal representation of $\widetilde{Sp}(W_1)$.

**Theorem 5.1.** 1. The even Weil representation embeds as

$$\omega_{V,n}^+ \hookrightarrow \chi_{V,\psi}^{-\frac{n+1}{2}}1_{GL(n, F)} \rtimes \omega_0.$$  

Moreover,

$$r_{(1,1,\ldots,1)}(\omega_{V,n}^+) = \chi_{V,\psi}^{-n+\frac{1}{2}} \otimes \chi_{V,\psi}^{-n+\frac{3}{2}} \otimes \cdots \otimes \chi_{V,\psi}^{-\frac{1}{2}} \otimes \omega_0,$$

and for $1 \leq k \leq n$,

$$r_k(\omega_{V,n}^+) = \chi_{V,\psi}(\nu^{k-n-\frac{1}{2}}) \otimes \omega_{V,n-k}^+.$$  

2. The odd Weil representation embeds as

$$\omega_{V,n}^- \hookrightarrow \chi_{V,\psi}^{-\frac{n-1}{2}}1_{GL(n-1, F)} \rtimes \Theta(sgnO_{V_0}, W_1).$$  

Moreover,

$$r_{(1,1,\ldots,1)}(\omega_{V,n}^-) = \chi_{V,\psi}^{-n+\frac{1}{2}} \otimes \chi_{V,\psi}^{-n+\frac{3}{2}} \otimes \cdots \otimes \chi_{V,\psi}^{-\frac{1}{2}} \otimes \Theta(sgnO_{V_0}, W_1),$$

and for $1 \leq k \leq n-1$, we have

$$r_k(\omega_{V,n}^-) = \chi_{V,\psi}(\nu^{k-n-\frac{1}{2}}) \otimes \omega_{V,n-k}^-.$$  

**Proof.** We first prove the claims about the even Weil representation. The embedding $\omega_{V,n}^+ \hookrightarrow \chi_{V,\psi}^{-\frac{n+1}{2}}1_{GL(n, F)} \rtimes \omega_0$ is known (for example, Example 5.4, p. 52 of [10]). Since

$$1_{GL(n, F)} \hookrightarrow \nu^{-\frac{n-1}{2}} \times \nu^{-\frac{n-3}{2}} \times \cdots \times \nu^{-\frac{n-3}{2}},$$

we have

$$\omega_{V,n}^+ \hookrightarrow \chi_{V,\psi}^{-n+\frac{1}{2}} \times \chi_{V,\psi}^{-n+\frac{3}{2}} \times \cdots \times \chi_{V,\psi}^{-\frac{1}{2}} \rtimes \omega_0.$$
First, we note that the representation \( \tau := \chi_{V,\psi} \nu^{-n+\frac{1}{2}} \times \chi_{V,\psi} \nu^{-n+\frac{3}{2}} \times \cdots \times \chi_{V,\psi} \nu^{-\frac{1}{2}} \times \omega_0 \) is regular, in the sense of ([20]) (we use ideas from this paper to work with Jacquet modules in the present context). More precisely, an irreducible representation \( \tau'' \) is regular if every Jacquet module of \( \tau'' \) is multiplicity one representation. Indeed, using the formula of Proposition 4.5, knowledge of Jacquet modules of \( \text{ind} \), and the fact that Weyl group of \( Sp(W_n) \) is the same as the Weyl group of \( Sp(W_n) \), we get that all the irreducible subquotients of \( r_{(1,1,\ldots,1)}(\tau) \) are of the form \( \chi_{V,\psi}^{\kappa_1 \nu^{e_1}} \otimes \chi_{V,\psi}^{\kappa_2 \nu^{e_2}} \otimes \cdots \otimes \chi_{V,\psi}^{\kappa_n \nu^{e_n}} \otimes \omega_0 \), where \( (e_0, \ldots, e_{n-1}) \in \{1,-1\}^n \) and \( p \) runs through all the permutations of the set \( \{0,1,2,\ldots,n-1\} \). All these subquotients are mutually different, so this induced representation, as well as \( \omega_{\psi,n}^+ \) is regular. To clarify this argument for the readers not accustomed to working with the extended parabolic induction we will explicitly calculate \( M^* (\chi_{V,\psi} \nu^{-n+\frac{1}{2}}) \). We have: 
\[ m^* (\chi_{V,\psi} \nu^{-n+\frac{1}{2}}) = \chi_{V,\psi} \nu^{-n+\frac{1}{2}} \otimes \omega_0 + \omega_0 \otimes \chi_{V,\psi} \nu^{-n+\frac{1}{2}}. \]

Then, 
\[ (\alpha \otimes m^*) \circ (\alpha \otimes m^*) \circ (\alpha \otimes m^*) = \chi_{V,\psi}^{-n+\frac{3}{2}} \chi_{V,\psi}^{\kappa_2} \otimes \omega_0 \otimes \omega_0 + \omega_0 \otimes \chi_{V,\psi} \nu^{-n+\frac{1}{2}} \otimes \omega_0, \]
so that 
\[ (m \otimes id) \circ (\alpha \otimes m^*) \circ (\alpha \otimes m^*) = \chi_{V,\psi}^{-n+\frac{3}{2}} \otimes \omega_0 + \chi_{V,\psi}^{\kappa_2} \nu^{-\frac{1}{2}} \otimes \omega_0 + \omega_0 \otimes \chi_{V,\psi} \nu^{-n+\frac{1}{2}}. \]

Let \( \pi \) be an irreducible subrepresentation of \( \tau \). Then, using Frobenius reciprocity, we get 
\[ 0 \neq \text{Hom}(\pi, \chi_{V,\psi} \nu^{-n+\frac{1}{2}} \times \chi_{V,\psi} \nu^{-n+\frac{3}{2}} \times \cdots \times \chi_{V,\psi} \nu^{-\frac{1}{2}} \times \omega_0) \cong \text{Hom}(r_{(1,1,\ldots,1)}(\pi), \chi_{V,\psi} \nu^{-n+\frac{1}{2}} \otimes \chi_{V,\psi} \nu^{-n+\frac{3}{2}} \otimes \cdots \otimes \chi_{V,\psi} \nu^{-\frac{1}{2}} \otimes \omega_0). \]

Since the multiplicity of \( \chi_{V,\psi} \nu^{-n+\frac{1}{2}} \otimes \chi_{V,\psi} \nu^{-n+\frac{3}{2}} \otimes \cdots \otimes \chi_{V,\psi} \nu^{-\frac{1}{2}} \otimes \omega_0 \) in \( r_{(1,1,\ldots,1)}(\tau) \) is equal to one, this subrepresentation \( \pi \) is unique. Our embedding gives \( \pi = \omega_{\psi,n}^+ \). Also, note that \( \omega_{\psi,n}^+ \) is the unique irreducible subquotient.
of $\tau$ which has $\chi_{V,\psi}\nu^{-n+\frac{1}{2}} \otimes \chi_{V,\psi}\nu^{-n+\frac{3}{2}} \otimes \cdots \otimes \chi_{V,\psi}\nu^{-\frac{1}{2}} \otimes \omega_0$ as a subquotient in its Jacquet module $r_{(1,1,\ldots,1)}(\omega_{\psi,n})$. Since $\chi_{V,\psi}\nu^{-n+\frac{1}{2}} \otimes \omega_{\psi,n-1}$ is a subrepresentation of $\chi_{V,\psi}\nu^{-n+\frac{1}{2}} \times \chi_{V,\psi}\nu^{-n+\frac{3}{2}} \times \cdots \times \chi_{V,\psi}\nu^{-\frac{1}{2}} \times \omega_0$, from the uniqueness of the irreducible subrepresentation of $\tau$ we get that

$$\omega^+_{\psi,n} \hookrightarrow \chi_{V,\psi}\nu^{-n+\frac{1}{2}} \otimes \omega^+_{\psi,n-1}. \quad (12)$$

In the same way,

$$\omega^+_{\psi,n} \hookrightarrow \langle \chi_{V,\psi}\nu^{-n+\frac{1}{2}}, \chi_{V,\psi}\nu^{-n+\frac{3}{2}} \rangle \otimes \omega^+_{\psi,n-2}. \quad (13)$$

Now, we have to prove that, actually, $r_{(1,1,\ldots,1)}(\omega^+_{\psi,n}) = \chi_{V,\psi}\nu^{-n+\frac{1}{2}} \otimes \chi_{V,\psi}\nu^{-n+\frac{3}{2}} \otimes \cdots \otimes \chi_{V,\psi}\nu^{-\frac{1}{2}} \otimes \omega_0$ and $r_k(\omega^+_{\psi,n}) = \chi_{V,\psi}\langle \nu^{-n+\frac{1}{2}}, \ldots, \nu^{k-n-\frac{1}{2}} \rangle \otimes \omega^+_{\psi,n-k}$. We accomplish that using induction on $n$. If $n = 1$, we know that the representation $\chi_{V,\psi}\nu^{-\frac{1}{2}} \otimes \omega_0$ reduces ([10], p. 89, [8]), is of length two, and $\omega^+_{\psi,1} \hookrightarrow \chi_{V,\psi}\nu^{-\frac{1}{2}} \otimes \omega_0$. Calculation of Jacquet modules gives $r_1(\omega^+_{\psi,1}) = \chi_{V,\psi}\nu^{-\frac{1}{2}} \otimes \omega_0$. We assume that $r_{(1,1,\ldots,1)}(\omega^+_{\psi,k}) = \chi_{V,\psi}\nu^{-k+\frac{1}{2}} \otimes \chi_{V,\psi}\nu^{-k+\frac{3}{2}} \otimes \cdots \otimes \chi_{V,\psi}\nu^{-\frac{1}{2}} \otimes \omega_0$, for every $k \leq n$ and $r_m(\omega^+_{\psi,k}) = \chi_{V,\psi}\langle \nu^{-k+\frac{1}{2}}, \ldots, \nu^{m-k-\frac{1}{2}} \rangle \otimes \omega^+_{\psi,k-m}$, for every $m \leq k \leq n$. Then, the representations on the right-hand sides of (12) and (13), (but replacing $n$ by $n + 1$) are denoted by $\pi_1$ and $\pi_2$, respectively. Now, we calculate $r_{n+1}(\pi_i)$, $i = 1, 2$. Using the formula from Proposition 4.5, we get

$$r_{n+1}(\pi_1) = \chi_{V,\psi}\langle \nu^{n+\frac{1}{2}} \times \langle \nu^{-n+\frac{1}{2}}, \ldots, \nu^{-\frac{1}{2}} \rangle \rangle \otimes \omega_0 + \chi_{V,\psi}\langle \nu^{-n-\frac{1}{2}} \times \langle \nu^{-n+\frac{1}{2}}, \ldots, \nu^{-\frac{1}{2}} \rangle \rangle \otimes \omega_0.$$ 

We also get

$$r_{n+1}(\pi_2) = \chi_{V,\psi}\langle \nu^{n+\frac{1}{2}} \times \nu^{-n-\frac{1}{2}} \rangle \times \chi_{V,\psi}\langle \nu^{-n+\frac{1}{2}}, \ldots, \nu^{-\frac{1}{2}} \rangle \otimes \omega_0 + \chi_{V,\psi}\langle \nu^{-n-\frac{1}{2}} \times \nu^{n+\frac{1}{2}} \rangle \times \chi_{V,\psi}\langle \nu^{-n+\frac{1}{2}}, \ldots, \nu^{-\frac{1}{2}} \rangle \otimes \omega_0 + \chi_{V,\psi}\langle \nu^{-n-\frac{1}{2}}, \nu^{n+\frac{1}{2}} \rangle \times \chi_{V,\psi}\langle \nu^{-n+\frac{1}{2}}, \ldots, \nu^{-\frac{1}{2}} \rangle \otimes \omega_0.$$ 

Note that $\pi_1$ and $\pi_2$ are both subrepresentations of $\tau' = \chi_{V,\psi}\nu^{-n-\frac{1}{2}} \otimes \chi_{V,\psi}\nu^{-n+\frac{1}{2}} \times \cdots \times \chi_{V,\psi}\nu^{-\frac{1}{2}} \otimes \omega_0$. Also, observe that the first subquotient of $r_{n+1}(\pi_1)$ and first two subquotients of $r_{n+1}(\pi_2)$ are mutually non-isomorphic irreducible representations. Observe that the second subquotient of $r_{n+1}(\pi_1)$ and the third subquotient of $r_{n+1}(\pi_2)$ have a common irreducible subquotient,
namely $\chi_{V,\psi}\langle[n^{-n+\frac{1}{2}}, \ldots, n^{-n+\frac{1}{2}}] \rangle \otimes \omega_0$. Since $\tau'$ is a regular representation, $r_{n+1}(\tau')$ is a multiplicity free representation, $r_{n+1}(\pi_1) \leq r_{n+1}(\pi_i)$, $i = 1, 2$, the common irreducible subquotient of $r_{n+1}(\pi_1)$ and $r_{n+1}(\pi_2)$ comes from the Jacquet module of a common irreducible subquotient of $\pi_1$ and $\pi_2$; call it $\pi$. We conclude that $r_{n+1}(\pi) = \chi_{V,\psi}\langle[n^{-n+\frac{1}{2}}, \ldots, n^{-n+\frac{1}{2}}] \rangle \otimes \omega_0$. This forces $r_{(1,1,\ldots,1)}(\pi) = \chi_{V,\psi}\nu^{-n+\frac{1}{2}} \otimes \nu^{-n+\frac{1}{2}} \otimes \ldots \otimes \nu^{-\frac{1}{2}} \otimes \omega_0$. But, we know that the only subquotient of $\tau'$ having that term in it’s Jacquet module is $\omega_{\psi,n+1}$. So $r_{n+1}(\omega_{\psi,n+1}) = \chi_{V,\psi}\langle[n^{-n+\frac{1}{2}}, \ldots, n^{-n+\frac{1}{2}}] \rangle \otimes \omega_0$. Now, applying the formula for $m^*(\chi_{V,\psi}\langle[n^{-n+\frac{1}{2}}, \ldots, n^{-n+\frac{1}{2}}] \rangle)$ (Proposition 3.4 of [21]), the rest of the claims follow.

To prove the analogous claims for $\omega_{-\psi,n}$, we just have to prove that we are in a similar situation as for the even Weil representation; namely we have to prove that the representation $\chi_{V,\psi}\nu^s \rtimes \Theta(sgn_{\nu_0}, W_1)$ reduces (that is the claim in the case $n = 2$). This is proved in ([8]), moreover $s = \pm \frac{3}{2}$ are the only real points of reducibility of the representations $\chi_{V,\psi}\nu^s \rtimes \Theta(sgn_{\nu_0}, W_1)$.

\[\square\]

References


