ON THE CUSPIDAL MODULAR FORMS FOR THE FUCHSIAN GROUPS OF
THE FIRST KIND

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Abstract. In this paper we study the construction and non–vanishing of cuspidal modular forms of weight $m \geq 3$ for arbitrary Fuchsian groups of the first kind. We give a spanning set for the space of cuspidal modular forms $S_m(\Gamma)$ of weight $m \geq 3$ in a uniform way which does not depend on the fact that $\Gamma$ has cusps or not.

1. Introduction

In this paper we discuss an application of our general theory of constructing (adelic) cuspidal automorphic forms via Poincaré series [7]. We use the results of [7] to construct and study non–vanishing of cuspidal modular forms for an arbitrary Fuchsian group of the first kind.

To explain our results, we recall some standard notation (see [6]). Let $X$ be the upper half–plane. Then the group $SL_2(\mathbb{R})$ acts on $X$ as follows:

$$g.z = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).$$

We let $\mu(g, z) = cz + d$.

Next, $SL_2(\mathbb{R})$–invariant measure on $X$ is defined by $dxdy/y^2$, where the coordinates on $X$ are written in a usual way $z = x + \sqrt{-1}y$, $y > 0$. A discrete subgroup $\Gamma \subset SL_2(\mathbb{R})$ is called a Fuchsian group of the first kind if its fundamental domain $F_\Gamma$ in $X$ has a finite volume i.e., $\Gamma$ has finite covolume in $SL_2(\mathbb{R})$. Then, adding finite number of points in $\mathbb{R} \cup \{\infty\}$ called cusps, $F_\Gamma$ can be compactified. In this way we obtain a compact Riemann surface $F_\Gamma^*$.

For an integer $m$, let $S_m(\Gamma)$ be the space of all modular forms of weight $m$ which are cuspidal i.e., this is a space of all holomorphic functions $f : X \rightarrow \mathbb{C}$ such that $f(\gamma.z) = \mu(\gamma, z)^m f(z)$ ($z \in X, \gamma \in \Gamma$) which are holomorphic and vanish at every cusp for $\Gamma$. One can use geometric considerations on the compact Riemannian surface $F_\Gamma^*$ to compute the dimension of the space $S_m(\Gamma)$ (see [6], Theorems 2.5.2, 2.5.3.).

The classical theory ([6], Section 2.6) gives the construction of elements and spanning set of the finite dimensional vector space $S_m(\Gamma)$, for $m \geq 3$, assuming that $F_\Gamma$ is not compact (i.e., $\Gamma$ has cusps). In this paper we give much more general construction of modular forms and a spanning set that does not depend if $F_\Gamma$ is compact or not. The following theorem is the main result of the paper:

**Theorem 1-1.** Let $m \geq 3$. Let $\Gamma \subset SL_2(\mathbb{R})$ be a discrete subgroup of finite covolume. Then we have the following:

(i) Let $f : X \rightarrow \mathbb{C}$ be a holomorphic function satisfying

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} y^{m/2} |f(z)| \frac{dxdy}{y^2} < \infty.$$
Then the series $\sum_{\gamma \in \Gamma} f(\gamma.z)\mu(\gamma, z)^{-m}$ converges uniformly and absolutely on compact sets in $X$ to an element of $S_m(\Gamma)$.

(ii) The holomorphic functions $(z + \sqrt{-1})^{-k-m}$ $(k \geq 0)$ on $X$ satisfy (1-2) and the corresponding modular forms

$$\beta_{k,m}(z) = \sum_{\gamma \in \Gamma} (\gamma.z + \sqrt{-1})^{-k-m} \mu(\gamma, z)^{-m}, \ k \geq 0,$$

span $S_m(\Gamma)$.

The proof of Theorem 1-1 is given in Section 3. It is obtained using the standard correspondence between automorphic forms on $SL_2(\mathbb{R})$ and modular forms on $X$ (see [2], Chapter 2, Proposition 2.1) from the main results of Section 2. In Section 2, we give a construction of cuspidal automorphic forms and a spanning set for certain spaces of cuspidal automorphic forms on $\Gamma \backslash SL_2(\mathbb{R})$ (see Lemmas 2-12 and 2-17). Lemma 2-12 is a reformulation of the general theory developed in ([7], Theorem 3-10) for $SL_2(\mathbb{R})$. Lemma 2-17 is a new result. Basically, Lemma 2-17 shows that the space $S_m(\Gamma)$ is spanned by the Poincaré series attached to $K$–finite matrix coefficients of a holomorphic discrete series $D_m$ of weight $m \geq 3$ (defined in [4], page 183). This lemma is the only place in the paper where we use the representation theory. Mostly, this is a representation theory of $SL_2(\mathbb{R})$ developed in [4]. But, we use more sophisticated ([3], Theorem 1) to give a quick computation of the matrix coefficients of $D_m$. In an early version of the paper we used more elementary methods to compute matrix coefficients.

Now, we relate our Theorem 1-1 to the classical situation ([6], Section 2.6). So, assume that $\infty$ is a cusp for $\Gamma$. Let $U_{p_{\infty}}$ be the group of upper–triangular unipotent matrices in $SL_2(\mathbb{R})$. Then $\Gamma_{\infty} = U_{p_{\infty}} \cap \Gamma$ is an infinite cyclic group ([1], 3.6). We write $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$, $h > 0$, for the generator of this group. It is well–known ([6], Corollary 2.6.11) that

$$\alpha_{l,m}(z) = \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} e^{2\pi l \sqrt{-1} \gamma.z/h} \mu(\gamma, z)^{-m}, \ l \geq 1$$

also span $S_m(\Gamma)$. Then we have the following (see Corollary 3-5):

$$\beta_{k,m}(z) = (-1)^{m+k-1} \cdot \frac{(2\pi \sqrt{-1})^{m+k}}{(m+k-1)!} \cdot \sum_{l=1}^{\infty} \frac{e^{-2\pi l/h}}{2 \cdot h^{m+k}} \alpha_{l,m}(z), \ k \geq 0,$$

where the convergence is absolute and uniform on compact sets in $X$.

Next, we study the non–vanishing of modular forms in Theorem 1-1 (i). In Section 4 we refine our general non–vanishing criterion (see [7], Theorem 4-1) having in mind $\Gamma$’s which have cusps (see Lemma 4-1 and Corollary 4-7).

In Section 5, we write them in our case in Lemma 5-1 (and in Lemma 5-2 which is for the case when $\Gamma$ has no cusps). As an application, we prove Theorem 1-3 in Section 5 stated below. In this theorem, we consider congruence subgroups. Let $N \geq 1$. Then we define standard congruence
subgroups
\[ \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); \ c \equiv 0 \pmod{N} \right\}; \]
\[ \Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); \ c \equiv 0, \ a, d \equiv 1 \pmod{N} \right\}; \]
\[ \Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); \ c, b \equiv 0, \ a, d \equiv 1 \pmod{N} \right\}. \]

It is well–known that they are discrete subgroup of \( SL_2(\mathbb{R}) \) of finite covolume (see [6]). The point \( \infty \) is their cusp.

**Theorem 1-3.** Let \( m \geq 3 \). Let \( f : X \to \mathbb{C} \) be a holomorphic function satisfying \( \int_{-\infty}^{\infty} \int_{0}^{\infty} y^{m/2} |f(z)| \frac{dx dy}{y^2} < \infty \). Then there exists \( N_0 \geq 1 \) such that the modular forms

\[ \sum_{\gamma \in \Gamma_N} f(\gamma z) \mu(\gamma, z)^{-m} \in S_m(\Gamma_N), \quad \Gamma_N \in \{ \Gamma(N), \Gamma_0(N), \Gamma_1(N) \}, \]

are non–trivial for \( N \geq N_0 \) assuming that \( m \) is even if \( -1 \in \Gamma_N \).

Thinking in terms of \( SL_2(\mathbb{R}) \), the non–vanishing criteria of Lemmas 5-1 and 5-2 use the Iwasawa decomposition (see (2-2)) to compute the integrals. The Cartan decomposition (see (2-14)) is not easily seen working on \( X \), but it is quite useful to study the non–vanishing of automorphic/modular forms. We illustrate this by Propositions 6-21 and 6-22 in Section 6 where the non–vanishing of another spanning set of \( S_m(\Gamma) \) given by (see Lemma 3-2)

\[ \sum_{\gamma \in \Gamma} \left( \gamma z - \sqrt{-1} \right)^k \left( \gamma z + \sqrt{-1} \right)^{-k-m} \mu(\gamma, z)^{-m}, \ k \geq 0, \]

is considered.

In fact, this is just the tip of the iceberg. More can be done on the lines of Sections 5 and 6 but we defer this for another occasion [11]. In the sequel to this paper we will study the non–vanishing of constructed forms in more detail as well as the action of Hecke operators using the methods of [10].

We mention here that we use compactly supported Poincaré series to study existence of Maass forms ([8], [9]).

I would like to thank Gordan Savin for his interest in my works ([7], [8], [9]). While we were both visitors at the Erwin Schrödinger Institute in Vienna, our discussions inspired me to start this project. He suggested to me that an elementary but nice Dragan Miličić’s idea would imply Lemma 2-17 (the text in the proof of Lemma 2-17 after Lemma 2-23 is inspired by that).

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2. Some Results on Automorphic Forms on \( SL_2(\mathbb{R}) \)

In this section we fix the notation and prove important results for the proof of Theorem 1-1 (see Lemmas 2-12 and 2-17.)

Let \( P_\infty = M_\infty A_\infty U_{p_\infty} \) be a minimal parabolic subgroup of \( G = SL_2(\mathbb{R}) \) consisting of upper triangular matrices. Explicitly, we have the following: \( M_\infty = \{ \pm 1 \}, A_\infty = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} ; a \in \mathbb{R}_{>0} \right\}, \)
$U_{P_{\infty}} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$. The maximal compact subgroup $K_{\infty}$ can be identified with $U(1)$ as follows:

$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mapsto \exp (\sqrt{-1}t) = \cos t + \sqrt{-1}\sin t$.

The unitary dual of $K_{\infty}$ can be identified with $\mathbb{Z}$ in the following way:

$\chi_m \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \exp (m \cdot \sqrt{-1}t), \ m \in \mathbb{Z}, \ t \in \mathbb{R}$.

We use the normalized Haar measure on $K_{\infty}$ given by

$$\int_{K_{\infty}} f(k) dk = \frac{1}{2\pi} \int_{0}^{2\pi} f \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} dt, \ f \in C_c^{\infty}(K_{\infty})$$

Let $X = \{ z \in \mathbb{C}; \text{Im}(z) > 0 \}$ be the upper half–plane. The group $SL_2(\mathbb{R})$ acts on $X$ as follows:

$g.z = \frac{az + b}{cz + d}, \ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$.

The stabilizer of $\sqrt{-1}$ is $K_{\infty}$. Thus $X$ is diffeomorphic to $SL_2(\mathbb{R})/K_{\infty}$ using the Iwasawa decomposition of $SL_2(\mathbb{R}) = U_{P_{\infty}} A_{\infty} K_{\infty}$:

$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mapsto g.\sqrt{-1} = x + y\sqrt{-1}$.

The measure on $SL_2(\mathbb{R})$ can be fixed as follows:

$$\int_{SL_2(\mathbb{R})} f(g) dg = \int_{-\infty}^{\infty} \int_{0}^{2\pi} f \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \frac{dxdy}{2y^2} dt, \ f \in C_c^{\infty}(SL_2(\mathbb{R}))$$

Thus, we see the following relation of the measures:

$$\int_{X} f(x + y\sqrt{-1}) \frac{dxdy}{y^2} = 2 \int_{SL_2(\mathbb{R})/K_{\infty}} f(g.\sqrt{-1}) dg, \ f \in C_c^{\infty}(X)$$

The stabilizer of a point $\overline{\mathbb{R}} = \mathbb{R} \cup \{ \infty \}$ is a (real) parabolic subgroup of $SL_2(\mathbb{R})$. The stabilizer of $\infty$ is $P_{\infty}$. If the stabilizer of $g.\infty$ is $gP_{\infty}g^{-1}$. We let $\mu(g) = cz + d$, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$.

The function $\mu$ satisfies the cocycle identity:

$$\mu(gg', z) = \mu(g, g'.z) \cdot \mu(g', z)$$

Let $\Gamma$ be a discrete subgroup of $SL_2(\mathbb{R})$ of finite covolume. The $\Gamma$–cuspidal parabolic subgroup for $\Gamma$ is a parabolic subgroup $P$ of $SL_2(\mathbb{R})$ such that $\Gamma \cap U_P$ contains a non–trivial element. Hence it is an infinite cyclic group ([1], 3.6). In particular, $\Gamma \cap U_P \setminus U_P$ is a compact group isomorphic to a unit circle. The group $\Gamma$ operates on the set of cuspidal parabolic subgroups by conjugation. If the volume of $\Gamma \setminus SL_2(\mathbb{R})/K_{\infty} = \Gamma \setminus X$, or equivalently, of $\Gamma \setminus SL_2(\mathbb{R})$ is finite, then set of equivalence classes of $\Gamma$–cuspidal parabolic subgroups is finite ([1], Theorem 3.14).
The space of cusp forms $A_{cusp}(\Gamma \backslash SL_2(\mathbb{R}))$ is defined in ([1], 5.5 and 7.8). Using ([1], 5.8), we can define it in the following way: it consists of all functions $\psi \in C^\infty(\Gamma \backslash SL_2(\mathbb{R}))$ satisfying the following conditions:

$\psi(\gamma g) = \psi(g)$, $\gamma \in \Gamma$, $g \in SL_2(\mathbb{R})$

$\psi$ is $K_\infty$-finite on the right i.e., the right translations of $\psi$ by $K_\infty$ span finite-dimensional space $\psi$ is $C$-finite on the right.

\[
\int_{\Gamma \backslash U_P \backslash U_P} \psi(ug) du = 0, \quad g \in SL_2(\mathbb{R}), \text{ for all } \Gamma \text{-cuspidal parabolic subgroups}.
\]

\[
\int_{\Gamma \backslash SL_2(\mathbb{R})} |\psi(g)| dg < \infty.
\]

Here $C$ is the Casimir operator and the action in the coordinates defined by (2-2) is given by (it is a half of what is called Casimir operator in ([4], page 198)

\[
2y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2) - 2y\partial^2/\partial x \partial t.
\]

Let $m \in \mathbb{Z}$, $m \geq 3$. The main result of this section describes the subspace $A_{cusp}(\Gamma \backslash SL_2(\mathbb{R}))_m$ consisting of all $\psi \in A_{cusp}(\Gamma \backslash SL_2(\mathbb{R}))$ satisfying the following conditions:

$\psi(gk) = \chi_m(k)\psi(g)$, $k \in K_\infty$, $g \in SL_2(\mathbb{R})$

(2-6)

$C.\psi = \left(\frac{m^2}{2} - m\right) \psi.$

First, applying the standard lifting procedure ([1], 5.14 or [2], Chapter 2) we can assign to a function $f : X \to \mathbb{C}$ the function $F_f : SL_2(\mathbb{R}) \to \mathbb{C}$ is defined by the following expression:

(2-7)

$F_f(g) = f(g, \sqrt{-1}) \mu(g, \sqrt{-1})^{-m}.$

Now, using Iwasawa decomposition (2-2), we obtain the following:

(2-8)

$F_f \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right) = y^{m/2} \exp(m t \sqrt{-1}) f(z).$

This shows that $F_f$ transforms on the right under $K_\infty$ as $\chi_m$

(2-9)

$F_f(gk) = \chi_m(k)F_f(g)$, $g \in SL_2(\mathbb{R})$, $k \in K_\infty$.

Assuming that $f$ is holomorphic on $X$, the Cauchy–Riemann equations show that

(2-10)

$C.F_f = \left(\frac{m^2}{2} - m\right) F_f.$

Applying (2-3), (2-8), and (2-9), we obtain the following:

(2-11)

$F_f \in L^1(SL_2(\mathbb{R}))$ if and only if $\int_{-\infty}^{\infty} \int_{0}^{\infty} y^{m/2} |f(z)| \frac{dxdy}{y^2} < \infty.$

At this point, ([7], Theorem 3-10) implies the following result:

**Lemma 2-12.** Let $m \geq 3$. Assume that $\Gamma \subset SL_2(\mathbb{R})$ is a discrete subgroup of finite covolume. Assume that $f : X \to \mathbb{C}$ is holomorphic function satisfying $\int_{-\infty}^{\infty} \int_{0}^{\infty} y^{m/2} |f(z)| \frac{dxdy}{y^2} < \infty$. Then the

$P_\Gamma(F_f)(g) = \sum_{\gamma \in \Gamma} F_f(\gamma \cdot g)$

converges absolutely and uniformly on compact sets to an element of $A_{cusp}(\Gamma \backslash SL_2(\mathbb{R}))_m$. 
Proof. ([7], Theorem 3-10) implies that the series converges absolutely and uniformly on compact sets to an element of \(\mathcal{A}_{\text{cusp}}(\Gamma\backslash SL_2(\mathbb{R}))\). Actually, the statement of ([7], Theorem 3-10) is given in adelic settings. Therefore, it is applicable directly only to congruence subgroups \(\Gamma\). But the general case has the same proof. We leave details to the reader.

Finally, we must show that it belongs to \(\mathcal{A}_{\text{cusp}}(\Gamma\backslash SL_2(\mathbb{R}))|_m\). But ([7], Theorem 3-10), shows that \(P_\Gamma\) commutes with the action of \(K_\infty\) and \(C\). Hence the claim follows from (2-9) and (2-10).

The reader should note that Lemma 2-12 is a more precise result than ([1], Proposition 6.1 and Theorem 8.9).

Next, one easily shows that the functions

\[
(2-13) \quad f_{k,m}(z) = (\gamma z - \sqrt{-1})^k (\gamma z + \sqrt{-1})^{-k-m}, \quad m \geq 3, \quad k \geq 0
\]

are holomorphic on \(X\) and \(\int_0^\infty f_{k,m}(z) \, \frac{dy \, dx}{y^2} < \infty\). This can be seen directly (adapting the argument on page 184 of [4]) or using the criterion (2-11) and the Cartan decomposition. Since this we need later on in the paper we present this approach (see Section 6). We write \(f_{k,m}\) for the function corresponding to \(f_{k,m}\) by (2-7).

We let \(A_+^\infty\) to be the set of all \(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in A_\infty\) such that \(t \geq 0\). Then we have the following:

\[
(2-14) \quad SL_2(\mathbb{R}) = K_\infty \cdot A_+^\infty \cdot K_\infty,
\]

and the corresponding integration formula ([4], page 139):

\[
(2-15) \quad \int_{SL_2(\mathbb{R})} \varphi(g) dg = \int_0^\infty \int_{K_\infty} \int_{K_\infty} \varphi \left( k_1 \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} k_2 \right) \sinh (2t) \, dk_1 \, dt \, dk_2.
\]

We have the following lemma:

**Lemma 2-16.**

(i) \(F_{k,m}(k_1 g k_2) = \chi_{-m-2k}(k_1) F_{k,m}(g) \chi_m(k_2), k_1, k_2 \in K_\infty, g \in SL_2(\mathbb{R}).\)

(ii) \(F_{k,m}(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}) = (\cosh t)^{-k-m}(\sinh t)^k/2^m \cdot (\sqrt{-1})^m, \text{ for } t \geq 0.\)

(iii) \(F_{k,m} \in L^1(SL_2(\mathbb{R})).\)

**Proof.** Using the Iwasawa decomposition, we can write

\[
\begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \cdot \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} (e^{-2t} \cos^2 (\alpha) + e^{2t} \sin^2 (\alpha))^{-1/2} & 0 \\ 0 & (e^{-2t} \cos^2 (\alpha) + e^{2t} \sin^2 (\alpha))^{1/2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\sin (\alpha) (e^{2t} - e^{-2t}) & 1 \end{pmatrix} = \begin{pmatrix} \cos \alpha & e^t \sin \alpha \\ -e^t \cos \alpha & e^t \sin \alpha \end{pmatrix}.
\]

The claim (i) and (ii) of the lemma follow from this identity by a straightforward and elementary but somewhat tedious computation. We remark that \(F_{k,m}\) transforms on the right as \(\chi_m\). The claim (iii) follows from (2-15); the details can be found in Section 6.

We show the following lemma:

**Lemma 2-17.** Let \(m \geq 3\). Assume that \(\Gamma \subset SL_2(\mathbb{R})\) be a discrete subgroup of finite covolume. Then the Poincaré series

\[
P_\Gamma(F_{k,m})(g) = \sum_{\gamma \in \Gamma} F_{k,m}(\gamma \cdot g), \quad k \geq 0,
\]

are holomorphic on \(\Gamma\backslash SL_2(\mathbb{R})\). Actually, the statement of ([7], Theorem 3-10) is given in adelic settings. Therefore, it is applicable directly only to congruence subgroups \(\Gamma\). But the general case has the same proof. We leave details to the reader.

Finally, we must show that it belongs to \(\mathcal{A}_{\text{cusp}}(\Gamma\backslash SL_2(\mathbb{R}))|_m\). But ([7], Theorem 3-10), shows that \(P_\Gamma\) commutes with the action of \(K_\infty\) and \(C\). Hence the claim follows from (2-9) and (2-10).
Lemma 2.20. Let \( \pi, D \) be the holomorphic discrete series of \( SL_2(\mathbb{R}) \) of weight \( m \geq 2 \). Its definition and what we need about \( \pi, D \) can be found in ([4], pages 181–184). Using, the computations in ([4], pages 119–121), we find that \( C \) acts on the space of \( K_\infty \)-finite vectors \( (D_m)_{K_\infty} \) in \( D_m \) as a scalar multiplication by \( \left( \frac{m^2}{2} - m \right) \). As a \( K_\infty \)-representation, we have the following:

\[
(2-18) \quad (D_m)_{K_\infty} = \oplus_{k=0}^{\infty} \chi_{m+2k}.
\]

We start with the following lemma:

**Lemma 2.19.** Let \( m \geq 2 \). Then \((\mathfrak{sl}_2(\mathbb{R}), K_\infty)\)-submodule of \( A_{\text{cusp}}(\Gamma) \) generated by a non-zero \( \psi \in A_{\text{cusp}}(\Gamma) \) is isomorphic to \( (D_m)_{K_\infty} \).

**Proof.** This module is semisimple of finite length i.e., it is equal to \( U_1 \oplus \cdots \oplus U_k \), where \( U_i \) are irreducible modules. Applying (2.6) and the classification of irreducible \((\mathfrak{sl}_2(\mathbb{R}), K_\infty)\)-modules ([4], pages 119–121) implies that all \( U_i \)'s are isomorphic to \((D_m)_{K_\infty}\). But, looking more closely at the action of \( \mathfrak{sl}_2(\mathbb{R}) \) and \( K_\infty \) in \( D_m \) ([4], pages 119–121), we conclude that \( \psi \) must be "killed" by the operator \( E^- \) (see the top of the page 116 in [4] for a definition of \( E^- \)). Then acting by \( \mathfrak{sl}_2(\mathbb{R}) \) and using explicit formulas ([4], pages 119), we find that in the module generated by \( \psi \), the representation \( \chi_m \) appears with the multiplicity one. Hence \( k = 1 \). □

In some early version of this paper we showed the next lemma by computing explicitly matrix coefficients from the realization of discrete series in certain induced representations.

**Lemma 2.20.** Let \( m \geq 2 \). Then \( F_{k,m} \) is a matrix coefficient of the representation \( (\pi, D) \).

**Proof.** First, we recall that \( F_{k,m} \) satisfies (2.9) and (2.10) with \( F_\ell \) replaced by \( F_{k,m} \). Then ([1], Corollary 2.22) and Lemma 2.16 (iii) imply that \( F_{k,m} \in L^2(SL_2(\mathbb{R})) \). In the unitary representation \( L^2(SL_2(\mathbb{R})) \) where \( SL_2(\mathbb{R}) \) acts by right–translations, the minimal closed subspace generated by \( F_{k,m} \) is isomorphic to \( D_m \). This follows from the fact that the corresponding \((\mathfrak{sl}_2(\mathbb{R}), K_\infty)\) is irreducible. The argument is similar to the one sketched in the proof of Lemma 2.19.

Next, since Lemma 2.16 (i) implies that \( F_{k,m} \) is \( K_\infty \)-finite on the left, there is \( \beta \in C^\infty_c(S_2(\mathbb{R})) \) such that

\[
F_{k,m}(g) = \int_{SL_2(\mathbb{R})} F_{k,m}(hg) \overline{\beta(h)} dh, \quad g \in SL_2(\mathbb{R}),
\]

applying ([3], Theorem 1). This proves the lemma. □

In particular, Lemma 2.20 implies the following corollary:

**Corollary 2.21.** The space of matrix coefficients of \( (\pi, D) \) \((m \geq 2)\) which transform on the right as \( \chi_m \) and which are \( K_\infty \)-finite on the left is spanned by the functions \( F_{k,m} \) \( k \geq 0 \).

Now, we complete the proof of Lemma 2.17. We assume that \( m \geq 3 \) from now on. Assume that the Poincaré series \( P_T(F_{k,m}) \) \((k \geq 0)\) does not span \( A_{\text{cusp}}(\Gamma) \). Then, since \( A_{\text{cusp}}(\Gamma) \) is finite dimensional ([1], Theorem 8.5), we obtain that there is non–zero \( \psi \in A_{\text{cusp}}(\Gamma) \) orthogonal to all of them i.e.,

\[
\int_{\Gamma} \psi(g) \overline{P_T(F_{k,m})}(g) dg = 0, \quad k \geq 0.
\]
Applying Corollary 2-21, we obtain the following:

\[
(2-22) \quad \int_{\Gamma \setminus SL_2(\mathbb{R})} \psi(g) P_{\Gamma}(c_{h,h'})(g) dg = 0, \quad h, h' \in (D_m)_{K\infty}, \quad \pi_m(k)h = \chi_m(k)h, \quad k \in K_{\infty},
\]

where we write \( c_{h,h'} \) for the matrix coefficient \( g \mapsto \langle \pi_m(g)h, h' \rangle \) of \((\pi_m, D_m)\). Here \( \langle \ , \ \rangle \) is a positive definite Hermitian form on \( H_m \) making it unitary representation \((\pi_m, D_m)\).

**Lemma 2-23.** If \( c_{h,h'} \) as in (2-22) but \( h' \) is not necessarily \( K_{\infty} \)–finite, then \( P_{\Gamma}(c_{h,h'}) \in A_{cusp}(\Gamma \setminus SL_2(\mathbb{R}))_m \).

**Proof.** This follows from ([7], Theorem 3-10) exactly as Lemma 2-12 follows from there. \( \square \)

Then, by the Dominated convergence theorem and Lemma 2-23, we obtain that (2-22) holds for all \( h' \in D_m \). This enables us to insert \( \pi_m(g_0)h' \) instead of \( h' \) in (2-22).

Next, we remark \( \psi \) being cusp form on \( \Gamma \setminus SL_2(\mathbb{R}) \) is rapidly decreasing in the direction of every cusp for \( \Gamma \). Hence it is bounded on \( SL_2(\mathbb{R}) \). Thus, \( \psi \pi_{m,\pi_m(g_0)h'} \) is in \( L^1(SL_2(\mathbb{R})) \). Thus, we have the following:

\[
0 = \int_{\Gamma \setminus SL_2(\mathbb{R})} \psi(g) P_{\Gamma}(c_{h,\pi_m(g_0)h})(g) dg = \int_{\Gamma \setminus SL_2(\mathbb{R})} \psi(g) \left( \sum_{\gamma \in \Gamma} \overline{c_{h,\pi_m(g_0)h'}}(\gamma \cdot g) \right) dg = \int_{\Gamma \setminus SL_2(\mathbb{R})} \left( \sum_{\gamma \in \Gamma} \psi(\gamma \cdot g) \overline{c_{h,\pi_m(g_0)h'}}(\gamma \cdot g) \right) dg = \int_{SL_2(\mathbb{R})} \psi(g) \overline{c_{h,\pi_m(g_0)h'}}(g) dg.
\]

Since \( \pi_m \) is a unitary representation and \( c_{h,h'}(g) = \langle \pi_m(g)h, h' \rangle \), this implies

\[
(2-24) \quad 0 = \int_{SL_2(\mathbb{R})} \psi(g) \overline{\langle \pi_m(g)h, \pi(g_0)h' \rangle} dg = \int_{SL_2(\mathbb{R})} \psi(g) \overline{\langle \pi_m(g_0^{-1})g, h' \rangle} dg = \int_{SL_2(\mathbb{R})} \psi(g_0g) \langle \pi_m(g)h, h' \rangle dg, \quad g_0 \in SL_2(\mathbb{R}).
\]

This means that the function \( \overline{c_{h,h'}} \in L^1(SL_2(\mathbb{R})) \) acts trivially on \( \psi \) in the unitary representation generated by \( \psi \) in \( L^2_{cusp}(\Gamma \setminus SL_2(\mathbb{R})) \). But by Lemma 2-19, this representation is unitary equivalent to \((\pi_m, D_m)\), and the \( K_{\infty} \)–type structure tells us that \( \psi \) is mapped to a scalar multiple of \( h \). Hence in \( D_m \)

\[
\int_{SL_2(\mathbb{R})} \overline{\langle \pi_m(g)h, h' \rangle} \pi_m(g)h dg = 0.
\]

Hence

\[
\int_{SL_2(\mathbb{R})} |\langle \pi_m(g)h, h' \rangle|^2 dg = 0
\]

which is not possible for all \( h' \in D_m \). This proves the lemma. \( \square \)

3. The proof of Theorem 1-1 and some consequences

We prove Theorem 1-1.

**Proof.** We start the proof with the following lemma. We use the notation of Section 2.

**Lemma 3-1.** The map \( f \mapsto F_f \) is an isomorphism of vector spaces \( S_m(\Gamma) \to A_{cusp}(\Gamma \setminus SL_2(\mathbb{R}))_m \).
Proof. This lemma is standard for the congruence subgroups (see [2], Chapter 2, Proposition 2.1). This proof works for general $\Gamma$ with a trivial modification. Alternatively, it also follows from ([1], 5.14, 7.2) and Lemma 2-19. □

As a next step, we use Lemma 3-1 to transfer the cuspidal automorphic forms $P_{\Gamma}(F_{k,m})$ defined in Lemma 2-17 to $S_m(\Gamma)$.

**Lemma 3-2.** The inverse of the map defined by Lemma 3-1 maps the Poincaré series $P_{\Gamma}(F_{k,m})$ onto the modular form

$$\sum_{\gamma \in \Gamma} (\gamma.z - \sqrt{-1})^k (\gamma.z + \sqrt{-1})^{-k-m} \mu(\gamma,z)^{-m}, \quad k \geq 0,$$

and the series converges uniformly and absolutely on compact sets in $X$. Finally, the collection of modular forms (3-3) span the space $S_m(\Gamma)$.

Proof. First, letting $z = g.\sqrt{-1}$ for $g \in SL_2(\mathbb{R})$ and using (2-8) we find that the transferred function is given by

$$P_{\Gamma}(F_{k,m})(g)\mu(g,\sqrt{-1})^m = \sum_{\gamma \in \Gamma} F_{k,m}(\gamma \cdot g)\mu(g,\sqrt{-1})^m.$$ Using (2-5), this is equal to

$$\sum_{\gamma \in \Gamma} (F_{k,m}(\gamma \cdot g)\mu(\gamma \cdot g,\sqrt{-1})^m) \mu(\gamma,g,\sqrt{-1})^{-m}$$

Since $F_{k,m}$ of the function $f_{k,m}$ defined by (2-13), we obtain finally (3-3). Now, we apply Lemma 2-17 to complete the proof of the lemma. □

**Lemma 3-4.** Let $m \geq 3$. Let $\Gamma \subset SL_2(\mathbb{R})$ be a discrete subgroup of finite covolume. Then the series

$$\sum_{\gamma \in \Gamma} (\gamma.z + \sqrt{-1})^{-k-m} \mu(\gamma,z)^{-m}, \quad k \geq 0,$$

converge uniformly and absolutely on compact sets in $X$ and they span the space $S_m(\Gamma)$.

Proof. The lemma follows from Lemma 3-2 by induction on $k \geq 0$. First, the particular case of Lemma 3-2 is the series $\sum_{\gamma \in \Gamma} (\gamma.z + \sqrt{-1})^{-m} \mu(\gamma,z)^{-m} \in S_m(\Gamma)$. Next, multiplying with $\mu(\gamma,z)^{-m}$ and then summing up over $\gamma \in \Gamma$ the following obvious identity:

$$(\gamma.z + \sqrt{-1})^{-k-m} = (\gamma.z - \sqrt{-1})^k (\gamma.z + \sqrt{-1})^{-m-k} - \sum_{i=1}^{k} \binom{k}{i} (-2\sqrt{-1})^{k-i} (\gamma.z + \sqrt{-1})^{-m-k+i}$$

we obtain the lemma. □

Now, we complete the proof of Theorem 1-1. First, (i) follows from Lemma 2-12 transferring the resulting form to $S_m(\Gamma)$ by the methods described in the proof of Lemma 3-2. (ii) follows from Lemma 3-4. This completes the proof of Theorem 1-1. □

It is well–known that if $-1 \in \Gamma$ and $m$ is odd then $S_m(\Gamma) = \{0\}$. This is consistent with our results since in this case all series identically zero.

Now, we give a more classical formulation of the main result (see [6], Chapter 2).
Let $\Gamma \subset SL_2(\mathbb{R})$ be a discrete subgroup of finite covolume. Assume that $\infty$ is a cusp for $\Gamma$. Then $\Gamma_\infty = U_{P_\infty} \cap \Gamma$ is an infinite cyclic group ([1], 3.6). We write $\left( \begin{array}{cc} 1 & h \\ 0 & 1 \end{array} \right)$, $h > 0$, for the generator of this group. It is well-known [6], Corollary 2.6.11) that

$$\alpha_{l,m}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e^{2\pi i \gamma z} \mu(\gamma, z)^{-m}, \quad l \geq 1$$

also span $S_m(\Gamma)$. We let

$$\beta_{k,m}(z) = \sum_{\gamma \in \Gamma} (\gamma, z + \sqrt{-1})^{-m-k} \mu(\gamma, z)^{-m}, \quad k \geq 0.$$

Then we have the following:

**Corollary 3-5.** Let $m \geq 3$. Let $\Gamma \subset SL_2(\mathbb{R})$ be a discrete subgroup of finite covolume such that $\infty$ is its cusp. Put $\Gamma_\infty = U_{P_\infty} \cap \Gamma$, and let $\left( \begin{array}{cc} 1 & h \\ 0 & 1 \end{array} \right)$, $h > 0$, be the generator of this group. Then

$$\beta_{k,m}(z) = \left( -1 \right)^{m+k-1} \frac{(2\pi \sqrt{-1})^{m+k}}{(m+k-1)!} \sum_{l=1}^{\infty} m+1 e^{-2\pi i / h} \alpha_{l,m}(z), \quad k \geq 0.$$

**Proof.** We use well-known identity $\pi \cot (\pi z) = \lim_{n \to \infty} \sum_{i=1}^{l} \frac{1}{\pi i}$ (the convergence is uniform on compact sets). Since $\pi \cot (\pi z) = \pi \sqrt{-1} e^{2\pi i / e^2 \sqrt{-1}},$ we can unfold the left hand side for $z \in X$:

$$2\pi \sqrt{-1} - \pi \sqrt{-1} \left( \frac{1}{1 - e^{2\pi i / e^2 \sqrt{-1}}} \right) = \lim_{l \to \infty} \sum_{i=1}^{l} \frac{1}{z + i}.$$

Taking the derivative $(m+k-1)$-times, we find

$$\sum_{l=1}^{\infty} \frac{1}{(z+l)^{m+k}} = \left( -1 \right)^{m+k-1} \frac{(2\pi \sqrt{-1})^{m+k}}{(m+k-1)!} \left[ \sum_{l=1}^{\infty} m+1 e^{-2\pi i / h} \right].$$

Substituting $(z + \sqrt{-1})/h$ we find finally

$$\sum_{l=-\infty}^{\infty} \frac{1}{(z + \sqrt{-1} + lh)^{m+k}} = \left( -1 \right)^{m+k-1} \frac{(2\pi \sqrt{-1})^{m+k}}{(m+k-1)!} \left[ \sum_{l=1}^{\infty} m+1 e^{-2\pi i / h} e^{2\pi i / e^2 \sqrt{-1} / h} \right].$$

Thus, we have the following:

$$\beta_{k,m}(z) = \sum_{\gamma \in \Gamma} (\gamma, z + \sqrt{-1})^{-m-k} \mu(\gamma, z)^{-m}$$

$$= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left( \sum_{l=-\infty}^{\infty} \frac{1}{(\gamma, z + \sqrt{-1} + lh)^{m+k}} \right) \mu(\gamma, z)^{-m}$$

$$= \left( -1 \right)^{m+k-1} \frac{(2\pi \sqrt{-1})^{m+k}}{(m+k-1)!} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left[ \sum_{l=1}^{\infty} m+1 e^{-2\pi i / h} e^{2\pi i / e^2 \sqrt{-1} / h} \right] \mu(\gamma, z)^{-m}$$

$$= \left( -1 \right)^{m+k-1} \frac{(2\pi \sqrt{-1})^{m+k}}{(m+k-1)!} \sum_{l=1}^{\infty} m+1 e^{-2\pi i / h} \alpha_{l,m}(z).$$
We can interchange the last two sums since the series converges absolutely and uniformly on compact sets

\[
\sum_{l=1}^{\infty} \sum_{\gamma \in \Gamma_1 \setminus \Gamma} l^{m+k-1} e^{-2\pi i l/h} e^{2\pi i \sqrt{-1} \gamma \cdot z/h} \cdot |\mu(\gamma, z)|^{-m} = \\
\sum_{l=1}^{\infty} \sum_{\gamma \in \Gamma_1 \setminus \Gamma} l^{m+k-1} e^{-2\pi i l/h} e^{-2\pi i k \cdot \Im(\gamma \cdot z)/h} \cdot |\mu(\gamma, z)|^{-m} \\
\leq \sum_{l=1}^{\infty} \sum_{\gamma \in \Gamma_1 \setminus \Gamma} l^{m+k-1} e^{-2\pi i l/h} \cdot |\mu(\gamma, z)|^{-m} \\
\leq \left( \sum_{l=1}^{\infty} l^{m+k-1} e^{-2\pi i l/h} \right) \cdot \left( \sum_{\gamma \in \Gamma_1 \setminus \Gamma} |\mu(\gamma, z)|^{-m} \right) < \infty,
\]

where we have used elementary identity \( \Im(\gamma \cdot z) = y/|\mu(\gamma, z)|^2 \), and the fact

\[
\sum_{\gamma \in \Gamma_1 \setminus \Gamma} |\mu(\gamma, z)|^{-m} < \infty
\]

which is proved in ([6], Theorem 2.6.6).

\[\square\]

4. Non–Vanishing Criteria for Certain Poincaré Series

In this section we use the integration theory for locally compact groups to prove the non–vanishing of certain Poincaré series [5]. The main result of this section Lemma 4-1 is the generalization and adaptation of ([7], Theorem 4-1) to our present needs.

Lemma 4-1. Let \( G \) be a locally compact unimodular (Hausdorff) topological group, and let \( \Gamma \subset G \) be its discrete subgroup and \( \Gamma_1 \subset \Gamma \) an arbitrary subgroup. Let \( \varphi \in L^1(\Gamma_1 \setminus G) \). Assume that there exists a compact neighborhood \( C \) in \( G \) such that the following conditions hold:

1. \( \Gamma \cap C \cdot C^{-1} \subset \Gamma_1 \), and
2. \( \int_{\Gamma_1 \setminus \Gamma_1 \cdot C} |\varphi(g)| \, dg > \frac{1}{2} \int_{\Gamma_1 \setminus G} |\varphi(g)| \, dg. \)

Then the Poincaré series

\[
P_{\Gamma_1 \setminus \Gamma}(\varphi)(g) \overset{\text{def}}{=} \sum_{\gamma \in \Gamma_1 \setminus \Gamma} \varphi(\gamma \cdot g)
\]

converges absolutely almost everywhere to a non–zero element of \( L^1(\Gamma \setminus G) \).

Proof. We write \( P_{\Gamma_1} \) when \( \Gamma_1 \) is trivial. It is a general fact from the integration theory that \( \psi \mapsto P_{\Gamma_1}(\psi) \) is an epimorphism \( C_c(G) \rightarrow C_c(\Gamma_1 \setminus \Gamma) \) see ([5], Chapter XII, Section 4, Theorem 4.1)), and the measure on \( \Gamma_1 \setminus G \) is fixed by the rule \( \int_{\Gamma_1 \setminus G} P_{\Gamma_1}(\psi)(g) \, dg = \int_{G} \psi(g) \, dg \). The same applies to \( \Gamma \). Since \( P_{\Gamma} = P_{\Gamma_1 \setminus \Gamma} \cdot P_{\Gamma_1} \), we obtain the following:

\[
\int_{\Gamma_1 \setminus G} P_{\Gamma_1}(\psi)(g) \, dg = \int_{G} \psi(g) \, dg = \int_{\Gamma \setminus G} P_{\Gamma}(\psi)(g) \, dg = \int_{\Gamma \setminus G} P_{\Gamma_1 \setminus \Gamma} \left( P_{\Gamma_1}(\psi) \right)(g) \, dg.
\]
This implies the following formula:

\[(4-2) \quad \int_{\Gamma \setminus G} P_{\Gamma \setminus \Gamma}(\varphi)(g) dg = \int_{\Gamma \setminus G} \varphi(g) dg, \quad \varphi \in L^1(\Gamma \setminus G).\]

Now, the fact that the Poincaré series \( \sum_{\gamma \in \Gamma \setminus \Gamma} \varphi(\gamma \cdot g) \) converges absolutely almost everywhere to an element of \( L^1(\Gamma \setminus G) \) follows from the identity (4-2) since it implies

\[\int_{\Gamma \setminus G} \left( \sum_{\gamma \in \Gamma \setminus \Gamma} |\varphi(\gamma \cdot g)| \right) dg = \int_{\Gamma \setminus G} |\varphi(g)| dg < \infty.\]

It remains to show that \( \int_{\Gamma \setminus G} |P_{\Gamma \setminus \Gamma}(\varphi)(g) dg \neq 0 \). To end this, we adapt the proof of ([7], Theorem 4-1). We begin with the following lemma:

**Lemma 4-3.** There exist \( \psi \in C_c(\Gamma \setminus G) \) such that \( \text{supp } \psi \subset \Gamma \setminus \Gamma \cdot C \) and \( \int_{\Gamma \setminus G} |\psi - \varphi| < \int_{\Gamma \setminus G} |\psi| \).

**Proof.** Replacing \( C \) by its interior does not affect the assumptions (1) and (2) of the theorem. Thus, in this proof we may assume that \( C \) is open with a compact closure. Then \( \Gamma \setminus \Gamma \cdot C \) is an open set with a compact closure. We denote it by \( C_1 \). Since \( C_1 \) is open, we may consider \( C_1(\Gamma \setminus G) \). It is well-known that \( C_1(\Gamma \setminus G) \) is dense in \( L^1(\Gamma \setminus G) \) (see [4], the definition of the Haar measure given in the first paragraph on the page 313 and Theorem 3.1). Hence, we can find a sequence \( \{\psi_n\}_{n \geq 1} \), where \( \psi_n \in C_c(C_1) \), such that \( \int_{C_1} |\psi_n(g) - \varphi(g)| dg \to 0 \) as \( n \to \infty \). In particular, \( \int_{C_1} |\psi_n(g)| dg \to \int_{C_1} |\varphi(g)| dg \) as \( n \to \infty \). Now, we compute

\[\int_{\Gamma \setminus G} |\psi_n(g)| dg - \int_{\Gamma \setminus G} |\psi_n(g) - \varphi(g)| dg = \int_{C_1} |\psi_n(g)| dg - \int_{C_1} |\psi_n(g) - \varphi(g)| dg \]

\[- \int_{\Gamma \setminus (G - C_1)} |\psi_n(g) - \varphi(g)| dg = \int_{C_1} |\psi_n(g)| dg - \int_{C_1} |\psi_n(g) - \varphi(g)| dg - \int_{\Gamma \setminus (G - C_1)} |\varphi(g)| dg.\]

Thus, as \( n \to \infty \), this approaches

\[\int_{C_1} |\varphi(g)| dg - \int_{\Gamma \setminus (G - C_1)} |\varphi(g)| dg = 2 \cdot \int_{C_1} |\varphi(g)| dg - \int_{\Gamma \setminus G} |\varphi(g)| dg > 0, \quad \text{by (2)}.\]

This proves the lemma. \( \square \)

To prove Theorem 4-2, we need one more lemma.

**Lemma 4-4.** Let \( \psi \) be given by Lemma 4-3. Then \( P_{\Gamma \setminus \Gamma}(\psi)(g) = \psi(g) \) for \( g \in \Gamma \cdot C \).

**Proof.** Let \( g \in \Gamma \cdot C \). We can write \( g = \gamma_1 \cdot c_1, \gamma_1 \in \Gamma, c_1 \in C \). Now, by definition, we have

\[\sum_{\gamma \in \Gamma} \psi(\gamma \cdot g) = \sum_{\gamma \in \Gamma \cap C^{-1}} \psi(\gamma \cdot g).\]

It must be \( \psi(\gamma \cdot g) \neq 0 \). Thus, we may write \( \gamma \cdot g = \gamma_2 \cdot c_2, \gamma_2 \in \Gamma, c_2 \in C \). Now, we have

\[\gamma \cdot (\gamma_1 \cdot c_1) = \gamma_2 \cdot c_2 \text{ which implies}\]

\[\gamma_2^{-1} \gamma \gamma_1 = c_2 c_1^{-1} \in \Gamma \cap C \cdot C^{-1}.\]

Thus, by (1), we obtain \( \gamma_2^{-1} \gamma \gamma_1 \in \Gamma \). Hence \( \gamma \in \Gamma \). This proves the lemma. \( \square \)
Now, we prove the theorem. We compute using Lemma 4-3
\[
\int_{\Gamma \backslash G} |\psi| > \int_{\Gamma \backslash G} |\varphi - \psi| = \int_{\Gamma \backslash G} |\varphi(g) - \psi(g)| \, dg = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma \backslash \Gamma} |\varphi(\gamma \cdot g) - \psi(\gamma \cdot g)| \, dg \geq \int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}(\varphi)(g) - P_{\Gamma \backslash \Gamma}(\psi)(g)| \, dg \geq \int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}(\psi)(g)| \, dg - \int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}(\varphi)(g)| \, dg.
\]
This implies the following:
\[(4-5) \quad \int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}(\varphi)(g)| \, dg > \int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}(\psi)(g)| \, dg - \int_{\Gamma \backslash G} |\psi|.
\]
Next, using Lemma 4-3 we obtain supp \(\psi \subset \Gamma \backslash \Gamma \cdot C\). Hence \(P_{\Gamma \backslash \Gamma}(\psi) \subset \Gamma \backslash \Gamma \cdot C\). Thus, we have the following:
\[(4-6) \quad \int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}(\psi)(g)| \, dg \geq \int_{\Gamma \backslash \Gamma \cdot C} |P_{\Gamma \backslash \Gamma}(\psi)(g)| \, dg = \int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}(\psi)(g)| \cdot \text{char}_{\Gamma \backslash \Gamma \cdot C}(g) \, dg.
\]
Arguing as in the proof of Lemma 4-4, we find that the characteristic functions are related by
\[\text{char}_{\Gamma \backslash \Gamma \cdot C} = P_{\Gamma \backslash \Gamma}(\text{char}_{\Gamma \backslash \Gamma \cdot C}).\]
The same lemma implies \(P_{\Gamma \backslash \Gamma}(\psi)(g) = \psi(g)\) for \(g \in \Gamma \cdot C\). Therefore, if we consider \(|P_{\Gamma \backslash \Gamma}(\psi)(\cdot)|\) as a function on \(G\) invariant by \(\Gamma\) on the left, then the usual integration implies
\[
\int_{\Gamma \backslash G} |\psi| = \int_{\Gamma \backslash \Gamma \cdot \Gamma \cdot C} |\psi(g)| \, dg = \int_{\Gamma \backslash \Gamma \cdot \Gamma \cdot C} |P_{\Gamma \backslash \Gamma}(\psi)(g)| \, dg = \\
\int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}(\psi)(g)| \cdot \text{char}_{\Gamma \backslash \Gamma \cdot \Gamma \cdot C}(g) \, dg = \\
\int_{\Gamma \backslash G} \left( \sum_{\gamma \in \Gamma \backslash \Gamma} |P_{\Gamma \backslash \Gamma}(\psi)(\gamma \cdot g)| \cdot \text{char}_{\Gamma \backslash \Gamma \cdot \Gamma \cdot C}(\gamma \cdot g) \right) \, dg = \\
\int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}(\psi)(g)| \cdot \left( \sum_{\gamma \in \Gamma \backslash \Gamma} \text{char}_{\Gamma \backslash \Gamma \cdot \Gamma \cdot C}(\gamma \cdot g) \right) \, dg = \int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}(\psi)(g)| \cdot \text{char}_{\Gamma \backslash \Gamma \cdot \Gamma \cdot C}(g) \, dg.
\]
Therefore, (4-5) and (4-6) imply
\[
\int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}(\varphi)(g)| \, dg > \int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}(\psi)(g)| \, dg - \int_{\Gamma \backslash G} |\psi| \geq \int_{\Gamma \backslash G} |\psi| - \int_{\Gamma \backslash G} |\psi| = 0.
\]
This proves the lemma.

**Corollary 4-7.** Let \(G\) be a locally compact unimodular (Hausdorff) topological group, and let \(\Gamma \subset G\) be its discrete subgroup and \(\Gamma_1 \subset \Gamma\) an arbitrary subgroup. Let \(\varphi \in L^1(\Gamma\backslash G)\). Assume that there exists a compact neighborhood \(C\) in \(G\) such that the following conditions hold:
1. \(\Gamma \cap C \cdot C^{-1} \subset \Gamma_1\), and
2. \(\int_{\Gamma \backslash \Gamma \cdot \Gamma \cdot C} |P_{\Gamma \backslash \Gamma}(\varphi)(g)| \, dg > \frac{1}{2} \int_{\Gamma \backslash G} |P_{\Gamma \backslash \Gamma}(\varphi)(g)| \, dg.
\]
Then the Poincaré series \(P_{\Gamma}(\varphi)(g)\) is a non-zero element of \(L^1(\Gamma \backslash G)\).

**Proof.** This follows directly from Lemma 4-1. \qed
5. Criteria for Non–Vanishing of Modular Forms and the Proof of Theorem 1.3

First, we apply Corollary 4.7 to write down a non–vanishing criterion assuming that $\Gamma$ has a cusp at $\infty$.

**Lemma 5.1.** Let $m \geq 3$. Let $\Gamma \subset SL_2(\mathbb{R})$ be a discrete subgroup of finite covolume such that $\infty$ is its cusp. Put $\Gamma_\infty = \Gamma \cap U_{P_\infty}$, and let $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$, $h > 0$, be the generator of this group. Assume that $m$ is even if $-1 \in \Gamma$. Let $f : X \to \mathbb{C}$ be a holomorphic function satisfying $\int_{-\infty}^{\infty} \int_{0}^{\infty} |f(z)| \frac{dxdy}{y^m} < \infty$. Then the series $\sum_{\gamma \in \Gamma} f(\gamma, z) \mu(\gamma, z)^{-m}$ is non–trivial modular form in $S_m(\Gamma)$ provided that there exists a compact set $C \subset [0, h] \times [0, \infty]$ such that the following two conditions hold:

1. $\gamma \cdot C \cap C \neq \emptyset$ implies $\gamma \in \{\pm 1\} \Gamma_\infty$, and
2. $\int \int_{C} y^{m/2} \left| \sum_{l=-\infty}^{\infty} f(z + l \cdot h) \right| \frac{dxdy}{y^m} > \frac{1}{2} \int \int_{0}^{\infty} y^{m/2} \left| \sum_{l=-\infty}^{\infty} f(z + l \cdot h) \right| \frac{dxdy}{y^m}$.

**Proof.** We define a complex function $F_f$ on $SL_2(\mathbb{R})$ by (2.7). Then, (2.11) implies that $F_f \in L^1(SL_2(\mathbb{R}))$.

Next, since $\Gamma_\infty = U_{P_\infty} \cap \Gamma$ is an infinite cyclic group, we can write a very simple formula for the measure on $\Gamma_\infty \setminus SL_2(\mathbb{R})$ (see (2.3)):

$$\int_{\Gamma_\infty \setminus SL_2(\mathbb{R})} \left( \sum_{\gamma \in \Gamma_\infty} \psi(\gamma \cdot g) \right) dg =$$

$$\int_{0}^{h} \int_{0}^{\infty} \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\infty} \psi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right) \frac{dxdy}{2y^2} dt, \psi \in C_c^\infty(SL_2(\mathbb{R})).$$

In order to apply Corollary 4.7, we compute using the definition of $F_{k,m}$ (see (2.13))

$$\int_{\Gamma_\infty \setminus SL_2(\mathbb{R})} \left| \sum_{\gamma \in \Gamma_\infty} F_f(\gamma \cdot g) \right| dg =$$

$$\int_{0}^{h} \int_{0}^{\infty} \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\infty} \left| \sum_{l=-\infty}^{\infty} F_f \left( \begin{pmatrix} 1 & x + l \cdot h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right) \right| \frac{dxdy}{2y^2} dt =$$

$$\frac{1}{2} \int_{0}^{h} \int_{0}^{\infty} y^{m/2} \left| \sum_{l=-\infty}^{\infty} f(z + l \cdot h) \right| \frac{dxdy}{y^2}.$$

We define $\tilde{C} \subset P_\infty$ using (2.2) as follows:

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \in \tilde{C} \quad \text{if and only if} \quad g \sqrt{-1} = x + y \sqrt{-1} \in C.$$

It is clear that $\tilde{C}$ and $\tilde{C}K_\infty$ are compact subsets in $SL_2(\mathbb{R})$.

Then, we have the following:

$$\int_{\Gamma_\infty \setminus \Gamma_\infty \setminus \tilde{C}K_\infty} \left| \sum_{\gamma \in \Gamma_\infty} F_f(\gamma \cdot g) \right| dg = \frac{1}{2} \int \int_{C} y^{m/2} \left| \sum_{l=-\infty}^{\infty} f(z + l \cdot h) \right| \frac{dxdy}{y^2},$$

since $C \subset [0, h] \times [0, \infty]$. Thus, the condition (2) of Corollary 4.7 is equivalent to (2) here.
Next, acting on \( \sqrt{-1} \in X \), we find that (1) is equivalent with
\[
\Gamma \cap (\tilde{C}K_\infty) \cdot (\tilde{C}K_\infty)^{-1} \supset \{ \pm 1 \} \Gamma_\infty, \quad \text{where the equality holds if and only if} \ -1 \in \Gamma.
\]
Thus, if \(-1 \notin \Gamma\), then (1) is equivalent with (1) of Corollary 4-7. In the other case, we can argue similarly. This completes the proof of the lemma. \( \square \)

Now, we prove Theorem 1-3.

**Proof.** We use Lemma 5-1 since all three series of congruence subgroups have \( \infty \) as their cusp, and we have the following:
\[
\Gamma_0(N) \cap U_{P_\infty} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\} \quad h = 1,
\]
\[
\Gamma_1(N) \cap U_{P_\infty} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\} \quad h = 1,
\]
\[
\Gamma(N) \cap U_{P_\infty} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in N\mathbb{Z} \right\} \quad h = N.
\]
Now, by the assumption \( \int_0^\infty \int_0^\infty y^{m/2} |f(z)| \frac{dx dy}{y^2} < \infty \), we obtain that
\[
\int_0^h \int_0^\infty y^{m/2} \left| \sum_{l=-\infty}^\infty f(z + l \cdot h) \right| \frac{dx dy}{y^2} \leq \int_0^h \int_0^\infty y^{m/2} \sum_{l=-\infty}^\infty |f(z + l \cdot h)| \frac{dx dy}{y^2} = \int_{-\infty}^\infty \int_0^\infty y^{m/2} |f(z)| \frac{dx dy}{y^2} < \infty.
\]
Hence, we can find \( \epsilon, \delta > 0 \) such that \( C = [0, h] \times [\epsilon, \delta] \) satisfies (2) of Lemma 5-1. Now, we select \( N_0 \) requiring the condition (1) of Lemma 5-1. To end this, we recall the following formulas.

Let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \). Then the imaginary part of \( \gamma. z \) is given by \( \text{Im}(\gamma. z) = \text{Im}(\gamma(z)) = \frac{y}{|\mu(\gamma, z)|^2} = \frac{y}{|cz + d|^2} \), for all \( z = x + \sqrt{-1}y \in X \).

Now, let \( \gamma \) be in \( \Gamma_N \). Then \( c \equiv 0 \pmod{N} \). If \( z \in C \) and \( \gamma. z \in C \), then \( y \) and \( \text{Im}(\gamma. z) \) belong to \( [\epsilon, \delta] \). Hence
\[
\delta/\epsilon \geq |cz + d|^2 = (cx + d)^2 + c^2y^2 \geq c^2\epsilon^2.
\]
Thus, we obtain
\[
|c| \leq \sqrt{\frac{\delta}{\epsilon^3}}.
\]
If we select, \( N \geq N_0 = \left\lceil \sqrt{\frac{\delta}{\epsilon^3}} \right\rceil + 1, \) then \( c = 0 \), and we obtain (1). This proves the theorem. \( \square \)

We end this section with a non–vanishing criterion which is useful when \( \Gamma \) has no cusps.

**Lemma 5-2.** Let \( m \geq 3 \). Let \( \Gamma \subset SL_2(\mathbb{R}) \) be a discrete subgroup of finite covolume. Assume that \( m \) is even if \(-1 \in \Gamma \). Let \( f : X \to \mathbb{C} \) be a holomorphic function satisfying \( \int_0^\infty \int_0^\infty y^{m/2} |f(z)| \frac{dx dy}{y^2} < \infty \). Then the modular form \( \sum_{\gamma \in \Gamma} f(\gamma. z) \mu(\gamma, z)^{-m} \) is non–trivial provided that there exists a compact set \( C \subset X \) such that the following two conditions hold:

1. \( \int_C y^{m/2} |f(z)| \frac{dx dy}{y^2} > \frac{1}{2} \int_{-\infty}^\infty \int_{-\infty}^\infty y^{m/2} |f(z)| \frac{dx dy}{y^2}, \) and
\( \gamma \cdot C \cap C = \emptyset \) for every \( \gamma \in \Gamma \), \( \gamma \neq \pm 1 \).

We should warn the reader that it is not necessarily \(-1 \in \Gamma \) in (2).

**Proof.** This has the proof similar to that of Lemma 5-1 \( \square \)

### 6. Non–Vanishing of Modular Forms II

Let \( \Gamma \subset SL_2(\mathbb{R}) \) be a discrete subgroup of finite covolume. Then Lemma 3-2 implies that

\[
\sum_{\gamma \in \Gamma} (\gamma \cdot z - \sqrt{-1})^k (\gamma \cdot z + \sqrt{-1})^{-k-m} \mu(\gamma, z)^{-m}, \quad k \geq 0,
\]

span \( S_m(\Gamma) \) for \( m \geq 3 \). By the same lemma, the modular form in (6-1) is obtained by transferring the Poincaré series \( P_\Gamma(F_{k,m}) \) to the modular forms on \( X \). We remind the reader that \( F_{k,m} \) is defined in the text after (2-13). Lemma 3-2 shows that the modular form in (6-1) is non–trivial if and only if \( P_\Gamma(F_{k,m}) \) is non–trivial. We use Corollary 4-7 and the Cartan decomposition for \( SL_2(\mathbb{R}) \) (see (2-14)).

First, we introduce the following compact sets:

\[
C_r = K_\infty \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : 0 \leq t \leq r \right\} K_\infty, \quad r \in \mathbb{R}_{>0}.
\]

Now, we prove the following lemma:

**Lemma 6-2.** If \( \chi_{m+2k} \) is not trivial on \( \Gamma \cap K_\infty \), then \( P_\Gamma(F_{k,m}) = 0 \). If \( \chi_{m+2k} \) is trivial on \( \Gamma \cap K_\infty \), then \( P_\Gamma(F_{k,m}) \neq 0 \) provided that

\[
\int_{SL_2(\mathbb{R})} |F_{k,m}(g)| \, dg > \frac{1}{2} \int_{C_r} |F_{k,m}(g)| \, dg \quad \text{and} \quad \Gamma \cap C_r \cdot C_r^{-1} \subset \Gamma \cap K_\infty.
\]

**Proof.** First, if \( \chi_{m+2k} \) is not trivial on \( \Gamma \cap K_\infty \)

\[
\sum_{\gamma \in \Gamma \cap K_\infty} F_{k,m}(\gamma \cdot g) = \left( \sum_{\gamma \in \Gamma \cap K_\infty} \chi_{-m-2k}(\gamma) \right) F_{k,m}(g) = 0,
\]

using Lemma 2-16 (i), by the well–known argument for the sum of values of a character over a finite group. Thus, if \( \chi_{m+2k} \) is not trivial on \( \Gamma \cap K_\infty \), then

\[
P_\Gamma(F_{k,m})(g) = \sum_{\gamma \in \Gamma} F_{k,m}(\gamma \cdot g) = \sum_{\delta \in \Gamma \cap K_\infty} \left( \sum_{\gamma \in \Gamma \cap K_\infty} F_{k,m}(\gamma \cdot \delta \cdot g) \right) = 0.
\]

Corollary 4-7 implies that \( P_\Gamma(F_{k,m}) \neq 0 \) provided that \( \Gamma \cap C_r \cdot C_r^{-1} \subset \Gamma \cap K_\infty \) and

\[
\int_{\Gamma \cap K_\infty \setminus SL_2(\mathbb{R})} \left| \sum_{\gamma \in \Gamma \cap K_\infty} F_{k,m}(\gamma \cdot g) \right| \, dg > \frac{1}{2} \int_{\Gamma \cap K_\infty \setminus C_r} \left| \sum_{\gamma \in \Gamma \cap K_\infty} F_{k,m}(\gamma \cdot g) \right| \, dg.
\]

Now, if \( \chi_{m+2k} \) is trivial on \( \Gamma \cap K_\infty \), then Lemma 2-16 (i) implies that \( F_{k,m}(\gamma \cdot g) = F_{k,m}(g) \), \( \gamma \in \Gamma \cap K_\infty \), \( g \in SL_2(\mathbb{R}) \). Since the characteristic functions are related by

\[
(#\Gamma \cap K_\infty) \text{char}_{\Gamma \cap K_\infty \setminus C_r}(g) = \sum_{\gamma \in \Gamma \cap K_\infty} \text{char}_{C_r}(\gamma \cdot g),
\]
we see that the inequality (6-3) becomes
\[ \int_{SL_2(\mathbb{R})} |F_{k,m}(g)| \, dg > \frac{1}{2} \int_{C_r} |F_{k,m}(g)| \, dg. \]

\[ \square \]

We have the following:
\[ \int_{C_r} |F_{k,m}(g)| \, dg = \frac{1}{2^{m-1}} \int_0^r (\cosh t)^{-k-m+1} (\sinh t)^{k+1} \, dt = \]
\[ \frac{1}{2^{m-1}} \int_0^\mu x^{k+1} (1 - x^2)^{m-4-x} \, dx, \quad \mu = \mu_r = \sinh (r) / \cosh (r), \quad r > 0. \]

using (2-15) and Lemma 2-16 for the first integral, and the substitution \( x = \sinh (t) / \cosh (t) \) for the second. Let us write \( I_{k,m}(\mu) \) for the integral in (6-4) multiplied by \( 2^{m-1} \). Then, the integration by parts implies the following recursive formula:
\[ (k + 2) I_{k,m}(\mu) - (m - 4) I_{k+2,m-2}(\mu) = \mu^{k+2} \left( 1 - \mu^2 \right)^{\frac{m-4}{2}}, \quad m \geq 4, \ k \geq 0. \]

This gives the following recursive formula
\[ I_{k,m}(1) = \frac{m-4}{k+2} I_{k+2,m-2}(1) \]

for
\[ I_{k,m}(1) = 2^{m-1} \int_{SL_2(\mathbb{R})} |F_{k,m}(g)| \, dg. \]

To apply Lemma 6-2, we need to determine \( \mu \in ]0, 1[ \) such that
\[ J_{k,m}(\mu) \overset{\text{def}}{=} I_{k,m}(\mu) - \frac{1}{2} I_{k,m}(1) > 0. \]

Combining (6-5) and (6-6), we find the following:
\[ (k + 2) J_{k,m}(\mu) - (m - 4) J_{k+2,m-2}(\mu) = \mu^{k+2} \left( 1 - \mu^2 \right)^{\frac{m-4}{2}}, \quad m \geq 4, \ k \geq 0. \]

**Lemma 6-9.** Fix \( k \geq 0 \) and \( m \geq 4 \). Let \( \mu \in ]0, 1[ \) such that \( J_{k+m-3,3}(\mu) > 0 \) (resp., \( J_{k+m-4,4}(\mu) > 0 \)) if \( m \) is odd (resp., \( m \) is even). Then \( J_{k,m}(\mu) > 0 \).

**Proof.** By (6-8), we obtain that \( J_{k+2,m-2}(\mu) > 0 \) implies \( J_{k,m}(\mu) > 0 \) for \( \mu \in ]0, 1[ \). Now, we iterate until we obtain the claim. \( \square \)

**Lemma 6-10.** Fix \( k \geq 0 \). Let \( m \geq 4 \) be even. Then, \( J_{k,m}(\mu) > 0 \) for \( \mu \in ]2^{-\frac{1}{k+m-2}}, 1[ \).

**Proof.** First, (6-4) implies the following:
\[ J_{k,4}(\mu) = \frac{1}{k+2} \left( \mu^{k+2} - \frac{1}{2} \right) > 0 \quad \text{if and only if} \quad \mu > 2^{-\frac{1}{k+2}}. \]

Now, we apply Lemma 6-9. \( \square \)

For odd \( m \) we have much better result.

**Lemma 6-12.** Let \( k \geq 0 \). Let \( m \geq 3 \) be odd. Then, \( J_{k,m}(\mu) > 0 \) for \( \mu > \frac{\sqrt{3}}{2} \).
Proof. The integral \( I_{k,3}(\mu) \) after substitution \( y = (1 - x^2)^{1/2} \) becomes
\[
I_{k,3}(\mu) = \int_{(1-\mu^2)^{1/2}}^{1} (1 - y^2)^{k/2} dy.
\]
Integrating by parts, we obtain
\[
(k + 1)J_{k,3}(\mu) - kJ_{k-2,3}(\mu) = \mu^k (1 - \mu^2)^{1/2}, \quad k \geq 2.
\]
Finally, we have the following:
\[
J_{0,3}(\mu) = \frac{1}{2} - (1 - \mu^2)^{1/2} > 0 \quad \text{if and only if} \quad \mu > \frac{\sqrt{3}}{2}.
\]
and
\[
2J_{1,3}(\mu) = \frac{\pi}{4} + \mu (1 - \mu^2)^{1/2} - \arcsin \sqrt{1-\mu^2} > 0 \quad \text{if} \quad \mu > \frac{\sqrt{3}}{2}.
\]
By induction, (6-13), (6-14), and (6-15) imply that \( J_{k,3}(\mu) > 0 \) for \( \mu > \frac{\sqrt{3}}{2} \). Now, Lemma 6-9 completes the proof. \( \square \)

Having completed determination of integrals, according to our Lemma 6-9, we need to study the following intersection:
\[
\Gamma \cap C_r \cdot C_{r^{-1}} = \Gamma \cap K_\infty \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} ; 0 \leq t \leq r \right\} K_\infty \left\{ \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} ; 0 \leq t \leq r \right\} K_\infty.
\]
In order to analyze this intersection, we let
\[
\|g\| = \text{tr}(g \cdot g^*)^{1/2} = \sqrt{a^2 + b^2 + c^2 + d^2}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).
\]
It is obvious that
\[
\|k_1 \cdot g \cdot k_2\| = \|g\|, \quad k_1, k_2 \in K_\infty.
\]
We show the following claim:

**Lemma 6-18.** \( \max_{g \in C_r \cdot C_{r^{-1}}} \|g\| = \sqrt{2} \cosh (4r) \).

**Proof.** Applying \( C_r \cdot C_{r^{-1}} = K_\infty \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} ; 0 \leq t \leq r \right\} K_\infty \left\{ \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} ; 0 \leq t \leq r \right\} K_\infty \) and
(6-17), we find that
\[
\max_{g \in C_r \cdot C_{r^{-1}}} \|g\| = \max_{0 \leq t, t' \leq r} \left| \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \left( \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} e^{-t'} & 0 \\ 0 & e^{t'} \end{pmatrix} \right) \right|
\]
\[
= \sqrt{2} \cdot \max_{0 \leq t, t' \leq r} \sqrt{\cosh(2(t-t')) \cdot \cos^2(\alpha) + \cosh(2(t+t')) \cdot \sin^2(\alpha)}
\]
\[
= \sqrt{2} \cdot \max_{0 \leq t, t' \leq r} \sqrt{\cosh(2(t-t')) + (\cosh(2(t+t')) - \cosh(2(t-t'))) \cdot \sin^2(\alpha)}
\]
\[
= \sqrt{2} \cdot \max_{0 \leq t, t' \leq r} \sqrt{\cosh(2(t+t'))}
\]
\[
= \sqrt{2} \cosh (4r).
\]
Lemma 6-19. We have the following:

(i) Let \( N \geq 2 \). Then, if \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N) \) is not diagonal, then \( \|g\| \geq \sqrt{N^2 + 2} \).

(ii) Let \( N = 1 \). Then, if \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N) \) is not in \( \Gamma(1) \cap K_{\infty} = \{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \} \), then \( \|g\| \geq \sqrt{N^2 + 2} \).

Proof. We prove (i), (ii) is analogous. By definition, \( a, d \equiv 1 \pmod{N} \). Since \( N \geq 2 \), we obtain \( a^2 + d^2 \geq 2 \). Also, by definition, \( b, c \equiv 0 \pmod{N} \). Thus, if \( g \) is not diagonal \( c^2 + b^2 \geq N^2 \). This proves the lemma.

Lemma 6-20. Let \( N \geq 1 \). Then, if \( N > 2 \sinh (2r) = \frac{4\mu}{1 - \mu^2} \), for some \( r > 0 \), then \( \Gamma(N) \cap C_r \cdot C_r^{-1} \subset \Gamma(N) \cap K_{\infty} \subset \{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \} \).

Proof. This follows from Lemmas 6-18 and 6-19.

Now, prove the following result:

Proposition 6-21. (i) Assume that \( m \geq 4 \) is even. Then the cuspidal modular form in (6-1) for \( \Gamma \subset \Gamma(N) \) is non–trivial if

\[
N > \frac{2^{2-\frac{1}{k+m-2}}}{1 - 2^{-\frac{2}{k+m-2}}}.
\]

(ii) Assume that \( m \geq 3 \) is odd. Then all cuspidal modular forms in (6-1) for \( \Gamma \subset \Gamma(N) \) are non–trivial for \( N \geq 14 \).

Proof. The function \( \frac{4\mu}{1 - \mu^2} \) is increasing on \( [0, 1] \). Now, we select \( \mu \in [2^{-\frac{1}{k+m-2}}, 1] \) such that

\[
\left\lfloor \frac{2^{\frac{1}{k+m-2}}}{1 - 2^{-\frac{2}{k+m-2}}} \right\rfloor + 1 > \frac{4\mu}{1 - \mu^2} > \frac{2^{\frac{1}{k+m-2}}}{1 - 2^{-\frac{2}{k+m-2}}}.
\]

Then \( J_{k,m}(\mu) > 0 \) by Lemma 6-10. Hence, if \( N \) satisfies the inequality stated in (i), we must have

\[
N \geq \left\lfloor \frac{2^{\frac{1}{k+m-2}}}{1 - 2^{-\frac{2}{k+m-2}}} \right\rfloor + 1 > \frac{4\mu}{1 - \mu^2}.
\]

Now, Lemma 6-20 implies that \( \Gamma(N) \cap C_r \cdot C_r^{-1} \subset \Gamma(N) \cap K_{\infty} \subset \{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \} \). But \( N \geq 3 \) for \( N \) which satisfies the displayed inequality in (i). Hence \( \Gamma(N) \cap K_{\infty} \) is trivial. In particular, \( \Gamma \cap K_{\infty} \) is trivial. Now, Lemma 6-2 completes the proof of (i). The proof of (ii) is similar. It uses Lemma 6-12, and \( 14 > \frac{4\mu}{1 - \mu^2} > 13 \) for \( \mu = \sqrt{3} \).
Let us consider the cuspidal modular form in \((6-1)\) for \(\Gamma \subset \Gamma(N)\). In proving Proposition 6-21, for fixed \(k\) and \(m\), we did not use the best possible solution of the inequality \(J_{k,m}(\mu) > 0\). Instead, we used a rather rough estimate given by Lemmas 6-9, 6-10, and 6-12. By its definition (see \((6-7)\)), the function \(J_{k,m}(\mu)\) is strictly increasing on \([0, 1]\), and it has a unique zero, say \(\mu_0\), in \([0, 1]\). The best possible solution would be \(\mu > \mu_0\), close to \(\mu_0\). It is quite likely that it is not possible to obtain a closed formula for \(\mu\) in general. But one can try to write a computer program based on formulas involved in the proof of Lemmas 6-9, 6-10, and 6-12 to compute to a sufficient accuracy so that we can determine the smallest \(N > \frac{4\mu_0}{1-\mu_0^2}\). Then Lemma 6-20 implies that \(\Gamma(N) \cap C_r \cdot C_r^{-1} \subset \Gamma(N) \cap K_\infty \subset \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}\), and we are left with seeing if \(\chi_{m+2k}\) is trivial on \(\Gamma \cap K_\infty\) which is a subgroup of \(\Gamma(N) \cap K_\infty\). This is completely easy. We illustrate this with the next proposition.

**Proposition 6-22.** Let \(\Gamma \subset \Gamma(N)\). Then the modular form \(\sum_{\gamma \in \Gamma} (\gamma.z + \sqrt{-1})^{-m} \mu(\gamma,z)^{-m} \) \((m \geq 3)\) is non–trivial if one of the following holds:

(i) \(m = 3\) and \(N \geq 14\)

(ii) \(m = 4\) and \(N \geq 8\)

(iii) \(m = 5, 6\) and \(N \geq 4\)

(iv) \(m = 7, 9, 11 \ldots\) and \(N \geq 3\) \((\text{the best possible since } m \text{ is odd})\)

(v) \(m = 8\) and \(N \geq 3\)

(vi) \(m = 10, 12, 14, \ldots, 26\) and \(N \geq 2\) \((\text{for } m = 10, 14, 18, 22, 26\text{ this is the best possible result since } 4 \ | m)\)

(vii) \(m = 30, 34, 38, \ldots, \) and \(N \geq 2\) \((\text{the best possible since } 4 \nmid m)\)

(viii) \(m = 28, 32, 36, \ldots\) and \(N \geq 1\).

The cases where we perhaps did not obtain the best possible result are \(m = 3, 4, 5, 6, 8, 12, 16, 20, 24\).

**Proof.** The inequality \(J_{0,m}(\mu) = \frac{1}{m-2} \left[ 1 - 2 \left( 1 - \mu^2 \right)^{(m-2)/2} \right] > 0\) implies \(\mu > \sqrt{1 - 2^{-\frac{2}{m-2}}}\). Then the condition in Lemma 6-20 is equivalent with:

\[ N > 4\lambda \sqrt{\lambda^2 - 1}, \quad \lambda = 2^{\frac{1}{m-2}}. \]

The computation of \(N\) is left to the reader as an exercise. To complete the proof we recall that \(\Gamma(N) \cap K_\infty\) is trivial for \(N \geq 3\), \(\Gamma(2) \cap K_\infty = \{ \pm 1 \}\), and \(\Gamma(1) \cap K_\infty\) given by Lemma 6-19 (ii). To apply Lemma 6-2, we remark that \(\chi_m\) is trivial on \(\Gamma(1) \cap K_\infty\) if and only if \(2 \mid m\) and \(\chi_m\) is trivial on \(\Gamma(1) \cap K_\infty\) if and only if \(4 \mid m\).

**References**


CUSP FORMS FOR $SL_2(\mathbb{R})$


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