ON THE CUSPIDAL MODULAR FORMS FOR THE FUCHSIAN GROUPS OF
THE FIRST KIND

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Abstract. In this paper we study the construction and non–vanishing of cuspidal modular forms
of weight $m \geq 3$ for arbitrary Fuchsian groups of the first kind. We give a spanning set for the
space of cuspidal modular forms $S_m(\Gamma)$ of weight $m \geq 3$ in a uniform way which does not depend
on the fact that $\Gamma$ has cusps or not.

1. Introduction

In this paper we study the construction and non–vanishing of cuspidal modular forms of weight
$m \geq 3$ for arbitrary Fuchsian groups of the first kind. To explain our results, we recall some
standard notation (see [6]). Let $X$ be the upper half–plane. Then the group $SL_2(\mathbb{R})$ acts on $X$ as
follows:

$$g.z = \frac{az + b}{cz + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).$$

We let $\mu(g, z) = cz + d$.

Next, $SL_2(\mathbb{R})$–invariant measure on $X$ is defined by $dxdy/y^2$, where the coordinates on $X$ are
written in a usual way $z = x + \sqrt{-1}y$, $y > 0$. A discrete subgroup $\Gamma \subset SL_2(\mathbb{R})$ is called a Fuchsian
group of the first kind if its fundamental domain has a finite volume i.e., $\Gamma$ has finite covolume in
$SL_2(\mathbb{R})$. Then, adding a finite number of points in $\mathbb{R} \cup \{\infty\}$ called cusps, the fundamental domain
can be compactified. In this way we obtain a compact Riemann surface $\mathcal{R}_\Gamma$.

For an integer $m$, let $S_m(\Gamma)$ be the space of all modular forms of weight $m$ which are cuspidal
i.e., this is a space of all holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ such that $f(\gamma.z) = \mu(\gamma, z)^m f(z)$
($z \in X, \gamma \in \Gamma$) which are holomorphic and vanish at every cusp for $\Gamma$. One can use geometric
considerations on the compact Riemann surface $\mathcal{R}_\Gamma$ to compute the dimension of the space $S_m(\Gamma)$
(see [6], Theorems 2.5.2, 2.5.3.).

The classical theory ([6], Section 2.6) gives the construction of elements and spanning set of the
finite dimensional vector space $S_m(\Gamma)$, for $m \geq 3$, assuming that a fundamental domain in $X$ is not
compact (i.e., $\Gamma$ has cusps). In this paper we give a much more general construction of modular
forms and a spanning set that does not depend if a fundamental domain in $X$ is compact or not.

We write $B_m \ (m \in \mathbb{Z})$ for the space of all holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ satisfying
$$\int_{-\infty}^{\infty} \int_{0}^{\infty} y^{m/2} |f(z)| \frac{dxdy}{y^2} < \infty.$$ The following theorem is one of the main results of the paper:

Theorem 1-1. Assume that $m \geq 3$. Let $\Gamma \subset SL_2(\mathbb{R})$ be a discrete subgroup of finite covolume.
Then we have the following:

(i) Let $f \in B_m$. Then the series $\sum_{\gamma \in \Gamma} f(\gamma.z)\mu(\gamma, z)^{-m}$ converges uniformly and absolutely on
compact sets in $X$ to an element of $S_m(\Gamma)$. 

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(ii) We have $(z + \sqrt{-1})^{-k-m} \in \mathcal{B}_m$ for $k \geq 0$, and the corresponding modular forms

$$
\beta_{k,m}(z) = \sum_{\gamma \in \Gamma} (\gamma \cdot z + \sqrt{-1})^{-k-m} \mu(\gamma) z^{-m}, \quad k \geq 0,
$$

span $S_m(\Gamma)$.

The proof of Theorem 1-1 is given in Section 4. It is obtained using the standard correspondence between automorphic forms on $SL_2(\mathbb{R})$ and modular forms on $X$ (see [2], Chapter 2, Proposition 2.1) from the main results of Sections 2 and 3. In Section 2, we give a construction of cuspidal automorphic forms related to Theorem 1-1 (i) (see Lemma 2-9). Lemma 2-9 sharpens ([1], Theorems 6.1, 6.2, and 8.9). The proof is modeled on the adelic proof of ([7], Theorem 3-10)). We remark that Lemma 2-9 and Theorem 1-1 (i) are valid for all $m$ but in Section 3 (see Lemma 3-15) we prove that $\mathcal{B}_m = \{0\}$ for $m < 3$.

In Section 3 we prove Lemma 3-1 which implies Theorem 1-1 (ii). Essentially, Lemma 3-1 shows that the space $S_m(\Gamma)$ is spanned by the Poincaré series attached to certain left and right $K$–finite matrix coefficients of a holomorphic discrete series, say $D_m$, of weight $m \geq 3$. (The representation $D_m$ is defined in [4], page 183. It is recalled in the first paragraph of the proof of Lemma 3-1.)

The proof of Lemma 3-1 is modeled on an unpublished result of Miličić which I learned from Savin. Miličić proved that the Poincaré series of the left and right $K$–finite matrix coefficients of an integrable discrete series $\pi$ of a semisimple group Lie group $G$ span the isotypic component of $\pi$ in $L^2(\Gamma \backslash G)$ where $\Gamma$ is a cocompact discrete subgroup of $G$. In a sense, in Lemma 3-1 we make the Miličić’s result explicit for $SL_2(\mathbb{R})$ computing $K$–finite matrix coefficients of holomorphic discrete series $D_m$ explicitly (see Lemma 3-5). (We are not aware of the existence of such result in the literature.) But, in fact, we use just the basic idea from the Miličić’s proof and we give a simple and natural proof of Lemma 3-1 based on some general results in [3] applied to the differentiable vectors in the Banach representation of $SL_2(\mathbb{R})$ on $L^1(SL_2(\mathbb{R}))$.

We remark that Section 3 is the only place in the paper where we use the representation theory. Mostly, this is the representation theory of $SL_2(\mathbb{R})$ developed in [4]. But we also use a more sophisticated although still basic results contained in the introductionary sections of [3].

In Section 4 we also relate our Theorem 1-1 to the classical situation ([6], Section 2.6). So, assume that $\infty$ is a cusp for $\Gamma$. Let $\Gamma_\infty$ be the stabilizer of $\infty$ in $\Gamma$. By ([6], Theorem 1.5.4), there exists real $h' > 0$ such that we have the following:

$$
(1-2) \quad \{\pm 1\} \Gamma_\infty = \{\pm 1\} \left\{ \begin{pmatrix} 1 & mh' \\ 0 & 1 \end{pmatrix} ; \ m \in \mathbb{Z} \right\}.
$$

We write $\Gamma_{U,\infty}$ for $\Gamma_\infty$ intersected with the group of upper triangular unipotent matrices in $SL_2(\mathbb{R})$. Obviously, $\Gamma_{U,\infty}$ is a cyclic group generated by $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ where $h$ is:

$$
(1-3) \quad h = \begin{cases} 
h' & \text{if } -1 \in \Gamma \\
h' & \text{if } -1 \notin \Gamma \text{ and } \Gamma_\infty = \Gamma_{U,\infty} \\
2h' & \text{if } -1 \notin \Gamma \text{ and } [\Gamma_\infty : \Gamma_{U,\infty}] = 2. 
\end{cases}
$$

If $h = 2h'$ the cusp is defined to be irregular. (Otherwise, it is defined to be regular.)

We assume that $\infty$ is a cusp for $\Gamma$ which is regular or not. Let $m \geq 3$ be an integer such that the following compatibility condition holds:

$$
(1-4) \quad m \text{ is even if } -1 \in \Gamma \text{ or if } \infty \text{ is irregular cusp for } \Gamma.
$$
Then in ([6], Corollary 2.6.11) the another spanning set for $S_m(\Gamma)$ is written. In the introduction of [11] this spanning set is rewritten as follows:
\[
\alpha_{l,m}(z) = \sum_{\gamma \in \Gamma_{\text{cusps}}} e^{2\pi i \sqrt{-1} \gamma \cdot z / h} \mu(\gamma, z)^{-m}, \quad l \geq 1.
\]
Now, under above assumptions, we have the following expansion (see Proposition 4-5):
\[
\beta_{k,m}(z) = \frac{(-1)^{m+k}}{(m+k-1)!} \cdot \frac{(2\pi \sqrt{-1})^{m+k}}{h^{m+k}} \cdot \sum_{l=1}^{\infty} (m+k-1)^{e-2\pi i l/h} \alpha_{l,m}(z), \quad k \geq 0,
\]
where the convergence is absolute and uniform on compact sets in $X$. Using Theorem 1-1 (ii), the first formula gives an alternative proof that $\alpha_{l,m}$ ($l \geq 1$) span $S_m(\Gamma)$ (see Corollary 4-6).

In Section 5 (see Lemma 5-1), we write down a non–vanishing criterion for the series introduced by Theorem 1-1. Lemma 5-1 is a particular case of ([11], Lemma 3-1). We remark that ([11], Lemma 3-1) is a consequence of a general criterion for locally compact groups given in ([7], Theorem 4-1). As an application, in the same section, we prove Theorem 1-6 stated below. In this theorem we consider congruence subgroups. Let $N \geq 1$. Then we define standard congruence subgroups
\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) ; \ c \equiv 0 \pmod{N} \right\} ;
\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) ; \ c \equiv 0, \ a, d \equiv 1 \pmod{N} \right\} ;
\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) ; \ c, b \equiv 0, \ a, d \equiv 1 \pmod{N} \right\} .
\]
It is well–known that they are discrete subgroup of $SL_2(\mathbb{R})$ of finite covolume (see [6]). Note that $\infty$ is a regular cusp for each of these three congruence subgroups.

**Theorem 1-6.** Assume that $m \geq 3$ and $f \in \mathcal{B}_m$. Let $\Gamma_N \in \{ \Gamma(N), \Gamma_0(N), \Gamma_1(N) \}$. Then there exists $N_0 \geq 1$ such that the modular form
\[
\sum_{\gamma \in \Gamma_N} f(\gamma \cdot z) \mu(\gamma, z)^{-m} \in S_m(\Gamma_N),
\]
is non–zero for $N \geq N_0$ assuming that $m$ is even, if $-1 \in \Gamma_N$, and $\sum_{l \in \mathbb{Z}} f(z + l) \neq 0$ if $\Gamma_N \in \{ \Gamma_0(N), \Gamma_1(N) \}$. (The case $\Gamma_N = \Gamma(N)$ already follows from [7], Theorem 0.8.)

Thinking in terms of $SL_2(\mathbb{R})$ instead of $X$, the non–vanishing criteria of Lemma 5-1 use the Iwasawa decomposition (see (2-2)) to compute the integrals. The Cartan decomposition (see (2-10)) is not easily seen working on $X$, but it is quite useful to study the non–vanishing of automorphic/modular forms. We illustrate this point by Propositions 6-23 and 6-24 in Section 6 where the non–vanishing of another spanning set of $S_m(\Gamma)$ given by (see Lemma 4-2)
\[
\sum_{\gamma \in \Gamma} (\gamma \cdot z - \sqrt{-1})^k (\gamma \cdot z + \sqrt{-1})^{-k-m} \mu(\gamma, z)^{-m}, \quad k \geq 0,
\]
is considered.

In fact, this is just the tip of the iceberg. More can be done on the lines of Sections 5 and 6 but we defer this for another occasion (see also [11]). In the sequel to this paper we will study the non–vanishing of constructed forms in more detail as well as the action of Hecke operators using the methods of [10].
We mention here that we use compactly supported Poincaré series to study existence of Maass forms ([8], [9]).

I would like to thank Gordan Savin for his interest in my works ([7], [8], [9]). While we were both visitors at the Erwin Schrödinger Institute in Vienna, our discussions inspired me to start this project. As I mentioned above, he explained to me the above mentioned result of Miličić which is an essential ingredient of the proof of Theorem 1-1 (ii). I would like to thank Joachim Schwermer and Marko Tadić, for some useful discussions. The work on the present paper has started while I was a visitor of the Erwin Schrödinger Institute in Vienna. I would like to thank the referee, who read the paper very carefully and helped to improve the style of presentation.

2. SOME RESULTS ON AUTOMORPHIC FORMS ON $SL_2(\mathbb{R})$

In this section we fix the notation and prove an important result for the proof of Theorem 1-1 (i) (see Lemma 2-9).

Let $P_\infty = M_\infty A_\infty U_{P_\infty}$ be a minimal parabolic subgroup of $G = SL_2(\mathbb{R})$ consisting of upper triangular matrices. Explicitly, we have the following: $M_\infty = \{ \pm 1 \}$, $A_\infty = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}_{>0} \right\}$, $U_{P_\infty} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$. The maximal compact subgroup $K_\infty$ can be identified with $U(1)$ as follows:

$$
\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \leftrightarrow \exp \left( \sqrt{-1}t \right) = \cos t + \sqrt{-1}\sin t.
$$

The unitary dual of $K_\infty$ can be identified with $\mathbb{Z}$ in the following way:

$$(2-1) \quad \chi_m \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \exp (m \cdot \sqrt{-1}t), \quad m \in \mathbb{Z}, \quad t \in \mathbb{R}.$$

We use the normalized Haar measure on $K_\infty$ given by

$$
\int_{K_\infty} f(k)dk = \frac{1}{2\pi} \int_0^{2\pi} f \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} dt, \quad f \in C^\infty(K_\infty).
$$

Let $X = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$ be the upper half-plane. The group $SL_2(\mathbb{R})$ acts on $X$ as follows:

$$
g.z = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).
$$

The stabilizer of $\sqrt{-1}$ is $K_\infty$. Thus $X$ is diffeomorphic to $SL_2(\mathbb{R})/K_\infty$ using the Iwasawa decomposition of $SL_2(\mathbb{R}) = U_{P_\infty} A_\infty K_\infty$:

$$(2-2) \quad g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mapsto g.\sqrt{-1} = x + y\sqrt{-1}.$$

The measure on $SL_2(\mathbb{R})$ can be fixed as follows ($f \in C_c^\infty(SL_2(\mathbb{R}))$):

$$(2-3) \quad \int_{SL_2(\mathbb{R})} f(g)dg = \int_{-\infty}^{\infty} \int_0^\infty \int_0^{2\pi} f \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \frac{dxdy dt}{y^2 2\pi}.$$

The stabilizer of a point $\bar{\mathbb{R}} = \mathbb{R} \cup \{ \infty \}$ is a (real) parabolic subgroup of $SL_2(\mathbb{R})$. The stabilizer of $\infty$ is $P_\infty$. We let

$$
\mu(g) = cz+d, \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).
$$
The function $\mu$ satisfies the cocycle identity:

\begin{align}
(2-4) \quad \mu(g g', z) &= \mu(g, g' z) \cdot \mu(g', z).
\end{align}

Let $\Gamma$ be a discrete subgroup of $SL_2(\mathbb{R})$ of finite covolume. The $\Gamma$-cuspidal parabolic subgroup for $\Gamma$ is a parabolic subgroup $P$ of $SL_2(\mathbb{R})$ such that $\Gamma \cap U_P$ contains a non-trivial element. (Here $U_P$ is the unipotent radical of $P$.) Hence, $\Gamma \cap U_P$ is an infinite cyclic group ([1], 3.6). In particular, $\Gamma \cap U_P \setminus U_P$ is a compact group isomorphic to the unit circle. The group $\Gamma$ operates on the set of cuspidal parabolic subgroups by conjugation. If the volume of $\Gamma \setminus SL_2(\mathbb{R})/K_{\infty} = \Gamma \setminus X$, or equivalently, of $\Gamma \setminus SL_2(\mathbb{R})$ is finite, then the set of equivalence classes of $\Gamma$-cuspidal parabolic subgroups is finite ([1], Theorem 3.14).

The space of cusps forms $\mathcal{A}_{\text{cusp}}(\Gamma \setminus SL_2(\mathbb{R}))$ is defined in ([1], 5.5 and 7.8). Using ([1], 5.8), we can define it in the following way: it consists of all functions $\psi \in C^\infty(\Gamma \setminus SL_2(\mathbb{R}))$ satisfying the following conditions:

$\psi(\gamma g) = \psi(g)$, $\gamma \in \Gamma$, $g \in SL_2(\mathbb{R})$

$\psi$ is $K_{\infty}$-finite on the right i.e., the right translations of $\psi$ by $K_{\infty}$ span finite-dimensional space

$\psi \in C$-finite on the right

\begin{align}
\int_{\Gamma \setminus U_P \setminus U_P} \psi(ug) du &= 0, \quad g \in SL_2(\mathbb{R}), \quad \text{for all } \Gamma \text{-cuspidal parabolic subgroups}
\end{align}

\begin{align}
\int_{\Gamma \setminus SL_2(\mathbb{R})} |\psi(g)|^2 dg &< \infty.
\end{align}

Here $C$ is the Casimir operator. Its action in the coordinates defined by (2-2) is given by (it is a half of what is called Casimir operator in ([4], page 198))

\begin{align}
2y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2) - 2y\partial^2/\partial x\partial t.
\end{align}

For $m \in \mathbb{Z}$, we write $\mathcal{A}_{\text{cusp}}(\Gamma \setminus SL_2(\mathbb{R}))_m$ for the subspace of $\mathcal{A}_{\text{cusp}}(\Gamma \setminus SL_2(\mathbb{R}))$ consisting of all $\psi \in \mathcal{A}_{\text{cusp}}(\Gamma \setminus SL_2(\mathbb{R}))$ satisfying the following conditions:

$\psi(gk) = \chi_m(k)\psi(g)$, $k \in K_{\infty}$, $g \in SL_2(\mathbb{R})$

\begin{align}
(2-5) \quad C\psi &= \left(\frac{m^2}{2} - m\right) \psi.
\end{align}

We write $B_m$ ($m \in \mathbb{Z}$) for the space of all holomorphic functions $f : X \rightarrow \mathbb{C}$ satisfying

\begin{align}
\int_{-\infty}^{\infty} \int_{0}^{\infty} y^{m/2} |f(z)| \frac{dz dy}{y^2} < \infty.
\end{align}

Applying the standard lifting procedure ([1], 5.14 or [2], Chapter 2) we can assign to a function $f : X \rightarrow \mathbb{C}$ the function $F_f : SL_2(\mathbb{R}) \rightarrow \mathbb{C}$ defined by the following expression:

\begin{align}
(2-6) \quad F_f(g) &= f(g, \sqrt{-1})\mu(g, \sqrt{-1})^{-m}.
\end{align}

Using the Iwasawa decomposition (2-2), we obtain the following:

\begin{align}
(2-7) \quad F_f \left( \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \left( y^{1/2} \begin{bmatrix} 0 & \cos t \\ 0 & \sin t \end{bmatrix} \right) \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \right) &= y^{m/2} \exp(mt\sqrt{-1})f(z).
\end{align}

We prove the following lemma:

**Lemma 2-8.** Let $f$ be a holomorphic function on $X$. Then we have the following:

(i) $F_f$ transforms on the right under $K_{\infty}$ as $\chi_m$ i.e., $F_f(gk) = \chi_m(k)F_f(g)$, $g \in SL_2(\mathbb{R})$, $k \in K_{\infty}$. 


(ii) $C.F_f = \left(\frac{m^2}{2} - m\right)F_f$.

(iii) If $f \in B_m$, then $F_f \in L^1(SL_2(\mathbb{R}))$.

(iv) We define $E^- \in \mathfrak{sl}_2(\mathbb{R})$ by

$$E^- = -2\sqrt{-1}ye^{-2it} \left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right) + \sqrt{-1}e^{-2it}\frac{\partial}{\partial t},$$

considered as a left-invariant vector field on $SL_2(\mathbb{R})$. (See the top of the page 116 in [4] for a definition of $E^-$.) Then $E^-.F = 0$.

Proof. (i) follows from (2-7). Since $f$ is holomorphic on $X$, the Cauchy–Riemann equations and (2-7) imply (ii). Next, (2-3) and (2-7) imply (iii). Finally, Cauchy–Riemann equations and (2-7) imply (iv). \[\square\]

The following lemma is the key fact for the proof of Theorem 1-1 (i):

**Lemma 2-9.** Assume that $\Gamma \subset SL_2(\mathbb{R})$ is a discrete subgroup of finite covolume. Let $f \in B_m$. Then the series

$$\sum_{\gamma \in \Gamma} |F_f(\gamma \cdot g)|$$

converges uniformly on compact sets, and

$$P_\Gamma(F_f)(g) = \sum_{\gamma \in \Gamma} F_f(\gamma \cdot g)$$

converges absolutely and uniformly on compact sets to an element of $A_{\text{cusp}}(\Gamma \backslash SL_2(\mathbb{R}))_m$.

Proof. Using Lemma 2-8, the fact that the series $\sum_{\gamma \in \Gamma} |F_f(\gamma \cdot g)|$ converges uniformly on compact sets under our assumption on $f$ is a sharper form of ([1], Theorem 6.1) but the proof follows the same lines. The details can be found in the proof of ([7], Theorem 3-10) on pages 216-217. The proof of ([7], Theorem 3-10) implies also that $P_\Gamma(F_f)$ is bounded and integrable on $\Gamma \backslash SL_2(\mathbb{R})$. This obviously implies that $P_\Gamma(F_f)$ is square–integrable on $\Gamma \backslash SL_2(\mathbb{R})$. Also, the argument used in the proof of ([7], Theorem 3-10) on page 217 for the proof of ([7], Theorem 3-10 (iii)) implies that $P_\Gamma$ commutes with the action of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ and the group $K_\infty$. Hence $\psi = P_\Gamma(F_f)$ satisfies (2-5). In view of above definition of the space $A_{\text{cusp}}(\Gamma \backslash SL_2(\mathbb{R}))_m$, it remains to show that

$$\int_{\Gamma \cap U_r \backslash U_r} \psi(ug)du = 0 \quad (g \in SL_2(\mathbb{R}))$$

for all $\Gamma$–cuspidal parabolic subgroups. But this follows from ([1], Lemma 8.8) using the standard argument (see the proof of ([1], Theorem 8.9)). \[\square\]

Now, we construct some examples of the functions $f \in B_m \quad (m \geq 3)$. In the next section we prove that $B_m = \{0\}$ for $m < 3$ (see Lemma 3-15). First, we review the Cartan decomposition for $SL_2(\mathbb{R})$. We let

$$A^+ = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in A_\infty : \ t \geq 0 \right\}. $$

Then we have the following Cartan decomposition (see [4], page 139):

$$SL_2(\mathbb{R}) = K_\infty \cdot A^+_\infty \cdot K_\infty,$$

and the corresponding integration formula ([4], page 139):

$$\int_{SL_2(\mathbb{R})} \varphi(g)dg = \int_0^\infty \int_{K_\infty} \int_{K_\infty} \varphi \left( k_1 \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} k_2 \right) \sinh(2t) \ dk_1 \ dt \ dk_2.$$
Next, for \( m \in \mathbb{Z} \), we define holomorphic functions on \( X \) by the following formula (see ([4], Lemma 2, page 183) for the motivation):

\[
(2-12) \quad f_{k,m}(z) = (z - \sqrt{-1})^k (z + \sqrt{-1})^{-k-m}, \quad k \geq 0.
\]

We write \( F_{k,m} \) for the function corresponding to \( f_{k,m} \) by (2-6). We have the following lemma:

**Lemma 2-13.** Let \( k \geq 0 \). Then we have the following:

1. \( F_{k,m}(k_1gk_2) = \chi_{m+2k}(k_1)F_{k,m}(g)\chi_{m}(k_2), \) \( k_1, k_2 \in K_{\infty}, g \in SL_2(\mathbb{R}) \).
2. \( F_{k,m} \left( \begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right) = (\cosh t)^{-k-m}(\sinh t)^k/2^m \cdot (\sqrt{-1})^m, \) for \( t \geq 0 \).
3. If \( m \geq 3 \), then \( F_{k,m} \in L^1(SL_2(\mathbb{R})) \).
4. If \( m \geq 3 \), then \( F_{k,m} \in B_m \). In particular, \( B_m \neq 0 \) for \( m \geq 3 \).
5. \( C.F_{k,m} = (m^2/2 - m)F_{k,m} \).
6. \( E^-F_{k,m} = 0 \).

**Proof.** Using the Iwasawa decomposition, we can write

\[
\left( \begin{array}{cc} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{array} \right) \left( \begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right) = \left( \begin{array}{cc} (e^{-2t} \cos^2(\alpha) + e^{2t} \sin^2(\alpha))^{-1/2} & 0 \\ 0 & (e^{-2t} \cos^2(\alpha) + e^{2t} \sin^2(\alpha))^{1/2} \end{array} \right) \cdot \left( \begin{array}{cc} e^{-2t} \cos^2(\alpha) + e^{2t} \sin^2(\alpha) & 1 \\ 1 & (e^{-2t} \cos^2(\alpha) + e^{2t} \sin^2(\alpha))^{1/2} \end{array} \right) \left( \begin{array}{cc} e^{-t} \cos \alpha & e^t \sin \alpha \\ -e^t \sin \alpha & e^{-t} \cos \alpha \end{array} \right).
\]

The claims (i) and (ii) of the lemma follow from this identity by a straightforward and elementary but somewhat tedious computation. The claim (iii) follows from (2-11) and (ii). The details can be found in Section 6 (see (6-3)). Next, (iv) follows from (iii), (2-3) and (2-7). Finally, (v) and (vi) follow from Lemma 2-8 (ii) and (iv), respectively. \( \square \)

### 3. SOME APPLICATIONS OF THE REPRESENTATION THEORY OF \( SL_2(\mathbb{R}) \)

In this section we continue with the notation from the previous section. The goal of this section is to prove Lemmas 3-1 and 3-15.

**Lemma 3-1.** Let \( m \geq 3 \). Assume that \( \Gamma \subset SL_2(\mathbb{R}) \) be a discrete subgroup of finite covolume. Then the Poincaré series

\[
P_{\Gamma}(F_{k,m})(g) = \sum_{\gamma \in \Gamma} F_{k,m}(\gamma \cdot g), \quad k \geq 0,
\]

span \( \mathcal{A}_{\text{cusp}}(\Gamma \backslash SL_2(\mathbb{R}))_{m} \).

**Proof.** In this proof we need some representation theory. Let \( (\pi_m, D_m) \) be the holomorphic discrete series of \( SL_2(\mathbb{R}) \) of weight \( m \geq 2 \). Its definition and properties can be found in ([4], pages 181–184). In particular, \( D_m \) is the Hilbert space of all holomorphic functions \( f : X \to \mathbb{C} \) satisfying:

\[
\int_{-\infty}^{\infty} \int_{0}^{\infty} y^m |f(z)|^2 \frac{dxdy}{y^2} < \infty.
\]

We write

\[
(3-2) \quad \langle f_1, f_2 \rangle = \int_{-\infty}^{\infty} \int_{0}^{\infty} y^m f_1(z)f_2(z) \frac{dxdy}{y^2}
\]

for the positive definite Hermitian form on \( D_m \). The group \( SL_2(\mathbb{R}) \) acts by the following unitary operators:

\[
\pi_m(g)f(z) = f(g^{-1}.z)\mu(g^{-1}, z), \quad g \in SL_2(\mathbb{R}), \quad f \in D_m.
\]
Also, using the computations in ([4], pages 119–121), we find that \( C \) acts on the space of \( K_\infty \)-finite vectors \((D_m)_{K_\infty}\) in \( D_m \) as a scalar multiplication by \( \left( \frac{m^2}{2} - m \right) \). As a \( K_\infty \)-representation, we have the following:

\[
(D_m)_{K_\infty} = \oplus_{k=0}^{\infty} \chi_{m+2k}.
\]

**Lemma 3-4.** Let \( m \geq 2 \). Then \((\mathfrak{sl}_2(\mathbb{R}), K_\infty)\)-module of \( \mathcal{A}_{cusp}(\Gamma \setminus SL_2(\mathbb{R})) \), generated by a non-zero \( \psi \in \mathcal{A}_{cusp}(\Gamma \setminus SL_2(\mathbb{R}))_m \) is isomorphic to \((D_m)_{K_\infty}\).

*Proof.* First, as in elementary ([3], Lemma 77 on page 89), (2-5) implies that the module generated by \( \psi \) is semisimple of finite length i.e., it is equal to \( U_1 \oplus \cdots \oplus U_k \), where \( U_i \) are irreducible modules. Applying again (2-5) and the classification of irreducible \((\mathfrak{sl}_2(\mathbb{R}), K_\infty)\)-modules ([4], pages 119–121), we obtain that all \( U_i \)'s are isomorphic to \((D_m)_{K_\infty}\). Since \((D_m)_{K_\infty}\) does not contain a \( K_\infty \)-type \( \chi_{m-2} \) (see (3-3)), the standard commutation formulas (see the second row of the displayed formulas on the top of the page 192 in [4]) show that \( E^- \psi = 0 \) (see Lemma 2-8 (iv) for a definition of \( E^- \)). Now, the same commutation formulas and the Poincaré–Birkhoff–Witt theorem show that in the module generated by \( \psi \), the character \( \chi_m \) appears with the multiplicity one. Hence \( k = 1 \). \( \square \)

In some early version of this paper we showed the next lemma by computing explicitly matrix coefficients from the realization of discrete series in certain induced representations. Here we give the following short proof:

**Lemma 3-5.** Let \( m \geq 2 \). Then \( F_{k,m} \) is a matrix coefficient of the representation \((\pi_m, D_m)\) which transforms on the right as \( \chi_m \) and on the left as \( \chi_{m+2k} \).

*Proof.* First, ([4], Lemma 2, page 183) implies that \( F_{k,m} \in D_m \). Hence, (2-3) and (2-7) imply that \( F_{k,m} \in L^2(SL_2(\mathbb{R})) \). Next, since \( F_{k,m} \) is \( K_\infty \)-finite on the left, and we have \( \mathcal{C} F_{k,m} = (m^2/2 - m)F_{k,m} \) (see Lemma 2-13 (v)), there is \( \beta \in C_c^\infty(SL_2(\mathbb{R})) \) such that

\[
F_{k,m}(g) = \int_{SL_2(\mathbb{R})} F_{k,m}(hg)\overline{\beta(h)}dh, \quad g \in SL_2(\mathbb{R}),
\]

applying ([3], Theorem 1).

In the unitary representation \( L^2(SL_2(\mathbb{R})) \), where \( SL_2(\mathbb{R}) \) acts by right–translations, the minimal closed subspace \( H \) generated by \( F_{k,m} \) is isomorphic to \( D_m \). This follows from the fact that the corresponding \((\mathfrak{sl}_2(\mathbb{R}), K_\infty)\)-module is irreducible. The argument is similar to the one sketched in the proof of Lemma 3-4 but is simpler since Lemma 2-13 (vi) holds. If we denote by \( \delta \) the orthogonal projection of \( \beta \) to \( H \), then we can write

\[
F_{k,m}(g) = \int_{SL_2(\mathbb{R})} F_{k,m}(hg)\overline{\beta(h)}dh = \int_{SL_2(\mathbb{R})} F_{k,m}(hg)\overline{\delta(h)}dh, \quad g \in SL_2(\mathbb{R}).
\]

Finally, Lemma 2-13 (i) implies the following:

\[
F_{k,m}(g) = \int_{K_\infty} \chi_{m-2k}(k)F_{k,m}(kg)dk = \int_{SL_2(\mathbb{R})} F_{k,m}(kg)\left( \int_{K_\infty} \chi_{m+2k}(\delta(k))dk \right)dh.
\]

Since, \( \delta_{k,m}(h) = \int_{K_\infty} \chi_{m-2k}(k)\delta(hk)dk \) is a \( K_\infty \)-finite vector in \( H \), the expression (3-6) proves the lemma. \( \square \)

In particular, Lemma 3-5 implies the following corollary:

**Corollary 3-7.** The space of matrix coefficients of \((\pi_m, D_m) \geq 2 \) which transform on the right as \( \chi_m \) and which are \( K_\infty \)-finite on the left is spanned by the functions \( F_{k,m} \) \( k \geq 0 \).
Now, we complete the proof of Lemma 3-1. We remark that \( \mathcal{A}_{\text{cusp}}(\Gamma \backslash SL_2(\mathbb{R}))_m \) is a finite dimensional Hilbert space under the inner product

\[
(\psi_1, \psi_2) = \int_{\Gamma \backslash SL_2(\mathbb{R})} \psi_1(g) \overline{\psi_2(g)} dg.
\]

(See ([1], Theorem 8.5).)

We assume that \( m \geq 3 \) from now on. By above remark, it is enough to show the following claim:

\[
\text{(3-8) If } \psi \in \mathcal{A}_{\text{cusp}}(\Gamma \backslash SL_2(\mathbb{R}))_m \text{ satisfies } (\psi, P_k(F_{k,m})) = 0 \text{ for all } k \geq 0, \text{ then } \psi = 0.
\]

First, applying Corollary 3-7, we obtain the following:

\[
(3-9) \quad \psi, P_l(c_{h,h'}) = \int_{\Gamma \backslash SL_2(\mathbb{R})} \psi(g) \overline{P_l(c_{h,h'})(g)} dg = 0, \quad h' \in (D_m)_{K_\infty},
\]

where we select and fix a non–zero \( h \in (D_m)_{K_\infty} \) such that \( \pi_m(k)h = \chi_m(k)h, \quad k \in K_\infty \). (We remark that \( h \) is unique up to a scalar (see (3-3).) Here we write \( c_{h,h'} \) for the matrix coefficient \( g \mapsto \langle \pi_m(g)h, h' \rangle \) of \( (\pi_m, D_m) \) where \( \langle , \rangle \) is a positive definite Hermitian form on \( D_m \) making the unitary representation \( (\pi_m, D_m) \) (see (3-2)).

Next, we remark that \( \psi \) being a cusp form on \( \Gamma \backslash SL_2(\mathbb{R}) \) is rapidly decreasing in the direction of every cusp for \( \Gamma ([1], \text{Corollary 7.9}) \). Hence it is bounded on \( SL_2(\mathbb{R}) \). Thus, \( \overline{c_{h,h'}} \) is in \( L^1(SL_2(\mathbb{R})) \).

Hence, we have the following:

\[
0 = \int_{\Gamma \backslash SL_2(\mathbb{R})} \psi(g) \overline{P_l(c_{h,h'})(g)} dg = \int_{\Gamma \backslash SL_2(\mathbb{R})} \psi(g) \left( \sum_{\gamma \in \Gamma} c_{h,h'}(\gamma \cdot g) \right) dg = \int_{\Gamma \backslash SL_2(\mathbb{R})} \psi(g) \overline{c_{h,h'}(g)} dg.
\]

(3-10)

We need a more precise result contained in the following lemma:

**Lemma 3-11.** Assume that \( h \) and \( h' \) are as in (3-9) and \( x \in SL_2(\mathbb{R}) \). Then

\[
\int_{SL_2(\mathbb{R})} \psi(xg) \overline{c_{h,h'}(g)} dg = 0.
\]

Assuming Lemma 3-11, we complete the proof of Lemma 3-1. Lemma 3-11 implies that the function \( \frac{\overline{c_{h,h'}}}{\mu} \in L^1(SL_2(\mathbb{R})) \) acts trivially on \( \psi \) in the unitary representation generated by \( \psi \) in \( L^2_{\text{cusp}}(\Gamma \backslash SL_2(\mathbb{R})) \). But, by Lemma 3-4, this representation is unitary equivalent to \( (\pi_m, D_m) \), and the \( K_\infty \)–type structure (see (3-3)) tells us that \( \psi \) is mapped to a scalar multiple of \( h \), say \( \mu h \), for some \( \mu \in \mathbb{C} \). Hence, in \( D_m \), we have the following:

\[
\mu \cdot \int_{SL_2(\mathbb{R})} \langle \pi_m(g)h, h' \rangle \pi_m(g)h \ dg = 0, \quad \text{for all } h' \in (D_m)_{K_\infty}.
\]

Hence

\[
\mu \cdot \int_{SL_2(\mathbb{R})} |\langle \pi_m(g)h, h' \rangle|^2 \ dg = 0, \quad \text{for all } h' \in (D_m)_{K_\infty}.
\]

This implies \( \mu = 0 \). This proves (3-8) and Lemma 3-1. It remains to prove Lemma 3-11.

We give the two proofs. The first proof is the one suggested by the referee (and it is due to Milčić also). Since \( h' \) is \( K_\infty \)–finite, we see that \( F(x) = \int_{SL_2(\mathbb{R})} \psi(xg) \overline{c_{h,h'}(g)} dg \) is a \( K_\infty \)–finite
vector in $L^2_{cusp}(\Gamma \backslash SL_2(\mathbb{R}))$. Hence it belongs to $\mathcal{A}_{cusp}(\Gamma \backslash SL_2(\mathbb{R}))$, and in particular it is a real analytic function on $SL_2(\mathbb{R})$. Let $X \in \mathfrak{sl}_2(\mathbb{R})$. Then, we have the following:

$$X.F(x) = \left. \frac{d}{dt} \right|_{t=0} F(x \exp(tX)) =$$

$$= \left. \frac{d}{dt} \right|_{t=0} \psi(x \exp(tX)g) \frac{\partial h}{\partial t}(g) dg =$$

$$= \left. \frac{d}{dt} \right|_{t=0} \psi(xg) c_{h,h'}(\exp(-tX))g) dg =$$

$$\int_{SL_2(\mathbb{R})} \psi(xg) c_{h,h'}(g) dg.$$

Combining this with the Poincaré–Birkhoff–Witt theorem and (3-10), we obtain $u.F(1) = 0$ for $u \in \mathcal{U}(\mathfrak{sl}_2)$ where $\mathcal{U}(\mathfrak{sl}_2)$ denotes the complexified enveloping algebra of $\mathfrak{sl}_2(\mathbb{R})$. Since $F$ is real analytic, $F(\exp X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n F(1) = 0$ for $X$ in a neighborhood of $0 \in \mathfrak{sl}_2(\mathbb{R})$. Hence $F = 0$. This completes the first proof.

Now, we give the second proof. Let $l$ denote the left action of $SL_2(\mathbb{R})$ on $L^1(SL_2(\mathbb{R}))$ i.e., $l(g)f(x) = f(g^{-1}x)$. This is a Banach representation of $SL_2(\mathbb{R})$. We put $F = c_{h,\pi_m(x)}h'$. Since $(\pi_m, D_m)$ is a unitary representation, we have the following:

$$F(g) = c_{h,\pi_m(x)}h'(g) = \langle \pi_m(g)h, \pi_m(x)h' \rangle = \langle \pi_m(x^{-1}g)h, h' \rangle = c_{h,h'}(x^{-1}g)$$

i.e., $F = l(x)c_{h,h'} \in L^1(SL_2(\mathbb{R}))$. For $n \in \mathbb{Z}$, we put

$$F_n(g) = \int_{K_\infty} \chi_{-n}(k) F(k^{-1}g) dk = \int_{K_\infty} \chi_{-n}(k) \langle \pi_m(k^{-1}g)h, \pi_m(x)h' \rangle dk =$$

$$= \langle \pi_m(g)h, h_n' \rangle, \text{ where } h_n' = \int_{K_\infty} \chi_n(k) \pi_m(kx) h' dk \in (D_m)_{K_\infty}.$$ 

Hence, by (3-10), we have the following:

$$\int_{SL_2(\mathbb{R})} \psi(g) F_n(g) dg = 0.$$

If we show that

$$\sum_{n \in \mathbb{Z}} F_n$$

converges absolutely to $F$ in $L^1(SL_2(\mathbb{R}))$,

then, since $\psi$ is bounded, by the Dominated convergence theorem we have the following:

$$\int_{SL_2(\mathbb{R})} \psi(xg) c_{h,h'}(g) dg = \int_{SL_2(\mathbb{R})} \psi(g) c_{h,h'}(x^{-1}g) dg =$$

$$= \int_{SL_2(\mathbb{R})} \psi(g) F(g) dg = \sum_{n \in \mathbb{Z}} \int_{SL_2(\mathbb{R})} \psi(g) F_n(g) dg = 0$$

which completes the proof of Lemma 3-11. It remains to show (3-12). Since $l(k)F_n = \chi_n(k)F_n$ for all $k \in K_\infty$, by ([3], Section 3, Lemma 5), we just need to show that $F$ is a differentiable vector
for $l$. Indeed, since $c_{h,h'}$ is \( K_\infty \)-finite on the left (and \( C \)-finite), there exists \( \beta \in C_c^\infty(SL_2(\mathbb{R})) \) such that
\[
c_{h,h'}(g) = \int_{SL_2(\mathbb{R})} \beta(y)c_{h,h'}(y^{-1}g)dy, \quad g \in SL_2(\mathbb{R}),
\]
applying ([3], Theorem 1). Thus, we have the following:
\[
F(g) = c_{h,h'}(x^{-1}g) = \int_{SL_2(\mathbb{R})} \beta(x^{-1}y)c_{h,h'}(y^{-1}g)dy, \quad g \in SL_2(\mathbb{R}),
\]
and this implies that \( F \) is smooth (see ([3], Section 2, Lemma 2)). This completes the second proof of Lemma 3-11.

In our case the proof of ([3], Section 3, Lemma 5) is essentially the classical elementary argument where one uses the partial integration to deal with the Fourier expansion of a \( 2\pi \)-periodic function \( f : \mathbb{R} \to \mathbb{C} \) of class \( C^2 \). We give some detail for reader’s convenience.

Let \( W \in sl_2(\mathbb{R}) \) be defined by \( W = \partial/\partial t \) in terms of coordinates (2-2) considered as a left invariant vector field.

Now, the computations in ([4], pages 114-115) show that \( l(\Omega)F_n = (n^2 + 1)F_n \), where \( \Omega = -W^2 + 1 \in \mathcal{U}(sl_2) \). Hence
\[
(3-13) \quad F_n(g) = \frac{1}{n^2 + 1} l(\Omega)F_n(g) = \frac{1}{n^2 + 1} \int_{K_\infty} \chi_{-n}(k)l(\Omega)F(k^{-1}g)dk.
\]
This implies that \( L^1 \)-norm of \( F_n \) satisfies
\[
(3-14) \quad |F_n|_1 \leq \frac{1}{n^2 + 1} |l(\Omega)F|_1.
\]
Since \( F \) is smooth, \( l(\Omega)F \in L^1(SL_2(\mathbb{R})) \). Now, by (3-14), the series \( \sum_{n \in \mathbb{Z}} |F_n|_1 \) converges. On the other hand, by (3-13), the series \( \sum_{n \in \mathbb{Z}} F_n \) converges absolutely and uniformly on compact sets. Hence, by the standard Fourier analysis on \( \mathbb{R} \), \( F(k^{-1}g) = \sum_{n \in \mathbb{Z}} \chi_n(k)F_n(g) \) (initially for fixed \( g \) and arbitrary \( k \in K_\infty \)). The proof of (3-12) is complete. \( \square \)

Finally, we prove the second main result of this section.

**Lemma 3-15.** Assume that \( m \in \mathbb{Z} \). Then \( B_m = \{0\} \) if and only if \( m < 3 \).

**Proof.** First, if \( m \geq 3 \), then \( B_m \neq \{0\} \) by Lemma 2-13 (iv). So, we assume that \( m < 3 \). Assume that \( B_m \neq \{0\} \). Let \( f \in B_m \), \( f \neq 0 \). Then \( F = F_f \) defined by (2-6) is non-zero and it satisfies the conclusions (i), (ii), and (iii) of Lemma 2-8. Hence, ([1], Corollary 2.22) implies that \( F \in L^2(SL_2(\mathbb{R})) \). Using again (i) and (ii) of Lemma 2-8, combined with elementary ([3], Lemma 77 on page 89), we obtain that that the minimal closed subset \( H \) of \( L^2(SL_2(\mathbb{R})) \) generated by \( F \) is a direct sum of finitely many irreducible representations \( U \) in the discrete series of \( SL_2(\mathbb{R}) \). Since \( F \) satisfies the conclusions (i) and (iv) of Lemma 2-8, every \( U \) contains a non-zero vector which transforms as \( \chi_m \) under \( K_\infty \) and is "killed" by \( E^- \). Since \( U \) is in the discrete series and \( m < 3 \), we have \( m = 2 \).

Next, in \( L^2(SL_2(\mathbb{R})) \), \( F = \sum_{n \in \mathbb{Z}} F_n \) where \( F_n(g) = \int_{K_\infty} \chi_{-n}(k)F(k^{-1}g)dk \). Hence, there exists \( n \) such that \( F_n \neq 0 \). Clearly, \( F_n \) satisfies all conclusions of Lemma 2-8 with \( m = 2 \). Since \( F_n \) is also \( K_\infty \)-finite on the left, we may conclude that \( F_n \) is a matrix coefficient of \( (\tau_2, D_2) \) which transforms on the right as \( \chi_2 \) and is \( K_\infty \)-finite on the left (see Lemma 3-5 for a similar proof). Applying ([3],
Theorem 1), we find that there is \( \beta \in C_c^\infty(SL_2(\mathbb{R})) \) such that
\[
F_n(g) = \int_{SL_2(\mathbb{R})} \beta(h) F_n(h^{-1}g) dh, \quad g \in SL_2(\mathbb{R}).
\]

Since the conclusion of Lemma 2-8 (iii) holds, we easily see that \( F_n \in L^1(SL_2(\mathbb{R})) \). Hence (3-16) implies that \( F_n \) is a differentiable vector for the left translation in \( L^1(SL_2(\mathbb{R})) \) ([3], Section 2, Lemma 2). Also, \( F_n \) belongs to the space described in Corollary 3-7 for \( m = 2 \). It is easy to see that this space is an irreducible \( (sl_2(\mathbb{R}), K_\infty) \)–module. Hence it is also generated by \( F_n \). We obtain that \( F_0, 2 \in L^1(SL_2(\mathbb{R})) \). This contradicts (6-4).

\[\Box\]

4. The Proof of Theorem 1-1 and Some Consequences

We start the proof with the following lemma. We use the notation introduced in Section 2.

**Lemma 4-1.** The map \( f \mapsto F_f \) defined by (2-6) is an isomorphism of vector spaces \( S_m(\Gamma) \to \mathcal{A}_{cusp}(\Gamma \backslash SL_2(\mathbb{R})).m \).

*Proof.* This lemma is standard for the congruence subgroups (see [2], Chapter 2, Proposition 2.1). This proof works for general \( \Gamma \) with a trivial modification. Alternatively, it also follows from ([1], 5.14, 7.2) and Lemma 3-4.

As a next step, we use Lemma 4-1 to transfer the cuspidal automorphic forms \( P_{\Gamma}(F_{k,m}) \) defined in Lemma 3-1 to \( S_m(\Gamma) \).

**Lemma 4-2.** The inverse of the map defined by Lemma 4-1 maps the Poincaré series \( P_{\Gamma}(F_{k,m}) \) onto the modular form
\[
\sum_{\gamma \in \Gamma} (\gamma.z - \sqrt{-1})^k (\gamma.z + \sqrt{-1})^{-k-m} \mu(\gamma,z)^{-m}, \quad k \geq 0,
\]
and the series in (4-3) converges uniformly and absolutely on compact sets in \( X \). Finally, the collection of modular forms (4-3) span the space \( S_m(\Gamma) \).

*Proof.* First, letting \( z = g.\sqrt{-1} \) for \( g \in SL_2(\mathbb{R}) \) and using (2-7) we find that the transferred function is given by
\[
P_{\Gamma}(F_{k,m})(g)\mu(g,\sqrt{-1})^m = \sum_{\gamma \in \Gamma} F_{k,m}(\gamma.g)\mu(\gamma.g,\sqrt{-1})^m.
\]

Using (2-4), this is equal to
\[
\sum_{\gamma \in \Gamma} (F_{k,m}(\gamma.g)\mu(\gamma.g,\sqrt{-1})^m) \mu(\gamma.g,\sqrt{-1})^{-m} = \sum_{\gamma \in \Gamma} f_{k,m}(\gamma.z)\mu(\gamma,z)^{-m}.
\]
Since \( f_{k,m} \) is defined by (2-12), we obtain (4-3). Now, we apply Lemma 3-1 to complete the proof of the lemma.

\[\Box\]

**Lemma 4-4.** Let \( m \geq 3 \). Let \( \Gamma \subset SL_2(\mathbb{R}) \) be a discrete subgroup of finite covolume. Then the series
\[
\sum_{\gamma \in \Gamma} (\gamma.z + \sqrt{-1})^{-k-m} \mu(\gamma,z)^{-m}, \quad k \geq 0,
\]
converges uniformly and absolutely on compact sets in \( X \) and they span the space \( S_m(\Gamma) \).
Proof. The lemma follows from Lemma 4-2 by induction on \( k \geq 0 \). First, the particular case of Lemma 4-2 is the series \( \sum_{\gamma \in \Gamma} (\gamma.z + \sqrt{-1})^{-m} \mu(\gamma, z)^{-m} \in S_m(\Gamma) \). Next, multiplying with \( \mu(\gamma, z)^{-m} \) and then summing up over \( \gamma \in \Gamma \) the following obvious identity:

\[
(-2\sqrt{-1})^k \cdot (\gamma.z + \sqrt{-1})^{-k-m} = (\gamma.z - \sqrt{-1})^k (\gamma.z + \sqrt{-1})^{-k-m} - \sum_{i=1}^{k} \binom{k}{i} (-2\sqrt{-1})^{k-i} (\gamma.z + \sqrt{-1})^{-k+i-m}
\]

we obtain the lemma.

Now, we complete the proof of Theorem 1-1. First, (i) follows from Lemma 2-9 transferring the resulting form to \( S_m(\Gamma) \) by the methods described in the proof of Lemma 4-2. Now, we prove (ii). Since \( f_0, m(z) = (z + \sqrt{-1})^{-m} \in B_m \) (see (2-12) and Lemma 2-13 (iv)), the identity

\[
(-2\sqrt{-1})^k \cdot (z + \sqrt{-1})^{-k-m} = (z - \sqrt{-1})^k (z + \sqrt{-1})^{-k-m} - \sum_{i=1}^{k} \binom{k}{i} (-2\sqrt{-1})^{k-i} (z + \sqrt{-1})^{-k+i-m}
\]

proves \( (z + \sqrt{-1})^{-k-m} \in B_m \) by induction on \( k \geq 0 \). Now, (ii) follows from Lemma 4-4. This completes the proof of Theorem 1-1.

It is well–know that if \(-1 \in \Gamma \) and \( m \) is odd then \( S_m(\Gamma) = \{0\} \). This is consistent with our results since in this case all series are identically zero.

Finally, we relate Theorem 1-1 to ([6], Corollary 2.6.11). So, we assume that \( \Gamma \subset SL_2(\mathbb{R}) \) is a discrete subgroup of finite covolume which has a cusp at \( \infty \). We maintain the notation introduced in (1-2), (1-3), and (1-5). We prove the following proposition:

**Proposition 4-5.** Assume that \( m \geq 3 \) satisfies \((1-4)\). Then

\[
\beta_{k,m}(z) = \frac{(-1)^{m+k}}{(m+k-1)!} \cdot \frac{(2\pi \sqrt{-1})^{m+k}}{h^{m+k}} \cdot \sum_{l=1}^{\infty} l^{m+k-1} e^{-2\pi l/h} \alpha_{l,m}(z), \quad k \geq 0.
\]

**Proof.** We use a well–known identity \( \pi \cot(\pi z) = \lim_{l \to \infty} \sum_{i=-l}^{l} \frac{1}{z+i} \) (the convergence is uniform on compact sets). Since \( \pi \cot(\pi z) = \pi \sqrt{-1} \frac{2e^{2\pi \sqrt{-1} z}}{e^{2\pi \sqrt{-1} z} e^{-2\pi \sqrt{-1} z}} \), we can unfold the left hand side for \( z \in X \):

\[
\pi \sqrt{-1} - 2\pi \sqrt{-1} \frac{1}{1 - e^{2\pi \sqrt{-1} z}} = \pi \sqrt{-1} - 2\pi \sqrt{-1} \left( \sum_{l=0}^{\infty} e^{2\pi l \sqrt{-1} z} \right) = \lim_{l \to \infty} \sum_{i=-l}^{l} \frac{1}{z+i}.
\]

Taking the derivative \((m+k-1)\)-times, we find

\[
\sum_{l=-\infty}^{\infty} \frac{1}{(z+l)^{m+k}} = \frac{(-1)^{m+k}}{(m+k-1)!} \cdot \frac{(2\pi \sqrt{-1})^{m+k}}{h^{m+k}} \left[ \sum_{l=1}^{\infty} l^{m+k-1} e^{-2\pi l/h} e^{2\pi l \sqrt{-1} z} \right]
\]

Substituting \( (z + \sqrt{-1})/h \) we find

\[
\sum_{l=-\infty}^{\infty} \frac{1}{(z + \sqrt{-1} + lh)^{m+k}} = \frac{(-1)^{m+k}}{(m+k-1)!} \cdot \frac{(2\pi \sqrt{-1})^{m+k}}{h^{m+k}} \left[ \sum_{l=1}^{\infty} l^{m+k-1} e^{-2\pi l/h} e^{2\pi l \sqrt{-1} z/h} \right]
\]
Thus, we have the following:

\[
\beta_{k,m}(z) = \sum_{\gamma \in \Gamma} (\gamma, z + \sqrt{-1})^{-m-k} \mu(\gamma, z)^{-m}
\]

\[
= \sum_{\gamma \in \Gamma \setminus \Gamma_{U, \infty}} \left( \sum_{l=-\infty}^{\infty} \frac{1}{(\gamma, z + \sqrt{-1} + lh)^{m+k}} \right) \mu(\gamma, z)^{-m}
\]

\[
= \frac{(-1)^{m+k}}{(m+k-1)!} \cdot \frac{(2\pi \sqrt{-1})^{m+k}}{h^{m+k}} \sum_{\gamma \in \Gamma \setminus \Gamma_{U, \infty}} \left( \sum_{l=1}^{\infty} l^{m+k-1} e^{-2\pi l/h} e^{2\pi l \sqrt{-1} \gamma z / h} \right) \mu(\gamma, z)^{-m}
\]

We need to justify the interchange the summation and show that the convergence is absolute and uniform on compact sets in \(X\). Let \(\Gamma_{\infty}\) be the stabilizer of \(\infty\) in \(\Gamma\). Then, the proof of ([6], Theorem 2.6.6) combined with ([6], Theorem 2.6.1) shows that \(\sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} |\mu(\gamma, z)|^{-m}\) converges uniformly on compact sets in \(X\). In particular, it is continuous on \(X\). Since \(\Gamma_{U, \infty}\) is a subgroup of \(\Gamma_{\infty}\) of index at most two ([6], Theorem 1.5.4), the same is true for \(\sum_{\gamma \in \Gamma_{U, \infty} \setminus \Gamma} |\mu(\gamma, z)|^{-m}\) as the following computation shows:

\[
\sum_{\gamma \in \Gamma_{U, \infty} \setminus \Gamma} |\mu(\gamma, z)|^{-m} = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \sum_{\delta \in \Gamma_{U, \infty} \setminus \Gamma_{\infty}} |\mu(\gamma, z)|^{-m} = (\# \Gamma_{U, \infty} \setminus \Gamma_{\infty}) \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} |\mu(\gamma, z)|^{-m}.
\]

(Since \(\mu(\delta \gamma, z) = \mu(\gamma, z) \mu(\delta, z)\) and \(\mu(\delta, z) = \pm 1\) using ([6], Theorem 1.5.4). (See also the paragraph where (1-2) is located in the introduction.))

Now, we have the following:

\[
|\alpha_{l,m}(z)| \leq \sum_{\gamma \in \Gamma_{U, \infty} \setminus \Gamma} |\mu(\gamma, z)|^{-m} (z \in X)
\]

since

\[
|e^{2\pi l \sqrt{-1} \gamma z / h}| = e^{-2\pi l \gamma z / h} = e^{-2\pi l y / h \cdot |\mu(\gamma, z)|^2} \leq 1 (z \in X).
\]

Finally, we have the following:

\[
\sum_{l=1}^{\infty} \sum_{\gamma \in \Gamma_{U, \infty} \setminus \Gamma} l^{m+k-1} e^{-2\pi l/h} \cdot |\mu(\gamma, z)|^{-m} = \left( \sum_{l=1}^{\infty} l^{m+k-1} e^{-2\pi l/h} \right) \cdot \left( \sum_{\gamma \in \Gamma_{U, \infty} \setminus \Gamma} |\mu(\gamma, z)|^{-m} \right),
\]

which combined with above remarks shows the claim. \(\square\)

We know that \(\alpha_{l,m} (l \geq 1)\) span \(S_m(\Gamma)\). This follows from ([6], Corollary 2.6.11) using a slight reformulation obtained in the introduction of [11]. Here we give a proof based on Theorem 1-1.

**Corollary 4-6.** Assume that \(m \geq 3\) satisfies (I-4). Then \(\alpha_{l,m} (l \geq 1)\) span \(S_m(\Gamma)\).

**Proof.** The fact would be an obvious consequence of Proposition 4-5 if it was not for infinite sums. We topologize \(S_m(\Gamma)\) by using the uniform convergence on compact sets in \(X\). Then \(S_m(\Gamma)\) is an Fréchet space (being a finite dimensional subspace of the Fréchet space of all holomorphic functions on \(X\)). Let \(Z\) be the span of all \(\alpha_{l,m} (l \geq 1)\). Since \(S_m(\Gamma)\) is finite dimensional space, \(Z\) is also finite dimensional and consequently closed in \(S_m(\Gamma)\). Hence, Proposition 4-5 implies that \(\beta_{k,m} \in Z\) for all \(k \geq 0\). Now, Theorem 1-1 implies the claim. \(\square\)
5. The Proof of Theorem 1-6

First, we work in the general set-up of Theorem 1-1 in order to write down a non–vanishing criterion for the series introduced there assuming that $\Gamma$ has a cusp at $\infty$. We maintain the notation introduced in (1-2), (1-3), and (1-5). We assume that $m$ satisfies (1-4).

Let $f : X \to \mathbb{C}$ be a holomorphic function satisfying $\int_{-\infty}^{\infty} \int_{0}^{\infty} y^{m/2} |f(z)| \frac{dxdy}{y^2} < \infty$. Then the series $\sum_{\gamma \in \Gamma} f(\gamma z) \mu(\gamma, z)^{-m}$ converges absolutely and uniformly on compact sets to an element of $S_m(\Gamma)$. In particular, the series

$$\sum_{\gamma \in \Gamma, \infty} f(\gamma z) \mu(\gamma, z)^{-m} = \# (\Gamma \cap \{\pm 1\}) \sum_{l=-\infty}^{\infty} f(z + l \cdot h')$$

converges absolutely uniformly on compact sets\(^1\). Hence it is a holomorphic function on $X$.

**Lemma 5-1.** Assume that $m \geq 3$ satisfies (1-4). Then the series $\sum_{\gamma \in \Gamma} f(\gamma z) \mu(\gamma, z)^{-m}$ is a non–zero modular form in $S_m(\Gamma)$ provided that there exists a compact set $C \subset \mathbb{R} \times \mathbb{R}$ such that the following two conditions hold:

1. $\gamma.C \cap C \neq \emptyset$ for $\gamma \in \Gamma$ implies $\gamma \in \Gamma_{\infty}$, and
2. $\int \int_C y^{m/2} |\sum_{l=-\infty}^{\infty} f(z + l \cdot h')| \frac{dxdy}{y^2} > \frac{1}{2} \int_{0}^{h'} \int_{0}^{h} y^{m/2} \sum_{l=-\infty}^{\infty} f(z + l \cdot h') \frac{dxdy}{y^2}$.

**Proof.** This is a particular case of ([11], Lemma 3-1). Indeed, as we explained above, the function $F(z) = \sum_{l=-\infty}^{\infty} f(z + l \cdot h')$ is a holomorphic function on $X$. It is obviously $h'$–periodic. This implies that the assumption (i) in ([11], Lemma 3-1) holds for $F$. Next, since by the assumption $\int_{-\infty}^{\infty} \int_{0}^{\infty} y^{m/2} |f(z)| \frac{dxdy}{y^2} < \infty$, we have the following (see (1-3)):

$$\int_{0}^{h} \int_{0}^{h} y^{m/2} |F(z)| \frac{dxdy}{y^2} \leq 2 \int_{0}^{h'} \int_{0}^{h} y^{m/2} |F(z)| \frac{dxdy}{y^2} \leq 2 \int_{-\infty}^{\infty} \int_{0}^{\infty} y^{m/2} |f(z)| \frac{dxdy}{y^2} < \infty.$$

Hence (ii) in ([11], Lemma 3-1) holds for $F$. Finally, since we can write

$$\sum_{\gamma \in \Gamma} f(\gamma z) \mu(\gamma, z)^{-m} = \# (\Gamma \cap \{\pm 1\}) \sum_{\gamma \in \Gamma, \infty} F(\gamma z) \mu(\gamma, z)^{-m},$$

the lemma follows from ([11], Lemma 3-1).

---

\(^1\)The uniform convergence on compact sets follows using the fact that $\sum_{\gamma \in \Gamma} |f(\gamma z) \mu(\gamma, z)^{-m}|$ converges uniformly on compact sets which holds since the Poincaré series $\sum_{\gamma \in \Gamma} |F(\gamma z)|$ (see Lemma 2-9) does (see the proof of Theorem 1-1).
Now, we prove Theorem 1-6. We use Lemma 5-1 since all three series of congruence subgroups has \( \infty \) as their cusp, and we have the following:

\[
\Gamma_0(N) \cap \infty = \{ \pm 1 \} \left\{ \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} ; \ l \in \mathbb{Z} \right\} \ h' = h = 1, \\
\{ \pm 1 \} \Gamma_1(N) \cap \infty = \{ \pm 1 \} \left\{ \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} ; \ l \in \mathbb{Z} \right\} \ h' = h = 1, \\
\{ \pm 1 \} \Gamma(N) \cap \infty = \{ \pm 1 \} \left\{ \begin{pmatrix} 1 & lN \\ 0 & 1 \end{pmatrix} ; \ l \in \mathbb{Z} \right\} \ h' = h = N.
\]

Thus, \( \infty \) is a regular cusp. Also, (1-4) means that \( m \geq 3 \) is even if \( -1 \in \Gamma \).

Now, by (5-2), we have

\[
\int_0^h \int_0^\infty y^{m/2} \left| \sum_{l=-\infty}^{\infty} f(z + l \cdot h') \right| \left| \frac{dxdy}{y^2} \right| < \infty.
\]

Hence, we can find \( \epsilon, \delta > 0 \) such that \( C = [0, h] \times [\epsilon, \delta] \) satisfies (2) of Lemma 5-1. Next, we select \( N_0 \) requiring the condition (1) of Lemma 5-1. To end this, we recall the following formulas. Let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \). Then the imaginary part of \( \gamma \cdot z \) is given by

\[
\text{Im}(\gamma \cdot z) = \frac{y}{|cz + d|^2},
\]

for all \( z = x + \sqrt{-1}y \in X \).

Now, let \( \gamma \) be in \( \Gamma_N \). Then \( c \equiv 0 \pmod{N} \). If \( z \in C \) and \( \gamma \cdot z \in C \), then \( y \) and \( \text{Im}(\gamma \cdot z) \) belong to \([\epsilon, \delta]\). Hence

\[
\frac{\delta}{\epsilon} \geq |cz + d|^2 = (cx + d)^2 + c^2y^2 \geq c^2 \epsilon^2.
\]

Thus, we obtain

\[
|c| \leq \frac{\delta}{c^2}.
\]

If we select \( N \geq N_0 = \left\lceil \sqrt{\frac{\delta}{c^2}} \right\rceil + 1 \), then \( c = 0 \), and we obtain (1). This proves the theorem.

### 6. Non–Vanishing of Modular Forms Via Cartan Decomposition

Let \( \Gamma \subset SL_2(\mathbb{R}) \) be a discrete subgroup of finite covolume. Then Lemma 4-2 implies that

\[
\sum_{\gamma \in \Gamma} (\gamma \cdot z - \sqrt{-1})^k (\gamma \cdot z + \sqrt{-1})^{-k-m} \mu(\gamma, z)^{-m}, \ k \geq 0,
\]

span \( S_m(\Gamma) \) for \( m \geq 3 \). By the same lemma, the modular form in (6-1) is obtained by transferring the Poincaré series \( P_1(F_{k,m}) \) to \( X \). We remind the reader that \( F_{k,m} \) is defined in the paragraph where (2-12) is contained. The proof of Lemma 4-2 shows that the modular form in (6-1) is non–zero if and only if \( P_1(F_{k,m}) \) is non–zero. We study this using ([11], Lemma 2-1) and the Cartan decomposition for \( SL_2(\mathbb{R}) \) (see (2-10)). We remark that ([11], Lemma 2-1) is a relatively straightforward analogue of ([7], Theorem 4-1). In fact Lemma 6-5 proved below is even more direct analogue of ([7], Theorem 4-1).

Before we state and prove Lemma 6-5, we introduce the following compact sets:

\[
C_r = K_\infty \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} ; \ 0 \leq t \leq r \right\} K_\infty, \ r \in \mathbb{R}_{>0}.
\]
We have the following:

\[
\int_{C_r} |F_{k,m}(g)| \, dg = \frac{1}{2m-1} \int_0^r (\cosh t)^{-k-m+1} (\sinh t)^{k+1} \, dt = \\
\frac{1}{2m-1} \int_0^\mu x^{k+1} (1 - x^2)^{\frac{m-4}{2}} \, dx, \ \mu = \mu_r = \sinh (r)/\cosh (r), \ r > 0.
\] (6-2)

using (2-11) and Lemma 2-13 (ii) for the first integral, and the substitution \( x = \sinh (t)/\cosh (t) \) for the second. This implies the following (for \( m \geq 3 \)):

\[
\int_{SL_2(\mathbb{R})} |F_{k,m}(g)| \, dg = \int_0^1 x^{k+1} (1 - x^2)^{\frac{m-4}{2}} \, dx \leq \int_0^1 (1 - x^2)^{-\frac{1}{2}} \, dx = \pi/2.
\] (6-3)

We need the following remark in the proof of Lemma 3-15:

\[
\int_{SL_2(\mathbb{R})} |F_{0,2}(g)| \, dg = \int_0^1 x (1 - x^2)^{-1} \, dx = \infty.
\] (6-4)

We remind the reader that we assume that \( m \geq 3 \) in this section.

**Lemma 6-5.** If \( \chi_{m+2k} \) is not trivial on \( \Gamma \cap K_\infty \), then \( P_\Gamma (F_{k,m}) = 0 \). If \( \chi_{m+2k} \) is trivial on \( \Gamma \cap K_\infty \), then \( P_\Gamma (F_{k,m}) \neq 0 \) provided that there exists \( r > 0 \) such that

\[
\int_{C_r} |F_{k,m}(g)| \, dg > \frac{1}{2} \int_{SL_2(\mathbb{R})} |F_{k,m}(g)| \, dg \quad \text{and} \quad \Gamma \cap C_r \cdot C_r^{-1} \subset \Gamma \cap K_\infty.
\]

**Proof.** First, if \( \chi_{m+2k} \) is not trivial on \( \Gamma \cap K_\infty \), then

\[
\sum_{\gamma \in \Gamma \cap K_\infty} F_{k,m}(\gamma \cdot g) = \left( \sum_{\gamma \in \Gamma \cap K_\infty} \chi_{-m-2k}(\gamma) \right) F_{k,m}(g) = 0,
\]

using Lemma 2-13 (i) and the well–known fact that the sum of values of a non–trivial character over a finite group is zero. Consequently

\[
P_\Gamma (F_{k,m}) (g) = \sum_{\gamma \in \Gamma} F_{k,m}(\gamma \cdot g) = \sum_{\delta \in \Gamma \cap K_\infty} \left( \sum_{\gamma \in \Gamma \cap K_\infty} F_{k,m}(\gamma \cdot \delta \cdot g) \right) = 0.
\]

Next, we assume that \( \chi_{m+2k} \) is trivial on \( \Gamma \cap K_\infty \). Then ([11], Lemma 2-1) implies that \( P_\Gamma (F_{k,m}) \neq 0 \) provided that \( \Gamma \cap C_r \cdot C_r^{-1} \subset \Gamma \cap K_\infty \) and

\[
\int_{\Gamma \cap K_\infty \setminus C_r} \sum_{\gamma \in \Gamma \cap K_\infty} F_{k,m}(\gamma \cdot g) \, dg > \frac{1}{2} \int_{\Gamma \cap K_\infty \setminus SL_2(\mathbb{R})} \sum_{\gamma \in \Gamma \cap K_\infty} F_{k,m}(\gamma \cdot g) \, dg.
\] (6-6)

Since \( \chi_{m+2k} \) is trivial on \( \Gamma \cap K_\infty \), Lemma 2-13 (i) implies that \( F_{k,m}(\gamma \cdot g) = F_{k,m}(g), \gamma \in \Gamma \cap K_\infty, g \in SL_2(\mathbb{R}) \). Finally, since \( (\Gamma \cap K_\infty)C_r = C_r \) and the characteristic functions are related by

\[
(\# \Gamma \cap K_\infty) \text{char}_{\Gamma \cap K_\infty \setminus C_r}(g) = \sum_{\gamma \in \Gamma \cap K_\infty} \text{char}_{C_r} (\gamma \cdot g), \ g \in SL_2(\mathbb{R}),
\]

we see that the inequality (6-6) becomes

\[
\int_{C_r} |F_{k,m}(g)| \, dg > \frac{1}{2} \int_{SL_2(\mathbb{R})} |F_{k,m}(g)| \, dg.
\]

\[\square\]
Let us write $I_{k,m}(\mu)$ for the integral in (6-2) multiplied by $2^{m-1}$. Then, the integration by parts implies the following recursive formula:

\begin{equation}
(k + 2)I_{k,m}(\mu) - (m - 4)I_{k+2,m-2}(\mu) = \mu^{k+2} (1 - \mu^2) \frac{m-4}{k+2}, \quad m \geq 4, \ k \geq 0.
\end{equation}

This gives the following recursive formula

\begin{equation}
I_{k,m}(1) = \frac{m-4}{k+2} I_{k+2,m-2}(1)
\end{equation}

for

\begin{equation}
I_{k,m}(1) = 2^{m-1} \int_{SL_2(\mathbb{R})} |F_{k,m}(g)| \, dg.
\end{equation}

To apply Lemma 6-5, we need to determine $\mu \in ]0, 1[$ such that

\begin{equation}
J_{k,m}(\mu) \overset{\text{def}}{=} I_{k,m}(\mu) - \frac{1}{2} I_{k,m}(1) > 0.
\end{equation}

Combining (6-7) and (6-8), we find the following:

\begin{equation}
(k + 2)J_{k,m}(\mu) - (m - 4)J_{k+2,m-2}(\mu) = \mu^{k+2} (1 - \mu^2) \frac{m-4}{k+2}, \quad m \geq 4, \ k \geq 0.
\end{equation}

**Lemma 6-11.** Fix $k \geq 0$ and $m \geq 4$. Let $\mu \in ]0, 1[$ such that $J_{k+m-3,3}(\mu) > 0$ (resp., $J_{k+m-4,4}(\mu) > 0$) if $m$ is odd (resp., $m$ is even). Then $J_{k,m}(\mu) > 0$.

**Proof.** By (6-10), we obtain that $J_{k+2,m-2}(\mu) > 0$ implies $J_{k,m}(\mu) > 0$ for $\mu \in ]0, 1[$. Now, we iterate until we obtain the claim. \qed

**Lemma 6-12.** Fix $k \geq 0$. Let $m \geq 4$ be even. Then, $J_{k,m}(\mu) > 0$ for $\mu \in ]2^{-\frac{1}{k+m-2}}, 1[$.

**Proof.** First, (6-2) implies the following:

\begin{equation}
J_{k,4}(\mu) = \frac{1}{k+2} \left( \mu^{k+2} - \frac{1}{2} \right) > 0 \quad \text{if and only if} \quad \mu > 2^{-\frac{1}{k+2}}.
\end{equation}

Now, we apply Lemma 6-11. \qed

For odd $m$ we have a better result.

**Lemma 6-14.** Let $k \geq 0$. Let $m \geq 3$ be odd. Then, $J_{k,m}(\mu) > 0$ for $\mu > \frac{\sqrt{3}}{2}$.

**Proof.** The integral $I_{k,3}(\mu)$ after substitution $y = (1 - x^2)^{1/2}$ becomes

\begin{equation}
I_{k,3}(\mu) = \int_{(1-\mu^2)^{1/2}}^1 (1 - y^2)^{k/2} \, dy.
\end{equation}

Integrating by parts, we obtain

\begin{equation}
(k + 1)J_{k,3}(\mu) - kJ_{k-2,3}(\mu) = \mu^{k} (1 - \mu^2)^{\frac{1}{2}}, \quad k \geq 2.
\end{equation}

Finally, we have the following:

\begin{equation}
J_{0,3}(\mu) = \frac{1}{2} - (1 - \mu^2)^{\frac{1}{2}} > 0 \quad \text{if and only if} \quad \mu > \frac{\sqrt{3}}{2}
\end{equation}

and

\begin{equation}
2J_{1,3}(\mu) = \frac{\pi}{4} + \mu (1 - \mu^2)^{\frac{1}{2}} - \arcsin \sqrt{1 - \mu^2} > 0 \quad \text{if} \quad \mu > \frac{\sqrt{3}}{2}.
\end{equation}
By induction, (6-15), (6-16), and (6-17) imply that $J_{k,3}(\mu) > 0$ for $\mu > \sqrt{3}/2$. Now, Lemma 6-11 completes the proof.

Having completed determination of integrals, according to our Lemma 6-5, we need to study the following intersection:

\((6-18)\quad \Gamma \cap C_r \cdot C_r^{-1} = \Gamma \cap K_\infty \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} ; 0 \leq t \leq r \right\} K_\infty \left\{ \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} ; 0 \leq t \leq r \right\} K_\infty.\)

In order to analyze that intersection, we let

\[ ||g|| = \sqrt{tr(g \cdot g^t)^{1/2}} = \sqrt{a^2 + b^2 + c^2 + d^2}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).\]

It is obvious that

\[(6-19)\quad ||k_1 \cdot g \cdot k_2|| = ||g||, \quad k_1, k_2 \in K_\infty.\]

We show the following claim:

**Lemma 6-20.** $\max_{g \in C_r \cdot C_r^{-1}} ||g|| = \sqrt{2 \cosh (4r)}$.

**Proof.** Applying

\[ C_r \cdot C_r^{-1} = K_\infty \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} ; 0 \leq t \leq r \right\} K_\infty \left\{ \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} ; 0 \leq t \leq r \right\} K_\infty, \]

and (6-19), we find that

\[
\max_{g \in C_r \cdot C_r^{-1}} ||g|| = \max_{0 \leq t, t' \leq r} \sqrt{\cosh (2(t-t')) \cdot \cos^2 (\alpha) + \cosh (2(t+t')) \cdot \sin^2 (\alpha)}
\]

\[
= \sqrt{2} \cdot \max_{0 \leq t, t' \leq r} \sqrt{\cosh (2(t-t')) \cdot \cos (2(t+t')) \cdot \sin^2 (\alpha)}
\]

\[
= \sqrt{2} \cdot \max_{0 \leq t, t' \leq r} \cosh (2(t+t'))
\]

\[
= \sqrt{2} \cosh (4r).
\]

\[\square\]

**Lemma 6-21.** We have the following:

(i) Let $N \geq 2$. Then, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$ is not diagonal, then $||g|| \geq \sqrt{N^2 + 2}$.

(ii) Let $N = 1$. Then, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ is not in

\[ \Gamma(1) \cap K_\infty = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \]

then $||g|| \geq \sqrt{N^2 + 2}$. 
Proof. We prove (i), (ii) is analogous. By definition, \(a, d \equiv 1 \pmod{N}\). Since \(N \geq 2\), we obtain \(a^2 + d^2 \geq 2\). Also, by definition, \(b, c \equiv 0 \pmod{N}\). Thus, if \(g\) is not diagonal, then \(c^2 + b^2 \geq N^2\). This proves the lemma. \(\square\)

**Lemma 6-22.** Let \(N \geq 1\). Then, if \(N > 2 \sinh(2r) = \frac{4\mu}{1 - \mu^2}\) (see (6-2)), for some \(r > 0\), then \(\Gamma(N) \cap C_r \cdot C_r^{-1} \subset \Gamma(N) \cap K_\infty \subset \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \).

**Proof.** This follows from Lemmas 6-20 and 6-21. \(\square\)

Now, prove the following result:

**Proposition 6-23.** (i) Assume that \(m \geq 4\) is even. Then the cuspidal modular form in (6-1) for \(\Gamma \subset \Gamma(N)\) is non–zero if

\[
N > \frac{2^{2 - \frac{1}{k + m - 2}}}{1 - 2^{-\frac{2}{k + m - 2}}}.
\]

(ii) Assume that \(m \geq 3\) is odd. Then all cuspidal modular forms in (6-1) for \(\Gamma \subset \Gamma(N)\) are non–zero for \(N \geq 14\).

**Proof.** The function \(\frac{4\mu}{1 - \mu^2}\) is increasing on \([0, 1]\). Now, we select \(\mu \in \left]2^{-\frac{1}{k + m - 2}}, 1\right]\) such that

\[
\left[ \frac{2^{2 - \frac{1}{k + m - 2}}}{1 - 2^{-\frac{2}{k + m - 2}}} \right] + 1 > \frac{4\mu}{1 - \mu^2} > \frac{2^{2 - \frac{1}{k + m - 2}}}{1 - 2^{-\frac{2}{k + m - 2}}}.
\]

Then \(J_{k,m}(\mu) > 0\) by Lemma 6-12. Hence, if \(N\) satisfies the inequality stated in (i), we must have

\[
N \geq \left[ \frac{2^{2 - \frac{1}{k + m - 2}}}{1 - 2^{-\frac{2}{k + m - 2}}} \right] + 1 \geq \frac{4\mu}{1 - \mu^2}.
\]

Now, Lemma 6-22 implies that \(\Gamma(N) \cap C_r \cdot C_r^{-1} \subset \Gamma(N) \cap K_\infty \subset \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \). But \(N \geq 3\) for \(N\) which satisfies the displayed inequality in (i). Hence \(\Gamma(N) \cap K_\infty\) is trivial. In particular, \(\Gamma \cap K_\infty\) is trivial. Now, Lemma 6-5 completes the proof of (i). The proof of (ii) is similar. It uses Lemma 6-14 and the fact that \(14 > \frac{4\mu}{1 - \mu^2} > 13\) for \(\mu = \frac{3}{2}\). \(\square\)

Let us consider the cuspidal modular form in (6-1) where \(\Gamma \subset \Gamma(N)\) for some \(N \geq 1\). In proving Proposition 6-23 we did not use the best possible solution for the inequality \(J_{k,m}(\mu) > 0\). Instead, we used a rather rough estimate given by Lemmas 6-11, 6-12, and 6-14. By its definition (see (6-9)), the function \(J_{k,m}(\mu)\) is strictly increasing on \([0, 1]\), and it has a unique zero, say \(\mu_0\), in \([0, 1]\). The best possible solution would be \(\mu > \mu_0\), close to \(\mu_0\). One can try to write a computer program based on formulas involved in the proof of Lemmas 6-11, 6-12, and 6-14 to compute \(\mu_0\) for each particular pair \(k, m\) to a sufficient accuracy. This can be used to determine the smallest \(N > \frac{4\mu_0}{1 - \mu_0^2}\). Then Lemma 6-22 implies that

\[
\Gamma \cap C_r \cdot C_r^{-1} \subset \Gamma \cap K_\infty \subset \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \;
\]

and, to apply Lemma 6-5, we are left with seeing if \(\chi_{m+2k}\) is trivial on \(\Gamma \cap K_\infty\) which is a subgroup of \(\Gamma(N) \cap K_\infty\).
We include one more example. We consider the cuspidal modular form in (6-1) for $k = 0$.

**Proposition 6-24.** Let $\Gamma \subset \Gamma(N)$. Then the modular form $\sum_{\gamma \in \Gamma} (\gamma, z + \sqrt{-1})^{-m} \mu(\gamma, z)^{-m}$ ($m \geq 3$) is non–zero if one of the following holds:

(i) $m = 3$ and $N \geq 14$
(ii) $m = 4$ and $N \geq 6$
(iii) $m = 5, 6$ and $N \geq 4$
(iv) $m = 7, 9, 11, \ldots$ and $N \geq 3$ (the best possible result since $m$ is odd)
(v) $m = 8$ and $N \geq 3$
(vi) $m = 10, 12, 14, \ldots, 26$ and $N \geq 2$ (for $m = 10, 14, 18, 22, 26$ this is the best possible result)
(vii) $m = 30, 34, 38, \ldots$ and $N \geq 2$ (the best possible result)
(viii) $m = 28, 32, 36, \ldots$ and $N \geq 1$.

The cases where we perhaps did not obtain the best possible result are $m = 3, 4, 5, 6, 8, 12, 16, 20, 24$.

**Proof.** The inequality

$$2 \cdot J_{0,m}(\mu) = \frac{1}{m-2} \left[ 1 - 2 \left( 1 - \mu^2 \right)^{(m-2)/2} \right] > 0$$

implies

$$\mu > \sqrt{1 - 2^{2/m - 2}}$$

Then, the condition in Lemma 6-22 is equivalent with:

$$N > 4\lambda \sqrt{\lambda^2 - 1}, \quad \lambda = 2^{1/m - 1/2}.$$  

The computation of $N$ is left to the reader as an exercise. To apply Lemma 6-5, we recall that $\Gamma(N) \cap K_\infty$ is trivial for $N \geq 3$, $\Gamma(2) \cap K_\infty = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, and $\Gamma(1) \cap K_\infty$ given by Lemma 6-21 (ii). We remark that $\chi_m$ is trivial on $\Gamma(2) \cap K_\infty$ if and only if $2|m$, and $\chi_m$ is trivial on $\Gamma(1) \cap K_\infty$ if and only if $4|m$. Hence, the claimed best results in (vi) and (vii) follow from the first part of Lemma 6-5.

**REFERENCES**

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