ON THE NON–VANISHING OF CERTAIN MODULAR FORMS

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Abstract. Let \( \Gamma \subset SL_2(\mathbb{R}) \) be a Fuchsian group of the first kind. In this paper we study the non–vanishing of the spanning set for the space of cuspidal modular forms \( S_m(\Gamma, \chi) \) of weight \( m \geq 3 \) constructed in ([5], Corollary 2.6.11). Our approach is based on the generalization of the non–vanishing criterion for \( L^1 \)-Poincaré series defined for locally compact groups and proved in ([6], Theorem 4.1). We obtain very sharp bounds for the non–vanishing of the spaces of cusp forms \( S_m(\Gamma, \chi) \) for general \( \Gamma \) having at least one cusp. We obtain explicit results for congruence subgroups \( \Gamma(N), \Gamma_0(N), \) and \( \Gamma_1(N) \) \( (N \geq 1) \).

1. Introduction

The non–vanishing of classical Poincaré series for \( \Gamma = SL_2(\mathbb{Z}) \) is discussed by Rankin in [10]. His method is based on the explicit formulas for their Fourier coefficients which are given as certain Kloosterman sums and their estimates. This approach was pursued further to the groups \( \Gamma_0(N) \) by Mozzochi in [8] and to the general Fuchsian group having a cusp at \( \infty \) by Lehner in [4]. Based on the theory of harmonic weak Maass forms and the program of Bruinier-Bringmann and Ono, Rhoades in [11] studied the non–vanishing and linear relations among classical Poincaré series for \( \Gamma_0(N) \) (included in [1]).

In this paper we study the non–vanishing of cuspidal modular forms for an arbitrary Fuchsian group of the first kind. Our approach is based on measure–theoretic arguments on locally compact groups (see Section 2) which is a generalization of ([6], Theorem 4-1).

To explain our results, we recall some standard notation (see [5]). Let \( X \) be the upper half–plane. Then the group \( SL_2(\mathbb{R}) \) acts on \( X \) as follows:

\[
g.z = \frac{az + b}{cz + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).
\]

We let \( \mu(g, z) = cz + d \). The function \( \mu \) satisfies the cocycle identity:

\[
(1-1) \quad \mu(gg', z) = \mu(g, g'.z) \cdot \mu(g', z).
\]

Next, \( SL_2(\mathbb{R}) \)–invariant measure on \( X \) is define by \( dx dy / y^2 \), where the coordinates on \( X \) are written in a usual way \( z = x + \sqrt{-1} y, \ y > 0 \). A discrete subgroup \( \Gamma \subset SL_2(\mathbb{R}) \) is called a Fuchsian group of the first kind if its fundamental domain \( \mathcal{F}_\Gamma \) in \( X \) has a finite volume. Then, adding a finite number of points in \( \mathbb{R} \cup \{ \infty \} \) called cusps, \( \mathcal{F}_\Gamma \) can be compactified. In this way we obtain a compact Riemann surface \( \mathcal{X}_\Gamma \).

As it is required by the construction of ([5], Section 2.6), we assume that \( \Gamma \) has at least one cusp. After conjugation with appropriate element of \( SL_2(\mathbb{R}) \), we may assume that \( \infty \) is one of the cusps.

Let \( \Gamma_\infty \) be the stabilizer of \( \infty \) in \( \Gamma \). By ([5], Theorem 1.5.4), there exists real \( h' > 0 \) such that we
have the following:

\[ \{\pm 1\} \Gamma_\infty = \{\pm 1\} \left\{ \begin{pmatrix} 1 & mh' \\ 0 & 1 \end{pmatrix}; \ m \in \mathbb{Z} \right\}. \]

We write \( \Gamma_{U,\infty} \) for \( \Gamma_\infty \) intersected with the group of upper triangular unipotent matrices in \( SL_2(\mathbb{R}) \). Obviously, \( \Gamma_{U,\infty} \) is a cyclic group generated by \( \begin{pmatrix} 1 & h' \\ 0 & 1 \end{pmatrix} \) where \( h \) is:

\[
(1-2) \quad h = \begin{cases} 
    h' & \text{if } -1 \in \Gamma \\
    h' & \text{if } -1 \notin \Gamma \text{ and } \Gamma_\infty = \Gamma_{U,\infty} \\
    2h' & \text{if } -1 \notin \Gamma \text{ and } [\Gamma_\infty : \Gamma_{U,\infty}] = 2.
\end{cases}
\]

If \( h = 2h' \) the cusp is defined to be irregular. (Otherwise, it is defined to be regular.)

We define \( \epsilon_\Gamma \) by

\[
(1-3) \quad \epsilon_\Gamma = \inf \left\{ |c| : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma - \Gamma_\infty \right\}.
\]

It is proved in ([5], Lemma 1.7.3) that

\[
(1-4) \quad h' \epsilon_\Gamma \geq 1.
\]

Let \( \chi : \Gamma \to \mathbb{C}^\times \) be a character of finite order and let \( m \) be an integer such that \( \chi(\gamma) \mu(\gamma, \cdot)^m \equiv 1 \) for \( \gamma \in \Gamma_\infty \) or equivalently

\[
(1-5) \quad \chi(\gamma) = (-1)^m \text{ if } -1 \in \Gamma \\
\chi \left( \begin{pmatrix} -1 & h' \\ 0 & -1 \end{pmatrix} \right) = (-1)^m \text{ if } \infty \text{ is not regular}.
\]

If \(-1 \in \Gamma \) or \( \infty \) is not regular, then the set of all possible \( m \)'s is not empty and they have the same parity. We call this parity the parity of \( \chi \).

We write \( \mathcal{S}_m(\Gamma, \chi) \) for the space of all cuspidal modular forms of weight \( m \) which transforms according to \( \chi \) i.e., \( f(\gamma, z) = \mu(\gamma, z)^m \chi(\gamma) f(z) \) (\( z \in X, \gamma \in \Gamma \)) (see [5], page 43). The space \( \mathcal{S}_m(\Gamma, \chi) \) is finite dimensional and, when \( \chi \) is trivial, one can use geometric considerations on the compact Riemann surface \( \mathfrak{H}_\Gamma \) to compute the dimension of the space \( \mathcal{S}_m(\Gamma, \chi) \) (see [5], Theorems 2.5.2, 2.5.3.). It is well–known ([5], Corollary 2.6.11) that the spanning set for \( \mathcal{S}_m(\Gamma, \chi) \) (\( m \geq 3 \)) is given by

\[
(1-6) \quad \alpha_{l,m,\chi}(z) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \overline{\chi(\gamma)} \mu(\gamma, z)^{-m} \exp \frac{2\pi i \sqrt{-1} \gamma.z}{h'}, \ l \geq 1.
\]

The natural question is when those functions are not zero and when \( \mathcal{S}_m(\Gamma, \chi) \) is non–trivial. (See [2], 3.2.) We consider those problems in the present paper.
First, it is more convenient to write the spanning set (1-5) in the following way. Using (1-1) and the second assumption in (1-4), we find

\[ \alpha_{l, m, \chi}(z) \overset{\text{def}}{=} \sum_{\gamma \in \Gamma} \overline{\chi(\gamma)} \mu(\gamma, z)^{-m} \exp \frac{2\pi l \sqrt{-1} \gamma.z}{h} \]

\[ = \sum_{\gamma \in \Gamma} \sum_{\delta \in \Gamma} \overline{\chi(\delta \gamma)} \mu(\delta \gamma, z)^{-m} \exp \frac{2\pi l \sqrt{-1} (\delta \gamma).z}{h} \]

\[ = \sum_{\gamma \in \Gamma} \chi(\gamma) \mu(\gamma, z)^{-m} \left( \sum_{\delta \in \Gamma} \exp \frac{2\pi l \sqrt{-1} \delta.z}{h} \right) = \begin{cases} 2 \cdot \alpha'_{l, m, \chi}(z), & \text{in the first case in (1-2)} \\ \alpha'_{l, m, \chi}(z), & \text{in the second case in (1-2)} \\ 0 & \text{if } l \text{ is odd, and } 2 \cdot \alpha'_{l/2, m, \chi}(z) & \text{if } l \text{ is even in the third case in (1-2).} \end{cases} \]

Having prepared the spanning set in this way, we may easily transfer them to the group $SL_2(\mathbb{R})$ and integrate over $\Gamma \setminus SL_2(\mathbb{R})$ (see Lemma 3-1 in Section 3). This is necessary in order to apply the non–vanishing criterion proved in Lemma 2-1. Lemma 2-1 is the main result of Section 2. It considers certain $L^1$–Poincaré series on a locally compact group and their non–vanishing. It is a generalization of (6), Theorem 4-1).

Now, we describe the non–vanishing of cuspidal modular forms given by (1-6) when $m$ is even.

Let

\[ P_k(x) = e^{-x} \sum_{i=0}^{k} \frac{x^i}{i!}, \quad k \geq 0. \]

The function $P_k$ is strictly decreasing on $]0, \infty[$ and there is a unique, say $x_k \in ]0, \infty[$, such that $P_k(x_k) = 1/2$.

**Theorem 1-7.** Let $\chi$ be a character of finite order of $\Gamma$ which is of the even parity if the first or the third case in (1-2) holds. Assume that $m \geq 4$ is even. Then, we have the following:

(i) We assume that $l \geq 1$ is an integer which is even if the third case in (1-2) holds. Then, if $\epsilon_{\Gamma} > \frac{2\pi l}{hx_{m/2}}$ or equivalently, $P_{m/2-2} (\frac{2\pi l}{h\epsilon_{\Gamma}}) > 1/2$, then the series $\alpha_{l, m, \chi}$ (see (1-6)) is non–zero.

(ii) $S_m(\Gamma, \chi)$ is non–trivial for all even integers $m \geq 16$. More precisely, we have the following:

\[ \begin{cases} m \geq 4 & \text{for } h'\epsilon_{\Gamma} \geq 10 \\ m \geq 6 & \text{for } h'\epsilon_{\Gamma} \geq 4 \\ m \geq 8 & \text{for } h'\epsilon_{\Gamma} \geq 3 \\ m \geq 10 & \text{for } h'\epsilon_{\Gamma} \geq 2 \\ m \geq 16 & \text{for } h'\epsilon_{\Gamma} \geq 1. \end{cases} \]

(iii) We assume that $l \geq 1$ is an integer which is even if the third case in (1-2) holds. Then, if $m \geq \frac{4\pi l}{h\epsilon_{\Gamma}} + 4$, then the series $\alpha_{l, m, \chi}$ is non–zero.
(iv) As \( m \to \infty \), the series \( \alpha_{l,m,\chi} \) are non-zero for \( 1 \leq l \leq \left\lfloor \frac{h\epsilon \Gamma}{2\pi} (m/2 - 4/3) \right\rfloor - 1 \), assuming that \( l \) is even if the third case in (1-2) holds. Here \( \left\lfloor x \right\rfloor \) is the largest integer \( \leq x \).

We prove Theorem 1-7 in Section 4. The main thing is part (i). It is a consequence of above mentioned results of Sections 2 and 3. (i) contains a simple criterion for non–vanishing of Poincaré series \( \alpha_{l,m,\chi} \) (see (1-6)) when we apply it in the form

\[
P_{m/2 - 2} \left( \frac{2\pi l}{h\epsilon \Gamma} \right) > \frac{1}{2}.
\]

Applying this to the non–vanishing of \( \alpha_{1,m,\chi} \) \( (l = 1, \) the first two cases in (1-2)) and \( \alpha_{2,m,\chi} \) \( (l = 2, \) in the last case in (1-2)), we obtain (ii). The congruence subgroups \( \Gamma(N) \), \( \Gamma_0(N) \), and \( \Gamma_1(N) \) \( (N \geq 1) \) are discussed in Example 4-7.

The estimate \( m \geq 16 \) in (ii) for "general \( h'\epsilon \Gamma \geq 1" \) (see (1-3)) is optimal since for \( \Gamma = SL_2(\mathbb{Z}) \) we have that \( S_{12}(\Gamma) \neq 0, S_{14}(\Gamma) = 0, \) and \( S_m(\Gamma) = 0 \) for any even integer \( m \geq 16 \).

The claims (iii) and (iv) of Theorem 1-7 also follow from (i) but they use the first form of the non–vanishing criterion \( \epsilon \Gamma > \frac{2\pi l}{h\epsilon_m / 2} \) and the following two claims (see Lemma 4-2):

\[
\begin{aligned}
x_k &> k \quad \text{for} \quad k \geq 0 \\
x_k &= k + 2/3 + o(1) \quad \text{as} \quad k \to \infty,
\end{aligned}
\]

respectively.

Now, we describe the non–vanishing of cuspidal modular forms given by (1-6) when \( m \) is odd. We let

\[
Q_k(x) = -2 \int_{0}^{\sqrt{x}} e^{-t^2} dt + e^{-x} \sum_{i=1}^{k} \frac{x^{i-1/2}}{(i-1/2)!}, \quad k \geq 1,
\]

where we write

\[
(n - 1/2)! \overset{def}{=} (1/2)(3/2) \cdots (n - 1/2), \quad n \geq 1.
\]

The function \( Q_k \) is strictly decreasing on \( [0, \infty] \) and there is a unique, say \( y_k \in [0, \infty] \), such that \( Q_k(y_k) = -\sqrt{\pi}/2 \). Lemma 5-3 implies that

\[
\begin{aligned}
y_k &> k - 1/2 \quad \text{for} \quad k \geq 1 \\
y_k &= k + 1/6 + o(1) \quad \text{as} \quad k \to \infty.
\end{aligned}
\]

Theorem 1-8. Let \( \chi \) be a character of finite order of \( \Gamma \) which is of the odd parity if the first or the third case in (1-2) holds. Assume that \( m \geq 3 \) is odd. Then, we have the following:

(i) We assume that \( l \geq 1 \) is an integer which is even if the third case in (1-2) holds. Let \( \chi \) be a character of finite order of \( \Gamma \) satisfying (1-4). Then, if

\[
\epsilon \Gamma > \frac{2\pi l}{h\Gamma} \overset{def}{=} \frac{2\pi l}{h\epsilon_m / 2},
\]

equivalently, \( Q_{m-1/2} \left( \frac{2\pi l}{h\epsilon \Gamma} \right) > -\sqrt{\pi}/2, \)

then the series \( \alpha_{l,m,\chi} \) (see (1-6)) is non–zero.

(ii) We assume that \( l \geq 1 \) is an integer which is even if the third case in (1-2) holds. Then, if

\[
m \geq \frac{4\pi l}{h\epsilon \Gamma} + 2,
\]

then the series \( \alpha_{l,m,\chi} \) is non–zero. Thus, \( S_m(\Gamma, \chi) \) is non–trivial for

\[
m \geq \frac{4\pi}{h\epsilon \Gamma} + 2.
\]

In particular, \( S_m(\Gamma, \chi) \) is non–trivial for all odd integers \( m \geq 15 \).
(iii) As $m \to \infty$, the series $\alpha_{l,m,\chi}$ are non–zero for $1 \leq l \leq \left\lfloor \frac{h_{\Gamma}}{2\pi} (m/2 - 1/3) \right\rfloor - 1$, assuming that $l$ is even if the third case in (1-2) holds.

In [7] we construct another spanning set for $S_m(\Gamma)$ which does not depend on the fact that $\Gamma$ has cusps or not. We study their non–vanishing by using Lemma 3-1 in Section 3.

2. A Non–Vanishing Criterion for Certain Poincaré Series

In this section we use the integration theory for locally compact groups to prove the non–vanishing of certain Poincaré series. The main result of this section, Lemma 2-1, is the generalization and adaptation of ([6], Theorem 4-1) to our present needs.

**Lemma 2-1.** Let $G$ be a locally compact unimodular (Hausdorff) topological group. Let $\Gamma \subseteq G$ be its discrete subgroup and $\Gamma_1 \subseteq \Gamma$ an arbitrary subgroup. We let $\chi : \Gamma \to \mathbb{C}^\times$ be an unitary character of $\Gamma$ trivial on $\Gamma_1$. Let $\varphi \in L^1(\Gamma_1 \backslash G)$. Assume that there exists a subgroup $\Gamma_2 \subseteq \Gamma$ such that $\Gamma_1$ is normal subgroup of $\Gamma_2$ of finite index and there exists a compact set $C$ in $G$ such that the following conditions hold:

1. $\varphi(\gamma g) = \overline{\chi(\gamma)} \varphi(g)$, for all $\gamma \in \Gamma_2$ and almost all $g \in G$,
2. $\Gamma \cap C \cdot C^{-1} \subseteq \Gamma_2$, and
3. $\int_{\Gamma_1 \backslash \Gamma_1 \cap C} |\varphi(g)| \, dg > \frac{1}{2} \int_{\Gamma_1 \backslash G} |\varphi(g)| \, dg$.

Then the Poincaré series defined by

$$P^{(\chi)}_{\Gamma_1 \backslash \Gamma}(\varphi)(g) \equiv \sum_{\gamma \in \Gamma_1 \backslash \Gamma} \chi(\gamma) \varphi(\gamma \cdot g)$$

converges absolutely almost everywhere to a non–zero element of $L^1(\Gamma \backslash G)$.

**Proof.** We let $P_\Gamma = P^{(\chi)}_{\Gamma_1 \backslash \Gamma}$, if $\Gamma_1$ and $\chi$ are trivial, and $P_{\Gamma_1 \backslash \Gamma} = P^{(\chi)}_{\Gamma_1 \backslash \Gamma}$ if $\chi$ is trivial.

It is a general fact from the integration theory that $\psi \mapsto P_{\Gamma_1}(\psi)$ is an epimorphism $C_c(G) \to C_c(\Gamma_1 \backslash G)$ (see ([3], Chapter XII, Section 4, Theorem 4.1)), and the measure on $\Gamma_1 \backslash G$ is fixed by the rule $\int_{\Gamma_1 \backslash G} P_{\Gamma_1}(\psi)(g) \, dg = \int_G \psi(g) \, dg$. The same applies to $\Gamma$. It is easy to check that $P_\Gamma = P_{\Gamma_1 \backslash \Gamma} P_{\Gamma_1}$.

Hence, we obtain the following:

$$\int_{\Gamma \backslash G} P_{\Gamma_1}(\psi)(g) \, dg = \int_{\Gamma_1 \backslash G} \psi(g) \, dg = \int_{\Gamma \backslash G} P_\Gamma(\psi)(g) \, dg = \int_{\Gamma \backslash G} P_{\Gamma_1 \backslash \Gamma}(P_{\Gamma_1}(\psi))(\gamma) \, dg.$$  

This implies the following integration formula:

$$\int_{\Gamma \backslash G} P_{\Gamma_1 \backslash \Gamma}(\varphi)(g) \, dg = \int_{\Gamma_1 \backslash \Gamma} \varphi(g) \, dg, \quad \varphi \in L^1(\Gamma_1 \backslash G).$$  

Now, let $\varphi \in L^1(\Gamma_1 \backslash G)$. Then, the expression for the Poincaré series $\sum_{\gamma \in \Gamma_1 \backslash \Gamma} \chi(\gamma) \varphi(\gamma \cdot g)$ does not depend on the choice of the representatives for $\Gamma_1 \backslash \Gamma$ since $\chi$ is trivial on $\Gamma_1$. Moreover, it converges absolutely almost everywhere since the identity (2-2) implies

$$\int_{\Gamma \backslash G} \left( \sum_{\gamma \in \Gamma_1 \backslash \Gamma} |\varphi(\gamma \cdot g)| \right) \, dg = \int_{\Gamma_1 \backslash G} |\varphi(g)| \, dg < \infty.$$

Hence

$$|P^{(\chi)}_{\Gamma_1 \backslash \Gamma}(\varphi)(g)| \leq \sum_{\gamma \in \Gamma_1 \backslash \Gamma} |\chi(\gamma) \varphi(\gamma \cdot g)| = \sum_{\gamma \in \Gamma_1 \backslash \Gamma} |\varphi(\gamma \cdot g)| < \infty \text{ almost everywhere on } G.$$
In particular, $P_{\Gamma \setminus G}^{(\chi)}(\varphi)$ is measurable on $\Gamma \setminus G$, and

\[ P_{\Gamma \setminus G}^{(\chi)}(\varphi)(\chi \cdot g) = \chi(\gamma) \cdot P_{\Gamma \setminus G}^{(\chi)}(\varphi)(g), \]

for all $\gamma \in \Gamma$ and almost everywhere for $g \in G$. Moreover, (2-3) and (2-4) imply

\[ \int_{\Gamma \setminus G} \left| P_{\Gamma \setminus G}^{(\chi)}(\varphi)(g) \right| \, dg \leq \int_{\Gamma \setminus G} \left( \sum_{\gamma \in \Gamma \setminus G} |\chi(\gamma)\varphi(\gamma \cdot g)| \right) \, dg < \infty. \]

Thus, $\left| P_{\Gamma \setminus G}^{(\chi)}(\varphi) \right| \in L^1(\Gamma \setminus G)$.

Now, let $\varphi \in L^1(\Gamma_1 \setminus G)$ such that (1), (2), and (3) hold. We must show that $\int_{\Gamma \setminus G} \left| P_{\Gamma \setminus G}^{(\chi)}(\varphi)(g) \right| \, dg \neq 0$. To end this, we adapt the proof of ([6], Theorem 4-1). We begin with the following lemma:

**Lemma 2-6.** There exist $\psi \in C_c(\Gamma_1 \setminus G)$ such that $\text{supp } \psi \subset \Gamma_1 \setminus \Gamma_2 \cdot C$, $\psi(\gamma g) = \overline{\chi(\gamma)}\psi(g)$, for all $\gamma \in \Gamma_2$ and $g \in G$, and $\int_{\Gamma_1 \setminus G} |\psi - \varphi| < \int_{\Gamma_1 \setminus G} |\psi|$.

**Proof.** Replacing $C$ by its interior does not affect the assumptions (2) and (3) of the lemma. Thus, in this proof we may assume that $C$ is open with a compact closure. Then $\Gamma_1 \setminus \Gamma_2 \cdot C$ is an open set with a compact closure. Since it is open, we may consider $C_c(\Gamma_1 \setminus \Gamma_2 \cdot C)$ as a subset of $C_c(\Gamma_1 \setminus G)$. It is well–known that $C_c(\Gamma_1 \setminus \Gamma_2 \cdot C)$ is dense in $L^1(\Gamma_1 \setminus \Gamma_2 \cdot C)$ (see [3], the definition of the Haar measure given in the first paragraph of the page 313 and Theorem 3.1). Hence, we can find a sequence $(\psi_n)_{n \geq 1}$, where $\psi_n \in C_c(\Gamma_1 \setminus \Gamma_2 \cdot C)$, such that $\int_{\Gamma_1 \setminus \Gamma_2 \cdot C} |\psi_n(g) - \varphi(g)| \, dg \to 0$ as $n \to \infty$. In particular, $\int_{\Gamma_1 \setminus \Gamma_2 \cdot C} |\psi_n(g)| \, dg \to \int_{\Gamma_1 \setminus \Gamma_2 \cdot C} |\varphi(g)| \, dg$ as $n \to \infty$. Now, we compute using (1)

\[ \int_{\Gamma_1 \setminus \Gamma_2 \cdot C} \left| \frac{1}{\#(\Gamma_1 \setminus \Gamma_2)} P_{\Gamma_1 \setminus \Gamma_2}^{(\chi)}(\psi_n)(g) - \varphi(g) \right| \, dg \leq \frac{1}{\#(\Gamma_1 \setminus \Gamma_2)} \sum_{\gamma \in \Gamma_1 \setminus \Gamma_2} \int_{\Gamma_1 \setminus \Gamma_2 \cdot C} |\psi_n(\gamma g) - \varphi(\gamma g)| \, dg \]

Since $\Gamma_1 \setminus \Gamma_2$ is a finite group, the left–hand side is equal to

\[ \int_{\Gamma_1 \setminus \Gamma_2 \cdot C} |\psi_n(g) - \varphi(g)| \, dg. \]

This shows that we can assume that all members of the sequence $(\psi_n)_{n \geq 1}$ satisfy $\psi_n(\gamma g) = \overline{\chi(\gamma)}\psi_n(g)$, for all $\gamma \in \Gamma_2$ and $g \in G$.

Now, we compute

\[ \int_{\Gamma_1 \setminus G} |\psi_n(g)| \, dg - \int_{\Gamma_1 \setminus G} |\psi_n(g) - \varphi(g)| \, dg = \int_{\Gamma_1 \setminus \Gamma_2 \cdot C} |\psi_n(g)| \, dg - \int_{\Gamma_1 \setminus \Gamma_2 \cdot C} |\psi_n(g) - \varphi(g)| \, dg \\
- \int_{\Gamma_1 \setminus G \setminus \Gamma_2 \cdot C} |\psi_n(g) - \varphi(g)| \, dg = \int_{\Gamma_1 \setminus \Gamma_2 \cdot C} |\psi_n(g)| \, dg - \int_{\Gamma_1 \setminus \Gamma_2 \cdot C} |\psi_n(g) - \varphi(g)| \, dg - \int_{\Gamma_1 \setminus G \setminus \Gamma_2 \cdot C} |\varphi(g)| \, dg. \]

Thus, as $n \to \infty$, this approaches

\[ \int_{\Gamma_1 \setminus \Gamma_2 \cdot C} |\varphi(g)| \, dg - \int_{\Gamma_1 \setminus G \setminus \Gamma_2 \cdot C} |\varphi(g)| \, dg = 2 \int_{\Gamma_1 \setminus \Gamma_2 \cdot C} |\varphi(g)| \, dg - \int_{\Gamma_1 \setminus G} |\varphi(g)| \, dg. \]

We remark that (1) implies that

\[ \int_{\Gamma_1 \setminus \Gamma_2 \cdot C} |\varphi(g)| \, dg = \#(\Gamma_1 \setminus \Gamma_2) \int_{\Gamma_1 \setminus \Gamma_2 \cdot C} |\varphi(g)| \, dg. \]
Proof. We can write
\[ \sum_{\gamma \in \Gamma \backslash \Gamma_{2}} |\varphi(g)| dg - \int_{\Gamma \backslash G} |\varphi(g)| dg \geq 2 \int_{\Gamma \backslash \Gamma_{2}} |\varphi(g)| dg - \int_{\Gamma \backslash G} |\varphi(g)| dg > 0. \]
This proves the lemma. \( \square \)

We need one more lemma.

Lemma 2-7. Let \( \psi \) be given by Lemma 2-6. Then \( P_{\Gamma \backslash \Gamma_{2}}^{(\psi)}(g) = \#(\Gamma \backslash \Gamma_{2}) \psi(g) \) for \( g \in \Gamma \cdot C \).

Proof. We can write \( g = \gamma_{1} \cdot c_{1}, \gamma_{1} \in \Gamma_{2}, c_{1} \in C \). Since, by definition, we have \( P_{\Gamma \backslash \Gamma}^{(\psi)}(g) = \sum_{\gamma \in \Gamma \backslash \Gamma} \chi(\gamma) \psi(\gamma \cdot g) \), to compute this expression, analyze when \( \psi(\gamma \cdot g) \neq 0 \). It must be \( \gamma \cdot g \in \Gamma \cdot C \).

Thus, we may write \( \gamma \cdot g = \gamma_{2} \cdot c_{2}, \gamma_{2} \in \Gamma_{2}, c_{2} \in C \). Now, we have \( \gamma \cdot (\gamma_{1} \cdot c_{1}) = \gamma_{2} \cdot c_{2} \) which implies
\[ \gamma_{2}^{-1} \gamma_{1} = c_{2}c_{1}^{-1} \in \Gamma \cap C \cdot C^{-1}. \]
Thus, by (2), we obtain \( \gamma_{2}^{-1} \gamma_{1} \in \Gamma_{2} \). Hence \( \gamma \in \Gamma_{2} \). Since \( \psi(g) = \overline{\chi(\gamma)} \psi(g) \), for all \( \gamma \in \Gamma_{2} \) and \( g \in G \), the lemma follows. \( \square \)

Now, we prove the lemma. We compute using Lemma 2-6
\[ \int_{\Gamma \backslash G} |\psi(g)| dg > \int_{\Gamma \backslash G} |\varphi(g) - \psi(g)| dg = \int_{\Gamma \backslash G} \left( \sum_{\gamma \in \Gamma \backslash \Gamma} |\varphi(\gamma \cdot g) - \psi(\gamma \cdot g)| \right) dg = \int_{\Gamma \backslash G} \left( \sum_{\gamma \in \Gamma \backslash \Gamma} |\chi(\gamma) \varphi(\gamma \cdot g) - \chi(\gamma) \psi(\gamma \cdot g)| \right) dg \geq \int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}^{(\chi)}(\varphi)(g) - P_{\Gamma \backslash \Gamma}^{(\chi)}(\psi)(g)| dg \geq \int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}^{(\chi)}(\psi)(g)| dg - \int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}^{(\chi)}(\varphi)(g)| dg. \]
This implies the following:
\[ (2-8) \quad \int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}^{(\psi)}(\varphi)(g)| dg > \int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}^{(\psi)}(\psi)(g)| dg - \int_{\Gamma \backslash G} |\psi|. \]
Next, we have the following:
\[ (2-9) \quad \int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}^{(\psi)}(\psi)(g)| dg \geq \int_{\Gamma \backslash \Gamma \cdot C} |P_{\Gamma \backslash \Gamma}^{(\psi)}(\psi)(g)| dg = \int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}^{(\psi)}(\psi)(g)| \text{char}_{\Gamma \backslash \Gamma \cdot C}(g) dg. \]

Arguing as in the proof of Lemma 2-7, we find that the characteristic functions are related by
\[ \#(\Gamma \backslash \Gamma_{2}) \text{char}_{\Gamma \backslash \Gamma \cdot C} = P_{\Gamma \backslash \Gamma}^{(\psi)}(\text{char}_{\Gamma \backslash \Gamma_{2}}). \]
The same lemma implies $P_{\Gamma_1 \Gamma}(\psi)(g) = \#(\Gamma_1 \setminus \Gamma_2)\psi(g)$ for $g \in \Gamma_2 \cdot C$. Therefore, if we consider $|P_{\Gamma_1 \Gamma}(\psi)(\cdot)|$ as a function on $G$ invariant by $\Gamma$ on the left, then the usual integration implies

$$\int_{\Gamma \setminus G} |\psi(g)| dg = \int_{\Gamma_1 \setminus \Gamma_2 \cdot C} |\psi(g)| dg = \frac{1}{\#(\Gamma_1 \setminus \Gamma_2)} \int_{\Gamma_1 \setminus \Gamma_2 \cdot C} |P_{\Gamma_1 \Gamma}(\psi)(g)| dg =$$

$$\frac{1}{\#(\Gamma_1 \setminus \Gamma_2)} \int_{\Gamma \setminus G} \left( \sum_{\gamma \in \Gamma_1 \setminus \Gamma} \left| P_{\Gamma_1 \Gamma}(\psi)(\gamma \cdot g) \right| \cdot char_{\Gamma_1 \setminus \Gamma_2 \cdot C}(\gamma \cdot g) \right) dg =$$

$$\frac{1}{\#(\Gamma_1 \setminus \Gamma_2)} \int_{\Gamma \setminus G} \left( \sum_{\gamma \in \Gamma_1 \setminus \Gamma} char_{\Gamma_1 \setminus \Gamma_2 \cdot C}(\gamma \cdot g) \right) dg = \int_{\Gamma \setminus G} |P_{\Gamma_1 \setminus \Gamma}(\psi)(g)| \cdot char_{\Gamma \setminus \Gamma \cdot C}(g) dg.$$

Therefore, (2-8) and (2-9) imply

$$\int_{\Gamma \setminus G} |P_{\Gamma_1 \setminus \Gamma}(\varphi)(g)| dg > \int_{\Gamma \setminus G} |P_{\Gamma_1 \setminus \Gamma}(\psi)(g)| dg - \int_{\Gamma \setminus G} |\psi(g)| dg > \int_{\Gamma \setminus G} |\psi(g)| dg - \int_{\Gamma \setminus G} |\psi(g)| dg = 0.$$

This proves the lemma. □

3. A Criterion for Non–Vanishing of Modular Forms

In this section we apply Lemma 2-1 to study a non–vanishing of modular forms.

**Lemma 3-1.** Let $\Gamma \subset SL_2(\mathbb{R})$ be a discrete subgroup of finite covolume such that $\infty$ is its cusp. We maintain the notation introduced in (1-2). Let $\chi : \Gamma \to \mathbb{C}^\times$ be a character of finite order and let $m$ be an integer such that (1-4) holds. Let $f : X \to \mathbb{C}$ be a holomorphic function satisfying the following two conditions:

(i) $f$ is periodic with period $h$ in the first two cases in (1-2) and with period $h/2$ in the last case in (1-2), and

(ii) $\int_0^h \int_0^\infty y^{m/2} |f(z)| \frac{dy}{y} < \infty$.

Then the series

$$\sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \chi(\gamma) f(\gamma z) \mu(\gamma, z)^{-m}$$

is non–trivial provided that there exists a compact set $C \subset [0, h] \times [0, \infty]$ such that the following two conditions hold:

(1) $\gamma \cdot C \cap C \neq \emptyset$ implies $\gamma \in \Gamma_{\infty}$, and

(2) $\int_C y^{m/2} |f(z)| \frac{dy}{y} > \frac{1}{2} \int_0^h \int_0^\infty y^{m/2} |f(z)| \frac{dy}{y}$.

**Proof.** We apply Lemma 2-1. So, we need to transfer the claim to the group $SL_2(\mathbb{R})$. The Iwasawa decomposition of $SL_2(\mathbb{R})$ implies that every $g \in SL_2(\mathbb{R})$ can be written uniquely in the following form:

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad x, y, t \in \mathbb{R}, \ y > 0.$$
We have \( g \sqrt{-1} = x + y \sqrt{-1} \in X \). The stabilizer of \( \sqrt{-1} \) we denote by \( K \). It is well–known that \( K \) is a maximal compact subgroup of \( SL_2(\mathbb{R}) \). We have

\[
K = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} : t \in \mathbb{R} \right\}.
\]

The Haar measure on \( SL_2(\mathbb{R}) \) can be fixed as follows:

\[
(3-3) \quad \int_{SL_2(\mathbb{R})} \psi(g) dg = \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{1}{2\pi} \int_{0}^{2\pi} \psi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right) \frac{dxdy}{y^2} dt, \quad \psi \in C_c^\infty(SL_2(\mathbb{R})).
\]

Now, we begin the proof of the lemma. We define the function \( F_f : SL_2(\mathbb{R}) \rightarrow \mathbb{C} \) by the following expression:

\[
(3-4) \quad F_f(g) = f(g \sqrt{-1}) \mu(g, \sqrt{-1})^{-m}.
\]

Using the Iwasawa decomposition (3-2), we obtain the following:

\[
(3-5) \quad F_f \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right) = y^{m/2} \exp(m \sqrt{-1} t) f(x + y \sqrt{-1}).
\]

It is clear from the discussion given by the paragraph containing (1-2) that \( \#(\Gamma_{U,\infty} \backslash \Gamma_{\infty}) \leq 2 \). Also, (1-4) shows that \( \chi \) is trivial on \( \Gamma_{U,\infty} \). Hence, the order of \( \chi \) is at most two i.e., \( \chi = \overline{\chi} \). Since by (i) \( f \) is \( h' \)–periodic in all cases, (1-1), (1-4), and (3-4) show

\[
(3-6) \quad F_f(\gamma_\infty g) = \overline{\chi(\gamma_\infty)} F_f(g), \quad g \in SL_2(\mathbb{R}), \quad \gamma_\infty \in \Gamma_{\infty},
\]

and

\[
(3-7) \quad F_f(\gamma_\infty g) = F_f(g), \quad g \in SL_2(\mathbb{R}), \quad \gamma_\infty \in \Gamma_{U,\infty}.
\]

We can write a very simple formula for the measure on \( \Gamma_{U,\infty} \backslash SL_2(\mathbb{R}) \) (see (3-3)):

\[
(3-8) \quad \int_{\Gamma_{U,\infty} \backslash SL_2(\mathbb{R})} \left( \sum_{\gamma_\infty \in \Gamma_{U,\infty}} \psi(\gamma_\infty \cdot g) \right) dg = \int_0^h \int_0^{\infty} \int_0^{2\pi} \psi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right) \frac{dxdy}{y^2} dt, \quad \psi \in C_c^\infty(SL_2(\mathbb{R})).
\]

We will apply Lemma 2-1 to \( G = SL_2(\mathbb{R}), \Gamma_1 = \Gamma_{U,\infty}, \Gamma_2 = \Gamma_{\infty}, \) and \( \varphi = F_f \). We have \( \varphi \in L^1(\Gamma_1 \backslash G) \) (see Lemma 2-1). Indeed, we compute using (3-5):

\[
\int_{\Gamma_{U,\infty} \backslash SL_2(\mathbb{R})} |F_f(g)| dg = \int_0^h \int_0^{\infty} \int_0^{2\pi} \left| F_f \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right) \right| \frac{dxdy}{y^2} dt = \int_0^h \int_0^{\infty} \left| F_f \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \right) \right| \frac{dxdy}{y^2} = \int_0^h \int_0^{y^{m/2}} |f(z)| \frac{dxdy}{y^2} < \infty,
\]
using (ii). Also, (3-6) and (3-7) imply all assumptions of Lemma 2-1 except (2) and (3). For (2) and (3) we need a compact set in $SL_2(\mathbb{R})$. We construct such set out of $C$.

First, we define $\tilde{C} \subset SL_2(\mathbb{R})$ using (3-2) as follows:

$$
\begin{pmatrix}
1 & x \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
y^{1/2} & 0 \\
0 & y^{-1/2}
\end{pmatrix}
\in \tilde{C}
$$

if and only if $g \cdot \sqrt{-1} = x + y\sqrt{-1} \in C$.

It is clear that $\tilde{C}$ and $\tilde{C}K$ are compact subsets in $SL_2(\mathbb{R})$. Next, we check the assumptions (2) and (3) of Lemma 2-1 for a compact set $\tilde{C}K$.

We have the following:

$$
(3-9)
\int_{\Gamma U,\infty} |F_f(g)| \, dg = \int_{\Gamma} y^{m/2} |f(z)| \frac{dxdy}{y^2},
$$

since $C \subset [0, h] \times ]-\infty, \infty[$. Hence, the assumption (2) of the lemma implies

$$
\int_{\Gamma U,\infty} |F_f(g)| \, dg > \frac{1}{2} \int_{\Gamma U,\infty} |F_f(g)| \, dg
$$

which is (3) of Lemma 2-1 in our case.

Next, we show the following:

$$
(3-10) \Gamma \cap (\tilde{C}K) \cdot (\tilde{C}K)^{-1} \subset \Gamma_{\infty}
$$

which is (2) of Lemma 2-1. Indeed, if $\gamma \in \Gamma \cap (\tilde{C}K) \cdot (\tilde{C}K)^{-1}$, then we can find $\tilde{c}_1, \tilde{c}_2 \in \tilde{C}$ and $k_1, k_2 \in K$ such that $\gamma \tilde{c}_1 k_1 = \tilde{c}_2 k_2$. Then, acting on $\sqrt{-1} \in X$, we find $\gamma c_1 = c_2$, where $c_1 = \tilde{c}_1 \cdot \sqrt{-1}$ and $c_2 = \tilde{c}_2 \cdot \sqrt{-1}$ are in $C$. Hence the assumption (1) implies $\gamma \in \Gamma_{\infty}$. This proves the claim (3-10).

Now, Lemma 2-1 implies the non–vanishing of

$$
\sum_{\gamma \in \Gamma U,\infty} \chi(\gamma) F_f(\gamma g).
$$

Finally, for $z = g \cdot \sqrt{-1}$, using (1-1) we have the following:

$$
\sum_{\gamma \in \Gamma U,\infty} \chi(\gamma) f(\gamma, z) \mu(\gamma, z)^{-m} = \mu(g, \sqrt{-1})^{m} \sum_{\gamma \in \Gamma U,\infty} \chi(\gamma) F_f(\gamma g).
$$

This proves the lemma.


4. The proof of Theorem 1-7

We let

$$
P_k(x) = e^{-x} \sum_{i=0}^{k} \frac{x^i}{i!}, \quad k \geq 0.
$$

Then $dP_k(x)/dx = -x^k e^{-x}/k!$. Hence

$$
(4-1) \int_{a}^{b} x^k e^{-x} \, dx = k! \cdot (P_k(a) - P_k(b)).
$$

Also, $P_k$ is strictly decreasing on $[0, \infty[$. Since $P_k(0) = 1$ and $P_k(\infty) = 0$. There is a unique $x_k \in [0, \infty[$ such that $P_k(x_k) = 1/2$. We have the following lemma:

**Lemma 4-2.**

(i) $\lim_{k} (x_k - (k + 2/3)) = 0$.

(ii) $x_k > k$ for all $k \geq 0$. 


The proof of (ii) begins with the following lemma:

**Lemma 4-3.** Let \((a_k)_{k \geq 0}\) be a strictly increasing sequence of positive real numbers. Then

\[
P_k(a_k) - P_{k+1}(a_{k+1}) = -\frac{a_{k+1}^{k+1}e^{-a_{k+1}}}{(k+1)!} + \frac{1}{k!} \int_{a_k}^{a_{k+1}} x^k e^{-x} \, dx.
\]

In addition, if \(a_k = k\) for all \(k \geq 0\), then the sequence \((P_k(a_k))\) \((k \geq 0)\) is strictly decreasing.

**Proof.** Applying (4-1), we find the following:

\[
(k + 1)! \cdot (1 - P_{k+1}(a_{k+1})) = \int_{a_k}^{a_{k+1}} x^{k+1} e^{-x} \, dx,
\]

and analogously for \(k\). Now, starting from the displayed equality and using partial integration, we obtain the first claim.

We remark that \(x^k e^{-x}\) is strictly decreasing for \(x > k\). So, if \(a_k = k\) for all \(k \geq 0\), then

\[
P_k(a_k) - P_{k+1}(a_{k+1}) = -\frac{a_{k+1}^{k+1}e^{-a_{k+1}}}{(k+1)!} + \frac{1}{k!} \int_{a_k}^{a_{k+1}} x^k e^{-x} \, dx > -\frac{a_{k+1}^{k+1}e^{-a_{k+1}}}{(k+1)!} + (a_{k+1} - a_k) \frac{a_{k+1}^k e^{-a_{k+1}}}{k!} = 0.
\]

Since \(P_k(k)\) is obviously non–negative, the limit \(\lim_{k \to \infty} P_k(k)\) exists. We show

\[
(4-4) \quad \lim_{k \to \infty} P_k(k) = 1/2.
\]

Then, since the sequence \((P_k(k))_{k \geq 0}\) is strictly decreasing, we obtain

\[
P_k(k) > \lim_{n \to \infty} P_n(n) = \frac{1}{2} = P_k(x_k).
\]

Hence \(x_k > k\). This proves (ii).

It remains to prove (4-4). First, applying (4-1), we find

\[
P_k(k) - \frac{1}{2} = P_k(k) - P_k(x_k) = \frac{1}{k!} \int_k^{x_k} x^k e^{-x} \, dx.
\]

Since the function \(x^k e^{-x}\) on \([0, \infty]\) has a maximum at \(x = k\), we find

\[
\left| P_k(k) - \frac{1}{2} \right| \leq \frac{k^k e^{-k}}{k!} |x_k - k|.
\]

Applying Lemma 4-3 (i) and the classical Stirling’s inequality:

\[
1 \leq \frac{k!}{\sqrt{2\pi k} k^{k+1/2} e^{-k}}, \quad k \geq 1,
\]

we obtain (4-4).

Now, we prove Theorem 1-7.
Proof. First, we prove (i). We select $\epsilon^{-1} < \epsilon < \frac{hx_m/2-2}{2\pi}$. Then

$$P_{m/2-2} \left( \frac{2\pi \epsilon}{h} \right) > P_{m/2-2} \left( \frac{x_m}{2} \right) = \frac{1}{2}.$$ 

Then we select $\delta > \epsilon$ such that

$$\left[ P_{m/2-2} \left( \frac{2\pi \epsilon}{h} \right) - P_{m/2-2} \left( \frac{2\pi \delta}{h} \right) \right] > \frac{1}{2}.$$ 

Now, we apply the non–vanishing criterion Lemma 3-1. Note that the assumption (i) of Lemma 3-1 is satisfied in all cases since by our assumption $l$ is even in the third case of (1-2).

We have the following:

$$\int_{0}^{h} \int_{\epsilon}^{\delta} y^{m/2} \left| e^{2\pi i \sqrt{-1}z/h} \right| \frac{dxdy}{y^2} = h \int_{\epsilon}^{\delta} y^{m/2} e^{-2\pi ly/h} dy = h \left( \frac{2\pi l}{h} \right)^{1-m/2} (m/2 - 1)! \left[ P_{m/2-2} \left( \frac{2\pi \epsilon}{h} \right) - P_{m/2-2} \left( \frac{2\pi \delta}{h} \right) \right].$$

Next, we have the following:

$$\int_{0}^{h} \int_{0}^{\infty} y^{m/2} \left| e^{2\pi i \sqrt{-1}z/h} \right| \frac{dxdy}{y^2} = h \left( \frac{2\pi l}{h} \right)^{1-m/2} (m/2 - 1)!.$$

Hence, (4-5) imply

$$\int_{0}^{h} \int_{\epsilon}^{\delta} y^{m/2} \left| e^{2\pi i \sqrt{-1}z/h} \right| \frac{dxdy}{y^2} - \frac{1}{2} \int_{0}^{h} \int_{0}^{\infty} y^{m/2} \left| e^{2\pi i \sqrt{-1}z/h} \right| \frac{dxdy}{y^2} > 0.$$

This is the condition (2) in Lemma 3-1.

Now, we recall the following formulas. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$. Then the imaginary part of $\gamma z$ is given by $\text{Im}(\gamma z) = y/|\mu(\gamma, z)|^2 = y/|cz + d|^2$, for all $z = x + \sqrt{-1}y \in X$. If $z \in C$ and $\gamma z \in C$, then $y$ and $\text{Im}(\gamma z)$ belong to $[\epsilon, \delta]$. Hence

$$\frac{y}{|cz + d|^2} = \frac{y}{(cx + d)^2 + c^2 y^2} \geq \epsilon.$$

Thus, we obtain

$$(cx + d)^2 \leq \frac{y}{\epsilon} - c^2 y^2.$$ 

In particular, we obtain the following:

$$\frac{y}{\epsilon} - c^2 y^2 \geq 0.$$

Hence

$$\frac{1}{\epsilon} \geq c^2 y \geq c^2 \epsilon.$$

Finally, we obtain

$$|c| \leq \frac{1}{\epsilon} < \epsilon r.$$

Now, ([5], Lemma 1.7.3) implies $\gamma \in \Gamma_{\infty}$ This proves (1) in Lemma 3-1. This completes the proof of (i). We prove (ii). First, we note that (i) can be used to easily test the non–vanishing of the modular forms (1-6). The condition

$$P_{m/2-2} \left( \frac{2\pi l}{h \epsilon r} \right) > \frac{1}{2}.$$
in Theorem 1-7 is equivalent to:

$$\log \left( \sum_{i=0}^{m/2-2} \frac{x^i}{i!} \right) > x - \log 2,$$

where

$$x = \frac{2\pi l}{h\epsilon_{\Gamma}}.$$

So, the algorithm is obvious. For fixed \( \Gamma \) and \( l \geq 1 \) (i.e., fixed \( x \)), find the least \( k \geq 0 \) such that

$$\log \left( \sum_{i=0}^{k} \frac{x^i}{i!} \right) > x - \log 2.$$

Then, for

$$m \geq 2(k + 2),$$

the Poincaré series in (1-6) is non–zero. We apply this to \( l = 1 \) in the first two cases of (1-2) and to \( l = 2 \) in the last case of (1-2). Obviously, the left–hand side of the inequality \( \log \left( \sum_{i=0}^{m/2-2} \frac{x^i}{i!} \right) - x > -\log 2 \) is strictly decreasing for \( x > 0 \). So, for example, to check that \( m \geq 4 \) for \( h'\epsilon_{\Gamma} \geq 10 \) it is enough to consider the case \( m = 4 \) and \( h'\epsilon_{\Gamma} = 10 \). The details are left to the reader. This proves (ii).

To prove (iii), we apply Lemma 4-2 (ii). If \( m \) satisfies the inequality in the statement of Theorem 1-7 (iii), then , by Lemma 4-2 (ii), we have

$$x_{m/2-2} > m/2 - 2 \geq \frac{2\pi l}{h\epsilon_{\Gamma}},$$

and we apply (i). This proves (iii). Finally, we prove (iv). By Lemma 4-2 (i), we have

$$x_{m/2-2} = m/2 - 2 + o(1),$$

where as usual \( o(1) \) means a sequence \( \alpha_m \) such that \( \alpha_m \to 0 \) as \( m \to \infty \). The Poincaré series (1-6) is non–trivial if

(4-6)$$1 \leq l < \frac{h\epsilon_{\Gamma}}{2\pi} x_{m/2-2} = \frac{h\epsilon_{\Gamma}}{2\pi} \left( m/2 - 4/3 \right) + o(1),$$

assuming that \( l \) is even if the third case in (1-2) is valid. Thus, for \( m \) large enough, to obtain the upper bound in (4-6), it is enough to require

$$l \leq \left[ \frac{h\epsilon_{\Gamma}}{2\pi} \left( m/2 - 4/3 \right) \right] - 1.$$

As an example, we consider the congruence subgroups \( (N \geq 1) \)

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); \ c \equiv 0 \ (mod \ N) \right\} \quad h = h' = 1, \epsilon_{\Gamma} = N;$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); \ c \equiv 0, a, d \equiv 1 \ (mod \ N) \right\} \quad h = h' = 1, \epsilon_{\Gamma} = N;$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); \ c, b \equiv 0, a, d \equiv 1 \ (mod \ N) \right\} \quad h = h' = N, \epsilon_{\Gamma} = N.$$
Example 4-7. Let $\Gamma \in \{\Gamma_0(N), \Gamma_1(N), \Gamma(N)\}$. Then the Poincaré series $\alpha_{1,m,\chi}$ (we take $l = 1$ in (1-6)) is non–zero in the following cases:

(i) Let $N \geq 2$ and $\Gamma = \Gamma_0(N)$ or $\Gamma = \Gamma_1(N)$. Then
   (1) $m \geq 4$ for $N \geq 10$
   (2) $m \geq 6$ for $4 \leq N \leq 9$
   (3) $m \geq 8$ for $N = 3$
   (4) $m \geq 10$ for $N = 2$.

(ii) Let $N \geq 2$ and $\Gamma = \Gamma(N)$. Then
   (1) $m \geq 4$ for $N \geq 4$
   (2) $m \geq 6$ for $2 \leq N \leq 3$

(iii) Let $N = 1$ and $\Gamma = \Gamma_0(1) = \Gamma_1(1) = \Gamma(1) = SL_2(\mathbb{Z})$. Then, $m \geq 16$.

5. The proof of Theorem 1-8

Theorem 1-8 has the proof which is analogous to that of Theorem 1-7. It is based on the following considerations and it is left to the reader. We let

$$Q_k(x) = -2 \int_0^{\sqrt{2\pi}} e^{-t^2} dt + e^{-x} \sum_{i=1}^{k} \frac{x^{i-1/2}}{(i-1/2)!}, \quad k \geq 1,$$

where we write

$$(n-1/2)! \overset{\text{def}}{=} (1/2)(3/2)\cdots(n-1/2), \quad n \geq 1.$$

Then

$$\frac{dQ_k(x)}{dx} = -x^{k-1/2}e^{-x}/(k-1/2)!.$$

Hence

$$\int_a^b x^{k-1/2}e^{-x}dx = (k-1/2)!(Q_k(a) - Q_k(b)).$$

Also, $Q_k$ is strictly decreasing on $]0, \infty[$. It is well–known that $\int_0^{\infty} e^{-x^2} dx = \sqrt{\pi}/2$. Hence $Q_k(\infty) = -\sqrt{\pi}$. We see $Q_k(0) = 0$ for $k \geq 0$. Thus, there is a unique $y_k \in ]0, \infty[$ such that $Q_k(y_k) = -\sqrt{\pi}/2$.

Lemma 5-3. (i) $\lim_{k \to \infty} (y_k - (k + 1/6)) = 0$.

(ii) $y_k > k - 1/2$ for all $k \geq 1$.

Proof. As in Lemma 4-2 (i), the proof relies on the same exercises in [9]. In fact, we explain how to modify them to obtain the proof. We use the standard Hardy’s notation $O$ and $o$. (See for example the notation section of [9].)

We write the generalized Stirling’s formula

$$\frac{\Gamma(x)}{\sqrt{2\pi}x^{x-1/2}e^{-x}} = e^{H(x)}, \quad x \in \mathbb{R}_{>0},$$

where $\Gamma$ is the usual $\Gamma$–function, and

$$0 < H(x) < \frac{1}{2x}.$$

Hence

$$\frac{\sqrt{2\pi}x^{x-1/2}e^{-x}}{\Gamma(x)} = e^{-H(x)} = 1 - H(x) + \frac{H(x)^2}{2!} - \cdots = 1 + O\left(\frac{1}{x}\right), \quad x \in \mathbb{R}_{>0}.$$
For $x = n + 1/2$, combining the functional equation $\Gamma(x + 1) = x\Gamma(x)$ and $\Gamma(1/2) = \sqrt{\pi}$, we find $\Gamma(n + 1/2) = (n - 1/2) \cdots (3/2)(1/2)\sqrt{\pi} = (n - 1/2)!\sqrt{\pi}$.

Thus, (5-4) becomes

$$\frac{\sqrt{2}(n + 1/2)^ne^{(n+1/2)}}{(n - 1/2)!} = 1 + O\left(\frac{1}{n}\right), \quad n \geq 1,$$

i.e., we have the following:

$$\frac{1}{(n - 1/2)!} = \frac{1}{\sqrt{2}(n + 1/2)^ne^{(n+1/2)}} \left(1 + O\left(\frac{1}{n}\right)\right)$$

$$= \frac{1}{(n + 1/2)^ne^{-1}\sqrt{2}(n - 1/2)^ne^{-(n-1/2)}} \left(1 + O\left(\frac{1}{n}\right)\right)$$

$$= \frac{1}{\sqrt{2}(n - 1/2)^ne^{-(n-1/2)}} \left(1 + O\left(\frac{1}{n}\right)\right),$$

since

$$\frac{(n - 1/2)^n}{(n + 1/2)^ne^{-1}} = \exp\left(n \log\left(1 - \frac{1}{n + 1/2}\right) + 1\right) = 1 + O\left(\frac{1}{n}\right)$$

by developing log and exp into their Taylor series around $x = 0$.

Now, following Polya–Szegő the following asimptotic formula is the key step:

**Lemma 5-6.** Let $\alpha, \beta \in \mathbb{R}$. Then we have the following

$$\frac{1}{(n - 1/2)!}\int_{0}^{(n-1/2) + \alpha\sqrt{n^{-1/2} + \beta}} x^{n-1/2}e^{-\frac{x^2}{2}}dx =$$

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{\alpha} e^{-\frac{x^2}{2}}dx + \frac{1}{2(n - 1/2)} \left[\beta - \frac{\alpha^2 + 2}{3}\right] + o\left((n - 1/2)^{-1/2}\right).$$

**Proof.** Using (5-5), the proof of this lemma goes in the same way as the proof of the claim given by the problem 210 in Part II, Chapter 5 in [9]. One needs to replace $n$ by $n - 1/2$ everywhere. The details are left to the reader as an exercise. \qed

Now, we are ready to complete the proof of Lemma 5-3 (i). Using (5-2), we find that

$$Q_k(x) = -\frac{1}{(k - 1/2)!}\int_{0}^{x} t^{k-1/2}e^{-t}dx.$$

We define $\alpha$ and $\beta$ in Lemma 5-6 by letting $\alpha = 0$ and $\beta = 2/3$. Finally, as in the proof of the claim given by the problem 211 in Part II, Chapter 5 in [9] we obtain $\lim_k (y_k - (k - 1/2) - 2/3) = 0$ which is (i).

We prove Lemma 5-3 (ii). Using (5-2) instead of (4-1), as in the proof of Lemma 4-3, we obtain the following formula:

$$Q_k(k - 1/2) - Q_{k+1}(k + 1/2) = -\frac{(k + 1/2)^{k+1/2}e^{-k-1/2}}{(k + 1/2)!} + \frac{1}{(k - 1/2)!}\int_{k-1/2}^{k+1/2} x^{k-1/2}e^{-x}dx.$$

The function $x^{k-1/2}e^{-x}$ on $]0, \infty[\]$ has a maximum at $x = k - 1/2$; it increases for $0 < x < k - 1/2$ and decreases for $x > k - 1/2$. Using this observation, (5-7) implies

$$Q_k(k - 1/2) - Q_{k+1}(k + 1/2) > 0$$
Thus the limit $\lim_{k \to \infty} Q_k(k - 1/2)$ exists. Now, we argue as in the proof Lemma 4-2 (i). First, we have the following:

$$\left| Q_k(k - 1/2) + \sqrt{\pi}/2 \right| = |Q_k(k - 1/2) - Q_k(y_k)| \leq \frac{(k - 1/2)^{k-1/2}e^{-k+1/2}}{(k-1/2)!} |y_k - k + 1/2|.$$  

Which implies that $\lim_{k \to \infty} Q_k(k - 1/2) = -\sqrt{\pi}/2$, and $y_k > k - 1/2$ using (5-5) and Lemma 5-3 (i). This proves Lemma 5-3 (ii) $\square$

REFERENCES


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