ON THE INNER PRODUCT OF CERTAIN AUTOMORPHIC FORMS AND APPLICATIONS

GORAN MUIČ

ABSTRACT. Let \( \Gamma \subset SL_2(\mathbb{R}) \) be a discrete subgroup such that the quotient \( \Gamma \backslash SL_2(\mathbb{R}) \) has a finite volume. In this paper we compute the Petersson inner product of automorphic cuspidal forms with Poincaré series constructed out of matrix coefficients of a holomorphic discrete series of lowest weight \( m \geq 3 \). Our approach is a refined method of Milić (see the introduction of [4]). We apply the result to give a new and representation–theoretic proofs of previous results [5], some of which were known to Petersson [6], and anyway not surprising to experts.

1. Introduction

The main virtue of the paper is to give a new and representation–theoretic proofs of previous results [5], some of which were known to Petersson [6], and anyway not surprising to experts. Before we introduce the main results of this paper, we fix some notation. A discrete subgroup \( \Gamma \subset SL_2(\mathbb{R}) \) is called a Fuchsian group of the first kind if the quotient \( \Gamma \backslash SL_2(\mathbb{R}) \) has a finite volume. Let \( K \) be the standard maximal compact subgroup of \( SL_2(\mathbb{R}) \). Its unitary characters are parameterized by \( \mathbb{Z} \), we write \( \chi_m \) for the character parameterized by \( m \in \mathbb{Z} \). Let \( C \) be the Casimir operator of the center of complexified universal enveloping algebra of \( gl_2(\mathbb{R}) \). Let \( m \geq 1 \). We write \( A_{cusp}(\Gamma \backslash SL_2(\mathbb{R}))(m) \) for the finite–dimensional subspace of the space of all cuspidal automorphic forms \( \psi \) for \( \Gamma \) satisfying:

\[
\psi(gk) = \chi_m(k)\psi(g), \quad k \in K, \quad g \in SL_2(\mathbb{R})
\]

\[C.\psi = \left( m^2/2 - m \right)\psi.
\]

It is well–known that this space is in one–to–one correspondence with space of cuspidal modular forms of weight \( m \) for \( \Gamma \) [1].

Let \( m \geq 3 \). Then we write \((\pi_m, D_m)\) for the holomorphic discrete series of lowest weight \( m \). In the standard Iwasawa decomposition of \( SL_2(\mathbb{R}) \) (see (2-1)), we define the function \((k \geq 0)\)

\[
F_{k,m} \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right) = y^{m/2} \exp(mt\sqrt{-1}) \frac{(z - \sqrt{-1})^k}{(z + \sqrt{-1})^{k+m}},
\]

where \( z = x + \sqrt{-1}y \). The function \( F_{k,m} \) is unique up to a scalar matrix coefficient of \((\pi_m, D_m)\) which transforms on the right (resp., left) under \( K \) as \( \chi_m \) (resp., \( \chi_{m+2k} \)). The short proof of this fact is given by ([4], Lemma 3-5) using some properties of Banach representations of \( SL_2(\mathbb{R}) \).

Next, it is well–known [1] that

\[ P_\Gamma(F_{k,m})(g) = \sum_{\gamma \in \Gamma} F_{k,m}(\gamma \cdot g) \]
converges absolutely and uniformly on compact sets to an element of $\mathcal{A}_{\text{cusp}}(\Gamma \backslash SL_2(\mathbb{R}))_m$. It is an unpublished observation of Miličić that cuspidal automorphic forms $F_k(\gamma, m)$, $k \geq 0$, span $\mathcal{A}_{\text{cusp}}(\Gamma \backslash SL_2(\mathbb{R}))_m$. (See [4], Lemma 3-1 for two proofs of this result.) The main result of the present paper is the following theorem (see Section 2):

**Theorem 1-1.** Let $m \geq 3$ and $k \geq 0$. Let $\psi \in \mathcal{A}_{\text{cusp}}(\Gamma \backslash SL_2(\mathbb{R}))_m$. Then, the Petersson inner product of $\psi$ and $F_k(\gamma, m)$ is given by

$$
\langle \psi, F_k(\gamma, m) \rangle = \frac{(\sqrt{-1})^m}{2^{m+k-2}(m-1)m \cdots (m+k-1)} (E^+)^k \cdot \psi(1),
$$

where $E^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

We apply Theorem 1-1 to give a new proof of ([5], Proposition 2.1). We need some more notation. Let $\chi$ be a character of $\Gamma$ of finite order. For an integer $m \geq 3$, let $S_m(\Gamma, \chi)$ be the space of all modular forms of weight $m$ which are cuspidal i.e., this is the space of all holomorphic functions $f : X \to \mathbb{C}$ such that $f(\gamma, z) = \mu(\gamma, z)^m \chi(\gamma) f(z)$ ($z \in X, \gamma \in \Gamma$) which are holomorphic and vanish at every cusp for $\Gamma$. The space $S_m(\Gamma, \chi)$ is a finite–dimensional Hilbert space under the Petersson inner product:

$$
\langle f_1, f_2 \rangle = \int_{\Gamma \backslash X} y^m f_1(z) \overline{f_2(z)} \frac{dxdy}{y^2}.
$$

**Corollary 1-2.** Let $\chi$ be a character of $\Gamma$ of finite order. Put $\epsilon_\Gamma = \#(\pm 1 \cap \Gamma)$. Assume that $m \geq 3$. Let $\xi \in X$. Then, the series $(k \geq 0)

$$
\Delta_{k, m, \xi, \chi}(z) = \frac{(m-1)m \cdots (m+k-1)(2\sqrt{-1})^m}{4\epsilon_\Gamma \pi} \sum_{\gamma \in \Gamma} (\gamma, z - \xi)^{-k-m} \mu(\gamma, z)^{-m} \chi(\gamma)^{-1},
$$

converges absolutely and uniformly on compact to an element of $S_m(\Gamma, \chi)$ which satisfies

$$
\langle f, \Delta_{k, m, \xi, \chi} \rangle = \frac{d^k f(z)}{dz^k} \big|_{z=\xi}, \quad f \in S_m(\Gamma, \chi), \quad k \geq 0.
$$

This immediately shows that (for fixed $m \geq 3$ and $\xi \in X$) the inner products $(f, \Delta_{k, m, \xi, \chi})$ $(k \geq 0)$ determine the coefficients of the power series expansion of the modular form $f$ centered at $\xi$. Obviously, this gives the interpretation of the family of modular forms $\Delta_{k, m, \xi, \chi}$ $(k \geq 0)$ which is analogous to the one for classical Poincaré series at cusps ([3], Theorem 2.6.10) where the Petersson inner products of classical Poincaré series with a modular form $f$ determine the Fourier coefficients of $f$ at a cusp.

The modular forms discussed in ([4], Theorem 1-1 (ii)) are essentially modular forms $\Delta_{k, m, \xi, \chi}$ attached to $\xi = \sqrt{-1}$ and trivial character $\chi$. Thus, Corollary 1-2 gives the interpretation of the modular forms discussed in ([4], Theorem 1-1 (ii)).

We should point out that the modular forms $\Delta_{k, m, \xi, \chi}$ for $k = 0$ were essentially known to Petersson [6]. In fact, in Section 4, we relate the results of the present paper (and [4]) to the work of Petersson [6] by giving a simple representation theoretic proof of one of his main results.

2. The proof of the main result

Let $X$ be the upper half–plane. Then the group $SL_2(\mathbb{R})$ acts on $X$ as follows:

$$
g.z = \frac{az + b}{cz + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).
$$
We let $\mu(g, z) = cz + d$. The function $\mu$ satisfies the cocycle identity $\mu(gg', z) = \mu(g, g'z) \cdot \mu(g', z)$. Next, $SL_2(\mathbb{R})$–invariant measure on $X$ is define by $dx dy/y^2$, where the coordinates on $X$ are written in a usual way $z = x + \sqrt{-1}y$, $y > 0$.

We continue by reviewing some notation and results following ([4], Section 2). The Iwasawa decomposition of $SL_2(\mathbb{R})$ implies that every $g \in SL_2(\mathbb{R})$ can be written uniquely in the following form:

\[
(2-1) \quad g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad x, y, t \in \mathbb{R}, \ y > 0.
\]

The stabilizer of $\sqrt{-1}$ we denote by $K$. It is well–known that $K$ is a maximal compact subgroup of $SL_2(\mathbb{R})$. We have

\[
K = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} : t \in \mathbb{R} \right\}.
\]

The set of characters of $K$ can be identified with $\mathbb{Z}$ using

\[
\chi_m \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = e^{\sqrt{-1}mt}, \quad m \in \mathbb{Z}, \ t \in \mathbb{R}.
\]

We define certain differential operators on $C^\infty(SL_2(\mathbb{R}))$ in terms of coordinates given by (2-1) (see [2], pages 115–116; the Casimir operator $C$ is half of (2) on page 195)

\[
(2-2) \quad \begin{cases} C = 2y^2(\partial^2 / \partial x^2 + \partial^2 / \partial y^2) - 2y\partial^2 / \partial x\partial t & \text{the Casimir operator} \\ E^- = -2\sqrt{-1}ye^{-2\sqrt{-1}t} \left( \frac{\partial}{\partial x} + \sqrt{-1} \frac{\partial}{\partial y} \right) + \sqrt{-1}e^{-2\sqrt{-1}t} \frac{\partial}{\partial t} \\ E^+ = 2\sqrt{-1}ye^{2\sqrt{-1}t} \left( \frac{\partial}{\partial x} - \sqrt{-1} \frac{\partial}{\partial y} \right) - \sqrt{-1}e^{2\sqrt{-1}t} \frac{\partial}{\partial t} \\ W = \frac{\partial}{\partial t}. \end{cases}
\]

They satisfy (see [2], pages 102, 195)

\[
(2-3) \quad \begin{cases} \{E^+, E^-\} = E^+E^- - E^-E^+ = -4\sqrt{-1}W \\ \{W, E^\pm\} = W E^\pm - E^\pm W = \pm 2\sqrt{-1}TE^\pm \\ C = \sqrt{-1}W - \frac{1}{2}W^2 + \frac{1}{2}E^+E^- \end{cases}
\]

The Haar measure on $SL_2(\mathbb{R})$ is given by

\[
(2-4) \quad \int_{SL_2(\mathbb{R})} \varphi(g) dg = \frac{1}{2\pi} \int_{-\infty}^\infty \int_0^\infty \int_0^{2\pi} \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right) dx dy y^2 dt,
\]

where $\varphi \in C^\infty_c(SL_2(\mathbb{R}))$. We define spaces $L^p(SL_2(\mathbb{R}))$ ($p \geq 1$) using this measure. The Hilbert space $L^2(SL_2(\mathbb{R}))$ has the following inner product:

\[
(2-5) \quad \langle \varphi, \psi \rangle_2 = \int_{SL_2(\mathbb{R})} \varphi(g) \overline{\psi(g)} dg.
\]

The group $SL_2(\mathbb{R})$ acts on $L^2(SL_2(\mathbb{R}))$ via the right translations. In this way we obtain the unitary representation $r$. The induced measure on $\Gamma \backslash SL_2(\mathbb{R})$ is given by

\[
(2-6) \quad \int_{\Gamma \backslash SL_2(\mathbb{R})} \left( \sum_{\gamma \in \Gamma} \psi(\gamma g) \right) dg = \int_{SL_2(\mathbb{R})} \psi(g) dg \quad \psi \in C^\infty_c(SL_2(\mathbb{R})).
\]
We write \( F \).

We recall ([4], Lemma 2-9).

Lemma 2-9. Let \( k \geq 0 \). Then we have the following:

(i) \( F_{k, m}(k_1 g k_2) = \chi_{m+2k}(k_1) F_{k, m}(g) \chi_{m}(k_2), \quad k_1, k_2 \in K, \ g \in SL_2(\mathbb{R}). \)

(ii) \( F_{k, m} \left( \begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right) = (\cosh t)^{-k-m}(\sinh t)^k / 2^m \cdot (\sqrt{-1})^m, \quad \text{for} \ t \geq 0. \)

(iii) If \( m \geq 3 \), then \( F_{k, m} \in L^1(SL_2(\mathbb{R})). \)

(iv) \( C.F_{k, m} = \left( \frac{m^2}{2} - m \right) F_{k, m}. \)

(v) \( E^{-}.F_{k, m} = 0. \)

There is a misprint in ([4], Lemma 2-13(i)). The statement there should be as in Lemma 2-9(i). We recall ([4], Lemma 2-9).
Lemma 2.10. Assume that $\Gamma \subset SL_2(\mathbb{R})$ is a discrete subgroup of finite covolume. Let $m \geq 3$ and $k \geq 0$. Then the series $P_1(F_{k,m})(g) = \sum_{\gamma \in \Gamma} F_{k,m}(\gamma \cdot g)$ converges absolutely and uniformly on compact sets to an element of $A_{\operatorname{cusp}}(\Gamma \backslash SL_2(\mathbb{R}))_m$.

The main result of this section is the following theorem:

**Theorem 2.11.** Let $m \geq 3$ and $k \geq 0$. Let $\psi \in A_{\operatorname{cusp}}(\Gamma \backslash SL_2(\mathbb{R}))_m$. Then, the Peterson inner product of $\psi$ and $P_1(F_{k,m})$ is given by

$$\langle \psi, P_1(F_{k,m}) \rangle = \frac{\pi(\sqrt{-1})^m}{2^{m+k-2}(m-1)m \cdots (m+k-1)} (E^+)^k \cdot \psi(1).$$

**Proof.** First, we prove that

$$\langle \psi, P_1(F_{k,m}) \rangle = \lambda_{k,m} \cdot (E^+)^k \cdot \psi(1),$$

where the constant $\lambda_{k,m}$ is given by

$$\lambda_{k,m} = \frac{\pi}{m-1} \cdot 2^{2k-2} \cdot k! \cdot m(m+1)(m+2) \cdots (m+k-1).$$

We compute the constant $\lambda_{k,m}$ in Lemma 3.2.

We begin the proof of (2.12) by the following lemma which lists additional properties of the functions $F_{k,m}$ (see also Lemma 2.9):

**Lemma 2.14.** Let $k \geq 0$ and $m \geq 2$. Then we have the following:

(i) $F_{k,m} \in L^2(SL_2(\mathbb{R}))$.

(ii) The minimal closed subspace generated by $F_{k,m}$ in $L^2(SL_2(\mathbb{R}))$ under the right translations of $SL_2(\mathbb{R})$ is an irreducible representation isomorphic to the holomorphic discrete series $(\pi_{0,m}, D_m)$ of lowest weight $m \geq 2$. (The representation $(\pi_{m}, D_m)$ is for example described in the proof of Lemma 3.1 in [4].)

(iii) For all $l \geq 0$

$$(E^-)^l(E^+)^l \cdot F_{k,m} = \left( (-1)^l 2^{2l} l! \cdot m(m+2) \cdots (m+l-1) \right) F_{k,m}.$$ 

(iv) In the action on $L^2(SL_2(\mathbb{R}))$, the (unbounded) operator $-E^-$ is the Hermitian contragredient of $E^+$.

**Proof.** (i) and (ii) are proved in the course of the proof of ([4], Lemma 3–5). (iii) is a consequence of the infinitesimal structure of the representation $(\pi_{m}, D_m)$ i.e., the explicit action of the unbounded linear operators given by (2.2) and (2.3) (and Lemma 2.9 (v)). This is standard and well–known (see [2], pages 119–120 for similar computations). We let

$$G_0 = F_{k,m}, \quad G_l = \frac{2^{-l}}{m(m+1) \cdots (m+l-1)} (E^+)^l \cdot F_{k,m}, \quad l \geq 1.$$ 

Using (see [2], page 119 (2) with $s = m - 1$) we find the following ($l \geq 0$):

$$\begin{align*}
W \cdot G_l &= \sqrt{-1}(m+2) G_l \\
E^+ \cdot G_l &= 2(m+l) G_{l+1} \\
E^- \cdot G_l &= (-2l) G_{l-1}, \quad G_{-1} = 0.
\end{align*}$$
Hence, we have the following:

\[
(E^{-})^{l}(E^{+})^{l}.F_{k,m} = 2^{l} \cdot m(m+1) \cdots (m+l-1) \times \\
\times (E^{-})^{l}.G_{l} = (-1)^{l}2^{2l}l! \cdot m(m+1) \cdots (m+l-1) \cdot G_{0}.
\]

This proves (iii). Finally, (iv) follows from the general fact about unitary representations using the description of the operators \(E^{\pm}\) given on ([2], pages 114–115).

The following lemma is the key point for the proof of (2-12):

**Lemma 2-15.** Let \((r, L^{2}(SL_{2}(\mathbb{R})))\) denote the unitary representation of \(SL_{2}(\mathbb{R})\) on \(L^{2}(SL_{2}(\mathbb{R}))\) by the right translations \(r\). Assume that \(m \geq 2\). Then we have the following:

\[
F_{k,m}(g) = \frac{(E^{+})^{k}.F_{k,m}(1)}{2^{2k} \cdot k! \cdot m(m+1) \cdots (m+k-1)} \langle r(g)F_{k,m}, (E^{+})^{k}.F_{k,m}\rangle_{2},
\]

for all \(g \in SL_{2}(\mathbb{R})\).

**Proof.** The function \(g \mapsto \langle r(g)F_{k,m}, (E^{+})^{k}.F_{k,m}\rangle_{2}\) is a matrix coefficient of the unitary representation generated by \(F_{k,m}\) in \(L^{2}(SL_{2}(\mathbb{R}))\). By Lemma 2-14 (ii), this is a matrix coefficient of \((\pi_{m}, D_{m})\). It is easy to check that

\[
\langle r(k_{1}gk_{2})F_{k,m}, (E^{+})^{k}.F_{k,m}\rangle_{2} = \chi_{m+2k}(k_{1}) \cdot \langle r(g)F_{k,m}, (E^{+})^{k}.F_{k,m}\rangle_{2} \cdot \chi_{m}(k_{2}),
\]

for all \(k_{1}, k_{2} \in K\) an \(g \in SL_{2}(\mathbb{R})\), using the description of the action of \(W\) given in the proof of Lemma 2-14. (We remind the reader that \(W\) spans the Lie algebra of \(K\).) But the space of matrix coefficients of \((\pi_{m}, D_{m})\) that transforms on the right as \(\chi_{m}\) and on the left as \(\chi_{m+2k}\) is one dimensional as the description of \(K\)-types of \((\pi_{m}, D_{m})\) shows (see for example [4], (3-3)). But then ([4], Lemma 3-5) shows that there exists a constant \(\mu\) such that

\[
(2-16) F_{k,m}(g) = \mu \langle r(g)F_{k,m}, (E^{+})^{k}.F_{k,m}\rangle_{2}, \text{ for all } g \in SL_{2}(\mathbb{R}).
\]

It remains to compute \(\mu\). If we \(E^{+}\)-differentiate the equation (2-16) \(k\) times at \(g = 1\), then we obtain

\[
(E^{+})^{k}.F_{k,m}(1) = \mu \langle (E^{+})^{k}.F_{k,m}, (E^{+})^{k}.F_{k,m}\rangle_{2}.
\]

Using Lemma 2-14 (iii) and (iv), we find the following:

\[
(E^{+})^{k}.F_{k,m}(1) = \mu \langle (E^{+})^{k}.F_{k,m}, (E^{+})^{k}.F_{k,m}\rangle_{2}
\]

\[
= \mu \langle F_{k,m}, (-E^{-})^{k}.(E^{+})^{k}.F_{k,m}\rangle_{2}
\]

\[
= \mu \cdot 2^{2k} \cdot k! \cdot m(m+1) \cdots (m+k-1) \langle F_{k,m}, F_{k,m}\rangle_{2}.
\]

This proves the lemma.

Let \(d(\pi_{m})\) be the formal degree of the holomorphic discrete series \((\pi_{m}, D_{m})\) of lowest weight \(m \geq 2\) defined with the respect to the Haar measure (2-4). It is defined via Schur’s orthogonality:

**Lemma 2-17.** Let \((\pi, D)\) be the unitary representation on the Hilbert space \(D\) with the inner product \((\ , \ )\). Assume that \((\pi, D)\) is unitarily equivalent to \((\pi_{m}, D_{m})\) where \(m \geq 2\). Then there exists \(d(\pi_{m}) > 0\) such that

\[
(2-18) \int_{SL_{2}(\mathbb{R})} |\langle \pi(g)x, y \rangle|^{2} dg = \frac{1}{d(\pi_{m})} \langle x, x \rangle \langle y, y \rangle, \ x, y \in D.
\]
We have the following:

\[ d(\pi_m) = \frac{m - 1}{4\pi} \]

Proof. The existence of the constant \( d(\pi_m) > 0 \) such that (2-18) holds is well-known. See for example ([7], Lemma 4.5.9.1). The deep fact due to Harish–Chandra is that \( d(\pi_m) \) is the Plancherel measure (corresponding to the Haar measure (2-4)) of the point in the unitary dual of \( SL_2(\mathbb{R}) \) which corresponds to \( (\pi_m, D_m) \). (see for example ([7], Theorem 7.2.1.2)). The explicit Plancherel formula for \( SL_2(\mathbb{R}) \) can be found in ([2], page 174). The Haar measure used there is a half of our Haar measure. Then the Plancherel measure there is the twice the Plancherel measure here. (See the paragraph in [7], Theorem 7.2.1.1.) □

Now, we complete the proof of (2-12). We remind the reader that

\[ \langle \psi, \varphi \rangle = \int_{\Gamma \backslash SL_2(\mathbb{R})} \psi(g)\overline{\varphi(g)}dg \]

is the inner product on \( L^2(\Gamma \backslash SL_2(\mathbb{R})) \) and that we write \( r_\Gamma \) for the right–regular representation of \( SL_2(\mathbb{R}) \) on \( L^2(\Gamma \backslash SL_2(\mathbb{R})) \).

In order to prove (2-12), we must compute the inner product

\[ \langle \psi, P_\Gamma(F_{k,m}) \rangle = \int_{\Gamma \backslash SL_2(\mathbb{R})} \psi(g)\overline{P_\Gamma(F_{k,m}(g))}dg = \int_{SL_2(\mathbb{R})} \psi(g)\overline{F_{k,m}(g)}dg. \]

The last equality holds since \( \psi \) is bounded (being a cusp form (see [1], Corollary 7.9)) and Lemma 2-9 (iii) is valid.

Let \( \mathcal{H} \subset L^2_{\text{cusp}}(\Gamma \backslash SL_2(\mathbb{R})) \) be the closed subspace generated by \( \psi \) under the the right translations \( r_\Gamma \) of \( SL_2(\mathbb{R}) \). By ([4], Lemma 3-4), \( \mathcal{H} \) is irreducible and isomorphic to the holomorphic discrete series \( (\pi_m, D_m) \) of lowest weight \( m \geq 3 \). We recall that ([4], (3-3)) and the proof of Lemma 3-4 in [4] imply that the infinitesimal structure of \( \mathcal{H} \) is the following:

\[ \mathcal{H}_K = \oplus_{l \geq 0} \mathbb{C} \left((E^+)^l, \psi \right) \]

(2-20)

(2-21)

where the vector \( (E^+)^l, \psi \) is non–zero and transforms under \( K \) as \( \chi_{m+2l} \), for all \( l \geq 0 \).

Next, and this is the key point (see also the proof of Lemma 2-15), ([4], Lemma 3-5) shows that there exists \( \mu \in \mathbb{C} - \{ 0 \} \) such that

\[ F_{k,m}(g) = \mu \cdot \langle r_\Gamma(g)\psi, (E^+)^k, \psi \rangle \]

(2-22)

since they are both non–zero matrix coefficients of \( (\pi_m, D_m) \) which transform on the right as \( \chi_m \) and on the left as \( \chi_{m+2k} \).

We consider the integral

\[ \varphi(x) = \int_{SL_2(\mathbb{R})} \psi(xg)\overline{F_{k,m}(g)}dg, \quad x \in SL_2(\mathbb{R}). \]

(2-23)

Obviously, by the defintion of the action of \( F_{k,m} \in L^1(SL_2(\mathbb{R})) \) on the unitary representation \( \mathcal{H} \), we have \( \varphi \in \mathcal{H} \). Since, we have the following:

\[ \varphi(xu) = \int_{SL_2(\mathbb{R})} \psi(xug)\overline{F_{k,m}(g)}dg = \int_{SL_2(\mathbb{R})} \psi(xg)\overline{F_{k,m}(u^{-1}g)}dg = \chi_{m+2k}(u)\varphi(x), \]

where \( x \in SL_2(\mathbb{R}) \) and \( u \in K \), applying Lemma 2-9 (i), (2-24) implies that there exists \( \lambda \in \mathbb{C} \) such that

\[ \varphi = \lambda \cdot (E^+)^k, \psi. \]
We compute $\lambda$ as follows:

$$
\lambda \cdot \langle (E^+)^k \psi, (E^+)^k \psi \rangle = \langle \phi, (E^+)^k \psi \rangle \\
= \langle r_T(F_{k,m}) \psi, (E^+)^k \psi \rangle = \int_{SL_2(\mathbb{R})} \langle r_T(g) \psi, (E^+)^k \psi \rangle \cdot F_{k,m}(g) dg \\
= \frac{\mu}{d(\pi_m)} \langle \psi, (E^+)^k \psi \rangle \\
= \frac{\mu}{d(\pi_m)} \langle \psi, (E^+)^k \psi \rangle,
$$

where the last line follows by using the Schur’s orthogonality relation (see Lemma 2-17). Hence,

$$
\lambda = \frac{\mu}{d(\pi_m)} \langle \psi, \psi \rangle.
$$

Combining with (2-24), we obtain

$$
(2-25) \quad \varphi = \frac{\mu}{d(\pi_m)} \langle \psi, (E^+)^k \psi \rangle.
$$

Hence, (2-20) and (2-23) imply

$$
(2-26) \quad \langle \psi, P_T(F_{k,m}) \rangle = \varphi(1) = \frac{\mu}{d(\pi_m)} \langle \psi, (E^+)^k \psi(1) \rangle.
$$

To complete the proof of the theorem, we must compute the scalar $\mu$ (see (2-22)). We write $\mathcal{H}_1$ for the irreducible subrepresentation of $L^2(SL_2(\mathbb{R}))$ generated by $F_{k,m}$ (see Lemma 2-14). Let $\Psi$ be a unitary isomorphism $\mathcal{H} \to \mathcal{H}_1$. Considering the $K$–types (see (2-21)), we see that we must have

$$
\Psi \psi = \eta F_{k,m},
$$

for some $\eta \in \mathbb{C} - \{0\}$. The scalar $\eta$ is easy to handle. It satisfies

$$
|\eta|^2 = \langle \psi, \psi \rangle \langle F_{k,m}, F_{k,m} \rangle.
$$

Also, we have the following:

$$
\langle r_T(g) \psi, (E^+)^k \psi \rangle = \langle \Psi (r_T(g) \psi), \Psi ((E^+)^k \psi) \rangle = \langle r(g) \Psi \psi, (E^+)^k \Psi \psi \rangle = \\
\langle r(g) (\eta F_{k,m} \psi), (E^+)^k \eta F_{k,m} \psi \rangle = |\eta|^2 \langle r(g) F_{k,m}, (E^+)^k F_{k,m} \rangle = \\
\frac{\langle \psi, \psi \rangle}{\langle F_{k,m}, F_{k,m} \rangle} \langle r(g) F_{k,m}, (E^+)^k F_{k,m} \rangle.
$$

Thus, using (2-22), we find the following:

$$
F_{k,m} = \mu \cdot \langle r_T(g) \psi, (E^+)^k \psi \rangle = \mu \frac{\langle \psi, \psi \rangle}{\langle F_{k,m}, F_{k,m} \rangle} \langle r(g) F_{k,m}, (E^+)^k F_{k,m} \rangle.
$$

Hence Lemma 2-15 implies

$$
\mu \langle \psi, \psi \rangle = \frac{(E^+)^k F_{k,m}(1)}{2^{2k} \cdot k! \cdot m(m+1) \cdots (m+k-1)}.
$$
Combining this with (2-26), we obtain

\[ \langle \psi, P_\Gamma(F_{k,m}) \rangle = \frac{\mathcal{H}(\psi, \psi)}{d(\pi_m)} \cdot (E^+)^k \psi(1) \]

(2-27)

\[ = \frac{(E^+)^k F_{k,m}(1)}{d(\pi_m) \cdot 2^{2k} \cdot k! \cdot m(m+1) \cdots (m+k-1)} \cdot (E^+)^k \psi(1). \]

This proves (2-12) for \( \psi \neq 0 \). The formula clearly is valid for \( \psi = 0 \). The constant \( \lambda_{k,m} \) is computed in Lemma 3-2. \( \square \)

3. TRANSFER TO UPPER–HALF PLANE AND THE PROOF OF COROLLARY 1-2

Let \( f \in S_m(\Gamma) \). Then the function defined by the following expression:

\[ F_f(g) = f(g \sqrt{-1}) \mu(g, \sqrt{-1})^{-m} \]

belongs to \( \mathcal{A}_{\text{cusp}}(\Gamma \setminus SL_2(\mathbb{R}))_m \). Moreover, the map \( f \mapsto F_f \) is an isomorphism of vector spaces \( S_m(\Gamma) \to \mathcal{A}_{\text{cusp}}(\Gamma \setminus SL_2(\mathbb{R}))_m \). This follows from ([4], Lemma 4-1). Using the Iwasawa decomposition (2-1) we obtain the following:

(3-1) 

\[ F_f \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 1 \\ 0 & y \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right) = y^{m/2} \exp (mt\sqrt{-1}) f(z). \]

We prove the following technical lemma which computes the constant \( \lambda_{k,m} \) (see (2-12)) and completes the proof of Theorem 2-11.

**Lemma 3-2.** Let \( f \) be a holomorphic function on the upper half plane. We define \( F_f \) by the formula (3-1). Then

\[ \frac{1}{2^k} (E^+)^k F_f(1) = \sum_{i=0}^{k} (2\sqrt{-1})^i \binom{k}{i} \prod_{j=i}^{k-1} (m+j) \frac{d^i f(z)}{dz^i} \bigg|_{z=\sqrt{-1}}. \]

Moreover, we have the following:

\[ \lambda_{k,m} = \frac{(\sqrt{-1})^{m+1}}{2^{m+k-2} (m-1)! \cdot (m+k-1)!}. \]

**Proof.** This is elementary. We just indicate the proof and leave details to the reader. Using (2-2) and (3-1) we find that \( (E^+)^k F_f(1) \) is equal to

\[ \left( 2\sqrt{-1} ye^{2it} \left( \frac{\partial}{\partial x} - \sqrt{-1} \frac{\partial}{\partial y} \right) - \sqrt{-1} e^{2it} \frac{\partial}{\partial t} \right)^k y^{m/2} \exp (mt\sqrt{-1}) f(z) \bigg|_{x=0, y=1, t=0}. \]

Now, one proceeds by induction on \( k \geq 0 \). For the formula for \( \lambda_{k,m} \) we use its definition (2-13), the first claim of the lemma, and the fact that \( F_{k,m} = F_{f_{k,m}} \). \( \square \)

Next, \( S_m(\Gamma) \) is a finite–dimensional Hilbert space under the inner product:

\[ \langle f_1, f_2 \rangle = \int_{\Gamma \setminus \mathbb{X}} y^m f_1(z) \overline{f_2(z)} \frac{dxdy}{y^2}. \]

We prove

**Lemma 3-3.** Let \( \epsilon_\Gamma = \# (\{ \pm 1 \} \cap \Gamma) \). Then, we have the following: \( \langle f_1, f_2 \rangle = \epsilon_\Gamma \langle F_{f_1}, F_{f_2} \rangle. \)
Lemma 3-5. Assume that $m \geq 3$. Then, we can restate Theorem 2-11 as follows:

\[ \Delta_{k,m}(z) \overset{\text{def}}{=} \frac{1}{2k \lambda_{k,m} \epsilon_{\Gamma}} \sum_{\gamma \in \Gamma} \left( \gamma, z - \sqrt{-1} \right)^k \left( \gamma, z + \sqrt{-1} \right)^{-k-m} \mu(\gamma, z)^{-m}. \]

Now, we can restate Theorem 2-11 as follows:

Lemma 3-5. Assume that $m \geq 3$. Then, for $f \in S_m(\Gamma)$, we have the following:

\[ \langle f, \Delta_{k,m} \rangle = \sum_{i=0}^{k} (2\sqrt{-1})^i \binom{k}{i} \prod_{j=i}^{k-1} (m + j) \frac{d^i f(z)}{dz^i} \bigg|_{z=\sqrt{-1}}. \]

Proof. It is proved in ([4], Lemma 3-3) that

\[ P_{\Gamma} \left( \frac{1}{2k \lambda_{k,m} \epsilon_{\Gamma}} F_{k,m} \right) = F_{\Delta_{k,m}}. \]

Now, using Theorem 2-11 and Lemma 3-3 we find that

\[ \langle f, \Delta_{k,m} \rangle = \epsilon_{\Gamma} \langle F_f, F_{\Delta_{k,m}} \rangle = \epsilon_{\Gamma} \frac{1}{2k \lambda_{k,m} \epsilon_{\Gamma}} \langle F_f, P_{\Gamma} (F_{k,m}) \rangle = \frac{1}{2k} (E^+)^k. \]

Now, we apply Lemma 3-2 to prove the lemma. \qed

Lemma 3-6. Assume that $m \geq 3$. Let $f \in S_m(\Gamma)$. Then we have the following ($k \geq 0$):

\[ \langle f, \Delta_{k,m,\sqrt{-1},1} \rangle = \frac{d^k f(z)}{dz^k} \bigg|_{z=\sqrt{-1}} \]

(Here 1 denotes the trivial character of $\Gamma$.)
Proof. This follows from Lemma 3-5 by rewriting the expression for $\Delta_{k,m}$ applying the binomial theorem to $(\gamma.z - \sqrt{-1})^k = ((\gamma.z + \sqrt{-1}) - 2\sqrt{-1})^k$. □

We remove the assumption that the point in which we compute the derivatives is $\sqrt{-1}$. First, we recall

**Lemma 3-7.** Let $g \in SL_2(\mathbb{R})$. Put $\Gamma' = g\Gamma g^{-1}$. Then the map

$$f \mapsto f|_m g^{-1} = \mu(g^{-1}, \cdot)^{-m} f(g^{-1} \cdot)$$

is an isomorphism of vector spaces $S_m(\Gamma') \rightarrow S_m(\Gamma)$ which preserves the inner products on them. The inverse map $S_m(\Gamma') \rightarrow S_m(\Gamma)$ is given by $f \mapsto f|_m g$

Proof. Indeed, it is a linear isomorphism by ([3], page 40, (2.1.18)). The formula for the inverse is immediate from the cocycle condition of $\mu$. The fact that the map preserves the inner products follows from the fact that the measure $\frac{\text{d}x \text{d}y}{y^2}$ is $SL_2(\mathbb{R})$–invariant (see [3], 1.4) and $\text{Im}(g.z) = \text{Im}(z)/|\mu(g,z)|^2$ (see [3], (1.1.7)). □

**Lemma 3-8.** Assume that $m \geq 3$. Let $\xi \in X$ be a fixed point and let $\Gamma$ be a discrete subgroup of $SL_2(\mathbb{R})$ of finite covolume. Let

$$g = \begin{pmatrix} 1 & \text{Re}(\xi) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Im}(\xi)^{1/2} & 0 \\ 0 & \text{Im}(\xi)^{-1/2} \end{pmatrix}$$

and

$$\Gamma' = g^{-1} \Gamma g.$$

Let $f \in S_m(\Gamma)$. Then we have the following:

$$\langle f, \Delta_{k,m,\sqrt{-1},1}'|_m g^{-1} \rangle = (\text{Im}(\xi))^{m/2 + k} \frac{df(z)}{dz^k} \big|_{z=\xi}, \quad k \geq 0.$$

(We write $\Delta_{k,m,\sqrt{-1},1}'$ instead of $\Delta_{k,m,\sqrt{-1},1}$ to indicate that this series is for $\Gamma'$.)

Proof. Put $\Delta = \Delta_{k,m,\sqrt{-1},1}'$. Using Lemmas 3-6 and 3-7, we compute

$$\langle f, \Delta|_m g^{-1} \rangle = \langle (f|_m g)|_m g^{-1}, \Delta|_m g^{-1} \rangle = \langle f|_m g, \Delta \rangle = \frac{d^k (f|_m g(w))}{dw^k} \big|_{w=\sqrt{-1}}.$$

It remains to compute the right–hand side. We write $z = g.w$ or $w = g^{-1}.z$. We have the following:

$$f|_m g(w) = \mu(g, w)^{-m} f(g.w) = \mu(g^{-1}, z)^m f(z),$$

since

$$\mu(g, w)\mu(g^{-1}, z) = \mu(g, g^{-1}.z)\mu(g^{-1}, z) = \mu(1, z) = 1$$

by the cocycle identity ([3], (1.1.5)). Using the definition of $g$, we find that

$$\mu(g^{-1}, z) = (\text{Im}(\xi))^{1/2}.$$

Hence

$$f|_m g(w) = (\text{Im}(\xi))^{m/2} f(z).$$

(From $z = g.w$ we see that ([3], (1.4.3))

$$\frac{dz}{dw} = \mu(g, w)^{-2} = \mu(g^{-1}, z)^2 = \text{Im}(\xi).$$
Now, by induction on $k$, we find the following:

$$
\frac{d^k (f|_mg(w))}{dw^k} = (Im(\xi))^{m/2+k} \frac{d^k f(z)}{dz^k}.
$$

Since obviously for $z = \xi$ we obtain

$$
w = g^{-1}z = g^{-1}\xi = \sqrt{-1}.
$$

the lemma follows. □

**Lemma 3-9.** Corollary 1-2 is valid for trivial $\chi$.

*Proof.* In view of Lemma 3-8, we need to show

$$
\Delta_{k,m,\xi,1} = (Im(\xi))^{-k-m/2} \Delta_{k,m,\sqrt{-1},1} |_{m} g^{-1}.
$$

Using their definitions this is straightforward. □

**Lemma 3-11.** Corollary 1-2 is valid for any $\chi$.

*Proof.* The map defined by

$$
f \mapsto \sum_{\gamma \in \Gamma'} \chi^{-1}(\gamma)f|_m \gamma
$$

is the projection from $S_m(\Gamma')$ onto $S_m(\Gamma, \chi)$. Here $\Gamma'$ is the kernel of $\chi$. The inner products on $S_m(\Gamma')$ and $S_m(\Gamma, \chi)$ are related by the following elementary formula ($f \in S_m(\Gamma), f_1 \in S_m(\Gamma, \chi)$):

$$
\langle f_1, \sum_{\gamma \in \Gamma'} \chi^{-1}(\gamma)f|_m \gamma \rangle = \langle f_1, f \rangle_{\Gamma'}.
$$

Now, the lemma follows noting that the modular form $\Delta_{k,m,\xi,\chi}$ is the image under the projection of the modular form $\Delta_{k,m,\xi,1}'$. □

4. **A Relation to the work of Petersson**

We relate our modular forms to those constructed by Petersson [6]. We assume that $m \geq 3$. Let $\xi \in X$. Among the three types of Poincaré series considered in [6], the elliptic type can be written as follows:

$$
\Phi_{k,m,\xi,\chi}(z) = \sum_{\gamma \in \Gamma} \frac{(\gamma.z - \xi)^k}{(\gamma.z - \xi)} \mu(\gamma, z)^{-m}\chi(\gamma)^{-1}.
$$

The second relation in (14) on page 41 (see also (2) on page 38) in [6]) explains the meaning of those forms. In more detail, the mapping $z \mapsto w = (z - \xi)/(z - \bar{\xi})$ is a holomorphic isomorphism of $X$ onto the unit disk $|w| < 1$ which maps $\xi$ onto 0. If $f$ is holomorphic function on $X$, then we can transfer the function $(z - \bar{\xi})^m f(z)$ to the unit disk and develop the resulting function $F(w)$ into the power series centered at 0:

$$
(z - \bar{\xi})^m f(z) = F(w) = \sum_{k=0}^{\infty} b_k(\xi, f) w^k = \sum_{k=0}^{\infty} b_k(\xi, f) \left(\frac{z - \bar{\xi}}{z - \xi}\right)^k
$$
Thus, we have the following expansion on $X$:

\[(4-2)\quad f(z) = \sum_{k=0}^{\infty} b_k(\xi, f) \frac{(z - \xi)^k}{(z - \xi)^{k+m}},\]

which is an analogue of the classical Fourier expansion of modular forms. Next, let $f \in S_m(\Gamma, \chi)$. Then, one of the main results in [6] proves that

\[(4-3)\quad \langle f, \Phi_{k,m,\xi,\chi} \rangle \sim b_k(\xi, f),\]

where $\sim$ means up to a constant which does not depend on $f$. We explain how this follows from our work [4] and how is related to the results of the present paper.

First, discussions like the ones in Lemmas 3-7, 3-8, and 3-9 allows us to assume that $\xi = \sqrt{-1}$ and $\chi$ is trivial. Then, using (3-4), (4-1) can be written as follows:

\[\langle f, \Phi_{k,m,\sqrt{-1},1} \rangle = 2^k \lambda_{k,m} \epsilon \Gamma \langle f, \tilde{\Delta}_{k,m} \rangle,\]

for all $f \in S_m(\Gamma)$. Next, we reprove (4-3). To accomplish this, we transfer the expansion (4-2) when $\xi = \sqrt{-1}$ to the group level (see the notation after (2-8) and the first paragraph in Section 3). We obtain the following:

\[(4-4)\quad F_f = \sum_{k=0}^{\infty} b_k(\sqrt{-1}, f) F_{k,m},\]

where the series converges uniformly on compact sets in $SL_2(\mathbb{R})$. Applying Lemma 3-3 and the first line of the proof of Lemma 3-5, we have the following:

\[\langle f, \Phi_{k,m,\sqrt{-1},1} \rangle = 2^k \lambda_{k,m} \epsilon \Gamma \langle f, \tilde{\Delta}_{k,m} \rangle = 2^k \lambda_{k,m} \epsilon \Gamma \langle f, P_\Gamma \left( \frac{1}{2^k \lambda_{k,m} \epsilon \Gamma} F_{k,m} \right) \rangle = \langle f, P_\Gamma (F_{k,m}) \rangle.\]

Using, (2-20) this can be further written as follows:

\[(4-5)\quad \langle f, \Phi_{k,m,\sqrt{-1},1} \rangle = \int_{SL_2(\mathbb{R})} F_f(g) \overline{F_{k,m}(g)} dg.\]

To compute the integral, we represent $SL_2(\mathbb{R})$ as an union of increasing sequence of compact sets $C_1 \subset C_2 \subset \cdots$ which satisfy $KC_i \subset C_i$. Then, for any $i \geq 1$, the fact $KC_i \subset C_i$ and Lemma 2-9 (i) implies that the functions $F_{l,m}$ are orthogonal in $L^2(C_i)$. Hence, the fact that the expansion (4-4) converges uniformly on $C_i$ implies the following:

\[\int_{C_i} F_f(g) \overline{F_{k,m}(g)} dg = \sum_{l=0}^{\infty} b_l(\sqrt{-1}, f) \int_{C_i} F_{l,m}(g) \overline{F_{k,m}(g)} dg = b_k(\sqrt{-1}, f) \int_{C_i} F_{k,m}(g) \overline{F_{k,m}(g)} dg.\]
Since, $F_f$ is bounded (being a cusp form) and $F_{k,m} \in L^1(SL_2(\mathbb{R}))$ (see Lemma 2-9 (iii)) we can take the limit $i \to \infty$ to obtain

$$\langle f, \Phi_{k,m,\sqrt{-1},1} \rangle = \int_{SL_2(\mathbb{R})} F_f(g) \overline{F_{k,m}(g)} dg = \lim_{i \to \infty} \int_{C_i} F_f(g) \overline{F_{k,m}(g)} dg = b_k(\sqrt{-1}, f) \lim_{i \to \infty} \int_{C_i} |F_{k,m}(g)|^2 dg = b_k(\sqrt{-1}, f) \|F_{k,m}\|_2^2.$$  

This is (4-3) for $\xi = \sqrt{-1}$.

5. Corrections to [4]

Corrections: The third sentence in the statement of Theorem 1-6 should be "Assume that $\sum_{l \in \mathbb{Z}} f(\cdot + l) \neq 0$ if $\Gamma_N \in \{\Gamma_0(N), \Gamma_1(N)\}$," in Lemma 2-13 (i) we should have $\chi_{m+2k}$ instead of $\chi_{-m-2k}$, in (3-6) $\chi_{m+2k}$ and $\chi_{-m-2k}$ should exchange positions, in the statement of Proposition 4-5 the exponent of $(-1)$ should be $m+k$, and $2 \cdot h^{m+k}$ is $h^{m+k}$, and in the proof the first displayed formula starts with $\pi \sqrt{-1} - 2\pi \sqrt{-1}$ (and remaining formulas in the part of the proof on page 1502 can be easily adjusted.)

References