The unitary dual of $p$–adic $G_2$

Goran Muić

Introduction

Let $G_2$ be a simply connected split simple group of type $G_2$ over a nonarchimedean field $F$ of characteristic zero. We determine here the set $\hat{G}_2$ of equivalence classes of irreducible unitarizable representations of $G_2$. The corresponding problem in the complex case was solved by Duflo in [D]. Recently, Vogan has solved the corresponding problem in the real case in [V1].

We expect that our results, besides being interesting by themselves, will play a role in both, local and global $\Theta$–correspondences ([GrSa1], [GrSa2], [MaSa], [Sa1], [Sa2])

Here is an outline of the paper. In the first section we establish notation and recall basic structure results for $G_2$ and basic facts from representation theory of $G_2$ and its standard Levi factors.

A. M. Aubert in [A], J. Bernstein, and P. Schneider- U. Stuhler in [ScSt] considered an involution $D_G$ on the Grothendieck group of representations of finite length of any reductive $F$–group $G$. In the second section we recall some facts about filtration of Jacquet modules and introduce the involution $D_{G_2}$, in $G_2$–setting.

In the third section we consider generalized and degenerate principal series. Theorem 3.1 describes reducibility points of generalized principal series $I_\gamma(s, \delta(\chi))$ and degenerate principal series $I_\gamma(s, \chi \circ \det)$ (see Section 1 for notation), where $\chi$ is an unitary character of $F^\times$ and $s$ a real number.

In the fourth section we obtain a classification of square integrable representations which are supported on the minimal parabolic subgroup and describe composition factors of $I_\gamma(s, \delta(\chi))$ and $I_\gamma(s, \chi \circ \det)$ (Propositions 4.1, 4.2, 4.3, 4.4 and Theorem 4.1). We use a method of Jacquet modules developed by M. Tadić. We refer to [Re] for equality of our square integrable representations, in the unramified case, to those obtained by well-known Kazhdan-Lusztig classification and to [GrSa1] for its importance in the local $\Theta$–correspondence. Tempered representations supported on the minimal parabolic subgroups were classified by Keys in [Ke] and in Theorems 3.1 and 4.1 of this paper.

In the fifth section we classify unitarizable Langlands quotients. Theorems 5.1 and 5.2 classify unitarizable non-tempered Langlands quotients which are supported on the minimal
parabolic subgroup. Theorem 5.3 classifies unitarizable Langlands quotients supported on maximal parabolic subgroups.

The sixth section is devoted to a proof of unitarizability of an interesting family of isolated unitary representations

$$J_{\beta}(1, \pi(\chi, \chi^{-1})),$$

where $\chi$ has order three (for the notation see Section 1). We give two proofs. The first uses a global method and is motivated by the proof of [Sp, Thm. 3.5.3] and the proof of [T1, Thm. A.8]. The second is local and is given under the restriction that $F$ has residual characteristic different from three. This proof is motivated by the paper of D. Barbasch and A. Moy [BM1] and papers of D. A. Vogan [V1] and [V2]. If we suppose that $\chi$ is unramified then $J_{\beta}(1, \pi(\chi, \chi^{-1}))$ is the image by Iwahori-Matsumoto involution of a square integrable representation $\pi(\chi)$ (see Proposition 4.1). Hence unitarizability is a consequence of [BM1] and [BM2].

It is interesting to note that any supercuspidal representation is trivially unitarizable and is always isolated in $\hat{G}_2$ (see [T4]).

I would like to express my gratitude to Prof. M. Tadić for his constant help and useful discussions during the preparation of this paper. Thanks are also due to Prof. G. Savin for his comments about representation theory of exceptional groups and $\Theta$–correspondences. I would like to thank S. Zampera for useful discussions about spectral decomposition.

1. Preliminaries

Let $F$ be a nonarchimedean field of characteristic zero. Denote by $O$ the ring of integers of $F$ and by $\wp$ its maximal ideal. Denote by $F_\wp$ the corresponding residual field and by $\nu$ normalized absolute value of $F$. Through this paper $\mathbf{R}$ denote the field of real numbers and $\mathbf{C}$ the field of complex numbers.

Denote by $T$ split torus and $B = TU$ a Borel subgroup of $G_2$. Denote by

$$\{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}.$$  

corresponding set of positive roots. Denote by $X(T)$ the group of rational characters of $T$. We have

$$X(T) = \mathbf{Z}(2\alpha + \beta) \oplus \mathbf{Z}(\alpha + \beta),$$

and the Positive Weyl chamber in $X(T) \otimes_{\mathbf{Z}} \mathbf{R}$ is

$$C^+ = \{s(2\alpha + \beta) + s'(\alpha + \beta), s > s' > 0\}.$$  

For any root $\gamma$ we denote by $w_\gamma$ the corresponding reflection in the Weyl group $W$. Set $\hat{w}_{2\alpha} = w_{3\alpha + 2\beta}$ and $\hat{w}_{\beta} = w_{2\alpha + \beta}$. Denote by $w_0$ the longest element in $W$. We identify $T \cong F^\times \times F^\times$ by

$$t \mapsto ((2\alpha + \beta)(t), (\alpha + \beta)(t)).$$
In this realization we have

\[
\begin{align*}
\alpha(t_1, t_2) &= t_1 t_2^{-1}, \quad \beta(t_1, t_2) = t_1^{-1} t_2^2, \\
w_\alpha(t_1, t_2) &= (t_2, t_1), \quad w_\beta(t_1, t_2) = (t_1, t_1 t_2^{-1}).
\end{align*}
\]

\[
\alpha^\vee (a) = (a, a^{-1}), \quad \beta^\vee (a) = (1, a)
\]

\[
\begin{align*}
(\alpha + \beta)^\vee(a) &= (a, a^2), \\
(2\alpha + \beta)^\vee(a) &= (a^2, a), \\
(3\alpha + \beta)^\vee(a) &= (a, 1), \quad (3\alpha + 2\beta)^\vee(a) = (a, a).
\end{align*}
\]

For

\[
\Lambda = s(2\alpha + \beta) + s'(\alpha + \beta) = (s, s') \in X(T) \otimes \mathbb{Z} \mathbb{C}
\]

we define unramified character of \( T \)

\[
t = (t_1, t_2) \mapsto \nu^s(t_1) \nu^{s'}(t_2).
\]

The modular character, \( \delta_0 \), corresponds to the sum of positive roots i.e. to \( 2\rho_0 = (4, 2) \).

Denote by \( M_\alpha \) and \( M_\beta \) standard Levi factors and by \( N_\alpha \) and \( N_\beta \) unipotent radicals of the maximal parabolic subgroups \( P_\alpha \) and \( P_\beta \) which correspond to \( \alpha \) and \( \beta \), respectively. We have \( M_\alpha \cong GL(2) \) under the map determined by

\[
t \mapsto \text{diag}((2\alpha + \beta)(t), (\alpha + \beta)(t))
\]

and such that \( \alpha \) corresponds to the standard positive root of \( GL(2) \). We have \( M_\beta \cong GL(2) \) under the map determined by

\[
t \mapsto \text{diag}((\alpha + \beta)(t), \alpha(t))
\]

and such that \( \beta \) corresponds to the standard positive root of \( GL(2) \), [Sh3, Lem. 2.1].

For \( \gamma \in \{ \alpha, \beta \}, s_1, s_2 \in \mathbb{C} \) and \( \chi_1, \chi_2 \) unitary characters of \( F^\times \), set

\[
I^\gamma(\nu^{s_1}\chi_1 \otimes \nu^{s_2}\chi_2) = \text{Ind}_{\Gamma((U \cap M_\gamma))}^{M_\gamma}(\nu^{s_1}\chi_1 \otimes \nu^{s_2}\chi_2).
\]

Similarly, write \( I(\cdot \otimes \cdot) \) for a normalized induction from \( B \) to \( G_2 \). We will write, also, \( I(\Lambda, \chi_1 \otimes \chi_2) \), where \( \Lambda = (s_1, s_2) \). Further, if we fix \( w \in W \) then for \( f \in I(\Lambda, \chi_1 \otimes \chi_2) \) we define

\[
A(\Lambda, \chi_1 \otimes \chi_2, w) f(g) = \int_{U_w} f(w^{-1}ng)dn,
\]

where \( U_w = U \cap wUw^{-1} \), \( U \) is the unipotent radical opposed to \( U \). This integral converges for \( Re(s_1) > Re(s_2) > 0 \), and continues to a meromorphic function in \( \Lambda = (s_1, s_2) \). Moreover, away from its poles it defines intertwining map

\[
A(\Lambda, \chi_1 \otimes \chi_2, w) : I(\Lambda, \chi_1 \otimes \chi_2) \rightarrow I(w(\Lambda), w(\chi_1 \otimes \chi_2)).
\]
If $\gamma$ is a simple root then the intertwining operator $A(\Lambda, \chi_1 \otimes \chi_2, w_\gamma)$ is induced from $GL(2)$—long intertwining operator which intertwines $I^\gamma(\Lambda, \chi_1 \otimes \chi_2)$ and $I^\gamma(w_\gamma(\Lambda), w_\gamma(\chi_1 \otimes \chi_2))$. It is well-known that this operator is holomorphic (and nonzero) unless $\Lambda \circ \gamma^\vee = 1$ and $(\chi_1 \otimes \chi_2) \circ \gamma^\vee = 1$. Further, it is an isomorphism unless $\Lambda \circ \gamma^\vee = \nu^{\pm 1}$ and $(\chi_1 \otimes \chi_2) \circ \gamma^\vee = 1$.

Now, we recall some basic facts about the Langlands quotient theorem in the $G_2$—setting, [BW, XI, Prop. 2.6 and Cor. 2.7]. For an admissible representation $\pi$ of $GL(2, F)$ and $\gamma \in \{\alpha, \beta\}$ denote

$$I_\gamma(s, \pi) = \text{Ind}_{F^\times_0}^{F^\times}((\nu^s \circ \det) \otimes \pi), \quad I_\gamma(\pi) = I_\gamma(0, \pi).$$

When $\pi$ is a tempered irreducible representation and $s > 0$, above representation has a unique irreducible quotient denoted by $J_\gamma(s, \pi)$. Further, we have a long intertwining operator

$$A(s, \pi, \tilde{w}_\gamma) : I_\gamma(s, \pi) \longrightarrow I_\gamma(-s, \tilde{\pi})$$

defined by

$$A(s, \pi, \tilde{w}_\gamma)f(g) = \int_{N_{A^\gamma}} f(\tilde{w}_\gamma^{-1}ng)dn,$$

with image isomorphic to $J_\gamma(s, \pi)$.

When $\chi_1, \chi_2$ are unitary and $s_1 > s_2 > 0$ representation $I(\nu^{s_1}\chi_1 \otimes \nu^{s_2}\chi_2)$ has a unique irreducible quotient denoted by $J(s_1, s_2, \chi_1, \chi_2)$. Further, the long intertwining operator

$$A(s_1, s_2, \chi_1, \chi_2, w_0) : I(\nu^{s_1}\chi_1 \otimes \nu^{s_2}\chi_2) \longrightarrow I(\nu^{-s_1}\chi_1^{-1} \otimes \nu^{-s_2}\chi_2^{-1}),$$

has image isomorphic to $J(s_1, s_2, \chi_1, \chi_2)$.

Now, we consider representations of $GL(2, F)$. For a character $\chi$ of $F^\times$ and any smooth representation $\pi$ of $GL(2, F)$ denote by $\chi\pi$ the twist of $\pi$ by one dimensional representation $\chi \circ \det$. Denote by $\delta(\chi)$ a unique irreducible subrepresentation of $\nu^{1/2}\chi \times \nu^{-1/2}\chi$ ([Ze]). It is a square integrable representation if $\chi$ is unitary. For unitary characters $\chi_1, \chi_2$ denote $\pi(\chi_1, \chi_2) = \chi_1 \times \chi_2$ a tempered irreducible representation. Now, we recall well-known facts about principal series representations of Levi factors of maximal parabolic subgroups of $G_2$. This is in fact $GL(2)$—theory.

**Proposition 1.1.** Suppose that $\chi, \chi_1$ and $\chi_2$ are characters of $F^\times$, and $\gamma \in \{\alpha, \beta\}$. Then

(i) The principal series $I^\gamma(\chi_1 \otimes \chi_2)$ of $M_\gamma$ reduces if and only if $(\chi_1 \otimes \chi_2) \circ \gamma^\vee = \nu^{\pm 1}$. If $I^\gamma(\chi_1 \otimes \chi_2)$ is irreducible it is isomorphic to $I^\gamma(w_\gamma(\chi_1 \otimes \chi_2))$.

(ii) The principal series $I^\alpha(\nu^{1/2}\chi \otimes \nu^{-1/2}\chi) \cdot I^\alpha(\nu^{-1/2}\chi \otimes \nu^{1/2}\chi)$ contains $\delta(\chi)$ and $\chi \circ \det$ as a unique irreducible subrepresentation (quotient) and quotient (subrepresentation), respectively.

(iii) The principal series $I^\beta(\nu^{-1/2}\chi \otimes \nu) \cdot I^\beta(\nu^{1/2}\chi \otimes \nu^{-1})$ contains $\delta(\chi)$ and $\chi \circ \det$ as a unique irreducible subrepresentation (quotient) and quotient (subrepresentation), respectively.
2. Jacquet modules for $G_2$

Let $G$ be a reductive group defined over $F$ then we denote by $R(G)$ the Grothendieck group of admissible representations of finite length of $G$. Let $P = MN$ be any $F$-parabolic subgroup of $G$ then we denote by $r^G_M$ the normalized Jacquet functor. As usual, we consider $r^G_M$ as a homomorphism $R(G) \rightarrow R(M)$. If $G = G_2$ we write $r_\alpha, r_\beta, r_\zeta$ if $M$ equals $M_\alpha, M_\beta, T$, respectively. Now, we recall ([BDK], [C]):

$$r_\alpha \circ I_\alpha(\pi) = \pi + w_{3\alpha + 2\beta}(\pi) + I_\alpha \circ w_{2\alpha + \beta} \circ r^M_T(\pi) + I^\alpha \circ w_\beta \circ r^M_T(\pi)$$

$$r_\beta \circ I_\alpha(\pi) = I^\beta \circ r^M_T(\pi) + I^\beta \circ w_{2\alpha + \beta} \circ r^M_T(\pi) + I^\beta \circ w_{3\alpha + \beta} \circ r^M_T(\pi),$$

if $\pi \in R(M_\alpha)$, and

$$r_\alpha \circ I_\beta(\pi) = I^\alpha \circ r^M_T(\pi) + I^\alpha \circ w_\alpha \circ r^M_T(\pi) + I^\alpha \circ w_\beta \circ r^M_T(\pi)$$

$$r_\beta \circ I_\beta(\pi) = \pi + w_{2\alpha + \beta}(\pi) + I^\beta \circ w_\alpha \circ r^M_T(\pi) + I^\beta \circ w_{3\alpha + \beta} \circ r^M_T(\pi),$$

if $\pi \in R(M_\beta)$.

We have an involutive endomorphism of $R(G_2)$

$$D_{G_2}(\pi) = I \circ r_\zeta(\pi) - I_\alpha \circ r_\alpha(\pi) - I_\beta \circ r_\beta(\pi) + \pi.$$ 

It follows from [A, Cor. 3.9] that if $\pi$ is irreducible and supported on $B$ then $D_{G_2}(\pi)$ is an irreducible representation, and if $\pi$ is supported on $P_\alpha$ or $P_\beta$ then $-D_{G_2}(\pi)$ is an irreducible representation. Furthermore, if $\pi$ is supercuspidal then $D_{G_2}(\pi) = \pi$.

Similarly, we have involutions on $R(M_\alpha)$ and $R(M_\beta)$, denoted by $D_\alpha$ and $D_\beta$, respectively. Let $\gamma \in \{\alpha, \beta\}$, then we have [A, Thm. 1.7 (3)]

$$D_{G_2} \circ I_\gamma = I_\gamma \circ D_{G_2}$$

$$r_\gamma \circ D_{G_2} = \bar{w}_\gamma \circ D_{G_2} \circ r_\gamma.$$ (2.2)

3. Reducibility of principal series

In this section we determine reducibility points of principal series representations, generalized principal series representations and degenerate principal series representations.

It is an result of Keys that unitary principal series $I(\chi_1 \otimes \chi_2)$ is reducible if and only if $\chi_1$ and $\chi_2$ are different characters of order two [K, Thm. G2]. It is of multiplicity one and length two.

Now, we consider a non-unitary principal series.

**Proposition 3.1.** A non-unitary principal series $I(\chi_1 \otimes \chi_2)$ is irreducible if and only if

$$\chi_1 \neq \nu^\pm, \chi_2 \neq \nu^\pm, \chi_1 \chi_2 \neq \nu^\pm, \chi_1 \chi_2^{-1} \neq \nu^\pm, \chi_1 \chi_2 \neq \nu^\pm, \chi_1 \chi_2 \neq \nu^\pm.$$
Proof. Set $\chi_i = \nu^{-s_i} \mu_i$, $\mu_i$ unitary and $s_i \in \mathbb{R}$. We may suppose that $s_1 \geq s_2 \geq 0$, since for $\Lambda = s_1(2\alpha + \beta) + s_2(\alpha + \beta)$ there exists $w \in W$ such that $w(\Lambda)$ is dominant, and we have

$$I(\Lambda, \mu_1 \otimes \mu_2) = I(w(\Lambda), w(\mu_1 \otimes \mu_2))$$

in $R(G_2)$. Following the same lines as ones in the proof of Theorem 3.14 in [SV] (see also the proof of Theorem 7.1 in [T2]) we conclude that $I(\chi_1 \otimes \chi_2)$ reduces if and only if there exists a root $\gamma$ such that

$$(\chi_1 \otimes \chi_2) \circ \gamma^\vee = \nu^\pm 1.$$ 

$\square$

Lemma 3.1. Let $\chi$ be an unitary character, $s \in \mathbb{R}$ and $\gamma \in \{\alpha, \beta\}$. Then we have

$$D_{G_2}(I_\gamma(s, \delta(\chi))) = I_\gamma(s, \chi \circ \det).$$

Semisimplifications are related by

$$I_\gamma(s, \delta(\chi)) = \sum_i n_i \pi_i \Rightarrow I_\gamma(s, \chi \circ \det) = \sum_i n_i D_{G_2}(\pi_i).$$

In particular, $I_\gamma(s, \delta(\chi))$ reduces if and only if $I_\gamma(s, \chi \circ \det)$ reduces.

Proof. This is a direct consequence of [A, Cor. 1.10 and 3.9]. $\square$

If we fix a non-trivial additive character $\psi_F$ of $F$ we can form a generic character $\psi : U \to \mathbb{C}$ which satisfies compatibility conditions described in [Sh6, p. 282.]. Let $\gamma \in \{\alpha, \beta\}$ and $\pi$ an irreducible generic ($\psi$-generic) representation of $M_\gamma$. If we fix $\psi$-generic functional $\lambda$ on $\pi$ we can form a $\psi$-generic functional

$$\lambda_\psi(s, \pi)(f) = \int_{N_\gamma} \lambda(f \nu^{-1}(n)) \overline{\psi(n)dn}$$

on $I_\gamma(s, \pi)$. Then we have a functional equation [Sh1, Thm. 3.1]

$$\lambda_\psi(s, \pi)(f) = C_\psi(s, \pi, \tilde{w}_\gamma) \lambda_\psi(-s, \tilde{w}_\gamma(\pi))(A(s, \pi, \tilde{w}_\gamma)f), \tag{3.3}$$

where $C_\psi(s, \pi, \tilde{w}_\gamma)$ is the local coefficient. It is a rational function in $q^{-s}$. Furthermore, we have [Sh2, Example 1]

$$C_\psi(s, \pi, w_{2\alpha+2\beta}) = \gamma(s, \pi, \psi_F) \gamma(2s, \omega_{\pi}, \psi_F) \gamma(3s, \omega_{\pi} \pi, \psi_F), \tag{3.4}$$

and [Sh3, Prop. 2.2]

$$C_\psi(s, \pi, w_{2\alpha+\beta}) = \gamma(s, \pi \times \Pi(\pi), \psi_F) \gamma(2s, \omega_{\pi}, \psi_F) / \gamma(s, \pi, \psi_F) \tag{3.5}$$
where \( \omega_\pi \) is the central character of \( \pi \), \( \Pi(\pi) \) is the Gelbart-Jacquet lift of \( \pi \) ([GeJa]), 
\( \gamma(s, \pi, \psi_F) \) is Jacquet–Langlands \( \gamma \)-factor ([JL]), and \( \gamma(s, \pi \times \Pi(\pi), \psi_F) \) is \( \gamma \)-factor attached to the pair \( \pi, \Pi(\pi) \) ([JPSS]).

The Gelbart - Jacquet lift of \( \delta(\chi) \) is the Steinberg representation of \( GL(3, F) \). It is a unique irreducible subrepresentation of \( \nu \times 1 \times \nu^{-1} \) ([Ze]). Then, using [JPSS] and [JL], we obtain the following equalities of local \( L \)-functions

\[
L(s, \delta(\chi) \times \Pi(\delta(\chi))) = L(s + 1/2, \chi)L(s + 3/2, \chi)
\]

\[
L(s, \delta(\chi)) = L(s + 1/2, \chi),
\]

where \( L(s, \chi) \) is Hecke–Tate local \( L \)-factor attached to character \( \chi \). Then, up to a monomial in \( q^{-s} \), we have the following equalities of local coefficients

\[
C_\psi(s, \delta(\chi), w_{2\alpha + 2\beta}) = \frac{L(3/2 - s, \chi^{-1})L(1 - 2s, \chi^{-2})L(3/2 - 3s, \chi^{-3})}{L(s + 1/2, \chi)L(2s, \chi)L(3s + 1/2, \chi^3)} \tag{3.6}
\]

\[
C_\psi(s, \delta(\chi), w_{2\alpha + \beta}) = \frac{L(5/2 - s, \chi^{-1})L(1 - 2s, \chi^{-2})}{L(s + 3/2, \chi)L(2s, \chi^2)}. \tag{3.7}
\]

Note \( L(s, \chi) = 1 \) unless \( \chi \) is unramified, and \( L(s, \chi) = (1 - q^{-s})^{-1} \) if \( \chi = \nu^s \). This implies that only possible real pole of \( L(s, \chi) \), \( \chi \) unitary, is \( s = 0 \), and occurs exactly when \( \chi = 1 \).

**Theorem 3.1.** Let \( \chi \) be an unitary character and \( s \in \mathbb{R} \).

(i) \( I_\alpha(s, \delta(\chi)) \) and \( I_\alpha(s, \chi \circ \det) \) reduces if and only if

\[ s = \pm 1/2, \chi^2 = 1 \; \text{or} \; s = \pm 3/2, \chi = 1 \; \text{or} \; s = \pm 1/2, \chi^3 = 1. \tag{3.8} \]

(ii) \( I_\beta(s, \delta(\chi)) \) and \( I_\beta(s, \chi \circ \det) \) reduces if and only if

\[ s = \pm 1/2, \chi^2 = 1 \; \text{or} \; s = \pm 5/2, \chi = 1. \tag{3.9} \]

**Proof.** Observe that [BDK, Lem. 5.4 (iii)]

\[
I_\gamma(s, \pi) = \operatorname{Ind}_{F_{\gamma}^s}^{G_\ast}(\nu^s \pi) = \operatorname{Ind}_{F_{\gamma}^s}^{G_\ast}((\tilde{\nu}_\gamma)(\nu^s \pi)) = I_\gamma(-s, \tilde{\pi})
\]

in \( R(G_\gamma) \) for any \( \pi \in R(M_\gamma) \). So, it is enough to consider \( s \geq 0 \).

Suppose \( s = 0 \). If \( \chi^2 \neq 1 \), a theorem of Bruhat [C, Thm. 6.6.1] implies irreducibility of \( I_\alpha(0, \delta(\chi)) \). Otherwise, we write for the corresponding Plancherel measure [Sh2, Example 1] and [Sh6, Cor. 3.6]

\[
\mu(s, \delta(\chi), w_{2\alpha + 2\beta}) = c^2 \frac{\gamma(s, \delta(\chi), \psi_F)\gamma(2s, \chi^2, \psi_F)\gamma(3s, \delta(\chi^3), \psi_F)}{\gamma(1 + s, \delta(\chi), \psi_F)\gamma(1 + 2s, \chi^2, \psi_F)\gamma(1 + 3s, \delta(\chi^3), \psi_F)}
\]
where $c$ is some positive constant. We see that $s = 0$ is a zero of Plancherel measure. Then $I_{\alpha}(0, \delta(\chi))$ is irreducible by the theory of $R$-groups. The irreducibility of $I_{\beta}(0, \delta(\chi))$ is proved by Shahidi [Sh3, Prop. 6.3].

Consider $s > 0$. It follows easily from [Sh1, Prop. 3.3.1(a)] that poles of the local coefficient $C_{\nu}(s, \delta(\chi), \tilde{\nu}_{\omega})$ (for $s > 0$) are points where Langlands quotient $J_{\alpha}(s, \delta(\chi))$ is not generic. These are, clearly, points of reducibility of $I_{\alpha}(s, \delta(\chi))$.

Then, using (3.6) and (3.7), it is easy to see that (3.8) and (3.9) are points of reducibility. We prove that $I_{\gamma}(s, \delta(\chi))$, $s > 0$, is irreducible otherwise.

Consider $\gamma = \alpha$. By Proposition 1.1 we see that $I_{\alpha}(s, \delta(\chi))$ is subrepresentation of the principal series $I(\nu^{1/2} \chi \otimes \nu^{1/2} \chi)$. Now, the defining integral formulas and analytic continuation imply

$$A(s + 1/2, s - 1/2, \chi, \chi, w_{3\alpha + 2\beta}) |_{I_{\alpha}(s, \delta(\chi))} = A(s, \delta(\chi), w_{3\alpha + 2\beta}).$$

Using a factorisation of $A(s + 1/2, s - 1/2, \chi, \chi, w_{3\alpha + 2\beta})$ for a reduced expression

$$w_{3\alpha + 2\beta} = w_{\beta} w_{\alpha} w_{\beta} x_w,$$

and properties of the long intertwining operator of $GL(2)$ (see Section 1), we see that away from

$$s = 1/2, \chi^2 = 1; \quad s = 3/2, \chi = 1; \quad s = 1/2, \chi^3 = 1; \quad s = 1/6, \chi^3 = 1$$

$A(s + 1/2, s - 1/2, \chi, \chi, w_{3\alpha + 2\beta})$ is holomorphic and an embedding. Hence $A(s, \delta(\chi), w_{3\alpha + 2\beta})$ is an embedding. Since its image is isomorphic to $J_{\alpha}(s, \delta(\chi))$ we conclude that $I_{\alpha}(s, \delta(\chi))$ is irreducible. The last case, $s = 1/6, \chi^3 = 1$ follows from

**Lemma 3.2.** $I_{\alpha}(1/6, \chi \circ \det) \cong J_{\alpha}(1/3, \pi(\chi^{-1}, \chi^{-1}))$.

**Proof.** Since $\pi(\chi, \chi^{-1})$ is induced representation of $GL(2, F)$, induction in stages implies

$$I_{\alpha}(1/3, \pi(\chi^{-1}, \chi^{-1})) \cong I(\nu^{1/3} \chi^{-1} \otimes \nu^{1/3} \chi^{-1}).$$

Since $w_{\beta}(\nu^{1/3} \chi^{-1} \otimes \nu^{1/3} \chi^{-1}) = \nu^{2/3} \chi \otimes \nu^{-1/3} \chi$, by Proposition 1.1 (i)

$$I(\nu^{1/3} \chi^{-1} \otimes \nu^{1/3} \chi^{-1}) \cong I(\nu^{2/3} \chi \otimes \nu^{-1/3} \chi).$$

Since $\nu^{1/6} \chi \circ \det$ is quotient of $I^0(\nu^{2/3} \chi \otimes \nu^{-1/3} \chi)$, induction in stages and (3.11) imply that $I_{\alpha}(1/6, \chi \circ \det)$ is quotient of $I_{\alpha}(1/3, \pi(\chi^{-1}, \chi^{-1}))$. Then $I_{\alpha}(1/6, \chi \circ \det)$ contains $J_{\alpha}(1/3, \pi(\chi^{-1}, \chi^{-1}))$ as a unique irreducible quotient. Furthermore, it is of multiplicity one in this induced representation (because it is of multiplicity one in $I_{\alpha}(1/3, \pi(\chi^{-1}, \chi^{-1}))$, [BW, Chapter XI]).

Next, since $w_{\beta}(\nu^{-1/3} \chi \otimes \nu^{2/3} \chi) = \nu^{-1/3} \chi^{-1} \otimes \nu^{2/3} \chi^{-1}$ by Proposition 1.1 we have

$$I_{\alpha}(1/6, \chi \circ \det) \hookrightarrow I(\nu^{-1/3} \chi \otimes \nu^{2/3} \chi) \cong I(\nu^{1/3} \chi^{-1} \otimes \nu^{-2/3} \chi^{-1}).$$

(3.12)
Since $J_\alpha(1/6, \delta(\chi))$ is generic and is a unique irreducible subrepresentation in

$$I_\alpha(-1/6, \delta(\chi^{-1})) \hookrightarrow I(\nu^{1/3}\chi^{-1} \otimes \nu^{-2/3}\chi^{-1}),$$

(3.12) implies that we have an embedding

$$I_\alpha(1/6, \chi \circ \text{det}) \hookrightarrow I(\nu^{1/3}\chi^{-1} \otimes \nu^{-2/3}\chi^{-1})/I_\alpha(-1/6, \delta(\chi^{-1})) \cong I_\alpha(-1/6, \chi^{-1}).$$

Then [BDK, Lem. 5.4 (iii)] implies

$$I_\alpha(1/6, \chi \circ \text{det}) \cong I_\alpha(-1/6, \chi^{-1} \circ \text{det}).$$

This means that $I_\alpha(1/6, \chi \circ \text{det})$ is selfcontragredient. Since $J_\alpha(1/3, \pi(\chi^{-1}, \chi^{-1}))$ is selfcontragredient (see Section 1) we conclude that it is a subrepresentation of $I_\alpha(1/6, \chi \circ \text{det})$, as well. This implies the lemma. □

The proof for $\gamma = \beta$ is similar, so we shall be brief. If we include $I_\beta(s, \delta(\chi))$ into the principal series $I(\nu^{s+1/2}\chi \otimes \nu)$ we obtain

$$A(s - 1/2, 1, \chi, 1, w_{2\alpha + \beta}) I_\beta(s, \delta(\chi)) = A(s, \delta(\chi), w_{2\alpha + \beta}).$$

Now, using the factorisation

$$w_{2\alpha + \beta} = w_{\alpha}w_{\beta}w_{\alpha}w_{\beta}w_{\alpha},$$

as in the proof of the first part we conclude that away from

$$s = 1/2, \chi^2 = 1; \ s = 5/2, \chi = 1; \ s = 3/2, \chi = 1$$

representation $I_\beta(s, \delta(\chi))$ is irreducible. The last case $s = 3/2, \chi = 1$ follows from

**Lemma 3.3.** $I_\beta(3/2, 1_{GL(2)}) \cong J_\alpha(1, \pi(1, 1)).$

**Proof.** The proof is analogous to that of Lemma 3.2. First

$$I_\alpha(1, \pi(1, 1)) \cong I(\nu \otimes \nu),$$

and since $I_\beta(3/2, 1_{GL(2)})$ is quotient of this representation we conclude that $J_\alpha(1, \pi(1, 1))$ is a unique irreducible quotient of $I_\beta(3/2, 1_{GL(2)})$. It is of multiplicity one in this induced representation. As in (3.12)

$$I_\beta(3/2, 1_{GL(2)}) \hookrightarrow I(\nu^2 \otimes \nu^{-1}) \cong I(\nu^{-1} \otimes \nu^2) \cong I(\nu \otimes \nu^{-2}) \cong I(\nu^{-2} \otimes \nu).$$

(3.13) Since $J_\beta(3/2, \delta(1))$ is generic and is a unique irreducible subrepresentation of $I_\beta(-3/2, \delta(1))$,

(3.13) implies

$$I_\beta(3/2, 1_{GL(2)}) \hookrightarrow I(\nu^{-2} \otimes \nu)/I_\beta(-3/2, \delta(1)) \cong I_\beta(-3/2, 1_{GL(2)}).$$

Now, to finish the proof one can follow the proof of Lemma 3.2. □
4. Discrete series

In this section we classify square integrable representations supported on a minimal parabolic subgroup. First we recall Casselman’s square integrability criterion [C, Thm. 6.5.1]. Suppose that \( \pi \) is an irreducible representation of \( G_2 \) supported on a minimal parabolic subgroup. Then \( \pi \) is square integrable if and only if for any irreducible subquotient \( \nu^{s_1} \chi_1 \otimes \nu^{s_2} \chi_2 \) of \( r_{\theta}(\pi) \) \((s_i \in \mathbb{R}, \chi_i \text{ unitary})\), we have

\[ s_1 + s_2 > 0 \quad \text{and} \quad 2s_1 + s_2 > 0. \]

**Proposition 4.1.** Suppose that \( \chi \) is a character of order two. Then we have

(i) The induced representation \( I(\nu \chi \otimes \chi) \) has a unique irreducible subrepresentation \( \pi(\chi) \).

We have \( r_{\theta}(\pi(\chi)) = \nu \chi \otimes \chi + \nu \otimes \chi + \chi \otimes \nu \) i.e. \( \pi(\chi) \) is square integrable.

(ii) \( I_{\alpha}(1/2, \delta(\chi)) = \pi(\chi) + J_{\alpha}(1/2, \delta(\chi)) \)

\( I_{\beta}(1/2, \delta(\chi)) = \pi(\chi) + J_{\beta}(1/2, \delta(\chi)) \)

\( I_{\alpha}(1/2, \chi \circ \det) = J_{\beta}(1, \pi(1, \chi)) + J_{\beta}(1/2, \delta(\chi)) \)

\( I_{\beta}(1/2, \chi \circ \det) = J_{\beta}(1, \pi(1, \chi)) + J_{\alpha}(1/2, \delta(\chi)) \) in \( R(G_2) \).

**Proposition 4.2.** Suppose that \( \chi \) is a character of order three. Then we have

(i) The induced representation \( I(\nu \chi \otimes \chi) \) has a unique irreducible subrepresentation \( \pi(\chi) \).

We have \( r_{\theta}(\pi(\chi)) = \nu \chi \otimes \chi + \nu^{-1} \otimes \chi^{-1} \) i.e. \( \pi(\chi) \) is square integrable. \( \pi(\chi) \cong \pi(\chi^{-1}) \).

(ii) \( I_{\alpha}(1/2, \delta(\chi)) = \pi(\chi) + J_{\alpha}(1/2, \delta(\chi)) \)

\( I_{\alpha}(1/2, \chi \circ \det) = J_{\beta}(1, \pi(\chi, \chi^{-1})) + J_{\alpha}(1/2, \delta(\chi^{-1})) \) in \( R(G_2) \).

Suppose that \( \chi \) is a character of order two or three. Then character \( \nu \chi \otimes \chi \) of \( T \) is regular (i.e. its stabilizer in Weyl group \( W \) is trivial). Denote by \( S \) the set of all roots \( \gamma \) such that \( (\nu \chi \otimes \chi) \circ \gamma^V = \nu \). Using results recalled in Section 1 it is easy to see that the cardinal number \(|S|\) of \( S \) is two. Then by [Ro], \( I(\nu \chi \otimes \chi) \) is representation of multiplicity one and of length \( 2|S| = 4 \). Further, it contains a unique irreducible subrepresentation. Denote this irreducible representation by \( \pi(\chi) \). Now, we are ready to prove Propositions 4.1 and 4.2.

**Proof of Proposition 4.1.** Since \( w_\gamma(\nu \chi \otimes \chi) = \nu \otimes \chi \) and \( w_\alpha(\nu \otimes \chi) = \chi \otimes \nu \) then by Proposition 1.1 (i) we obtain

\[ I(\nu \chi \otimes \chi) \cong I(\nu \otimes \chi) \cong I(\chi \otimes \nu). \quad (4.14) \]

Induction in stages, using (4.14) and Proposition 1.1, implies that \( I_{\gamma}(1/2, \delta(\chi)) \), \( \gamma \in \{\alpha, \beta\} \), are subrepresentations of \( I(\nu \chi \otimes \chi) \), and, consequently, contain \( \pi(\chi) \) as a subrepresentation. Again, induction in stages and (4.14) imply that \( r_{\alpha}(\pi(\chi)) \) contains \( I^\alpha(\nu \otimes \chi) \) as an irreducible quotient. Now, if we compare

\[ r_{\alpha}(I_{\alpha}(1/2, \delta(\chi))) = \nu^{1/2} \delta(\chi) + \nu^{-1/2} \delta(\chi) + I^\alpha(\nu \otimes \chi) + I^\alpha(\nu \chi \otimes \nu^{-1}) \]
and
\[ r_\alpha(I_\beta(1/2, \delta(\chi))) = \nu^{1/2} \delta(\chi) + \nu^{1/2} \chi \circ \det + I^\alpha(\nu \otimes \chi) + I^\alpha(\nu \otimes \nu^{-1} \chi) \]
we conclude
\[ r_\alpha(\pi(\chi)) = \nu^{1/2} \delta(\chi) + I^\alpha(\nu \otimes \chi). \]

Then the transitivity of Jacquet modules implies (i). Since \( I(\nu \chi \otimes \chi) \) is of length four it is easy to complete the proof of (ii) using Lemma 3.1. □

Proof of Proposition 4.2. The proof is analogous to that of Proposition 4.1. First, \( w_\beta(\nu \chi \otimes \chi) = \nu \chi^{-1} \otimes \chi^{-1} \) and induction in stages imply
\[ I(\nu \chi \otimes \chi) \cong I(\nu \chi^{-1} \otimes \chi^{-1}). \]
This implies that \( \pi(\chi) \) is subrepresentation of \( I_\alpha(1/2, \delta(\chi)) \) and \( I_\alpha(1/2, \delta(\chi^{-1})) \). Now, if we compare
\[ r_\alpha(I_\alpha(1/2, \delta(\chi))) = \nu^{1/2} \delta(\chi) + \nu^{1/2} \delta(\chi^{-1}) + \nu^{-1/2} \delta(\chi^{-1}) + \nu^{1/2} \chi^{-1} \circ \det + I^\alpha(\nu \chi \otimes \nu^{-1} \chi) \]
and analogous formula for \( r_\alpha(I_\alpha(1/2, \delta(\chi^{-1}))) \) we conclude
\[ r_\alpha(\pi(\chi)) = \nu^{1/2} \delta(\chi) + \nu^{1/2} \delta(\chi^{-1}). \]
Now, to finish the proof one can follow the proof of Proposition 4.1. □

Proposition 4.3. Suppose that \( \chi = 1 \). Then we have

(i) The induced representation \( I(1 \otimes \nu) \) contains exactly two irreducible subrepresentations \( \pi(1) \) and \( \pi(1)^\prime \). We have \( r_\theta(\pi(1)) = 1 \otimes \nu \), \( r_\theta(\pi(1)^\prime) = 1 \otimes \nu + 2(\nu \otimes 1) \) i.e. \( \pi(1), \pi(1)^\prime \) are square integrable, and \( \pi(1) \neq \pi(1)^\prime \).

(ii) \( I_\alpha(1/2, \delta(1)) = \pi(1)^\prime + J_\alpha(1/2, \delta(1)) + J_\beta(1/2, \delta(1)) \)
\( I_\beta(1/2, \delta(1)) = \pi(1) + \pi(1)^\prime + J_\beta(1/2, \delta(1)) \)
\( I_\alpha(1/2, 1_{GL(2)}) = \pi(1) + J_\beta(1, \pi(1, 1)) + J_\beta(1/2, \delta(1)) \)
\( I_\beta(1/2, 1_{GL(2)}) = J_\beta(1, \pi(1, 1)) + J_\beta(1/2, \delta(1)) + J_\alpha(1/2, \delta(1)) \) in \( R(G_2) \).

Proof. First, we write semisimplifications of Jacquet modules (see Section 2)
\[ r_\alpha(I_\alpha(1/2, \delta(1))) = 2\nu^{1/2} \delta(1) + \nu^{-1/2} \delta(1) + \nu^{1/2} \circ \det + I^\alpha(\nu \otimes \nu^{-1}) \] (4.15)
\[ r_\alpha(I_\alpha(1/2, 1_{GL(2)})) = 2\nu^{-1/2} \circ \det + \nu^{1/2} \circ \det + \nu^{-1/2} \delta(1) + I^\alpha(\nu \otimes \nu^{-1}) \] (4.16)
\[ r_\alpha(I_\beta(1/2, \delta(1))) = 2\nu^{1/2} \delta(1) + 2\nu^{1/2} \circ \det + I^\alpha(\nu \otimes \nu^{-1}) \] (4.17)
\[ r_\alpha(I_\beta(1/2, 1_{GL(2)})) = 2\nu^{-1/2} \circ \det + 2\nu^{-1/2} \delta(1) + I^\alpha(\nu \otimes \nu^{-1}) \] (4.18)
By induction in stages from suitable maximal parabolics we conclude that $I_\beta(1/2, \delta(1))$ and $I_\alpha(1/2, 1_{GL(2)})$ are subrepresentations in $I(1 \otimes \nu)$. Now, using the relation

$$I_\alpha(1/2, \delta(1)) + I_\alpha(1/2, 1_{GL(2)}) = I(1 \otimes \nu)$$

in $R(G_2)$ together with (4.15), (4.16) and (4.17) we conclude

$$I_\beta(1/2, \delta(1)) \cap I_\alpha(1/2, 1_{GL(2)}) \neq 0.$$

Denote this representation by $\pi(1)$. Comparing (4.16) and (4.17) it follows that $\pi(1)$ is of length at most two. Further, by Proposition 1.1 we have

$$J_\beta(1/2, \delta(1)) \hookrightarrow I_\beta(-1/2, \delta(1)) \hookrightarrow I(\nu^{-1} \otimes \nu) \cong I(\nu \otimes \nu^{-1}). \quad (4.19)$$

This implies that $r_\alpha(J_\beta(1/2, \delta(1)))$ contains $I^*(\nu \otimes \nu^{-1})$ as an irreducible quotient. Then

$$r_\alpha(\pi(1)) = \nu^{1/2} \circ \det \quad \text{and} \quad r_\beta(\pi(1)) = \nu^{1/2} \delta(1). \quad (4.20)$$

Then $\pi(1)$ is irreducible and square integrable.

By Proposition 1.1 $I_\beta(1/2, 1_{GL(2)})$ is subrepresentation in $I(\nu \otimes \nu^{-1})$. We must have (see 4.19)

$$I_\beta(-1/2, \delta(1)) \cap I_\beta(1/2, 1_{GL(2)}) \neq 0, \quad (4.21)$$

or otherwise

$$I_\beta(-1/2, \delta(1)) \hookrightarrow I(\nu \otimes \nu^{-1})/I_\beta(1/2, 1_{GL(2)}) \cong I_\beta(1/2, \delta(1)).$$

This implies that $J_\beta(1/2, \delta(1))$ is subrepresentation in $I_\beta(1/2, \delta(1))$. Then $I_\beta(1/2, \delta(1))$ is irreducible. This is a contradiction with Theorem 3.1.

Using (4.17), (4.18) and (4.21) we obtain

$$r_\alpha(J_\beta(1/2, \delta(1))) = I^*(\nu^{-1} \otimes \nu), \quad (4.22)$$

$$r_\beta(J_\beta(1/2, \delta(1))) = \nu^{-1/2} \delta(1) + \nu^{1/2} \circ \det. \quad (4.23)$$

Next, (2.2) and (4.20) imply

$$r_\alpha(D_{G_2}(\pi(1))) = \nu^{-1/2} \delta(1). \quad (4.24)$$

Since $\pi(1)$ is subquotient of $I_\alpha(1/2, 1_{GL(2)})$ then $D_{G_2}(\pi(1))$ is subquotient of $I_\alpha(1/2, \delta(1))$. Then (4.24) implies

$$D_{G_2}(\pi(1)) = J_\alpha(1/2, \delta(1)).$$

Using

$$r_\beta(I_\beta(1/2, \delta(1))) = 2\nu^{1/2} \delta(1) + \nu^{-1/2} \delta(1) + \nu^{1/2} \circ \det + I_\beta(\nu \otimes 1),$$
(4.20) and (4.23) we obtain

$$r_\beta(I_\beta(1/2, \delta(1)) - \pi(1) - J_\beta(1/2, \delta(1))) = \nu^{1/2}\delta(1) + I_\beta(\nu \otimes 1).$$

(4.25)

From (4.25) we conclude that there exists $\pi(1)'$, an irreducible subquotient of

$$I_\beta(1/2, \delta(1))/\pi(1) \twoheadrightarrow I_\alpha(1/2, \delta(1)),$$

(4.26) such that $r_\beta(\pi(1)')$ contains $I_\beta(\nu \otimes 1)$ as an irreducible subquotient. (4.17), (4.21), and (4.24) imply

$$r_\alpha(I_\beta(1/2, \delta(1)) - \pi(1) - J_\beta(1/2, \delta(1))) = 2\nu^{1/2}\delta(1) + \nu^{1/2} \circ \det.$$

(4.27)

By definition and (4.27) we see that $r_\alpha(\pi(1)')$ contains $\nu^{1/2}\delta(1)$ as a composition factor twice. Then (4.15) implies that $\pi(1)'$ is subrepresentation of left-hand side in (4.26). (4.25) implies that $\pi(1)'$ is square integrable, and (4.21) implies

$$\pi(1)' \not\subset \pi(1).$$

(4.28)

Denote by $\Pi$ the inverse image of $\pi(1)'$ under the canonical morphism

$$I_\beta(1/2, \delta(1)) \longrightarrow I_\beta(1/2, \delta(1))/\pi(1).$$

Then in $R(G_2)$ we have

$$\Pi = \pi(1) + \pi(1)'.$$

(4.29)

This implies that $\Pi$ is tempered representation ([Si]). Since square integrable representations are projective objects in the category of all tempered representations [Si, Lem. 5.4.1.4], (4.29) and (4.28) imply

$$\Pi \cong \pi(1) \oplus \pi(1)'.$$  

This implies that $\pi(1)'$ is a subrepresentation of $I_\beta(1/2, \delta(1))$. Then $r_\beta(\pi(1)'')$ is right-hand side in (4.25). Now, using Lemma 3.1, (ii) follows. Obviously, $\pi(1)$ and $\pi(1)'$ are irreducible subrepresentations of $I(1 \otimes \nu)$. If we consider $r_\emptyset(I(1 \otimes \nu))$ we obtain (i). This completes the proof of the proposition. □

Denote by $St_{G_2}$ and $1_{G_2}$ the Steinberg and the trivial representation of $G_2$, respectively.

**Proposition 4.4.** Suppose that $\chi = 1$. Then we have

(i) $I_\alpha(3/2, \delta(1)) = St_{G_2} + J_\alpha(3/2, \delta(1))$

$\quad I_\beta(5/2, \delta(1)) = St_{G_2} + J_\beta(5/2, \delta(1))$

$\quad I_\alpha(3/2, 1_{GL(2)}) = 1_{G_2} + J_\beta(5/2, \delta(1))$

$\quad I_\beta(5/2, 1_{GL(2)}) = 1_{G_2} + J_\alpha(3/2, \delta(1))$ in $R(G_2)$.

(4.30)

(ii) $r_\emptyset(St_{G_2}) = \nu^2 \otimes \nu.$
Proof. Note that
\[ \text{Ind}_{\mathbb{G}_2}^G(\delta_{\frac{1}{2}}) = I(\nu^2 \otimes \nu). \]
It is well-known that the length of this representation is four. It contains \( \text{St}_{\mathbb{G}_2}(1_{\mathbb{G}_2}) \) as a unique irreducible subrepresentation (quotient). Since \( I_\alpha(3/2, \delta(1)) \) and \( I_\beta(5/2, \delta(1)) \) are subrepresentations in \( I(\nu^2 \otimes \nu) \) the proposition easily follows. □

Theorem 4.1. (i) Suppose that \( \chi \) is a character of order two or three of \( F^\times \). Then the induced representation \( I(\nu \chi \otimes \chi) \) contains a unique irreducible subrepresentation \( \pi(\chi) \). \( \pi(\chi) \) is square integrable. The only equivalences among these representations are \( \pi(\chi) \cong \pi(\chi^{-1}) \).

(ii) The induced representation \( I(1 \otimes \nu) \) contains exactly two irreducible subrepresentations \( \pi(1) \) and \( \pi(1)' \). They are non-equivalent and square integrable.

(iii) Suppose that \( \pi \) is a square integrable representation supported on a minimal parabolic subgroup. Then \( \pi \) is either the Steinberg representation or it is equivalent to exactly one of the representations considered in (i) and (ii).

Proof. It is a direct consequence of Theorem 3.1 and Propositions 4.1, 4.2, 4.3 and 4.4. □

5. Classification of unitary representations

Let \( G \) be any reductive \( F \)-group and \( P = MN \) its \( F \)-parabolic subgroup. Denote by \( \text{Unr}(M) \) the group of unramified characters. It is naturally isomorphic (as a topological group) to a direct product of finitely many copies of \( C^\times \). For any irreducible representation \( \pi \) of \( M \) and \( \chi \in \text{Unr}(M) \) denote \( I(\chi, \pi) = \text{Ind}_{\mathbb{G}_2}^G(\chi \otimes \pi) \).

Lemma 5.1. Under above assumptions we have:

(i) The set of all \( \chi \in \text{Unr}(M) \) such that \( I(\chi, \pi) \) has an unitarizable irreducible subquotient, is compact.

(ii) Let \( S \subseteq \text{Unr}(M) \) be a connected set. Suppose that for all \( \chi \in S \) representation \( I(\chi, \pi) \) is an irreducible unitarizable representation. Then any irreducible subquotient of \( I(\chi, \pi) \), \( \chi \in \text{cl}(S) \), is unitarizable (\( \text{cl}(S) \) denote the closure of the set \( S \)).

(iii) Suppose that \( \pi \) is Hermitian, and \( I(0, \pi) \) irreducible and unitarizable. Then \( \pi \) is unitarizable.

Proof. (i) is a special case of [T4, Thm. 4.5]. (ii) follows directly from [T1, Thm. 2.7]. (iii) was proved by Speh. □
Theorem 5.1. Suppose that $\chi$ is an unitary character and $s > 0$ a real number. Let $\gamma \in \{\alpha, \beta\}$. Then

(i) $J_{\gamma}(s, \delta(\chi))$ is unitarizable if and only if $\chi^2 = 1$ and $s \leq 1/2$.

(ii) Suppose that $\pi$ is tempered (non-square integrable) irreducible representation of $GL(2)$. Then $J_{\alpha}(s, \pi)$ is unitarizable if and only if one of the following conditions is satisfied

(\textit{\textit{\textit{i}}}i) $\pi \cong \pi(\chi, \chi^{-1})$ and $s \leq 1/2$.

(\textit{\textit{\textit{i}}}i) $\pi \cong \pi(1, \chi)$, $\chi$ has order two and $s \leq 1/3$.

$J_{\beta}(s, \pi)$ is unitarizable if and only if one of the following conditions is satisfied

(\textit{\textit{\textit{i}}}i) $\pi \cong \pi(\chi, \chi^{-1})$, $\chi^3 \neq 1$ and $s \leq 1/2$.

(\textit{\textit{\textit{i}}}i) $\pi \cong \pi(\chi, \chi^{-1})$, $\chi^3 = 1$, and $s \leq 1/2$ or $s = 1$.

(\textit{\textit{\textit{i}}}i) $\pi \cong \pi(1, \chi)$, $\chi$ has order two, and $s \leq 1$.

Proof. For a tempered $\pi$, $J_{\gamma}(s, \pi)$ is Hermitian if and only if $\pi \cong \hat{w}_{\gamma}(\pi)$ ($\cong \hat{\pi}$). We form one parameter family of Hermitian representations

$$X_s = I_{\gamma}(s, \pi), \quad s \geq 0,$$

realized on the same space (the compact picture of induced representation) $X$. We choose a polynomial $P(s)$ with real coefficients such that $A(s) = P(s)A(s, \pi, \hat{w}_{\gamma})$ is holomorphic and non-zero for $s \geq 0$. Furthermore, if we choose unitary isomorphism $\rho : \hat{w}_{\gamma}(\pi) \rightarrow \pi$ we obtain a continuous family of Hermitian forms

$$< f, g >_s = \int_{G_{2}(\mathbb{C})} (pA(s)f(k), g(k))dk, \quad f, g \in X,$$

on $X$, where $(\cdot, \cdot)$ is a positive definite Hermitian form on the space of representation $\pi$. $<, >_s$ is $X_s$-invariant and nondegenerate on $J_{\gamma}(s, \pi)$. If $X_0$ is irreducible then we can (and will) assume that $<, >_0$ is positive definite. In this case $<, >_s$ is positive definite in a neighborhood of $s = 0$, until $X_s$ becomes reducible.

$J_{\gamma}(s, \delta(\chi))$ is Hermitian if and only if $\chi^2 = 1$. Using Theorem 3.1 and above discussion we conclude that $<, >_s$ is positive definite for $s < 1/2$. Hence, representations $J_{\gamma}(s, \delta(\chi))$ are unitarizable for $s \leq 1/2$. Suppose that $\chi$ is a character has order two. Then continuous family of representation $X_s$, $s > 1/2$ is irreducible and, consequently, never unitary. The same is true for $\chi = 1$ and $s > 3/2$ if $\gamma = \alpha$ (or $s > 5/2$ if $\gamma = \beta$). Let $\chi = 1$. Consider $\gamma = \alpha$. Then, according to [BW, XI, Thm. 4.5], $J_{\alpha}(3/2, \delta(1))$ is not unitarizable (see Proposition 4.4). Using Lemma 5.1 (ii) we conclude that $X_s$, $1/2 < s < 3/2$, is not unitarizable. The case $\gamma = \beta$ is proved analogously. This proves (i).
To prove (ii) set \( \pi = \pi(\chi_1, \chi_2) \). \( J_\gamma(s, \pi) \) is Hermitian if and only if \( \chi_1 \chi_2 = 1 \) or \( \chi_1^2 = \chi_2^2 = 1 \). Suppose that \( \chi_1, \chi_2 \) are different characters of order two. Then, using Proposition 3.1, we conclude that \( X_s = I_s(\pi) \) is irreducible for \( s > 0 \), and never unitary by Lemma 5.1 (i). Consider \( \pi = \pi(1, \chi) \), where \( \chi \) has order two. \( I_\alpha(s, \pi(1, \chi)) = I(\nu^* \otimes \nu^* \chi) \) reduces if and only if \( s = 1/3 \) or \( s = 1 \). It is not unitarizable for \( s > 1 \). We have

\[
I_\alpha(1, \pi(1, \chi)) = I_\beta(3/2, \delta(\chi)) + I_\beta(3/2, \chi).
\]

Hence \( J_\alpha(1, \pi(1, \chi)) \cong I_\beta(3/2, \chi \circ \det) \), nonunitarizable by Lemma 5.2. It follows that \( J_\alpha(s, \pi(1, \chi)) \) is not unitarizable for \( 1/3 < s \leq 1 \). Unitarizability of \( J_\beta(s, \pi(1, \chi)) \) one can treat analogously (\( s = 1 \) is only point of reducibility).

Unitarizability of \( J_\alpha(s, \pi(\chi, \chi^{-1})) \), \( \chi \neq 1 \) and \( J_\beta(s, \pi(\chi, \chi^{-1})) \), \( \chi^3 \neq 1 \) are treated similarly.

We denote by \( \pi_{s, \chi} \) the induced representation \( \nu^s \chi \times \nu^{-s} \chi \), \( s \geq 0 \) of \( GL(2, F) \). We know that \( \pi_{s, \chi} \) is irreducible and Hermitian for \( s \neq 1/2 \). Further, it is unitarizable if and only if \( 0 \leq s < 1/2 \).

Suppose that \( \chi^3 = 1 \). Since \( \pi_{s, \chi} \) is induced we obtain

\[
I_\alpha(\pi_{s, \chi}) = I(\nu^s \chi \otimes \nu^{-s} \chi).
\]

Further, \( w_\beta(\nu^s \chi \otimes \nu^{-s} \chi) = \chi^2 \otimes \nu^s \chi^{-1} \) and \( w_\alpha(\chi^2 \otimes \nu^s \chi^{-1}) = \nu^s \chi^{-1} \otimes \chi^2 \) together with induction in stages imply

\[
I_\alpha(\pi_{s, \chi}) \cong I(\chi^2 \otimes \nu^s \chi^{-1}) \cong I(\nu^s \chi^{-1} \otimes \chi^2) = I_\beta(s, \pi(\chi, \chi^{-1})),
\]

for \( s \neq 1/2, 1 \) (Proposition 3.1). Now, \( J_\beta(s, \pi(\chi, \chi^{-1})) \) is unitarizable for \( s \leq 1/2 \) and nonunitarizable for \( s > 1/2, s \neq 1 \), by Lemma 5.1(iii) \( (\pi_{s, \chi} \) is nonunitarizable for \( s > 1/2 \) \). Unitarizability of \( J_\beta(1, \pi(1,1)) \) is Corollary 5.1. Unitarizability of \( J_\beta(1, \pi(\chi, \chi^{-1})) \), where \( \chi \) has order three, is Theorem 6.2.

As above we have

\[
I_\beta(\pi_{s, 1}) = I(\nu^{-s} \otimes \nu^{2s}) \cong I(\nu^{2s} \otimes \nu^{-s}) \cong I(\nu^s \otimes \nu^s) \cong I_\alpha(s, \pi(1,1)),
\]

for \( s \neq 1/3, 1/2, 1 \) (Proposition 3.1). We conclude that \( J_\alpha(s, \pi(1,1)) \) is unitarizable for \( s \leq 1/2 \) and, using Lemma 5.1, nonunitarizable for \( s > 1/2, s \neq 1 \). If \( s = 1 \) then

\[
I(\nu \otimes \nu) = I_\alpha(3/2, \delta(1)) + I_\alpha(3/2, 1_{GL(2)}),
\]

and Theorem 3.1 imply \( J_\alpha(1, \pi(1,1)) \cong I_\alpha(3/2, 1_{GL(2)}) \). Hence \( J_\alpha(1, \pi(1,1)) \) is not unitarizable by Lemma 5.2. \( \square \)

**Lemma 5.2.** Suppose that \( \chi \) is a character such that \( \chi^2 = 1 \) and \( s > 0 \). Let \( \gamma \in \{\alpha, \beta\} \).
Then \( I_\gamma(s, \chi \circ \det) \) is unitarizable (away from points of reducibility) if and only if \( s < 1/2 \).
Proof. For any irreducible representation $\pi$ denote by $\overline{\pi}$ its complex conjugate representation ([T1, 2.1]). Then $\pi$ is Hermitian if and only if $\pi \cong \overline{\pi}$. Assume that $I_\gamma(s, \chi \circ \det)$ is irreducible. Then we have $I_\gamma(s, \chi \circ \det) \cong I_\gamma(-s, \chi \circ \det)$ (see the proof of Theorem 3.1). This means that $I_\gamma(s, \chi \circ \det)$ is Hermitian. We continue as in the first part of the proof of Theorem 5.1. We choose a rational function $P(s)$ such that $A(s) = P(s)A(s, \pi, \overline{\psi})$ is holomorphic and nonzero away from points of reducibility. Then we can form continuous family of Hermitian forms $\langle \cdot, \cdot \rangle_s$ as in the proof of Theorem 5.1. Since $I_\gamma(0, \chi \circ \det)$ is irreducible we can continue as in the proof of Theorem 5.1 (i). □

Corollary 5.1. $J_\beta(1, \pi(1, 1))$ is unitarizable.

Proof. Using Proposition 4.3(ii) we see that $J_\beta(1, \pi(1, 1))$ is subquotient of $I_n(1/2, 1_{GL(2)})$. It is unitarizable by Lemma 5.1 (ii) and Lemma 5.2. □

Theorem 5.2. Suppose that $\chi_1, \chi_2$ are unitary characters and $s_1 > s_2 > 0$ real numbers. $J(s_1, s_2, \chi_1, \chi_2)$ is unitarizable if and only if one of the following conditions is satisfied

(i) $\chi_1 = \chi_2 = 1$, and $2s_1 + s_2 \leq 1$ or $s_1 + 2s_2 \geq 1$, $s_1 + s_2 \leq 1$ or $s_1 = 2, s_2 = 1$ (trivial representation).

(ii) $\chi_1 = 1$, $\chi_2$ has order two, and $s_1 + 2s_2 \leq 1$ or $s_1 = 1$.

(iii) $\chi_1$ has order two, $\chi_2 = 1$, and $2s_1 + s_2 \leq 1$.

(iv) $\chi_1 = \chi_2$ has order two, and $s_1 + s_2 \leq 1$.

Proof. $J(s_1, s_2, \chi_1, \chi_2)$ is Hermitian if and only if $\chi_1^2 = \chi_2^2 = 1$. In this case we look at two parameter family of Hermitian representations

$$X_{s_1, s_2} = I(s_1, s_2, \chi_1, \chi_2), \quad s_1 > s_2 > 0,$$

realized on the same space $X$ (the compact picture of induced representation). Furthermore, we have a continuous family of Hermitian forms

$$\langle f, g \rangle_{s_1, s_2} = \int_{G_2(O)} A(s_1, s_2, \chi_1, \chi_2, w_0) f(k) g(k) dk, \quad f, g \in X.$$ 

It is $X_{s_1, s_2}$–invariant and nondegenerate on $J(s_1, s_2, \chi_1, \chi_2)$. If we fix $s_1, s_2$ we can form one parameter family of Hermitian representations $X_t = X_{ts_1, ts_2}$, $t \geq 0$. Further, we choose a polynomial $P(t)$ with real coefficients such that $A(t) = P(t)A(ts_1, ts_2, \chi_1, \chi_2, w_0)$ is holomorphic and non-zero for any $t \geq 0$. Then we have a continuous family of Hermitian forms

$$\langle f, g \rangle_t = \int_{G_2(O)} A(t) f(k) g(k) dk, \quad f, g \in X.$$
If $X_0$ is irreducible we assume that $<,>_0$ is positive definite. In this case $<,>_t$ is positive definite in a neighborhood of $t = 0$, until $X_t$ becomes reducible.

If $\chi_1, \chi_2$ are different characters has order two then $I(\nu s\chi_1 \otimes \nu t\chi_2)$ is irreducible for $s_1 > s_2 > 0$. Hence $J(s_1, s_2, \chi_1 \otimes \chi_2)$ is never unitarizable. Otherwise we have several cases:
(i) $\chi_1 = \chi_2 = 1$. Then $I(\nu s\otimes \nu t\chi_2)$ reduces if and only if one of the following conditions is satisfied

$$2s_1 + s_2 = 1, \ s_1 + 2s_2 = 1, \ s_1 + s_2 = 1, \ s_1 = 1, \ s_1 - s_2 = 1, \ s_2 = 1$$

(denoted by $S_1, \ldots, S_6$ in the diagram).

\[ \begin{array}{c}
\text{Fig. Spherical unitary dual with real infinitesimal character} \\
\end{array} \]

Then we have several continuous family of Hermitian representations:

(i) $2s_1 + s_2 < 1$. Since $I(1 \otimes 1)$ is irreducible, $J(s_1, s_2, 1, 1) \cong I(\nu s\otimes \nu t\chi_2)$ is unitarizable using discussion at the beginning of the proof. Using Lemma 5.1 (ii), $J(s_1, s_2, 1, 1)$ is also unitarizable for $2s_1 + s_2 = 1$.

(ii) $2s_1 + s_2 > 1, s_1 + 2s_2 < 1$. There is no unitarizability in this region by Lemma 5.1 (ii). More precisely, points $(s, 0), 1/2 < s < 1$, are in the closure of this region, and $I(\nu s \otimes 1) \cong I_\beta(s, \pi(1, 1))$ contains $J_\beta(s, \pi(1, 1))$ which is not unitarizable (Theorem 5.1 (ii)).
(iii) $s_1 + 2s_2 > 1, s_1 + s_2 < 1$. To prove unitarizability of representations $J(s_1, s_2, 1, 1)$ we fix $s, 1/3 < s < 1/2$, and consider continuous family of induced representations

$$X_t = I_\beta(t, \pi_{s,1}) \cong I(\nu^{-s} \otimes \nu^{2s}), 0 \leq t < s - 1/3$$

(see the proof of Theorem 5.1 (ii) for definition of $\pi_{s,1}$). These representations are irreducible and Hermitian. Since $X_0$ is irreducible and unitarizable representation, as in the proof of Lemma 5.2 we conclude that any representation $X_t$ is unitarizable. Now, fix $t, 0 < t < s - 1/3$. Then $w_\alpha(\nu^{-s} \otimes \nu^{2s}) = \nu^{2s} \otimes \nu^{-s}$ and $w_\beta(\nu^{2s} \otimes \nu^{-s}) = \nu^{s+t} \otimes \nu^{-s+t}$ imply

$$I_\beta(t, \pi_{s,1}) \cong I(\nu^{-s} \otimes \nu^{2s}) \cong I(\nu^{2s} \otimes \nu^{-s}) \cong I(\nu^{s+t} \otimes \nu^{-s+t}) \cong J(s + t, s - t, 1, 1).$$

Next, point $(s + t, s - t)$ is in our region.

(i4) $s_1 + s_2 = 1$ or $s_1 + 2s_2 = 1$. Then $J(s_1, s_2, 1, 1)$ is unitarizable because of (iii).

(i5) $s_1 + s_2 > 1, s_1 < 1$. There are no unitarizability here because points $(s, s), 1/2 < s < 1$, are in the closure of this region, and $J_\alpha(s, \pi(1, 1))$ is not unitarizable (see (i2)).

(i6) $s_1 = 1$. Then $0 < s_2 < 1$, and, using Proposition 1.1 and Theorem 3.1,

$$J(1, s_2, 1, 1) \cong I_\beta(s_2 + 1/2, 1_{GL(2)}).$$

Hence $J(1, s_2, 1, 1)$ is not unitarizable by Lemma 5.2.

(i7) $s_1 > 1, s_1 - s_2 < 1, s_2 < 1$. There is no unitarizability in this region because of (i6).

(i8) Regions $s_1 - s_2 > 1, s_2 < 1; s_1 - s_2 < 1, s_2 > 1$ and $s_1 - s_2 > 1, s_2 < 1$ are unbounded.

There is no unitarizability here by Lemma 5.1 (i).

(i9) $s_2 = 1, s_1 \neq 2$. Then, by Proposition 1.1 (iii) and Theorem 3.1 we obtain

$$J(s_1, s_2, 1, 1) \cong I_\beta(s_1 + 1/2, 1_{GL(2)}).$$

Hence there is no unitarizability here.

(i10) $s_1 - s_2 = 1, s_2 \neq 1$. Then, by Proposition 1.1 (ii) and Theorem 3.1 we obtain

$$J(s_1, s_2, 1, 1) \cong I_\alpha(s_2 + 1/2, 1_{GL(2)}).$$

There is no unitarizability in this region.

(i11) $s_1 = 1, s_2 = 2$. Then $J(2, 1, 1, 1) \cong 1_G$, is an unitarizable representation.

The other cases are treated similarly (but are much simpler):

(ii) $\chi_1 = 1, \chi_2$ has order two.

(iii) $\chi_1$ has order two, $\chi_2 = 1$.

(iv) $\chi_1 = \chi_2$ has order two. □
Denote by \( W_F \) the Weil group of the field \( F \). Let \( \rho = \pi(\tau) \) be any unitary supercuspidal representation of \( GL(2, F) \), where

\[
\tau : W_F \rightarrow GL(2, \mathbb{C})
\]

is attached admissible homomorphism. Then \( \det \tau = \omega_\rho \) (the central character of \( \rho \)) via class field theory (see [Sh3, Section 1] for more details).

**Theorem 5.3.** Suppose that \( \rho \) is an unitary supercuspidal representation of \( GL(2, F) \). Let \( \gamma \in \{\alpha, \beta\} \). Then the Langlands quotient \( J_\gamma(s, \rho) \) is unitarizable if and only if \( \rho \cong \tilde{\rho} \) and one of the following conditions is satisfied:

1. \( \omega_\rho = 1 \) and \( 0 < s \leq 1/2 \).
2. \( \gamma = \beta, \Im(\tau) \cong S_3 \) (the symmetric group) and \( 0 < s \leq 1 \).

If \( I_\gamma(s_0, \rho) \), \( s_0 > 0 \), reduces then it has unique irreducible subrepresentation \( \pi_\gamma(\rho, s_0) \). That subrepresentation is square integrable. For a different representations \( \rho \) we obtain different \( \pi(\rho, s_0) \) (\( s_0 \) is uniquely determined by \( \rho \)). If \( I_\gamma(0, \rho) \) reduces then it has length two and is of multiplicity one.

**Proof.** The only non-trivial part of this theorem is to calculate the points of reducibility of generalized principal series representations \( I_\gamma(s, \rho) \), \( s \in \mathbb{R} \). It is well-known that \( I_\gamma(s, \rho) \) reduces for some \( s_0 \in \mathbb{R} \) if and only if \( \tilde{\rho} \cong \rho \). Then \( s_0 \) is a unique point of reducibility (on the real axis). If \( \gamma = \alpha \) then the points of reducibility are calculated in [Sh5, Prop. 6.2] using explicit formula for the Plancherel measure \( \mu(s, \rho, w_{3\alpha + 2\beta}) \) (see also [Sh6, Cor. 7.6]) and in [GrSa1, Cor. 5.5] using \( \Theta \)-correspondence of the dual pair

\[
PGL(3, F) \times G_2 \subseteq H,
\]

where \( H \) is split adjoint group of type \( E_6 \), through the minimal representation of \( H \) (see [Sa1, Thm. 2.2]). If \( \gamma = \beta \) the points of reducibility are calculated in [Sh6, Prop. 8.2] using explicit formula for the Plancherel measure \( \mu(s, \rho, w_{2\alpha + \beta}) \). \( \square \)

### 6. Unitarizability of \( J_\beta(1, \pi(\chi, \chi^{-1})) \)

In this section we prove that \( J_\beta(1, \pi(\chi, \chi^{-1})) \), where \( \chi \) has order three, is an unitary representation. Then it is an isolated point in the unitary dual of \( G_2 \). It is a direct consequence of the description of topology on the dual space based on an application of the Bernstein center [T4, Thm. 2.2].

Let \( K \) be an algebraic number field. For each place \( v \) of \( K \) denote its completion by \( K_v \). Denote by \( \mathcal{O}_v \) the ring of integers of \( K_v \), if \( K_v \) is nonarchimedean. Let \( \mathbb{A} \) be the ring of adeles of \( K \) and \( v \) absolute value of idele class group \( \mathbb{A}^\times \).
Let $G_2$ be a split simple group defined over $K$ of type $G_2$. Let $G_{2,v} = G_2(K_v)$. Let $G_2(A)$ denote the $A$-points of $G_2$. For each archimedean place $v$ choose a maximal compact subgroup $Q_v$. If $v$ is finite set $Q_v = G_2(O_v)$. Then $Q = \prod_v Q_v$ is a maximal compact subgroup of $G_2(A)$.

We consider

$$L^2(G_2(K) \setminus G_2(A))$$

the space of classes of measurable functions $f : G_2(K) \setminus G_2(A) \to \mathbb{C}$ such that

$$\int_{G_2(K) \setminus G_2(A)} |f(g)|^2 \, dg < +\infty.$$ 

It is a unitary representation where $G_2(A)$ acts by right translations. It is well-known that we have a Hilbert space decomposition

$$L^2(G_2(K) \setminus G_2(A)) = L^2_d \oplus L^2_e,$$

where $L^2_d$ is a direct sum of irreducible unitary representations, each occurring with finite multiplicity, and $L^2_e$ is a direct integral of irreducible unitary representations.

Let $\mu : K^\times \setminus A^\times \to \mathbb{C}^\times$ be a character of order three. Denote by $I(\mu)$ the space of continuous right $Q$-finite complex valued functions $\Psi$ on $G_2(A)$ satisfying

$$\Psi(utg) = (\mu \otimes \mu)(t)\Psi(g), \quad t \in T(A), u \in U(A), g \in G_2(A).$$

Then we form an induced representation $I(\nu^{s+1/2}\chi \otimes \nu^{s-t/2}\chi)$, $s, t \in \mathbb{C}$, on the space of functions of the form

$$f_{s,t}(g) = \exp <(s + t/2, s - t/2) + \rho_0, H(g) > \Psi(g)$$

(for undefined notation see [Sh4, p. 551] or [Ki, p. 139]).

We form an Eisenstein series

$$E(g, f_{s,t}) = \sum_{\eta \in B(K) \setminus G_2(K)} f_{s,t}(\eta g)$$

It converges absolutely for $Re(s + t/2, s - t/2) \in \mathbb{C}^+ + \rho_0$ and extends to a meromorphic function of $(s, t)$. It is an automorphic form and its constant term along $B$ is given by

$$E_0(g, f_{s,t}) = \int_{U(K) \setminus U(A)} E(ug, f_{s,t}) \, du = \sum_{w \in W} A(s + t/2, s - t/2, \mu, \mu, w)f_{s,t}(g),$$

where for sufficiently regular $\Lambda = (s + t/2, s - t/2)$,

$$A(s + t/2, s - t/2, \mu, \mu, w)f_{s,t}(g) = \int_{U_w(A)} f_{s,t}(w^{-1}ug) \, du,$$
where \( U_w = U \cap w \bar{U} w^{-1}, \bar{U} \) is the unipotent radical opposed to \( U \). \( A(s + t/2, s - t/2, \mu, \mu, w) \) is an intertwining operator (for a Hecke algebra of \( G_2(\mathbb{A}) \)) from \( I(\Lambda, \mu \otimes \mu) \) to \( I(w(\Lambda), w(\mu \otimes \mu)) \).

Set

\[
R_1 = \text{Res}_{s=1/2} \text{Res}_{\sigma=1} A(s + t/2, s - t/2, \mu, \mu, w_0), \\
R_2 = \text{Res}_{s=1/2} \text{Res}_{\sigma=1} A(s + t/2, s - t/2, \mu, \mu, w_{2n+\beta}).
\]

Computations in [Za] show that automorphic form

\[
E'(g, f_{s, t}) = (s - 1/2)(t - 1)E(g, f_{s, t})
\]

is holomorphic and nonzero at \((s, t) = (1/2, 1)\) and \( R_1, R_2 \) are nonzero intertwining operators. Moreover, its constant term \( E'_0(g, f_{s, t}) \) along \( B \) satisfies

\[
E'_0(g, f_{1/2, 1}) = R_1 f_{1/2, 1}(g) + R_2 f_{1/2, 1}(g). \tag{6.30}
\]

Using Langlands square integrability criteria [Ki, p.143] we conclude that \( E'(g, f_{1/2, 1}) \) is a square integrable automorphic form. Furthermore,

\[
f \mapsto E'(g, f) \tag{6.31}
\]

defines a Hecke algebra intertwining map \( I(\nu \mu \otimes \mu) \longrightarrow I^2_d \). It is easy to see that

\[
I(\nu^{-1} \mu^{-1} \otimes \mu^{-1}) \supset \text{Im} R_1 \cong \otimes_v J_\beta(1, \pi(\mu_v, \mu_v^{-1})), \tag{6.32}
\]
\[
I(\nu^{-1} \mu \otimes \mu) \supset \text{Im} R_2 \cong \otimes_v J_\beta(1, \pi(\mu_v, \mu_v^{-1})). \tag{6.33}
\]

(6.32) and (6.33) imply that (6.31) has image isomorphic to \( \otimes_v J_\beta(1, \pi(\mu_v, \mu_v^{-1})) \).

Finally, we need [AT, p.103, Thm. 5].

**Theorem 6.1.** Let \( K \) be a global field, \( n \) positive integer, \( S \) a finite set of places of \( K \). For \( v \in S \), let \( \chi_v \) be a character of \( K_v^\times \) of order dividing \( n \). Then there exists a character \( \mu \) of \( \mathbb{A}^\times \), trivial on \( K_v^\times \), of order dividing \( 2n \), with \( \mu_v = \chi_v \) at \( v \in S \).

**Theorem 6.2.** Suppose that \( \chi \) is a character of order three. Then the Langlands quotient \( J_\beta(1, \pi(\chi, \chi^{-1})) \) is unitarizable.

**The first proof.** Choose global field \( K \) having \( F \) as its completion at finite place \( w \). Choose character \( \mu \) of \( \mathbb{A}^\times \), trivial on \( K_v^\times \), of order dividing six such that \( \mu_w = \chi \). Since \( \mu^2 \neq 1 \), we see that \( \mu \) has order three or six. If \( \mu \) is of order three, then \( J_\beta(1, \pi(\chi, \chi^{-1})) \) is unitarizable being a factor of global unitarizable representation \( \otimes_v J_\beta(1, \pi(\mu_v, \mu_v^{-1})) \) at place \( w \). Otherwise, \( \mu^2 \) has order three. Then, representation \( J_\beta(1, \pi(\mu_w^2, \mu_w^{-2})) \) is unitarizable and

\[
J_\beta(1, \pi(\mu_w^2, \mu_w^{-2})) = J_\beta(1, \pi(\chi^{-1}, \chi)) = J_\beta(1, \pi(\chi, \chi^{-1})).
\]
The second proof. Now, we present local proof based on ideas of [BM1] and [V2]. We shall assume that the residual characteristic of $F$ is different from three. Then every character $\chi$ which has order three on $F^\times$ is trivial on the first congruence subgroup $1 + \wp$. Set $\chi' = \chi |_{\wp}$. Then in our parametrization $\chi \otimes \chi |_{U_1} = \chi' \otimes \chi'$. Using reduction homomorphism

$$G_2(O) \longrightarrow G_2(F_q)$$

we can identify $\chi' \otimes \chi'$ as a character of $T(F_q)$. Then we can consider $\chi' \otimes \chi'$ as an character of $B(F_q)$ trivial on $U(F_q)$. Iwahori subgroup $B$ is the preimage of $B(F_q)$ under the reduction homomorphism. We consider $\chi_0 = \chi' \otimes \chi'$ as a one dimensional representation of $B$. Denote by $K_1$ the kernel of the reduction homomorphism.

Following [V2], for any admissible representation $(\pi, X)$ of finite length of $G_2$ we define a formal $G_2(O)$-character $\Theta_{G_2(O)}(\pi)$ as

$$\sum_{\delta \in G_2(O)} m(\delta)\delta,$$

where the sum ranges over all classes of irreducible representation of $G_2(O)$ and where $m(\delta)$ denotes the multiplicity of $\delta$ in $\pi |_{G_2(O)}$. We also set

$$\Theta'_{G_2(O)}(\pi) = \sum_{\delta} m(\delta)\delta, \quad (6.34)$$

where the sum ranges over all irreducible representations

$$\delta \subset \text{Ind}_{B(F_q)}^{G_2(F_q)} (\chi_0)$$

considered as a representation of $G_2(O)$ under the reduction homomorphism. Suppose that $\pi$ is a Hermitian representation with non-degenerate $G_2$-invariant Hermitian $\langle \cdot, \cdot \rangle$. For any representation $\delta \in \hat{G}_2(O)$ fix a positive-definite Hermitian form on the space $V_\delta$ of $\delta$. Then finite dimensional vector space

$$X^\delta = \text{Hom}_{G_2(O)}(V_\delta, X)$$

acquires a non-degenerate Hermitian form; write $(p(\delta), q(\delta))$ for its signature. The signature of $\langle \cdot, \cdot \rangle$ is the pair of formal sums

$$\left( \sum_{\delta \in \hat{G}_2(O)} p(\delta)\delta, \sum_{\delta \in \hat{G}_2(O)} q(\delta)\delta \right).$$

We have $m(\delta) = p(\delta) + q(\delta)$, for any $\delta \in \hat{G}_2(O)$. In the proof of Theorem 5.1 we considered one parameter family of Hermitian representations

$$X_s = I(\nu^s \chi \otimes \chi) = I_{\beta}(s, \pi(\chi, \chi^{-1})), \quad s \geq 0,$$
with Hermitian form $<\cdot,\cdot>_s$, which is positive-definite at $s = 0$. $s = 1/2, 1$ are only points of reducibility of $X_s$ for $s \geq 0$. To prove the theorem we need to know how the signature of $<\cdot,\cdot>_s$ changes from $s = 0$ to $s = 1$. It does not change over intervals where the form $<\cdot,\cdot>_s$ is nondegenerate. At points of reducibility we consider the Jantzen filtration [V2, Section 3]. Let $s_0 \in \{1/2, 1\}$. Then the Jantzen filtration is the sequence of subspace

$$X = X^{0}_{s_0} \supseteq X^{1}_{s_0} \supseteq \cdots \supseteq X^{N}_{s_0} = \{0\},$$

($X$ denote the compact picture of induced representations $X_s$) defined as follows. The space $X^n_{s_0}$ is the space of vectors $v \in X$ for which there is a neighborhood $U$ of $s_0$ and an analytic function

$$f_v : U \longrightarrow X$$

satisfying

(i) $f_v$ takes values in a finite dimensional $G_2(\mathcal{O})$-subspace of $X$

(ii) $f_v(s_0) = v$

(iii) $\forall v' \in X$ the function $s \mapsto <f_v(s), v'>_s$ vanishes at $s_0$ to order at least $n$.

Define Hermitian form $<\cdot,\cdot>_n$ on $X^n_{s_0}$ by the formula

$$<v, v'> = \lim_{s \rightarrow s_0} \frac{1}{(s - s_0)^n} <f_v(s), f_{v'}(s)>_s.$$

Jantzen has shown that the form $<\cdot,\cdot>_n$ has radical exactly $X^{n+1}_{s_0}$ [V2, Thm. 3.2]. Write $(p_n, q_n)$, for the signature character on $X^n_{s_0}/X^{n+1}_{s_0}$. Vogan showed [V2, Prop. 3.3]

**Theorem 6.3.** For $s - s_0$ small positive, $<\cdot,\cdot>_s$ has signature character

$$\left(\sum_{n \text{ even}} p_n, \sum_{n \text{ odd}} q_n\right)$$

and for $s - s_0$ small negative

$$\left(\sum_{n \text{ even}} p_n + \sum_{n \text{ odd}} q_n, \sum_{n \text{ odd}} p_n + \sum_{n \text{ even}} q_n\right).$$

Note that $X^n_{s_0}$ are subrepresentations of $X_{s_0}$, $<\cdot,\cdot>_n$ is $G_{\mathcal{O}}$-invariant and

$$X^{0}_{s_0}/X^{1}_{s_0} \cong J_\beta(s_0, \pi(\chi, \chi^{-1})).$$

Now, we calculate the signature character $(p_0, q_0)$ of $J_\beta(1, \pi(\chi, \chi^{-1}))$. To do this we calculate the signature character of $X_s$, $1/2 < s < 1$, using Jantzen filtrations in $s = 1/2$ and $s = 1$. Using Jantzen filtration at $s = 1/2$ we obtain
Lemma 6.1. The signature character of $X_s$, $1/2 < s < 1$, is given by

\[
(\Theta_{G_2(O)}(J_\beta(1/2, \pi(\chi, \chi^{-1}))), \Theta_{G_2(O)}(I_\alpha(0, \delta(\chi)))).
\] (6.35)

Proof. $w_\beta w_\alpha w_\beta(\nu^{1/2} \otimes \chi) = \nu^{1/2} \otimes \nu^{-1/2} \chi$ and Proposition 1.1 (ii) imply

\[X_{1/2} = I_\alpha(0, \delta(\chi)) + I_\alpha(0, \chi \circ \det)\] (6.36)
in $R(G_2)$. Then Theorem 3.1 implies that $X_{1/2}$ is of length two. Further, by the proof of Theorem 5.1 representations $X_s$ are not unitarizable for $1/2 < s < 1$. Now, it is easy to complete the proof using Theorem 6.3. □

Using Jantzen filtration at $s = 1$ we obtain

Lemma 6.2. The signature character of $X_s$, $1/2 < s < 1$, is given by

\[
(\rho_0 + \Theta_{G_2(O)}(J_\alpha(1/2, \delta(\chi))), \rho_0 + \Theta_{G_2(O)}(I_\alpha(0, \delta(\chi)))).
\] (6.37)

Now, if we compare (6.35) and (6.37) we conclude $\rho_0 = 0$. Hence $J_\beta(1, \pi(\chi, \chi^{-1}))$ is unitarizable, and the theorem is proved.

Proof of the lemma. Note that in $R(G_2)$ we have (Proposition 4.2)

\[X_1 = \pi(\chi) + J_\alpha(1/2, \delta(\chi)) + J_\beta(1, \pi(\chi, \chi^{-1})) + J_\alpha(1/2, \delta(\chi^{-1})).\]

Hermitian contragredient of $J_\alpha(1/2, \delta(\chi))$ is $J_\alpha(1/2, \delta(\chi^{-1}))$. Then, using [V2, Lem. 3.9] we conclude that there exists $n > 0$ such that $X_1^n / X_1^{n+1}$ has $J_\alpha(1/2, \delta(\chi))$ and $J_\alpha(1/2, \delta(\chi^{-1}))$ as irreducible subquotients.

Since $\pi(\chi)$ is a unique irreducible subrepresentation of $X_1$ (Theorem 4.1 (i)) it is not irreducible subquotient of $X_1^n / X_1^{n+1}$. Hence $n < N - 1$. Now, using Theorem 6.3 and [V2, Lem. 3.9], we conclude that the signature character of $1/2 < s < 1$ is given by

\[
(\rho_0 + \Theta_{G_2(O)}(J_\alpha(1/2, \delta(\chi))) + \Theta_{G_2(O)}(\pi(\chi)), \rho_0 + \Theta_{G_2(O)}(J_\alpha(1/2, \delta(\chi))) = (\rho_0 + \Theta_{G_2(O)}(J_\alpha(0, \delta(\chi))), \rho_0 + \Theta_{G_2(O)}(J_\alpha(1/2, \delta(\chi)))
\] (6.38)
or

\[
(\rho_0 + \Theta_{G_2(O)}(J_\alpha(1/2, \delta(\chi))), \rho_0 + \Theta_{G_2(O)}(\pi(\chi)) + \Theta_{G_2(O)}(J_\alpha(1/2, \delta(\chi))) = (\rho_0 + \Theta_{G_2(O)}(J_\alpha(0, \delta(\chi))), \rho_0 + \Theta_{G_2(O)}(I_\alpha(0, \delta(\chi)))
\] (6.39)

Now we prove that (6.38) does not occur. If it does, then (6.35) and (6.38), using (6.36), imply

\[\rho_0 = \Theta_{G_2(O)}(I(\chi \otimes \chi)) - 2\Theta_{G_2(O)}(I_\alpha(0, \delta(\chi))).\] (6.40)

(6.40) implies

\[\Theta'_{G_2(O)}(I(\chi \otimes \chi)) - 2\Theta'_{G_2(O)}(I_\alpha(0, \delta(\chi))) \geq 0,\] (6.41)
where ‘$\geq$’ is the usual order in the Grothendieck group $R(G_2(\mathcal{O}))$.

Denote by $\delta(\chi_0)$ the twist by $\chi_0 \circ \det$ of the Steinberg representation of $GL(2, F_q)$. Then (6.41) implies that
\[
\text{Ind}_{B(F_q)}^{G_2(F_q)}(\chi_0) - 2\text{Ind}_{P_{1}(F_q)}^{G_2(F_q)}(\delta(\chi_0)) =
\text{Ind}_{P_{2}(F_q)}^{G_2(F_q)}(\chi_0 \circ \det) - \text{Ind}_{P_{1}(F_q)}^{G_2(F_q)}(\delta(\chi_0))
\]
is positive in $R(G_2(F_q))$. This is a contradiction. □

References

References


Department of Mathematics, University of Zagreb, Bijenicka cesta 30, 10000 Zagreb, Croatia
E-mail: gnuic@eomath.math.hr