ON AN ALGEBRAIC APPROACH TO THE ZELEVINSKY CLASSIFICATION FOR CLASSICAL $p$–ADIC GROUPS

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Abstract. In this paper we classify irreducible, admissible representations of $p$–adic classical groups in a purely algebraic way. We present a new and simple algorithm for constructing the classifying data which is also useful in other contexts where the computation with Jacquet modules or constant terms of Eisenstein series appears. We construct and study non–standard intertwining operators. They are defined in a purely algebraic way although their rationality follows from [22].

INTRODUCTION

The Langlands classification for reductive $p$–adic groups is obtained using, partly, analytical methods, such as asymptotic of the matrix coefficients and standard integral intertwining operators (see for example [8], Chapter XI, Theorems 2.10 and 2.11). Applying the duality [1] we would obtain the dual classification which should be called the Zelevinsky classification. It seems that the details of such approach are not yet written.

On the other hand, in a beautiful paper [27], Zelevinsky gave completely algebraic classification of irreducible representations of $p$-adic general linear groups. Inspired by the Zelevinsky work we give completely algebraic classification of irreducible representations of $p$–adic classical groups. In fact, our approach is explicit and it gives an algorithm for the construction of the classifying datum of an irreducible representation.

This paper can be regarded as a generalization of [17] which treats the case of unramified representations. We use notation introduced there, and [17] maybe considered as a main source of examples of various classes of representations studied here.

We remark that A. Minguez have written a completely algebraic proof of the Langlands classification for general linear groups in his thesis [11] in a similar spirit.

Although, as we remarked above, it is possible to write down some sort of Zelevinsky classification for general reductive groups, in our opinion, our algorithm that uses combinatorics of Zelevinsky segments is what makes the case of the classical groups appealing and worthwhile to study separately. The ideas used here are based on techniques that were successfully applied to the study of unitarity (see for example [10], [17], [19], [23], [9]) and to the explicit determination of $Θ$–correspondence (see [14], [15], [18], [20], [21]). We expect that techniques introduced here will be useful in further study of a more complicated induced representations that appear in the classification of unitary duals of $p$–adic classical groups (see [12]) and as well as study of $Θ$–correspondence. Moreover, we hope that this type of argument can be extended to the case of the metaplectic groups, and also, since we have not used matrix coefficients explicitly, to the modular representations of the $p$–adic classical groups [11].

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Techniques that we use here are intentionally basic. We rely on a Hopf algebra approach of Zelevinsky [27] and its extension to a Hopf module approach for classical groups due to Tadić [24] which is based on the Geometric lemma of Bernstein and Zelevinsky ([7], 2.12). To prove that certain induced representations have equivalent composition series and to construct intertwining operators, we use the description of a contragredient representation due to Waldspurger ([14], Chapter 4, II.1) which is based entirely on the results of [6] and which generalizes similar formula of Gelfand and Kazhdan for general linear groups ([6], Theorem 7.3).

In Section 1, the reader can find the description of the structure of the groups that we consider as well as the description of original Zelevinsky classification [27] (see Theorem 1-1) and a particular case of the formula for Jacquet modules due to Tadić [24] for classical groups that we use in the paper (see Theorem 1-4). After being acquainted with the basic notation in Section 1, the reader may proceed to read Section 4 directly where the main results are stated. We begin this section by giving a definition of strongly negative and negative representations (see Definition 4-1). This is a generalization of the similar notion introduced for unramified representations in [17]. In fact, by duality [1], strongly negative and negative representations corresponds to representations in discrete series and tempered representations, respectively. One may apply the duality and results of [13] to construct strongly negative and negative representations. We do not use this in our paper since we would like to keep our paper based on basic techniques while the validity of results of [13] to construct strongly negative and negative representations. We do not use this in our paper (see Theorem 1-4). After being acquainted with the basic notation in Section 1, the reader may proceed to read Section 4 directly where the main results are stated. We begin this section by giving a definition of strongly negative and negative representations (see Definition 4-1). This is a generalization of the similar notion introduced for unramified representations in [17]. In fact, by duality [1], strongly negative and negative representations corresponds to representations in discrete series and tempered representations, respectively. One may apply the duality and results of [13] to construct strongly negative and negative representations. We do not use this in our paper since we would like to keep our paper based on basic techniques while the validity of results of [13] to construct strongly negative and negative representations.

**Theorem.** Let \( \sigma \in \text{Irr}(G_n) \) be a negative representation. Then there exists a sequence of segments \( \Delta_1, \Delta_2, \ldots, \Delta_k \) such that \( e(\Delta_i) = 0 \), \( i = 1, \ldots, k \) and an irreducible strongly negative representation \( \sigma_{\text{sn}} \) such that

\[
\sigma \hookrightarrow (\Delta_1) \times (\Delta_2) \times \cdots \times (\Delta_k) \times \sigma_{\text{sn}}.
\]

(We allow empty sequence of segments here. Then \( k = 0 \).) If, in addition, \( \sigma \hookrightarrow (\Delta'_1) \times (\Delta'_2) \times \cdots \times (\Delta'_{k'}) \times \sigma'_{\text{sn}} \) for some other sequence of segments \( \Delta'_1, \Delta'_2, \ldots, \Delta'_{k'} \) such that \( e(\Delta'_i) = 0 \), \( i = 1, \ldots, k' \), and for some strongly negative representation \( \sigma'_{\text{sn}} \), then the sequence \( (\Delta'_1, \Delta'_2, \ldots, \Delta'_{k'}) \) is a ~-permutation of the sequence \( (\Delta_1, \Delta_2, \ldots, \Delta_k) \) and \( \sigma'_{\text{sn}} \simeq \sigma_{\text{sn}} \).

**Theorem.** Suppose that \( \Delta_1, \ldots, \Delta_k \) is a sequence of segments satisfying \( e(\Delta_1) \geq \cdots \geq e(\Delta_k) > 0 \). (We allow empty sequence here; in this case \( k = 0 \).) Let \( \sigma_{\text{neg}} \) be a negative representation. Then we have the following:

(i) The induced representation \( (\Delta_1) \times (\Delta_2) \times \cdots \times (\Delta_k) \times \sigma_{\text{neg}} \) has a unique irreducible subrepresentation (we call it Zelevinsky subrepresentation); we will denote it by \( (\Delta_1, \ldots, \Delta_k; \sigma_{\text{neg}}) \).

(ii) We have \( (\Delta_1, \ldots, \Delta_k; \sigma_{\text{neg}}) \hookrightarrow (\Delta_1) \times (\Delta_2, \ldots, \Delta_k; \sigma_{\text{neg}}) \).
(iii) The representation $\langle \Delta_1, \ldots, \Delta_k; \sigma_{\text{neg}} \rangle$ appears with the multiplicity one in the composition series of $\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_k \rangle \rtimes \sigma_{\text{neg}}$.
(iv) The representation $\langle \Delta_1, \ldots, \Delta_k; \sigma_{\text{neg}} \rangle$ is negative if and only if $k = 0$; in that case $\langle \emptyset; \sigma_{\text{neg}} \rangle \simeq \sigma_{\text{neg}}$.
(v) The induced representation $\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \rtimes \sigma_{\text{neg}}$ has a unique maximal proper subrepresentation; the corresponding quotient is $\langle \Delta_1, \ldots, \Delta_k; \sigma_{\text{neg}} \rangle$.
(vi) $\langle \Delta_1, \ldots, \Delta_k; \sigma_{\text{neg}} \rangle \simeq \langle \Delta'_1, \ldots, \Delta'_k; \sigma'_{\text{neg}} \rangle$ if and only if $(\Delta_1, \ldots, \Delta_k)$ is a permutation of $(\Delta'_1, \ldots, \Delta'_k)$ and $\sigma_{\text{neg}} \simeq \sigma'_{\text{neg}}$.

**Theorem.** If $\sigma \in \text{Irr}(G_n)$, then there exists a sequence of segments $\Delta_1, \ldots, \Delta_k$ satisfying $e(\Delta_1) \geq \cdots \geq e(\Delta_k) > 0$ and a negative representation $\sigma_{\text{neg}}$ such that $\sigma \simeq \langle \Delta_1, \ldots, \Delta_k; \sigma_{\text{neg}} \rangle$.

Main results of the paper are proved in Sections 5, 6, 7, 8. We prove the existence of classifying data for both negative (see the first part of Theorem 4-3) and arbitrary representations (see Theorem 4-6) in a rather novel way. Both proofs are similar and based on our "Principle of maximality" (see Theorem 3-3 and Remark 3-12) proved in Section 3. In Section 6 we prove the uniqueness of classifying data (see the second part of Theorem 4-3). It is based on a combinatorial argument which comes from the computation of Jacquet modules but it is well–organized so it serves as a model for some similar computations such as the proof of Theorem 4-5 given in Section 7. In addition to the proof of Theorem 4-5, Section 7 contains two important results (see Lemmas 7-3 and 7-4) that give additional information on the induced representation (that is, the classifying datum) of an irreducible representation. We again cite here Lemma 7-3.

**Lemma.** Suppose that $\Delta_1, \ldots, \Delta_k$ is a sequence of segments satisfying $e(\Delta_1) \geq \cdots \geq e(\Delta_k) > 0$.

(We allow empty sequence here; in this case $k = 0$.) Let $\sigma_{\text{neg}}$ be a negative representation. The numbers $i_1 < i_2 < \cdots < i_l$ are defined by the following condition:

$$e(\Delta_1) = \cdots = e(\Delta_{i_1}) > e(\Delta_{i_1+1}) = \cdots = e(\Delta_{i_2}) > \cdots > e(\Delta_{i_l+1}) = \cdots = e(\Delta_k) > 0.$$ 

The irreducible representation

$$\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_{i_1} \rangle \otimes \langle \Delta_{i_1+1} \rangle \times \cdots \times \langle \Delta_{i_2} \rangle \otimes \cdots \otimes \langle \Delta_{i_l+1} \rangle \times \cdots \times \langle \Delta_k \rangle \rtimes \sigma_{\text{neg}}$$

appears with the multiplicity one in the appropriate Jacquet module of $\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \rtimes \sigma_{\text{neg}}$.

We emphasize that above Lemma uniquely characterizes the Zelevinsky subrepresentation by its specific Jacquet module.

Lemmas 7-3 and 7-4 should be important for various computations with the classification. For example, we use them in Section 9 to study certain intertwining operators. In Section 2, the intertwining operators are initially constructed using mentioned description of the contragredient due to Waldspurger in a purely algebraic fashion (see Theorem 2-6). In general, they are not identical to the intertwining operators constructed by analytic methods ([26]) or by geometric methods ([22]). After constructing intertwining operators in Section 2, we obtain more precise results in Section 9 such as uniqueness up to a non–zero scalar (see Lemma 9-1) and factorization (see Theorem 9-4).

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1. Preliminaries

Let \( \mathbb{Z}, \mathbb{R}, \) and \( \mathbb{C} \) be the ring of rational integers, the field of real numbers, and the field of complex numbers, respectively. Let \( F \) be a non–Archimedean field of characteristic different from \( 2 \). We write \( \nu \) for the normalized absolute value of \( F \).

Let \( G \) be an \( l \)–group (see [6]). We will consider smooth representations of \( G \) on complex vector spaces. We call them shortly representations. If \( \sigma \) is a representation of \( G \), then we write \( V_\sigma \) for its space. Its contragredient representation will be denoted by \( \tilde{\sigma} \) and the corresponding non–degenerate canonical pairing will be denoted by \( \langle \cdot, \cdot \rangle : V_\sigma \times V_\sigma \to \mathbb{C} \). If \( \sigma_1 \) and \( \sigma_2 \) are representations of \( G \), then we write \( \text{Hom}_G(\sigma_1, \sigma_2) \) for the space of all \( G \)–intertwining maps \( \sigma_1 \to \sigma_2 \). We say that \( \sigma_1 \) and \( \sigma_2 \) are equivalent, \( \sigma_1 \simeq \sigma_2 \), if there is a bijective \( \varphi \in \text{Hom}_G(\sigma_1, \sigma_2) \). Let \( \text{Irr}(G) \) be the set of equivalence classes of irreducible admissible representations of \( G \). Let \( R(G) \) be the Grothendieck group of the category \( \mathcal{M}_{\text{adm.fin.leng}}(G) \) of all admissible representations of finite length of \( G \). If \( \sigma \) is an object of \( \mathcal{M}_{\text{adm.fin.leng}}(G) \), then we write \( s.s.(\sigma) \) for its semi–simplification in \( R(G) \). Frequently, in computations we will write shortly \( \sigma \) instead of \( s.s.(\sigma) \). If \( G \) is the trivial group, then we write its unique irreducible representation as \( 1 \).

Next, we shall fix the notation for the general linear group \( \text{GL}(n, F) \) of type \( n \times n \) with entries in \( F \). Let \( I_n \) be the identity matrix in \( \text{GL}(n, F) \). Let \( \bar{g} \) be the transposed matrix of \( g \in \text{GL}(n, F) \). The transposed matrix of \( g \in \text{GL}(n, F) \) with respect to the second diagonal will be denoted by \( \bar{g} \).

If \( \chi \) is a character of \( F^\times \) and \( \pi \) is a representation of \( \text{GL}(n, F) \), then the representation \( (\chi \circ \det \otimes \pi) \) of \( \text{GL}(n, F) \) will be written as \( \chi \pi \).

We fix the minimal parabolic subgroup \( P_{\text{min}}^{\text{GL}_n} \) of \( \text{GL}(n, F) \) consisting of all upper triangular matrices in \( \text{GL}(n, F) \). A standard parabolic subgroup \( P \) of \( \text{GL}(n, F) \) is a parabolic subgroup containing \( P_{\text{min}}^{\text{GL}_n} \). There is a one–to–one correspondence between the set of all ordered partitions of \( n, \alpha = (n_1, \ldots, n_k) \) \( (n_i \in \mathbb{Z}_{>0}) \), and the set of standard parabolic subgroups of \( \text{GL}(n, F) \) attached to a partition \( \alpha \) a parabolic subgroup \( P_\alpha \) consisting of all block–upper triangular matrices:

\[
p = (p_{ij})_{1 \leq i,j \leq k}, \quad p_{ij} \text{ is a matrix of type } n_i \times n_j, \quad p_{ij} = 0 \quad (i > j).
\]

The parabolic subgroup \( P_\alpha \) admits a Levi decomposition \( P_\alpha = M_\alpha N_\alpha \), where

\[
M_\alpha = \{ \text{diag}(g_1, \ldots, g_k); \ g_i \in \text{GL}(n_i, F) \ (1 \leq i \leq k) \},
\]

\[
N_\alpha = \{ p \in P_\alpha; \ p_{ii} = I_{n_i} \ (1 \leq i \leq k) \}.
\]

Let \( \pi_i \) be a representation of \( \text{GL}(n_i, F) \) \((1 \leq i \leq k)\). Then we consider \( \pi_1 \otimes \cdots \otimes \pi_k \) as a representation of \( M_\alpha \) as usual:

\[
\pi_1 \otimes \cdots \otimes \pi_k(\text{diag}(g_1, \ldots, g_k)) = \pi_1(g_1) \otimes \cdots \otimes \pi_k(g_k),
\]

and extend it trivially across \( N_\alpha \) to the representation of \( P_\alpha \) denoted by the same letter. Then we form (normalized) induction written down as follows (see [7], [27]):

\[
\pi_1 \times \cdots \times \pi_k = i_{\alpha, \alpha}(\pi_1 \otimes \cdots \otimes \pi_k) := \text{Ind}_{P_\alpha}^{\text{GL}(n, F)}(\pi_1 \otimes \cdots \otimes \pi_k)
\]

In this way we obtain the functor \( \mathcal{M}_{\text{adm.fin.leng}}(M_\alpha) \xrightarrow{i_{\alpha, \alpha}} \mathcal{M}_{\text{adm.fin.leng}}(\text{GL}(n, F)) \) and a group homomorphism \( R(M_\alpha) \xrightarrow{i_{\alpha, \alpha}} R(\text{GL}(n, F)) \). Next, if \( \pi \) is a representation of \( \text{GL}(n, F) \), then we form the normalized Jacquet module \( r_{\alpha,n}(\pi) \) of \( \pi \) (see [7]). It is a representation of \( M_\alpha \). In this way obtain a functor \( \mathcal{M}_{\text{adm.fin.leng}}(\text{GL}(n, F)) \xrightarrow{r_{\alpha,n}} \mathcal{M}_{\text{adm.fin.leng}}(M_\alpha) \) and a group homomorphism \( R(\text{GL}(n, F)) \xrightarrow{r_{\alpha,n}} R(M_\alpha) \). The functors \( i_{\alpha, \alpha} \) and \( r_{\alpha,n} \) are related by the Frobenius reciprocity:

\[
\text{Hom}_{\text{GL}(n, F)}(\pi, i_{\alpha, \alpha}(\pi_1 \otimes \cdots \otimes \pi_k)) \simeq \text{Hom}_{M_\alpha}(r_{\alpha,n}(\pi), \pi_1 \otimes \cdots \otimes \pi_k).
\]
We list some additional basic properties of induction:
\[
\pi_1 \times (\pi_2 \times \pi_3) \simeq (\pi_1 \times \pi_2) \times \pi_3,
\]
\[
\pi_1 \times \pi_2 \text{ and } \pi_2 \times \pi_1 \text{ have the same composition series,}
\]
if \( \pi_1 \times \pi_2 \) is irreducible, then \( \pi_1 \times \pi_2 \simeq \pi_2 \times \pi_1 \),
\[
\chi(\pi_1 \times \pi_2) \simeq (\chi \pi_1) \times (\chi \pi_2), \quad \text{for a character } \chi \text{ of } F^\times,
\]
\[
\pi_1 \times \pi_2 \simeq \tilde{\pi}_1 \times \tilde{\pi}_2.
\]

We take \( GL(0, F) \) to be the trivial group (we consider formally the unique element of this group as a \( 0 \times 0 \) matrix and the determinant map \( GL(0, F) \to F^\times \) as a map \( 1 \mapsto 1 \)). We extend \( \times \) formally as follows: \( \pi \times 1 = 1 \times \pi =: \pi \) for every representation \( \pi \) of \( GL(n, F) \). The listed properties hold in this extended setting. We also let \( r_{(0)} \), \( 0(1) = 1 \). We let
\[
\begin{align*}
R(GL) &= \oplus_{m \geq 0} R(GL(m, F)) \\
Irr(GL) &= \oplus_{m \geq 0} Irr(GL(m, F)) \quad \text{(disjoint union)}.
\end{align*}
\]

If \( \pi \in Irr(GL) \), then we define \( m_\pi \in \mathbb{Z}_{\geq 0} \) by \( \pi \in Irr(GL(m_\pi, F)) \). If \( \pi_1, \pi_2 \in R(GL) \), then we write \( \pi_1 \geq \pi_2 \) if \( \pi_1 - \pi_2 \) is a linear combination of irreducible representations with non-negative coefficients.

The Abelian group \( R(GL) \) has a structure of graded Hopf \( \mathbb{Z} \)-algebra where the multiplication is given by \( m(\pi_1, \pi_2) = \pi_1 \times \pi_2 \) and the comultiplication given by \( m^*(\pi) = \sum_{k=0}^n r_{(k-n,k)}(\pi) \) (see [27], 1.7).

We say that \( \rho \in Irr(GL) \) is supercuspidal if \( r_{\alpha,m_\rho}(\rho) = 0 \) for all \( \alpha \neq (m_\rho) \). A segment \( \Delta \) is a set of the form \( [\rho, \nu^k \rho] := \{\rho, \ldots, \nu^k \rho\} \). The induced representation \( \rho \times \cdots \times \nu^k \rho \) contains the unique irreducible subrepresentation denoted by \( \langle \Delta \rangle \). (See [27], 2.2 and 3.3.) We let \( 1 = \langle \emptyset \rangle \).

All other irreducible representations are classified in terms of those representations. More precisely, we have the following theorem (see [27], Theorems 4.1 and 6.1):

**Theorem 1-1.**
1. Let \( \Delta_1, \ldots, \Delta_k \) be a sequence of segments. Then \( \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_k \rangle \) is reducible if and only there are indices \( i, j \), such that the segments \( \Delta_i \) and \( \Delta_j \) are linked
   (that is, \( \Delta_i \cup \Delta_j \) is a segment but \( \Delta_i \not\subset \Delta_j \) and \( \Delta_j \not\subset \Delta_i \)).
2. Let \( \Delta_1, \ldots, \Delta_k \) be a sequence of segments such that if \( i < j \), then the segment \( \Delta_i \) does not precede the segment \( \Delta_j \). (The segment \( \Delta = [\rho, \nu^k \rho] \) precedes the segment \( \Delta' = [\rho', \nu^{k'} \rho'] \) if they are linked and there exists \( l \in \mathbb{Z}_{>0} \) such that \( \rho' \simeq \nu^l \rho \).) Then the induced representation \( \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_k \rangle \) has the unique irreducible subrepresentation. We denote it by \( \langle \Delta_1, \ldots, \Delta_k \rangle \). It appears in the composition series with multiplicity one.
3. Conversely, if \( \sigma \in Irr(GL) \), then there is, up to a permutation, a unique sequence of segments \( \Delta_1, \ldots, \Delta_k \), satisfying the assumption of (ii), such that \( \sigma \simeq \langle \Delta_1, \ldots, \Delta_k \rangle \).
4. \( \langle \Delta_1, \ldots, \Delta_k \rangle \simeq \langle \Delta_1', \ldots, \Delta_k' \rangle \) if and only if the sequences \( \Delta_1, \ldots, \Delta_k \) and \( \Delta_1', \ldots, \Delta_k' \) are equal up to a permutation.

We use the following standard reformulation of the Zelevinsky classification. It is well-known that every irreducible supercuspidal representation \( \rho \) can be written uniquely in the form \( \nu^{\epsilon(\rho)} \rho^a \) where \( \epsilon(\rho) \in \mathbb{R} \) and \( \rho^a \) is unitary and supercuspidal representation. Then every segment can be written in the form that we use in this paper \( \Delta = [\nu^{-\alpha} \rho, \nu^\beta \rho] \), where \( \alpha, \beta \in \mathbb{R} \) such that \( \alpha + \beta \in \mathbb{Z}_{\geq 0} \) and \( \rho \) is unitary and supercuspidal representation. We let \( \epsilon(\Delta) = -\frac{\alpha + \beta}{2} \).

It is easy to check that if \( \epsilon(\Delta) = \epsilon(\Delta') \), then the segments cannot be linked. Therefore, \( \langle \Delta \rangle \times \langle \Delta' \rangle \simeq \langle \Delta' \rangle \times \langle \Delta \rangle \) (see Theorem 1-1 (i)). It is easy to check that \( \Delta \) precedes \( \Delta' \) if and only if
they are linked and \( e(\Delta) < e(\Delta') \). Now, it is easy to see that Theorem 1-1 (ii) is equivalent to the following statement:

(ii) Let \( \Delta_1, \ldots, \Delta_k \) be a sequence of segments such that \( e(\Delta_1) \geq \cdots \geq e(\Delta_k) \). Then the induced representation \( \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_k \rangle \) has the unique irreducible subrepresentation.

In this form we use (ii) in the paper.

The following technical lemma will be important for computations of Jacquet modules for classical groups later in the paper.

**Lemma 1-2.** Let \( \Delta'_1, \ldots, \Delta'_k \) be a sequence of segments such that \( \Delta'_i \) and \( \Delta'_j \) are not linked for all \( i, j \). (Then \( \langle \Delta'_1 \rangle \times \cdots \times \langle \Delta'_k \rangle \) is irreducible by Theorem 1-1 (ii).) Let \( \psi_1, \ldots, \psi_l \) be a sequence of segments and \( \langle \Delta_1, \ldots, \Delta_k \rangle \) an irreducible representation written in the Zelevinsky classification (see Theorem 1-1 (ii)). If \( \langle \psi_1 \rangle \times \cdots \times \langle \psi_l \rangle \times \langle \Delta_1, \ldots, \Delta_k \rangle \geq \langle \Delta'_1 \rangle \times \cdots \times \langle \Delta'_k \rangle \) then \( \langle \Delta_1, \ldots, \Delta_k \rangle \cong \langle \Delta'_1 \rangle \times \cdots \times \langle \Delta'_k \rangle \).

**Proof.** We apply the involution \( t \) (see [27], 9.12) that carries irreducible representations into irreducible representations (see [1], Theorem 2.3) and commutes with induction. It implies the following:

\[
\langle \psi_1 \rangle^t \times \cdots \times \langle \psi_l \rangle^t \times \langle \Delta_1, \ldots, \Delta_k \rangle^t \geq \langle \Delta'_1 \rangle^t \times \cdots \times \langle \Delta'_k \rangle^t.
\]

Hence \( \langle \Delta_1, \ldots, \Delta_k \rangle^t \) must be non–degenerate. Now, ([27], Theorem 9.7) implies \( \langle \Delta_1, \ldots, \Delta_k \rangle^t \cong \langle \Delta'_1 \rangle^t \times \cdots \times \langle \Delta'_k \rangle^t \). Hence the claim. \( \square \)

Now, we fix the basic notation for classical groups. Let

\[
J_n = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots \\
1 & 0 & \cdots & 0 & 0
\end{bmatrix} \in GL(n, F).
\]

The symplectic group (of rank \( n \geq 1 \)) is defined as follows:

\[
\text{Sp}(2n, F) = \left\{ g \in \text{GL}(2n, F); \ g \cdot \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix}^t \cdot g = \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} \right\}.
\]

Next, the split orthogonal groups and special odd-orthogonal groups (both of rank \( n \geq 1 \)) are defined by

\[
\text{SO}(n, F) = \left\{ g \in \text{SL}(n, F); \ g \cdot J_n \cdot g = J_n \right\}
\]

\[
\text{O}(n, F) = \left\{ g \in \text{GL}(n, F); \ g \cdot J_n \cdot g = J_n \right\}.
\]

We take \( \text{Sp}(0, F), \text{SO}(0, F), \text{O}(0, F) \) to be the trivial groups (we consider their unique element formally as \( 0 \times 0 \) matrix).

Orthogonal groups can be non–split. To describe them let us fix \( a_1, \ldots, a_u \in F^\times \) such that the form \( \sum_{i=1}^u a_i x_i^2 \) does not represent zero non–trivially over \( F \). Then \( u \leq 4 \). Let us fix the matrix:

\[
I(a_1, \ldots, a_u) := \begin{bmatrix}
a_1 & 0 & \cdots & 0 \\
0 & a_2 & \cdots & 0 \\
\vdots \\
0 & \cdots & 0 & a_u
\end{bmatrix}.
\]
We let
\[
O(n, a_1, \ldots, a_u, F) = \begin{cases} 
\{ g \in \text{GL}(2n + u, F) ; \quad g \cdot \begin{bmatrix} 0 & 0 & J_n \\
0 & I(a_1, \ldots, a_u) & 0 \\
J_n & 0 & 0 \end{bmatrix} \cdot g = \begin{bmatrix} 0 & 0 & J_n \\
0 & I(a_1, \ldots, a_u) & 0 \\
J_n & 0 & 0 \end{bmatrix} \} .
\end{cases}
\]

In the sequel, we fix one of the following four series of the groups:
\[
G_n = \text{Sp}(2n, F), \quad n \geq 0 \\
G_n = O(2n, F), \quad n \geq 0 \\
G_n = \text{SO}(2n + 1, F), \quad n \geq 0 \\
G_n = O(n, a_1, \ldots, a_u, F), \quad n \geq 0.
\]

Let \( n > 0 \). There is a one–to–one correspondence between the set of all finite sequences of positive integers of total mass \( \leq n \) and the set of standard parabolic subgroups of \( G_n \) defined as follows. For \( \alpha = (m_1, \ldots, m_k) \) of the total mass \( m := \sum_{i=1}^{k} m_i \leq n \), we let
\[
P^\alpha_G := \begin{cases} 
P_{(m_1, \ldots, m_k, 2(n-m), m_k, m_k-1, \ldots, m_1)} \cap G_n ; 
G_n = \text{Sp}(2n, F), \quad O(2n, F) \\
P_{(m_1, \ldots, m_k, 2(n-m)+1, m_k, m_k-1, \ldots, m_1)} \cap G_n ; 
G_n = \text{SO}(2n + 1, F) \\
P_{(m_1, \ldots, m_k, 2(n-m)+u, m_k, m_k-1, \ldots, m_1)} \cap G_n ; 
G_n = O(n, a_1, \ldots, a_u, F).
\end{cases}
\]

(The middle term \( 2(n-m) \) is omitted if \( m = n \).) The parabolic subgroup \( P^\alpha_G \) admits a Levi decomposition \( P_\alpha = M^\alpha_G N^\alpha_G \), where
\[
M^\alpha_G = \{ \text{diag}(g_1, \ldots, g_k, g_\alpha^{-1}, \ldots, g_1^{-1}) ; \quad g_i \in \text{GL}(m_i, F) \ (1 \leq i \leq k), \quad g \in G_{n-m} \}
\]
\[
N^\alpha_G = \{ p \in P^\alpha_G ; \quad p_{ii} = I_{m_i} \ \forall i \}.
\]

Let \( \pi \) be a representation of \( \text{GL}(m_i, F) \) \((1 \leq i \leq k)\). Let \( \sigma \) be a representation of \( G_{n-m} \). Then we consider \( \pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma \) as a representation of \( M^\alpha_G \) as usual:
\[
\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma(\text{diag}(g_1, \ldots, g_k, g_\alpha^{-1}, \ldots, g_1^{-1})) = \pi_1(g_1) \otimes \cdots \otimes \pi_k(g_k) \otimes \sigma(g),
\]
and extend it trivially across \( N^\alpha_G \) to the representation of \( P^\alpha_G \) denoted by the same letter. Then we form (normalized) parabolic induction written as follows:
\[
\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma = \text{Ind}_{P^\alpha_G}^{G_n}(\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma) := \text{Ind}_{P^\alpha_G}^{G_n}(\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma).
\]

In this way we obtain a functor \( M_{\text{adm.fin.leng}}(M^\alpha_G) \xrightarrow{\text{Ind}_{P^\alpha_G}^{G_n}} M_{\text{adm.fin.leng}}(G_n) \) and a group homomorphism \( R(M^\alpha_G) \xrightarrow{\text{Jacq}_{\alpha,n}} R(G_n) \). Next, if \( \pi \) is a representation of \( G_n \), then we form the normalized Jacquet module \( \text{Jacq}_{\alpha,n}(\pi) \) of \( \pi \). It is a representation of \( M^\alpha_G \). In this way obtain a functor \( M_{\text{adm.fin.leng}}(G_n) \xrightarrow{\text{Jacq}_{\alpha,n}} M_{\text{adm.fin.leng}}(M^\alpha_G) \) and a group homomorphism \( R(G_n) \xrightarrow{\text{Jacq}_{\alpha,n}} R(M^\alpha_G) \). Here Frobenius reciprocity tells:
\[
\text{Hom}_{G_n}(\pi, \text{Ind}_{P^\alpha_G}^{G_n}(\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma)) \simeq \text{Hom}_{M^\alpha_G}(\text{Jacq}_{\alpha,n}(\pi), \pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma).
\]

Further
\[
\pi_1 \times (\pi_2 \otimes \sigma) \simeq (\pi_1 \times \pi_2) \otimes \sigma, \\
\pi \times \sigma \simeq \pi \otimes \sigma.
\]
We let
\[
\begin{aligned}
R(G) &= \bigoplus_{n \geq 0} R(G_n) \\
\text{Irr}(G) &= \bigoplus_{n \geq 0} \text{Irr}(G_n) \quad \text{(disjoint union)}.
\end{aligned}
\]
If \(\sigma \in \text{Irr}(G)\), then we define \(m_\sigma \in \mathbb{Z}_{\geq 0}\) by \(\sigma \in \text{Irr}(G_{m_\sigma})\). If \(\sigma_1, \sigma_2 \in R(G)\), then we write \(\sigma_1 \geq \sigma_2\) if \(\sigma_1 - \sigma_2\) is a linear combination of irreducible representations with non-negative coefficients.

Following Tadić ([24], Section 3.2), we define \(\mu^* : R(G) \to R(GL) \otimes R(G)\) using the formula
\[
\mu^*(\sigma) = \sum_{k=0}^n \text{Jacq}_{(k,n)}(\sigma), \quad \sigma \in \text{Irr}(G_n).
\]
Also, let \(\kappa : R(GL) \otimes R(GL) \to R(GL) \otimes R(GL)\) be defined by \(\kappa(x \otimes y) = y \otimes x\). We extend the contragredient \(\sim\) to an automorphism of \(R(GL)\) in a natural way. Finally, we let
\[
M^*(\sigma) = (m \otimes \text{id}) \circ (\sim \otimes m^*) \circ \kappa \circ m^* : R(GL) \to R(GL) \otimes R(GL).
\]
We also define the action of \(R(GL) \otimes R(GL)\) on \(R(GL) \otimes R(GL)\) by \((x \otimes y) \times (x_1 \otimes y_1) = x \times x_1 \otimes y \otimes y_1\) on irreducible representations. The following theorem is the cornerstone of all of our computations with Jacquet modules:

**Theorem 1-3.** Let \(\pi \in \text{R(GL)}\) and \(\sigma \in R(G)\). Then we have \(\mu^*(\pi \times \sigma) = M^*(\pi) \times \mu^*(\sigma)\).

In particular, if \(\pi = \langle \Delta \rangle\), \(\Delta = [\nu^{-\alpha}\rho, \nu^\beta \rho]\), where \(\rho\) is a supercuspidal representation, and \(\alpha, \beta \in \mathbb{R}\) such that \(\alpha + \beta \in \mathbb{Z}_{\geq 0}\). Then
\[
M^*\langle \Delta \rangle = \sum_{i=0}^{\alpha + \beta + 1} \sum_{j=0}^{i} \langle [\nu^{-\beta} \rho, \nu^{-i + \alpha} \rho] \times [\nu^{-\alpha} \rho, \nu^{j - \alpha} \rho] \rangle \otimes \langle [\nu^{j - \alpha} \rho, \nu^{j - \alpha - 1} \rho] \rangle.
\]

This follows by a direct computation from the formula (see [27], Proposition 3.4; it is denoted by \(c(\langle \Delta \rangle)\) there):
\[
m^*(\langle \Delta \rangle) = \sum_{k=0}^{\alpha + \beta + 1} \langle [\nu^{-\alpha} \rho, \nu^{k - \alpha} \rho] \rangle \otimes \langle [\nu^{k - \alpha} \rho, \nu^{\beta} \rho] \rangle.
\]

Often we use the following consequence of Theorem 1-3:

**Theorem 1-4.** Let \(\sigma\) be an admissible representation of finite length of \(G_n\). We decompose into irreducible representations (with repetitions possible) \(\mu^*(\sigma) = \sum_{\pi, \sigma_1} \pi \otimes \sigma_1\). Then
\[
\mu^*(\langle \Delta \rangle \times \sigma) = \sum_{i=0}^{\alpha + \beta + 1} \sum_{j=0}^{\alpha} \langle [\nu^{-\beta} \rho, \nu^{-i + \alpha} \rho] \times [\nu^{-\alpha} \rho, \nu^{j - \alpha} \rho] \rangle \times \pi \otimes \langle [\nu^{j - \alpha} \rho, \nu^{j - \alpha - 1} \rho] \rangle \times \sigma_1.
\]

**2. A Result of Waldspurger and its Applications**

In this section we use the description of the contragredient representation of an irreducible representation in \(\text{Irr}(G)\) due to Waldspurger ([14], Chapter 4, II.1), which is based entirely on the results of [6], to prove equalities of semi-simplifications of certain induced representations, and to construct certain intertwining operators. Our approach is motivated by the proof of the commutativity of the multiplication \(\times\) in \(R(GL)\) ([27], Theorem 1.9). First, we prove the following result:
Theorem 2.1. Assume that \( \pi \in \text{Irr}(GL(m, F)) \) and \( \sigma \in \text{Irr}(G_n) \). Then we have the following equality in \( R(G) \):

\[
\pi \times \sigma = \tilde{\pi} \times \sigma.
\]

Proof. We recall the result of Waldspurger mentioned above. First, if \( G_n \) is an orthogonal group, then \( \sigma \simeq \tilde{\sigma} \). On the other hand, if \( G_n = Sp(2n, F) \), then for each element \( \eta \in GSp(2n, F) \) of similitude \(-1\) we have \( \sigma^\eta \simeq \tilde{\sigma} \).

Let us decompose \( \pi \times \sigma = \sum m_i \rho_i \) in \( R(G) \) into irreducible representations. Now, for \( G_n \) orthogonal, we have the following:

\[
\tilde{\pi} \times \sigma = \tilde{\pi} \times \tilde{\sigma} = \tilde{\pi} \times \sigma = \sum m_i \tilde{\rho}_i = \sum m_i \rho_i = \pi \times \sigma,
\]

which proves the claim.

For \( G_n = Sp(2n, F) \), we proceed as follows. We choose an element of the form \( \eta = (id, \eta') \in GL(m, F) \times GSp(2n, F) \), identified with the Levi subgroup of the appropriate maximal parabolic subgroup of \( GSp(2n + 2m, F) \), where \( \eta' \) is an element (with similitude equal to \(-1\)). Thus, we have the following:

\[
(\pi \times \sigma)^\eta = \pi \times \sigma' = \pi \times \tilde{\sigma} = \sum m_i \rho_i^\eta = \sum m_i \tilde{\rho}_i = \tilde{\pi} \times \sigma = \tilde{\pi} \times \tilde{\sigma}.
\]

Now, by interchanging \( \sigma \) with \( \tilde{\sigma} \), we prove the claim. \( \square \)

Remark 2.2. In the case of the group of \( F \)-points of a general Zariski connected reductive group, the analogue of the result follows from \([5\text{, Lemma 5.4 (iii)}]\). But the proof there uses full power of the Langlands classification \([8\text{, Chapter XI, Theorems 2.10 and 2.11}]\) and analytic properties of intertwining operators as well as results on generic irreducibility of induced representations. We use completely (and more elementary) algebraic approach.

We proceed as follows:

Definition 2.3. We say that a sequence of representations in \( \text{Irr}(GL) \) \( (\pi'_1, \ldots, \pi'_k) \) is a \( \sim \)-permutation of the sequence \( (\pi_1, \ldots, \pi_k) \) if the following holds:

(i) \((i_1, \ldots, i'_k)\) is a permutation of \((1, 2, \ldots, k)\) (in particular, \(k = k'\)), and

(ii) \(\forall j \in \{1, \ldots, k\}\) we have \(\pi'_{i_j} \in \{\pi_{i_j}, \pi_{\tilde{i_j}}\}\).

Remark 2.4. If \( \Delta = [\nu^{-\alpha} \rho, \nu^{\beta} \rho] \) is a segment, then we let \( \tilde{\Delta} = [\nu^{-\beta} \tilde{\rho}, \nu^{\alpha} \tilde{\rho}] \). This is again a segment. We have \( \langle \tilde{\Delta} \rangle \simeq \langle \Delta \rangle \). (See \([27\text{, Proposition 3.3}]\). We can make analogous definition of \( \sim \)-permutations of two sequences of segments. Then, the sequence \((\Delta'_1, \Delta'_2, \ldots, \Delta'_k)\) is a \( \sim \)-permutation of the sequence \((\Delta_1, \Delta_2, \ldots, \Delta_k)\) if and only if the sequence \((\langle \Delta'_1 \rangle, \langle \Delta'_2 \rangle, \ldots, \langle \Delta'_k \rangle)\) is a \( \sim \)-permutation of the sequence \((\langle \Delta_1 \rangle, \langle \Delta_2 \rangle, \ldots, \langle \Delta_k \rangle)\).

The following is a direct consequence of Theorem 2.1:

Corollary 2.5. For every \( \sim \)-permutation \((\pi'_1, \ldots, \pi'_k)\) of \((\pi_1, \ldots, \pi_k)\) and \( \sigma \in \text{Irr}(G) \), we have the following equality of semisimplifications in \( R(G) \):

\[
\pi'_{i_1} \times \pi'_{i_2} \times \cdots \times \pi'_{i_k} \times \sigma = \pi_1 \times \pi_2 \times \cdots \times \pi_k \times \sigma.
\]
Proof. It is obvious that equality in Theorem 2-1 can be extended directly to all \( \sigma \in R(G) \). Now, the claim follows from it by induction on \( k \) and commutativity of \( \times \). \qed

Now, we can prove the existence of certain intertwining operators between the induced representations in a purely algebraic fashion. In fact, it seems that such result is not known in general but it should also follow from the analytic theory of the intertwining operators ([26], [22]).

**Theorem 2-6.** Assume that \( \pi_i \in \text{Irr}(\text{GL}(m_i, F)) \) \( (i = 1, \ldots, k) \), and \( \sigma \in \text{Irr}(G_n) \). Let \( m = m_1 + \cdots + m_k \) and \( l = m + n \). Then the following holds:

(i) Every irreducible quotient of \( \pi_1 \times \pi_2 \times \cdots \times \pi_k \) is an irreducible subrepresentation of \( \pi_k \times \pi_{k-1} \times \cdots \times \pi_1 \). In particular, \( \text{Hom}_{\text{GL}(m, F)}(\pi_1 \times \pi_2 \times \cdots \times \pi_k, \pi_k \times \pi_{k-1} \times \cdots \times \pi_1) \neq 0 \)

(ii) Every irreducible quotient of \( \pi_1 \times \pi_2 \times \cdots \times \pi_k \times \sigma \) is an irreducible subrepresentation of \( \tilde{\pi}_1 \times \tilde{\pi}_2 \times \cdots \times \tilde{\pi}_k \times \sigma \). In particular, \( \text{Hom}_{G_l}(\pi_1 \times \pi_2 \times \cdots \times \pi_k \times \sigma, \tilde{\pi}_1 \times \tilde{\pi}_2 \times \cdots \times \tilde{\pi}_k \times \sigma) \neq 0 \).

Proof. Firstly, we prove (i). Let \( \pi \) denote an irreducible quotient of the representation \( \pi_1 \times \pi_2 \times \cdots \times \pi_k \), that is, there is an epimorphism

\[
\pi_1 \times \pi_2 \times \cdots \times \pi_k \twoheadrightarrow \pi.
\]

Consequently, we have \( \tilde{\pi} \hookrightarrow \tilde{\pi}_1 \times \tilde{\pi}_2 \times \cdots \times \tilde{\pi}_k \). Then, after applying the automorphism \( s : \text{GL}(m, F) \to \text{GL}(m, F) \) (see [27], the proof of Theorem 1.9) we obtain

\[
\pi \simeq s(\tilde{\pi}) \hookrightarrow s(\tilde{\pi}_1 \times \tilde{\pi}_2 \times \cdots \times \tilde{\pi}_k) \simeq s(\tilde{\pi}_k) \times s(\tilde{\pi}_{k-1}) \times \cdots \times s(\tilde{\pi}_1) \simeq \pi_k \times \pi_{k-1} \times \cdots \times \pi_1.
\]

Since \( \pi \), simultaneously, it is a quotient of \( \pi_1 \times \pi_2 \times \cdots \times \pi_k \) and a subrepresentation of \( \pi_k \times \pi_{k-1} \times \cdots \times \pi_1 \), we immediately prove the claim. Now, we prove claim (ii) in a fashion similar to the proof of (i). Again, let \( \tau \) be an irreducible quotient of the representation \( \pi_1 \times \pi_2 \times \cdots \times \pi_k \times \sigma \) and then we have \( \tilde{\tau} \hookrightarrow \tilde{\pi}_1 \times \tilde{\pi}_2 \times \cdots \times \tilde{\pi}_k \times \tilde{\sigma} \). Again, we use the result of Waldspurger explained in the proof of Theorem 2-1. (See also the notation introduced there.)

If \( G_n \) is orthogonal, then since \( \tau \simeq \tilde{\tau} \) and \( \sigma \simeq \tilde{\sigma} \) we have \( \tau \hookrightarrow \tilde{\pi}_1 \times \tilde{\pi}_2 \times \cdots \times \tilde{\pi}_k \times \tilde{\sigma} \). Again, we use the result of Waldspurger explained in the proof of Theorem 2-1. (See also the notation introduced there.)

If \( G_n \) is symplectic, then we have

\[
\tilde{\tau}' \hookrightarrow \tilde{\pi}_1 \times \tilde{\pi}_2 \times \cdots \times \tilde{\pi}_k \times \tilde{\sigma}'.
\]

This means

\[
\tau \hookrightarrow \tilde{\pi}_1 \times \tilde{\pi}_2 \times \cdots \times \tilde{\pi}_k \times \sigma.
\]

This completes the proof of (ii). \qed

3. Principle of Maximality

The main result of this section is the cornerstone of our approach to the classification of irreducible representations.

We start by reviewing some results of [7]. (See [7], Theorems 2.5 and 2.9 for (Zariski) connected \( G_n \). If \( G_n \) is an orthogonal group, then one needs a simple Mackey style argument to obtain the result ([13], Section 16).)

**Lemma 3-1.** Let \( \sigma \in \text{Irr}(G_n) \). Then there exists a standard parabolic subgroup \( P = MN \subset G_n \), where

\[
M \simeq \text{GL}(m_1, F) \times \cdots \times \text{GL}(m_k, F) \times G_{n'}, \quad n = m_1 + \cdots + m_k + n',
\]
and a supercuspidal representation
\[ \rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma' \in \text{Irr}(M) \]
such that
\[ (3-2) \quad \sigma \leftarrow \rho_1 \times \cdots \times \rho_k \times \sigma'. \]
Moreover, if \( \rho_1, \ldots, \rho_i \in \text{Irr}(GL) \) and \( \sigma'' \in \text{Irr}(G) \) are supercuspidal representations such that \( \sigma \leftarrow \rho_1 \times \cdots \times \rho_i \times \sigma'' \), then \( (\rho_1^3, \ldots, \rho_i^3) \) is a \( \sim \)-permutation of \( (\rho_1, \ldots, \rho_k) \) and \( \sigma'' \simeq \sigma' \).

Now, we state and prove the main result of this section.

**Theorem 3-3.** Let \( \sigma \in \text{Irr}(G_n) \). Let us fix some embedding of the form (3-2) where we relax the requirement that \( \sigma' \) is supercuspidal. Consider all possible embeddings of the following form:
\[ (3-4) \quad \sigma \leftarrow \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_l \rangle \times \sigma', \]
where we have the following equality of the multisets:
\[ (3-5) \quad \Delta_1 + \cdots + \Delta_l = \{ \rho_1, \rho_2, \ldots, \rho_k \}. \]
(Such embedding clearly exist; for example we can take \( l = k \), \( \Delta_1 = \{ \rho_1 \}, \ldots, \Delta_l = \{ \rho_l \} \).) To the embedding (3-4) we attach an \( n - n' \)-tuple in \( \mathbb{R}^{n-n'} \) as follows:
\[ (3-6) \quad \{ e(\Delta_1), \ldots, e(\Delta_1), e(\Delta_2), \ldots, e(\Delta_2), \ldots, e(\Delta_l), \ldots, e(\Delta_l) \}, \]
here \( e(\Delta_i) \) appears exactly \( M_i \) times where \( M_i \) is defined by \( \langle \Delta_i \rangle \in \text{Irr}(GL(M_i, F)) \). Clearly, the set of all embeddings (3-4) is finite. Then, if (3-4) is such that (3-6) is maximal with respect to the lexicographic ordering on \( \mathbb{R}^{n-n'} \), then
\[ (3-7) \quad e(\Delta_1) \geq e(\Delta_2) \geq \ldots \geq e(\Delta_l). \]

**Proof.** Indeed, assume that for some \( 2 \leq i_0 \leq l \) we have \( e(\Delta_{i_0-1}) < e(\Delta_{i_0}) \). Then there is a non-zero intertwining operator \( \langle \Delta_{i_0-1} \rangle \times \langle \Delta_{i_0} \rangle \rightarrow \langle \Delta_{i_0} \rangle \times \langle \Delta_{i_0-1} \rangle \) (see Theorem 2-6 (i)) which is in fact unique up to a non-zero scalar multiple (see [27], Proposition 4.6). Therefore, by induction in stages we obtain the following sequence of maps (see (3-4)):
\[
\begin{align*}
\sigma & \leftarrow \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_{i_0-1} \rangle \times \langle \Delta_{i_0} \rangle \times \cdots \times \langle \Delta_l \rangle \times \sigma' \\
& \leftarrow \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_{i_0-1} \rangle \times \langle \Delta_{i_0} \rangle \times \cdots \times \langle \Delta_l \rangle \times \sigma'.
\end{align*}
\]
If the composition of those two maps is non-zero, we obtain an embedding
\[ \sigma \leftarrow \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_{i_0} \rangle \times \langle \Delta_{i_0-1} \rangle \times \cdots \times \langle \Delta_l \rangle \times \sigma' \]
such that its \( n - n' \)-tuple is larger than (3-6). This is a contradiction since \( e(\Delta_{i_0-1}) < e(\Delta_{i_0}) \).
Therefore, the composition of the maps must be zero. This forces that \( \sigma \) is in the kernel which is of the form
\[ (3-8) \quad \sigma \leftarrow \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_{i_0} \rangle \times \pi \times \langle \Delta_{i_0+1} \rangle \times \cdots \times \langle \Delta_l \rangle \times \sigma', \]
where \( \pi \) is the kernel of \( \langle \Delta_{i_0-1} \rangle \times \langle \Delta_{i_0} \rangle \rightarrow \langle \Delta_{i_0} \rangle \times \langle \Delta_{i_0-1} \rangle \).

To describe the kernel \( \pi \), we use again our assumption \( e(\Delta_{i_0-1}) < e(\Delta_{i_0}) \). Let \( \Delta_{i_0-1} = [\nu^{-\alpha_{i_0-1}} \rho_{i_0-1}, \nu^{\beta_{i_0-1}} \rho_{i_0-1}] \) and \( \Delta_{i_0} = [\nu^{-\alpha_{i_0}} \rho_{i_0}, \nu^{\beta_{i_0}} \rho_{i_0}] \), where \( \rho_{i_0-1} \) and \( \rho_{i_0} \) are unitarizable supercuspidal representations, and real numbers \( \alpha_{i_0-1}, \alpha_i, \beta_{i_0-1}, \beta_i \) such that \( \alpha_{i_0-1} + \beta_{i_0-1}, \alpha_i + \beta_i \in \mathbb{Z}_{\geq 0} \). The existence of a non-zero kernel forces
\[ (3-9) \quad \text{segments } \Delta_{i_0-1} \text{ and } \Delta_{i_0} \text{ are linked (see Theorem 1-1).} \]
Since these segments are linked, $\rho_{i_0-1} \simeq \rho_{i_0}$ and $\alpha_{i_0-1} - \alpha_{i_0} \in \mathbb{Z}$. Next, we have the following:

$$\frac{-\alpha_{i_0-1} + \beta_{i_0-1}}{2} = e(\Delta_{i_0-1}) < e(\Delta_{i_0}) = \frac{-\alpha_{i_0} + \beta_{i_0}}{2}$$

This and (3-9) implies that $\beta_{i_0} > \beta_{i_0-1}$ and $\alpha_{i_0-1} > \alpha_{i_0}$. Now, we have the following:

$$\begin{align*}
\left\{ \frac{-\alpha_{i_0-1} + \beta_{i_0-1}}{2} < \frac{-\alpha_{i_0-1} + \beta_{i_0}}{2} = e((\nu^{-\alpha_{i_0-1}} \rho_{i_0}, \nu^{\beta_{i_0}} \rho_{i_0})) \right\} \\
\left\{ \frac{-\alpha_{i_0-1} + \beta_{i_0-1}}{2} < \frac{-\alpha_{i_0-1} + \beta_{i_0}}{2} = e((\nu^{-\alpha_{i_0}} \rho_{i_0}, \nu^{\beta_{i_0-1}} \rho_{i_0})) \right\}
\end{align*}$$

(3-10)

Now, ([27], Proposition 4.6) implies that the kernel $\pi$ is of the form

$$\pi \simeq \langle [\nu^{-\alpha_{i_0-1}} \rho_{i_0}, \nu^{\beta_{i_0}} \rho_{i_0}] \rangle \times \langle [\nu^{-\alpha_{i_0}} \rho_{i_0}, \nu^{\beta_{i_0-1}} \rho_{i_0}] \rangle.$$

Combining this with (3-8), we obtain

$$\sigma \leftarrow \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_{i_0-2} \rangle \times \langle [\nu^{-\alpha_{i_0-1}} \rho_{i_0}, \nu^{\beta_{i_0}} \rho_{i_0}] \rangle \times \langle [\nu^{-\alpha_{i_0}} \rho_{i_0}, \nu^{\beta_{i_0-1}} \rho_{i_0}] \rangle \times \langle \Delta_{i_0+1} \rangle \times \cdots \times \langle \Delta_l \rangle \times \sigma'. $$

Since (3-10) holds, this embedding contradicts the choice of (3-4). \qed

**Remark 3-11.** Taking (3-4) such that (3-6) is a minimum, we would obtain in (3-7) reverse inequality:

$$e(\Delta_1) \leq e(\Delta_2) \leq \ldots \leq e(\Delta_l).$$

**Remark 3-12.** There are variations of our principle with exactly the same proof as the one above. We mention the following two:

(i) Relax the requirement that $\rho_i$ are supercuspidal and replace (3-5) by the following:

$$\Delta_1 + \cdots + \Delta_l = \text{supp}(\rho_1) + \text{supp}(\rho_2) + \ldots + \text{supp}(\rho_k).$$

Here, for $\pi \in \text{Irr(GL)}$, supp($\pi$) denotes the unique multiset of supercuspidal representations such that $\pi$ is an irreducible subquotient of $\times_{\rho \in \text{supp}(\pi)} \rho$. (See [27], Proposition 1.10.)

(ii) Let $\epsilon \in \mathbb{R}$ be given. Assume that we have an embedding of the form $\sigma \leftarrow \langle \Psi_1 \rangle \times \cdots \times \langle \Psi_k \rangle \times \sigma'$, where $\Psi_i$ are some segments such that

$$e(\Psi_i) > \epsilon, \ i = 1, \ldots, k, $$

and $\sigma'$ is an irreducible representation. Then we look at the family of the embeddings (3-4) satisfying

$$\left\{ \begin{array}{l}
\Delta_1 + \cdots + \Delta_l = \text{supp}(\Psi_1) + \text{supp}(\Psi_2) + \ldots + \text{supp}(\Psi_k) \\
e(\Delta_i) > \epsilon, \ i = 1, \ldots, l
\end{array} \right.$$

instead of (3-5). The tuple attached to the maximal embedding satisfies the following:

$$e(\Delta_1) \geq e(\Delta_2) \geq \ldots \geq e(\Delta_l) > \epsilon.$$

instead of (3-7).
4. Zelevinsky Type Classification for Classical Groups

In this section we state the theorems that give the classification of irreducible representations in \( \text{Irr}(G) \). This is a sort of the classification that is dual to the usual Langlands classification ([8], Chapter XI, Theorems 2.10 and 2.11) in the sense of [1]. It is completely analogous to the Zelevinsky classification (see Theorem 1-1).

The classification is given in terms of strongly negative representations generalizing the approach adopted in ([17]). We start by an appropriate definition.

**Definition 4-1.** Let \( \sigma \in \text{Irr}(G_n) \). Then \( \sigma \) is a strongly negative (resp., negative) representation if and only if for every embedding \( \sigma \hookrightarrow \rho_1 \times \rho_2 \times \cdots \rho_t \times \sigma_{sc} \), where \( \rho_i, i = 1, \ldots t \) and \( \sigma_{sc} \) are irreducible supercuspidal representations, we have the following:

\[
e(\rho_1)m_{\rho_1} < 0 \quad (\text{resp.,} \quad \leq 0)
\]
\[
e(\rho_1)m_{\rho_1} + e(\rho_2)m_{\rho_2} < 0 \quad (\text{resp.,} \quad \leq 0)
\]
\[
\vdots
\]
\[
e(\rho_1)m_{\rho_1} + e(\rho_2)m_{\rho_2} + \cdots + e(\rho_t)m_{\rho_t} < 0 \quad (\text{resp.,} \quad \leq 0).
\]

The existence of at least one embedding into the representation induced by supercuspidal representations is guaranteed by Lemma 3-1. The duality of [1] takes strongly negative representations (resp., negative) to discrete series (resp., tempered) representations. Tempered representations (and discrete series) play an important role in the harmonic analysis on a reductive group (see [26]). Their asymptotic properties guarantee that they are unitary and this is used in their investigation. In part, our investigation of negative representations parallels that of tempered representations but it is completely algebraic and the proofs are simpler than the usual for tempered representations (see for example [26], Section III). We do not have the unitarity at our disposal. The unitarity of negative representations is much more subtle than that of tempered representations ([9], [12], [19]).

**Example 4-2.** The trivial representation \( 1_{G_n} \) is an example of a strongly negative representation (see [17]). The paper [17] contains more interesting examples of strongly negative representations which are unramified. All of them can be obtained applying the duality [1] and the results of [13].

Now, we describe the negative representations in terms of the strongly negative representations.

**Theorem 4-3.** Let \( \sigma \in \text{Irr}(G_n) \) be a negative representation. Then there exists a sequence of segments \( \Delta_1, \Delta_2, \ldots \Delta_k \) such that \( e(\Delta_i) = 0, \ i = 1, \ldots k \) and an irreducible strongly negative representation \( \sigma_{sn} \) such that

\[
\sigma \hookrightarrow (\Delta_1) \times (\Delta_2) \times \cdots \times (\Delta_k) \times \sigma_{sn}.
\]

(We allow empty sequence of segments here. Then \( k = 0 \).) If, in addition, \( \sigma \hookrightarrow (\Delta'_1) \times (\Delta'_2) \times \cdots \times (\Delta'_{k'}) \times \sigma'_{sn} \) for some other sequence of segments \( \Delta'_1, \Delta'_2, \ldots \Delta'_{k'} \) such that \( e(\Delta'_i) = 0, \ i = 1, \ldots k' \), and for some strongly negative representation \( \sigma'_{sn} \), then the sequence \( (\Delta'_1, \Delta'_2, \ldots, \Delta'_{k'}) \) is a \( \sim \)-permutation of the sequence \( (\Delta_1, \Delta_2, \ldots, \Delta_k) \) and \( \sigma'_{sn} \simeq \sigma_{sn} \).
Theorem 4-3 does not tell us how to build negative representations from strongly negative for we do not know if all irreducible subrepresentations of \( \langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{sn} \) are negative. We take care of this problem in the next theorem:

**Theorem 4-4.** Assume that \( \Delta_1, \Delta_2, \ldots, \Delta_k \) are segments such that \( e(\Delta_i) = 0, \ i = 1, \ldots k \) and \( \sigma_{sn} \) is an irreducible strongly negative representation. Then all irreducible subquotients of the representation \( \langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{sn} \) are negative.

Finally, we present the classification of all other representations in terms of negative representations.

**Theorem 4-5.** Suppose that \( \Delta_1, \ldots, \Delta_k \) is a sequence of segments satisfying \( e(\Delta_1) \geq \ldots \geq e(\Delta_k) > 0 \). (We allow empty sequence here; in this case \( k = 0 \).) Let \( \sigma_{neg} \) be a negative representation. Then we have the following:

(i) The induced representation \( \langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg} \) has a unique irreducible subrepresentation (we call it Zelevinsky subrepresentation); we will denote it by \( \langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle \).

(ii) We have \( \langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle \hookrightarrow \langle \Delta_1 \rangle \times \langle \Delta_2, \ldots, \Delta_k; \sigma_{neg} \rangle \).

(iii) The representation \( \langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle \) appears with the multiplicity one in the composition series of \( \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_k \rangle \rtimes \sigma_{neg} \).

(iv) The representation \( \langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle \) is negative if and only if \( k = 0 \); in that case \( \langle \emptyset; \sigma_{neg} \rangle \simeq \sigma_{neg} \).

(v) The induced representation \( \langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg} \) has a unique maximal proper subrepresentation; the corresponding quotient is \( \langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle \).

(vi) \( \langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle \simeq \langle \Delta'_1, \ldots, \Delta'_k; \sigma'_{neg} \rangle \) if and only if \( \langle \Delta_1, \ldots, \Delta_k \rangle \) is a permutation of \( \langle \Delta'_1, \ldots, \Delta'_k \rangle \) and \( \sigma_{neg} \simeq \sigma'_{neg} \).

Finally, we have the following theorem which ends the classification of irreducible representations in terms of strongly negative representations:

**Theorem 4-6.** If \( \sigma \in \text{Irr}(G_n) \), then there exists a sequence of segments \( \Delta_1, \ldots, \Delta_k \) satisfying \( e(\Delta_1) \geq \ldots \geq e(\Delta_k) > 0 \) and a negative representation \( \sigma_{neg} \) such that \( \sigma \simeq \langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle \).

The existence of classifying data, that is, the first part of Theorem 4-3 and Theorem 4-6, is an immediate consequence of our principle of maximality (see Theorem 3-3). It is given in Section 5. In Section 6 we prove the uniqueness of classifying data (see the second part of Theorem 4-3). Theorem 4-5 is proved in Section 7. Theorem 4-4 is proved in Section 8 as a consequence of Theorems 4-3 and 4-6.

5. **Existence of Classifying Data**

In this section we prove the existence of the classifying data in Theorems 4-3 and 4-6 using our principle of maximality. We start by Theorem 4-3.

Let \( \sigma \in \text{Irr}(G_n) \) be negative representation. If \( \sigma \) is strongly negative, then we need to take \( k = 0 \), an empty sequence of segments, and \( \sigma_{sn} = \sigma \). Thus, we have proved the existence of the data in this case. So, assume that \( \sigma \) is negative, but not strongly negative. Then, by definiton (see Definition...
4-1), we can find irreducible supercuspidal representations $\rho_i$, $i = 1, \ldots, t$ and $\sigma_{sc}$ such that
\begin{equation}
\sigma \hookrightarrow \rho_1 \times \rho_2 \times \cdots \times \rho_t \times \sigma_{sc}
\end{equation}
and, for some $1 \leq k \leq t$, we have
\begin{equation}
\sum_{i=1}^{k} m_{\rho_i} e(\rho_i) = 0.
\end{equation}
Clearly (5-1) and induction in stages implies that there exists an irreducible subquotient $\sigma'$ of $\rho_k \times \cdots \times \rho_1 \times \sigma_{sc}$ such that
\begin{equation}
\sigma \hookrightarrow \rho_1 \times \rho_2 \times \cdots \times \rho_k \times \sigma'.
\end{equation}
Now, we apply our maximality principle (see Theorem 3-3). We obtain an embedding
\begin{equation}
\sigma \hookrightarrow (\Delta_1) \times \cdots \times (\Delta_l) \times \sigma'
\end{equation}
such that
\begin{equation}
\left\{ \Delta_1 + \cdots + \Delta_l = \{ \rho_1, \rho_2, \ldots, \rho_k \} \right. \\
e(\Delta_1) \geq e(\Delta_2) \geq \ldots \geq e(\Delta_l).
\end{equation}
We introduce real numbers $\alpha_i$, $\beta_i$, $i = 1, \ldots, l$, and unitarizable supercuspidal representations $\rho'_i$, $i = 1, \ldots, l$ such that $\alpha_i + \beta_i \in \mathbb{Z}_{\geq 0}$ and $\Delta_i = [\nu^{-\alpha_i} \rho'_i, \nu^{\beta_i} \rho'_i]$.

Let $M = \max\{m_{\rho_i} \cdot (\beta_i + \alpha_i + 1) | i = 1, \ldots, l\}$. Clearly, $M \in \mathbb{Z}_{\geq 0}$. Now, we compute using (5-4)
\begin{equation}
0 = \sum_{i=1}^{k} m_{\rho_i} e(\rho_i) = \sum_{i=1}^{l} \sum_{\rho \in \Delta_i} m_{\rho} e(\rho) = \\
\sum_{i=1}^{l} \sum_{j=-\alpha_i}^{\beta_i} j \cdot m_{\rho'_i} = \sum_{i=1}^{l} m_{\rho'_i} \cdot (\beta_i + \alpha_i + 1) \frac{-\alpha_i + \beta_i}{2} = \\
\sum_{i=1}^{l} m_{\rho'_i} \cdot (\beta_i + \alpha_i + 1) \cdot e(\Delta_i) \leq M \cdot \sum_{i=1}^{l} e(\Delta_i) \leq l \cdot M \cdot e(\Delta_1).
\end{equation}
On the other hand, (5-3), $\Delta_i \hookrightarrow \nu^{-\alpha_i} \rho'_i \times \cdots \times \nu^{\beta_i} \rho'_i$ (see Theorem 1-1), and the induction in stages implies that
\begin{equation}
\sigma \hookrightarrow \nu^{-\alpha_1} \rho'_1 \times \cdots \times \nu^{\beta_1} \rho'_1 \times \cdots \times \sigma_{sc}.
\end{equation}
Hence, the definition of negativity, implies that
\begin{equation}
0 \geq \sum_{j=-\alpha_1}^{\beta_1} j \cdot m_{\rho'_1} = m_{\rho'_1} \cdot (\beta_1 + \alpha_1 + 1) \cdot e(\Delta_1).
\end{equation}
Hence, (5-5) implies that $e(\Delta_1) = 0$. Now, we again exploit (5-3). By induction in stages, we can find an irreducible subquotient $\sigma_1$ of $\langle \Delta_2 \rangle \times \cdots \times \langle \Delta_l \rangle \times \sigma'$ such that
\begin{equation}
\sigma \hookrightarrow \langle \Delta_1 \rangle \times \sigma_1.
\end{equation}
We claim that $\sigma_1$ is negative. This enables us to iterate\footnote{Here and further, we use freely the expression "iterate" to mean the proof given by (the obvious) induction. In this way we stress the constructive nature of our proofs and minimize the notation.} the construction i.e. repeat the same procedure with $\sigma_1$ in place of $\sigma$. Clearly, after finitely many steps we will obtain a strongly negative representation where the procedure stops. (See the beginning of this proof.)
In order to prove that $\sigma_1$ is negative, we take any embedding $\sigma_1 \hookrightarrow \rho''_1 \times \cdots \times \rho''_{t''} \rtimes \sigma_{sc}$, where all appearing representations are supercuspidal. Then, by induction in stages, we obtain $\sigma \hookrightarrow \nu^{-\alpha_1}_1 \rho'_1 \times \cdots \times \nu^{\beta_1}_1 \rho'_1 \times \rho^n_1 \times \rho''_2 \times \cdots \rho''_t \rtimes \sigma_{sc}$.

Now, the negativity of $\sigma$ and $e(\Delta_1) = 0$ imply
\[
\sum_{i=1}^{s} m_{\rho''_i} e(\rho''_i) = \sum_{j=-\alpha_1}^{\alpha_1} m_{\rho'_i} \cdot j + \sum_{i=1}^{s} m_{\rho''_i} e(\rho''_i) \leq 0, \text{ for all } s = 1, \ldots, t''.
\]
Hence $\sigma_1$ is negative. This completes the proof of the first part of Theorem 4-3.

Now, we prove Theorem 4-6 (assuming Theorem 4-5 which will be proved in Section 7). The proof is very similarly to the previous one. So, we will be brief.

First, if $\sigma$ is negative, we can take $\sigma_{neg} = \sigma$, $k = 0$, and an empty sequence of segments. Assume that $\sigma$ is not negative. Then, by definition, there exists an embedding $\sigma \hookrightarrow \rho_1 \times \rho_2 \times \cdots \times \rho_t \rtimes \sigma_{sc}$, where $\rho_i$ and $\sigma_{sc}$ are supercuspidal, and $k \in \{1, 2, \ldots, t\}$ such that
\[
\sum_{i=1}^{j} m_{\rho_i} e(\rho_i) > 0.
\]
Using this instead of (5-2), as in the proof above we can argue to conclude that there exists a segment $\Delta_1$ such that $e(\Delta_1) > 0$ and an irreducible representation $\sigma_1$ such that $\sigma \hookrightarrow \langle \Delta_1 \rangle \times \sigma_1$.

Again, we can iterate this construction until we obtain a negative representation. This results in an embedding $\sigma \hookrightarrow \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_l \rangle \times \sigma_{neg}$, for some negative representation $\sigma_{neg}$, where
\[
e(\Delta_i) > 0, \text{ for all } i = 1, \ldots, l.
\]
If, in addition, we have the following:
\[
e(\Delta_1) \geq e(\Delta_2) \geq \cdots \geq e(\Delta_l)
\]
we are done. If not, we apply Remark 3-12 (ii) with $\epsilon = 0$ to this embedding. This completes the proof of Theorem 4-6 (assuming Theorem 4-5).

6. **Uniqueness of Classifying Data for Negative Representations**

In this section we complete the proof of Theorem 4-3. Assume that $\sigma$ is a negative representation such that
\[
\sigma \hookrightarrow \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_s \rangle \times \sigma_{sn}\]
\[
\sigma \hookrightarrow \langle \Delta'_1 \rangle \times \cdots \times \langle \Delta'_l \rangle \times \sigma'_{sn},
\]
where $e(\Delta_i) = 0$, $e(\Delta'_j) = 0$, for all $i, j$, and $\sigma_{sn}$, $\sigma'_{sn}$ are strongly negative. Then we need to prove the following:
\[
(\Delta_1, \ldots, \Delta_s) \text{ is a } \sim \text{-permutation of } (\Delta'_1, \ldots, \Delta'_l) \text{ and } \sigma'_{sn} \simeq \sigma_{sn}.
\]
We will use Jacquet modules through Theorem 1-4. The remainder of this section is devoted to the proof of (6-2).
We introduce real numbers $l_i$, $l_i'$, and unitarizable supercuspidal representations $\rho_i$ and $\rho_i'$ such that $\Delta_i = [\nu^{-l_i} \rho_i, \nu^{l_i'} \rho_i]$ and $\Delta_i' = [\nu^{-l_i} \rho_j', \nu^{l_i'} \rho_j']$, where $2l_i \in \mathbb{Z}_{\geq 0}$, $i = 1, \ldots, s$, $2l_j' \in \mathbb{Z}_{\geq 0}$, $j = 1, \ldots, t$.

First, we note that the segments $\Delta_i$ and $\Delta_j$ (resp., $\Delta_i'$ and $\Delta_j'$) are not linked. Therefore, the representations $\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_s \rangle$ and $\langle \Delta_1' \rangle \times \cdots \times \langle \Delta_t' \rangle$ are irreducible. (See Theorem 1-1.) Because of that, we can arrange

\begin{equation}
\begin{cases}
l_1 \geq l_2 \geq \cdots \geq l_s \\
l_1' \geq l_2' \geq \cdots \geq l_t'.
\end{cases}
\end{equation}

Now, applying Frobenius reciprocity to (6-1), we obtain (see the notation introduced before Theorem 1-4)

\begin{equation}
\mu^* (\sigma) \geq \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_s \rangle \otimes \sigma_{sn}
\end{equation}

\begin{equation}
\mu^* (\sigma) \geq \langle \Delta_1' \rangle \times \cdots \times \langle \Delta_t' \rangle \otimes \sigma'_{sn}.
\end{equation}

Next, since $\sigma$ is an irreducible subquotient of $\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_s \rangle \times \sigma_{sn}$, the second formula in (6-4) implies

\begin{equation}
\mu^* (\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_s \rangle \times \sigma_{sn}) \geq \langle \Delta_1' \rangle \times \cdots \times \langle \Delta_t' \rangle \otimes \sigma'_{sn}.
\end{equation}

Applying Theorem 1-4 several times, we obtain that there are indices $0 \leq \beta_i \leq \alpha_i \leq 2l_i + 1$, $i = 1, \ldots, s$, and an irreducible constituent $\delta_1 \otimes \sigma_1 \leq \mu^* (\sigma_{sn})$ such that

\begin{equation}
\prod_{i=1}^s \langle [\nu^{-l_i} \tilde{\rho}_i, \nu^{-\alpha_i+l_i} \tilde{\rho}_i] \times [\nu^{-l_i} \rho_i, \nu^{\beta_i-l_i-1} \rho_i] \rangle \times \delta_1 \geq \langle \Delta_1' \rangle \times \cdots \times \langle \Delta_t' \rangle
\end{equation}

\begin{equation}
\prod_{i=1}^s \langle [\nu^{\beta_i-l_i} \rho_i, \nu^{\alpha_i-l_i-1} \rho_i] \rangle \times \sigma_1 \geq \sigma'_{sn}.
\end{equation}

The main object is to construct two injective mappings, say, $\phi$ and $\psi$ such that $\phi : \{1, 2, \ldots, t\} \rightarrow \{1, 2, \ldots, s\}$ and $\psi : \{1, 2, \ldots, s\} \rightarrow \{1, 2, \ldots, t\}$ having property $\Delta_i' \in \{\Delta_{\phi(i)}, \Delta_{\psi(i)}\}$, $1 \leq i \leq t$, and $\Delta_i \in \{\Delta_{\phi(i)}, \Delta_{\psi(i)}\}$, $1 \leq i \leq s$.

First, we will construct $\phi$ from the first inequality in (6-6). Let us write $\delta_1$ in the Zelevinsky classification (see Theorem 1-1) as follows $\delta_1 \simeq \langle \Delta_1'', \ldots, \Delta_t'' \rangle$. Then the first condition in (6-6) and Lemma 1-2 implies

\begin{equation}
\delta_1 \simeq \langle \Delta_1'' \rangle \times \cdots \times \langle \Delta_t'' \rangle.
\end{equation}

Next, comparing the supercuspidal supports of both sides in the first inequality in (6-6), we obtain the following equality of multisets:

\begin{equation}
\sum_{i=1}^s \left( [\nu^{-l_i} \tilde{\rho}_i, \nu^{-\alpha_i+l_i} \tilde{\rho}_i] + [\nu^{-l_i} \rho_i, \nu^{\beta_i-l_i-1} \rho_i] \right) + \sum_{k=1}^t \Delta_k' = \sum_{j=1}^t \Delta_j'.
\end{equation}

Supercuspidal representation $\nu^{l_i'} \rho_i'$, which is the endpoint of $\Delta_i'$, must appear also on the left-hand side of the above relation. It can happen in the following three ways:

- Assume $\nu^{l_i'} \rho_i' \in [\nu^{-l_i} \tilde{\rho}_i, \nu^{-\alpha_i+l_i} \tilde{\rho}_i]$ for some $i$. Then, we conclude $\rho_i' \simeq \tilde{\rho}_i$. We remark that $l_i'$ (resp., $-l_i'$) is the largest (resp., smallest) exponent on the right-hand side of (6-8) (see (6-3)). This fact has several consequences that we present now. First, $\nu^{l_i'} \rho_i'$ must be the endpoint of $[\nu^{-l_i} \tilde{\rho}_i, \nu^{-\alpha_i+l_i} \tilde{\rho}_i]$. In particular, we have $l_i' = -\alpha_i + l_i$. Hence, $l_i' = -\alpha_i + l_i \leq l_i$.

Next, the segment $[\nu^{-l_i} \tilde{\rho}_i, \nu^{-\alpha_i+l_i} \tilde{\rho}_i]$ is non-empty, and this forces $-l_i$ to appear as an
exponent on the right-hand side of (6-8). This implies \(-l_i \geq -l'_i\). This means \(l'_i \geq l_i\). Since, we also have \(l_i \geq l'_i\), we obtain \(l'_i = l_i\). Since \(l'_i = -\alpha_i + l_i\) and \(0 \leq \beta_i \leq \alpha_i\), we have \(\alpha_i = \beta_i = 0\), and in the sum on the left-hand side of (6-8) \([\nu^{-l'_i} \rho_i, \nu^{-\alpha_i+l'_i} \rho_i] + [\nu^{-l_i} \rho_i, \nu^{\beta_i-l_i-1} \rho_i]\) the first segment is equal to \([\nu^{-l'_i} \rho_i, \nu^{l'_i} \rho_i] = [\nu^{-l_i} \rho_i, \nu^{l_i} \rho_i]\), while the other one is empty.

We let \(\phi(1) = i\) and let \(\Delta'_i = \Delta_{\phi(i)}\), and we can cancel these segments from the relation (6-8) to obtain new equality of multisets

\[(6-9) \quad \sum_{r=1, r \neq \phi(1)}^{s} [\nu^{-l'_r} \rho_r, \nu^{-\alpha_r+l'_r} \rho_r] + [\nu^{-l_r} \rho_r, \nu^{\beta_r-l_r-1} \rho_r] + \sum_{k=1}^{l} \Delta''_{k} = \sum_{j=2}^{s} \Delta'_j.\]

- Assume \(\nu^{\alpha'_i} \rho'_i \in [\nu^{-l_i} \rho_i, \nu^{\beta_i-l_i-1} \rho_i]\) for some \(i\). The discussion is analogous to the previous case. We record just the end result. We must have \(\alpha_i = \beta_i = 2l_i + 1\). We may let \(\phi(1) = i\) and \(\Delta'_i = \Delta_{\phi(i)}\). The equality (6-9) holds.

- Assume \(\nu^{\alpha'_i} \rho'_i \in \Delta''_k\), for some \(k\). Then there exist real numbers \(\alpha\) and \(\beta\) in such a way that \(\Delta'_k = [\nu^{-\alpha} \rho'_i, \nu^{\beta} \rho'_i]\). Again, since \(l'_i\) (resp., \(-l'_i\)) is the largest (resp., smallest) exponent on the right-hand side of (6-8), we obtain \(l'_i = \beta\) (resp., \(-\alpha \geq -l'_i\)). Hence \(\Delta''_k = [\nu^{-\alpha} \rho'_i, \nu^{\beta} \rho'_i]\) is non-empty and \(\alpha \leq l'_i\). Now, as we can permute the segments in (6-7), the induction in stages implies

\[\delta_1 \hookrightarrow \nu^{-\alpha} \rho'_1 \times \cdots \times \nu^{\beta} \rho'_1 \times \cdots,\]

where all unwritten representations are supercuspidal. In particular, by Frobenius reciprocity, \(\nu^{-\alpha} \rho'_1 \otimes \cdots \otimes \nu^{\beta} \rho'_1 \otimes \cdots\) appears in the appropriate Jacquet module of \(\delta_1\). Since \(\mu^*(\sigma_{sn}) \geq \delta_1 \otimes \sigma_1\), the representation \(\nu^{-\alpha} \rho'_1 \otimes \cdots \otimes \nu^{\beta} \rho'_1 \otimes \cdots \otimes \sigma_{sc}\) is an irreducible subquotient of appropriate Jacquet module of \(\sigma_{sn}\). Then it must be a quotient ([6], Theorem 2.4; for orthogonal groups one needs a simple argument based on Mackey theory). Hence, the Frobenius reciprocity implies the following:

\[\sigma_{sn} \hookrightarrow \nu^{-\alpha} \rho'_1 \times \cdots \times \nu^{\beta} \rho'_1 \times \cdots \otimes \sigma_{sc},\]

where all unwritten representations are supercuspidal. But we have demonstrated above that \(\alpha \leq l'_i\). Hence

\[\sum_{i=\alpha}^{l'_i} m_{\rho'_i} \cdot i \geq 0.\]

This contradicts the strong negativity of \(\sigma_{sn}\).

We iterate this construction until we construct \(\phi\). In the end, it is clear that the last case never occurs, i.e. \(\sum \Delta''_k = \emptyset\), which means \(\delta_1 = 1\) and \(\sigma_1 = \sigma_{sn}\). We summarize the conclusion as follows. Because of our canceling process, the mapping \(\phi\) is obviously an injective mapping, and we have the following:

\[(6-10) \quad \alpha_{\phi(i)} = \beta_{\phi(i)} = 0 \quad (\Delta'_i = \Delta_{\phi(i)}) \quad \text{or} \quad \alpha_{\phi(i)} = \beta_{\phi(i)} = 2l_{\phi(i)} + 1 \quad (\Delta'_i = \Delta_{\phi(i)}), \quad \text{for} \ 1 \leq i \leq t.\]

Since the situation is symmetric, we construct the mapping \(\psi: \{1, 2, \ldots, s\} \rightarrow \{1, 2, \ldots, t\}\) which would be also injective. This forces \(s = t\). Now, (6-10) implies that \((\Delta_1, \ldots, \Delta_t)\) is a \(\sim\)-permutation of \((\Delta_1, \ldots, \Delta_s)\). Because of (6-10), the second condition in (6-6) is simply \(\sigma_{sn} \simeq \sigma'_{sn}\). This proves our claim (6-2).
7. Proof of Theorem 4-5

In this section we prove Theorem 4-5. All claims are trivially true if \( k = 0 \). So, we assume that \( k > 0 \). Let us define numbers \( i_1 < i_2 < \cdots < i_l \) by the following condition (see the assumption in Theorem 4-5):

\[
(7-1) \quad e(\Delta_1) = \cdots = e(\Delta_{i_1}) > e(\Delta_{i_1+1}) = \cdots = e(\Delta_{i_2}) > \cdots > e(\Delta_{i_2+1}) = \cdots = e(\Delta_k) > 0.
\]

We show that our representation \( (\Delta_1, \ldots, \Delta_k; \sigma_{neg}) \) can be characterized as the unique irreducible subquotient of \( (\Delta_1) \times (\Delta_2) \times \cdots \times (\Delta_k) \times \sigma_{neg} \) which contains the irreducible representation

\[
(7-2) \quad \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_{i_1} \rangle \otimes \langle \Delta_{i_1+1} \rangle \times \cdots \times \langle \Delta_{i_2} \rangle \otimes \cdots \otimes \langle \Delta_{i_2+1} \rangle \times \cdots \times \langle \Delta_k \rangle \otimes \sigma_{neg}
\]

as an irreducible subquotient in its appropriate Jacquet module. We remark that the irreducibility of the representation follows from Theorem 1-1.

More precisely, we prove the following lemma:

**Lemma 7-3.** The irreducible representation \((\Delta_1, \ldots, \Delta_k; \sigma_{neg})\) can be characterized as the unique irreducible subquotient of \((\Delta_1) \times (\Delta_2) \times \cdots \times (\Delta_k) \times \sigma_{neg}\) which contains the irreducible representation

\[
(\Delta_1) \times \cdots \times (\Delta_{i_1}) \otimes (\Delta_{i_1+1}) \times \cdots \times (\Delta_{i_2}) \otimes \cdots \otimes (\Delta_{i_2+1}) \times \cdots \times (\Delta_k) \otimes \sigma_{neg}
\]

as an irreducible subquotient in its appropriate Jacquet module. We remark that the irreducibility of the representation follows from Theorem 1-1.

Lemma 7-3 immediately proves claims (i) and (ii) of Theorem 4-5. The claim (iii) is obviously follows directly from (i) and Lemma 7-3. (vi) is a direct consequence of (i), (iii), and Theorem 2-6 (ii). We prove (iv). Since \( k > 0 \), it is enough to show that \( (\Delta_1, \ldots, \Delta_k; \sigma_{neg}) \) is not negative. Let us write \( \Delta_1 \) in our usual form \([\nu^{-\alpha_1} \rho_i, \nu^{\beta_i} \rho_i]\) where \( \rho_i \) is unitary, and \( \alpha_i \) and \( \beta_i \) are real. Then since \( (7-2) \) is a subquotient of some Jacquet module of \((\Delta_1, \ldots, \Delta_k; \sigma_{neg})\), we see that some other Jacquet module of the same representation contains \( \nu^{-\alpha_i} \rho \otimes \cdots \otimes \nu^{\beta_i} \rho \otimes \cdots \), where all unwritten representations are supercuspidal, as a subquotient. Then it must be a quotient (see the assumption in Mackey theory). Hence, the Frobenius reciprocity implies the following:

\[
\sigma \leftrightarrow \nu^{-\alpha_1} \rho \times \cdots \times \nu^{\beta_i} \rho \times \cdots,
\]

where all unwritten representations are supercuspidal. But, since \( e(\Delta_1) > 0 \), we have

\[
\sum_{i=\alpha_i}^{\beta_i} m_{\rho_i} \cdot i > 0.
\]

Thus, \( (\Delta_1, \ldots, \Delta_k; \sigma_{neg}) \) cannot be negative. This proves (iv). We prove (vi) after the proof of Lemma 7-3 since both claims have similar proofs.

Now, we prove Lemma 7-3. We write the segments \( \Delta_1, \ldots, \Delta_k \) explicitly as usual \( \Delta_i = [\nu^{-\alpha_i} \rho_i, \nu^{\beta_i} \rho_i] \), where \( \alpha_i, \beta_i, i = 1, 2, \ldots, k \) are real numbers satisfying \( \alpha_i + \beta_i \in \mathbb{Z}\geq 0, i = 1, \ldots, k \), and \( \rho_i, i = 1, \ldots, k \) are unitarizable supercuspidal representations. We remark that \( e(\Delta_i) = (-\alpha_i + \beta_i)/2 \). Thus, \( (7-1) \) implies

\[
-\alpha_1 + \beta_1 = \cdots = -\alpha_{i_1} + \beta_{i_1} > -\alpha_{i_1+1} + \beta_{i_1+1} = \cdots = -\alpha_{i_2} + \beta_{i_2} > \cdots
\]

We prove the following lemma:

**Lemma 7-4.** If \( \mu^*(\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg}) \geq \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_{i_1} \rangle \otimes \tau \) for some irreducible representation \( \tau \), then \( \tau \) is an irreducible subquotient of \( \langle \Delta_{i_1+1} \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg} \). Moreover, the
multiplicity of \( (\Delta_1) \times \cdots \times (\Delta_i) \otimes \tau \) in \( \mu^*((\Delta_1) \times (\Delta_2) \times \cdots \times (\Delta_k) \times \sigma_{neg}) \) is exactly the multiplicity of \( \tau \) in \( (\Delta_{i+1}) \times \cdots \times (\Delta_k) \times \sigma_{neg} \).

**Proof.** Again, we apply Theorem 1-4. The assumption of the lemma implies that there exist an irreducible constituent \( \delta_2 \otimes \sigma_2 \) of \( \mu^*(\sigma_{neg}) \), and indices \( 0 \leq b_i \leq a_i \leq a_i + \beta_i + 1 \), \( i = 1, \ldots, k \), such that the following holds:

\[
(7-5) \quad \prod_{i=1}^{k} \left[ [\nu^{\beta_i} \rho_i, \nu^{-a_i+\alpha_i} \rho_i] \right] \times \left[ [\nu^{-a_i} \rho_i, \nu^{b_i-a_i-1} \rho_i] \right] \times \delta_2 \geq \prod_{j=1}^{i_1} \left[ [\nu^{-\alpha_j} \rho_j, \nu^{\beta_j} \rho_j] \right].
\]

Since the right–hand side is irreducible, we can assume \( \beta_1 \geq \beta_2 \geq \ldots \geq \beta_{i_1} \), and because of \(-\alpha_1 + \beta_1 = -\alpha_2 + \beta_2 = \ldots = -\alpha_{i_1} + \beta_1 \) we obtain \( \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{i_1} \). Again, we write the equality of the multisets that we analyze

\[
(7-6) \quad \sum_{i=1}^{k} \left[ [\nu^{\beta_i} \rho_i, \nu^{-a_i+\alpha_i} \rho_i] + [\nu^{-a_i} \rho_i, \nu^{b_i-a_i-1} \rho_i] \right] + \text{supp}(\delta_2) = \sum_{j=1}^{i_1} [\nu^{-\alpha_j} \rho_j, \nu^{\beta_j} \rho_j].
\]

We observe that representation \( \nu^{\beta_1} \rho_1 \) appears on the right–hand side, so it must appear on the left–hand side. Now we analyze each possibility in the exactly same way as in the cases in Section 6. In doing so, if we assume that \( \nu^{\beta_1} \rho_1 \in \text{supp}(\delta_2) \) or \( \nu^{\beta_1} \rho_1 \in [\nu^{\beta_i} \rho_i, \nu^{-a_i+\alpha_i} \rho_i] \) for some \( i \), we obtain a contradiction. Thus, the only possibility is \( \nu^{\beta_1} \rho_1 \in [\nu^{-\alpha_i} \rho_i, \nu^{b_i-a_i-1} \rho_i] \) for some \( i \). This forces \( \rho_1 \simeq \rho_i \) and, further, it has some simple consequences

(i) \( i \in \{1, 2, \ldots, i_1\} \)
(ii) \( \alpha_1 = \alpha_i, \beta_1 = \beta_i \), which also means \( a_i = b_i = a_i + \beta_i + 1 \).

We conclude that \( \Delta_1 = \Delta_i \). We might as well take \( i = 1 \) (because we can permute the segments). Then, on the left–hand side of the equation (7-6) for \( i = 1 \) the first segment is empty, the second is actually \( \Delta_1 \), so we can cancel \( \Delta_1 \) on left–hand and right–hand sides. Now, we can iterate this procedure for \( \Delta_2 \) and so on, until the segment \( \Delta_{i_1} \) (including). In the end, we cancel all the segments on the right–hand side of (7-6), so we conclude that \( \delta_2 = 1 \), and, for \( i > i_1 \) all the segments appearing on the left hand–side must be trivial, i.e. \( b_i = 0, a_i = a_i + \beta_i + 1, i = i_1 + 1, \ldots, k \). This forces \( \sigma_2 = \sigma_{neg} \). Again, we apply Theorem 1-4 to the assumption of the lemma. At this moment we know all the \( a_i \)'s and all the \( b_i \)'s. We obtain

\[
\prod_{j=i_1+1}^{k} \left[ [\nu^{-\alpha_j} \rho_j, \nu^{\beta_j} \rho_j] \right] \times \sigma_{neg} \geq \tau.
\]

The multiplicity claim is also clear. This proves the lemma. \( \square \)

We now continue with the proof of the Lemma 7-3. We prove Lemma 7-3 by induction on \( k \). (Here we allow \( k = 0 \).) If \( k = 0 \) the claim trivially holds since \( \sigma_{neg} \) appears in the composition series of \( \sigma_{neg} \) exactly once. Assume that the lemma holds for all numbers \( < k \). We prove it for \( k \). We will use parts (i)–(ii) of Theorem 4-5 if the number of segments is \( < k \) since we have showed that the validity of Lemma 7-3 implies them.

Now, the transitivity and the exactness of Jacquet modules implies that for every irreducible constituent of the form of (7-2) of the appropriate Jacquet module, say \( Jacq_i \), of \( (\Delta_1) \times (\Delta_2) \times \cdots \times (\Delta_k) \times \sigma_{neg} \) we can find \( \tau \) as in Lemma 7-4 such that in the appropriate Jacquet module of the right–hand side of

\[
\mu^*((\Delta_1) \times (\Delta_2) \times \cdots \times (\Delta_k) \times \sigma_{neg}) \geq (\Delta_1) \times \cdots \times (\Delta_{i_1}) \otimes \tau
\]
we obtain the desired subquotient of $Jacq$. As the inductive assumption forces that
\[ \tau \simeq (\Delta_{i_1+1}, \Delta_{i_1+2}, \ldots, \Delta_k; \sigma_{neg}) \]
we obtain the claim from the fact that $(\Delta_{i_1+1}, \Delta_{i_1+2}, \ldots, \Delta_k; \sigma_{neg})$ appears in $(\Delta_{i_1+1}) \times (\Delta_{i_1+2}) \times \cdots \times (\Delta_k) \rtimes \sigma_{neg}$ with the multiplicity one (which is a consequence of our inductive assumption).
This proves Lemma 7-3. Consequently, Theorem 4-5 (i)-(v) is proved.

It remains to prove Theorem 4-5 (vi). So, assume that $\Delta_1, \ldots, \Delta_k$ and $\Delta'_1, \ldots, \Delta'_k$ are segments, and $\sigma_{neg}$ and $\sigma'_{neg}$ are negative representations such that
\[ e(\Delta_1) \geq e(\Delta_2) \geq \cdots \geq e(\Delta_k) > 0, \quad e(\Delta'_1) \geq e(\Delta'_2) \geq \cdots \geq e(\Delta'_k) > 0 \]
(7-7)
\[ (\Delta_1, \ldots, \Delta_k; \sigma_{neg}) \simeq (\Delta'_1, \ldots, \Delta'_k; \sigma'_{neg}). \]
(7-8)

Let us write (see (7-1))
\[ e(\Delta'_1) = \cdots = e(\Delta'_{i_1}) > e(\Delta'_{i_1+1}) = \cdots = e(\Delta'_{i_2}) > \cdots > e(\Delta'_{i_3}) = \cdots = e(\Delta'_{k'}) > 0. \]
(7-9)
The assertion of Theorem 4-5 (vi) will be proved if we establish that
\[ (\Delta'_1, \ldots, \Delta'_{i_1}) \text{ is a permutation of } (\Delta_1, \ldots, \Delta_{i_1}). \]
(7-10)
Indeed, if (7-10) holds, the Theorem 1-1 implies that
\[ (\Delta_1) \times \cdots \times (\Delta_{i_1}) \simeq (\Delta'_1) \times \cdots \times (\Delta'_{i_1}). \]
(7-11)
Applying Theorem 4-5 (ii) several times we obtain
\[ \begin{cases} 
(\Delta_1, \ldots, \Delta_k; \sigma_{neg}) \hookrightarrow (\Delta_1) \times \cdots \times (\Delta_{i_1}) \times (\Delta_{i_1+1}, \ldots, \Delta_k; \sigma_{neg}) \\
(\Delta'_1, \ldots, \Delta'_k; \sigma'_{neg}) \hookrightarrow (\Delta'_1) \times \cdots \times (\Delta'_{i_1}) \times (\Delta'_{i_1+1}, \ldots, \Delta'_k; \sigma'_{neg}) 
\end{cases} \]
(7-12)
Applying (7-8) and (7-11) to this, we find
\[ \begin{cases} 
(\Delta_1, \ldots, \Delta_k; \sigma_{neg}) \hookrightarrow (\Delta_1) \times \cdots \times (\Delta_{i_1}) \times (\Delta_{i_1+1}, \ldots, \Delta_k; \sigma_{neg}) \\
(\Delta'_1, \ldots, \Delta'_k; \sigma'_{neg}) \hookrightarrow (\Delta'_1) \times \cdots \times (\Delta'_{i_1}) \times (\Delta'_{i_1+1}, \ldots, \Delta'_k; \sigma'_{neg}). 
\end{cases} \]

Therefore, the Frobenius reciprocity implies
\[ \mu^*((\Delta_1, \ldots, \Delta_k; \sigma_{neg})) \geq (\Delta_1) \times \cdots \times (\Delta_{i_1}) \otimes (\Delta_{i_1+1}, \ldots, \Delta_k; \sigma_{neg}), \]
\[ (\Delta'_1) \times \cdots \times (\Delta'_{i_1}) \otimes (\Delta'_{i_1+1}, \ldots, \Delta'_k; \sigma'_{neg}). \]

Hence, the exactness of Jacquet modules implies that both representations on the left hand side are also $\leq \mu^*((\Delta_1) \times \cdots \times (\Delta_k) \rtimes \sigma_{neg})$. Now, Lemma 7-4 and Theorem 4-5 (iii) implies that
\[ (\Delta_{i_1+1}, \ldots, \Delta_k; \sigma_{neg}) \simeq (\Delta'_{i_1+1}, \ldots, \Delta'_k; \sigma'_{neg}) \]
which enables us to prove Theorem 4-5 (vi) by induction on $k$.

It remains to prove the claim (7-10). We use the isomorphism (7-8) and the second embedding in (7-12). As a consequence we obtain
\[ \mu^*((\Delta_1) \times \cdots \times (\Delta_k) \rtimes \sigma_{neg}) \geq (\Delta'_1) \times \cdots \times (\Delta'_{i_1}) \otimes (\Delta'_{i_1+1}, \ldots, \Delta'_k; \sigma'_{neg}). \]
Again, we apply Theorem 1-4. We proceed as in the analysis of (6-5). But here the discussion is simpler since the positivity (see (7-7)) forces that only the second among the three cases of that
analysis can appear. In more words, we can construct an injection \( \phi : \{1, \ldots, i'\} \to \{1, \ldots, k\} \) such that \( \Delta_i' = \Delta_{\phi(i)} \). Similarly, working with

\[
\mu^*(\langle \Delta_i' \rangle \times \cdots \times \langle \Delta_k' \rangle \times \sigma_{neg}') \geq \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_{i_1} \rangle \otimes \langle \Delta_{i_1+1}, \ldots, \Delta_k; \sigma_{neg}\rangle,
\]
we can construct an injection \( \psi : \{1, \ldots, i'\} \to \{1, \ldots, k'\} \) such that \( \Delta_i = \Delta_{\psi(i)}' \). We are done if we show that the image of \( \phi \) (resp., \( \psi \)) is in \( \{1, \ldots, i'_1\} \) (resp., \( \{1, \ldots, i'_2\} \)). But this is easy. We use (7-1) and (7-9). We have

\[
\text{we show that the image of } \phi \text{ (resp., } \psi \text{) is in } \{1, \ldots, i'_1\} \text{ (resp., } \{1, \ldots, i'_2\} \).
\]

8. Proof of Theorem 4-4

We prove the theorem by induction on \( k \). If \( k = 0 \), then the induced representation reduces just to \( \sigma_{sn} \) which is negative since it is strongly negative. We assume that the claim holds for all \( k < k \). We prove it for \( k \). Let \( \sigma \) be an irreducible subquotient of \( \langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{sn} \). Then there is an irreducible subquotient \( \sigma_1 \) of \( \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{sn} \) such that \( \sigma \) is an irreducible subquotient of \( \langle \Delta_1 \rangle \times \sigma_1 \). Since \( \sigma_1 \) is negative by the inductive assumption, it is enough to prove the following lemma:

**Lemma 8-1.** Let \( \Delta \) be a segment such that \( e(\Delta) = 0 \). Let \( \sigma_1 \) be a negative representation. Then all irreducible subquotients of \( \langle \Delta \rangle \times \sigma_1 \) are negative.

**Proof.** Assume that \( \sigma \) is a non-negative irreducible subquotient of the representation \( \langle \Delta \rangle \times \sigma_1 \). We write \( \Delta = [\nu^{-\alpha_1}\rho_1, \nu^{\alpha_1}\rho_1] \), where \( \rho \) is unitarizable and \( 2\alpha \in \mathbb{Z}_{\geq 0} \). Applying Theorem 4-6, we may write \( \sigma = \langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle \) with the apology for an overuse of the symbols \( \Delta_1, \ldots, \Delta_k \). (They have different meaning here than in the statement of Theorem 4-4.) Next, Theorem 4-5 (ii) implies \( \sigma \hookrightarrow \langle \Delta_1 \rangle \times \langle \Delta_2, \ldots, \Delta_k; \sigma_{neg} \rangle \). Hence the Frobenius reciprocity implies

\[
\mu^*(\sigma) \geq \langle \Delta_1 \rangle \otimes \langle \Delta_2, \ldots, \Delta_k; \sigma_{neg} \rangle.
\]

Thus, the exactness of Jacquet functor implies

\[
\mu^*(\langle \Delta \rangle \times \sigma_1) \geq \langle \Delta_1 \rangle \otimes \langle \Delta_2, \ldots, \Delta_k; \sigma_{neg} \rangle.
\]

Again, we analyze this using Theorem 1-4. We introduce \( \alpha_1, \beta_1 \) such that \( \Delta_1 = [\nu^{-\alpha_1}\rho_1, \nu^{\beta_1}\rho_1] \), for some unitarizable supercuspidal representation \( \rho_1 \). We have \( e(\Delta_1) = -\frac{\alpha_1 + \beta_1}{2} > 0 \). Applying Theorem 1-4, there exist \( i, j \) such that \( 0 \leq j \leq i \leq 2\alpha + 1 \) such that

\[
([\nu^{-\alpha_1}\rho_1, \nu^{-i+\alpha_1}\rho_1]) \times ([\nu^{-\alpha_1}\rho_1, \nu^{j-\alpha_1-1}\rho_1]) \times \delta \geq ([\nu^{-\alpha_1}\rho_1, \nu^{\beta_1}\rho_1]),
\]

where \( \delta \) and \( \tau \) are irreducible representations with \( \delta \otimes \tau \leq \mu^*(\sigma_{neg}'). \) Again, we obtain the equality of the multisets

\[
[\nu^{-\alpha_1}\rho_1, \nu^{-i+\alpha_1}\rho_1] + [\nu^{-\alpha_1}\rho_1, \nu^{j-\alpha_1-1}\rho_1] + \text{supp}(\delta) = [\nu^{-\alpha_1}\rho_1, \nu^{\beta_1}\rho_1].
\]

Then we analyze all the possible positions of the representation \( \nu^{\beta_1}\rho_1 \) on the left hand side applying the discussion which is similar to the one after (6-8) in Section 6. We omit the details. We obtain contradiction in each case. This proves the lemma. \( \Box \)
9. Some Further Results on Intertwining Operators

In this section we give a more precise form of the Theorem 2-6 (ii) when we combine it with the results of Section 4. We start by the following lemma:

**Lemma 9-1.** Suppose that $\Delta_1, \ldots, \Delta_k$ is a sequence of segments satisfying $e(\Delta_1) \geq \ldots \geq e(\Delta_k) > 0$. (We allow empty sequence here; in this case $k = 0$.) Let $\sigma_{neg}$ be a negative representation. Let $n = \sum_i m(\Delta_i) + m_{\sigma_{neg}}$. Then the intertwining space

$$\text{Hom}_{G_n}(\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg}; \langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg})$$

is one-dimensional.

**Proof.** Let $A$ be a non-zero intertwining operator. Then its image is a subrepresentation of $\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg}$. As Theorem 4-5 (i) implies, $\langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle$ must be a subrepresentation of the image. On the other hand, we have that the image is isomorphic to $\langle \tilde{\Delta}_1 \rangle \times \langle \tilde{\Delta}_2 \rangle \times \cdots \times \langle \tilde{\Delta}_k \rangle / \ker A$. Using Theorem 4-5 (iii) and (v), we see that the image must be irreducible and isomorphic to $\langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle$. This proves the lemma. \hfill \Box

The proof of the lemma shows that every intertwining operator is trivial on the unique maximal proper subrepresentation of $\langle \tilde{\Delta}_1 \rangle \times \langle \tilde{\Delta}_2 \rangle \times \cdots \times \langle \tilde{\Delta}_k \rangle \times \sigma_{neg}$. Let us fix some non-zero intertwining operator $J(\Delta_1, \ldots, \Delta_k; \sigma_{neg})$ in (9-2). By the lemma, it is unique up to a scalar multiple. Using analytic techniques [26] or geometric techniques [22] it is possible to normalize the intertwining operator. We do not need this here.

In the case of the Langlands classification, this lemma is due to to Milićić ([8], Corollary 2.7).

We remark that the similar statement is implicitly contained in [27]. More precisely, we have the following result:

**Lemma 9-3.** Let $k > 0$. Suppose that $\Delta_1, \ldots, \Delta_k$ is a sequence of segments satisfying $e(\Delta_1) \geq \ldots \geq e(\Delta_k)$. Let $m = \sum_i m(\Delta_i)$. Then the intertwining space

$$\text{Hom}_{GL(m,F)}(\langle \Delta_k \rangle \times \langle \Delta_{k-1} \rangle \times \cdots \times \langle \Delta_1 \rangle; \langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle)$$

is one-dimensional.

**Proof.** This follows from Theorem 1-1 and Theorem 2-6 (i) using arguments similar to that of Lemma 9-1. \hfill \Box

Again, we denote some non-zero intertwining operator by $J(\Delta_1, \ldots, \Delta_k)$. For us, the case $k = 2$ is especially important. We recall (see Theorem 1-1) that $J(\Delta_1, \Delta_2)$ is an isomorphism if and only if the segments $\Delta_1$ and $\Delta_2$ are not linked. In addition, if $\Delta_1$ and $\Delta_2$ are linked, then the assumption $e(\Delta_1) \geq e(\Delta_2)$ forces that the kernel of $J(\Delta_1, \Delta_2)$ is isomorphic to $\langle \Delta_1 \cup \Delta_2 \rangle \times \langle \Delta_1 \cap \Delta_2 \rangle$. (See [27], Proposition 4.6.)

Let us introduce some notation. Assume that $\Psi_1, \ldots, \Psi_l$ is an arbitrary sequence of segments. Let $\sigma \in \text{Irr}(G)$ be an irreducible representation. Then if $e(\Psi_i) \leq e(\Psi_{i+1})$, the intertwining operator $J(\Psi_{i+1}, \Psi_i) : \langle \Psi_i \rangle \times \langle \Psi_{i+1} \rangle \rightarrow \langle \Psi_{i+1} \rangle \times \langle \Psi_i \rangle$ is well-defined up to a non-zero constant, and, by the induction in stages, it induces an intertwining operator:

$$J(\Psi_{i+1}, \Psi_i; \sigma) : \langle \Psi_i \rangle \times \cdots \times \langle \Psi_i \rangle \times \langle \Psi_{i+1} \rangle \times \cdots \times \langle \Psi_1 \rangle \times \langle \Psi_1 \rangle \times \cdots \times \langle \Psi_i \rangle \times \sigma \rightarrow \langle \Psi_1 \rangle \times \cdots \times \langle \Psi_{i+1} \rangle \times \langle \Psi_i \rangle \times \cdots \times \langle \Psi_1 \rangle \times \sigma.$$
Now, we go back to the settings of Lemma 9-1. We prove the following result:

**Theorem 9-4.** Let $k > 0$. Suppose that $\Delta_1, \ldots, \Delta_k$ is a sequence of segments satisfying $e(\Delta_1) \geq \cdots \geq e(\Delta_k) > 0$. Let $\sigma_{neg}$ be a negative representation. Then, up to a non-zero constant, the intertwining operator $J(\Delta_1, \ldots, \Delta_k; \sigma_{neg})$ is equal to the composition:

$$J(\Delta_2; \ldots, \Delta_k; \sigma_{neg}) J(\Delta_1, \widetilde{\Delta}; \sigma_{neg}) \cdots J(\Delta_{k-1}; \sigma_{neg}) J(\Delta_{k}; \sigma_{neg}) \cdots J(\Delta_1, \sigma_{neg}).$$

**Proof.** By definition of $J(\Delta_1, \ldots, \Delta_k; \sigma_{neg})$, it takes the unique irreducible quotient of $\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg}$ to the unique irreducible subrepresentation $\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg}$ which is by definition $\langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle$. So, by Corollary 2-5 and Theorem 4-5 (iii), it is enough to prove that $\langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle$ is not in the kernel of any of the intertwining operators appearing in above decomposition.

We start by $J(\Delta_2; \ldots, \Delta_k; \sigma_{neg})$ which in the setting of above decomposition must be interpreted as an intertwining operator

$$(9-5) \quad \langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg} \to \langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg},$$

which is induced from the intertwining operator

$$\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg} \to \langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg},$$

defined by Lemma 9-1. The image of (9-5) is isomorphic to $\langle \Delta_1 \rangle \times \langle \Delta_2, \ldots, \Delta_k; \sigma_{neg} \rangle$. Now, Theorem 4-5 (ii) implies that $\langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle \to \langle \Delta_1 \rangle \times \langle \Delta_2, \ldots, \Delta_k; \sigma_{neg} \rangle$. On the other hand, applying Corollary 2-5 and Theorem 4-5 (iii), $\langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle$ cannot be the subquotient of the kernel.

To discuss the kernels of other intertwining operators, we use important characterization of $\langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle$ given by Lemma 7-3: $\langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle$ is the unique irreducible subquotient of $\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg}$ which contains (7-2) in its Jacquet module.

The intertwining operator $J(\Delta_1; \sigma_{neg})$ which must be interpreted as the intertwining operator:

$$\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg} \to \langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg},$$

has the image $\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg}$. We show it contains (7-2) in appropriate Jacquet module. Applying Corollary 2-5, it is enough to show the same claim for $\langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \langle \Delta_1; \sigma_{neg} \rangle$. Now, the claim follows from Theorem 1-4. Indeed, Theorem 4-5 (ii) implies that $\langle \Delta_1; \sigma_{neg} \rangle \to \langle \Delta_1 \rangle \times \sigma_{neg}$. Hence the Frobenius reciprocity implies $\mu^*(\langle \Delta_1; \sigma_{neg} \rangle \geq \langle \Delta_1 \rangle \otimes \sigma_{neg}$.

Now, Theorem 1-4 implies the following:

$$\mu^*(\langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \times \langle \Delta_1; \sigma_{neg} \rangle \geq \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_k \rangle \otimes \sigma_{neg}.$$

Now, we apply the transitivity of Jacquet modules to obtain the claim.

Next, we consider the intertwining operator $J(\Delta_i, \Delta_i; \sigma_{neg})$, $1 \leq i \leq k$. It has a non-trivial kernel if and only if the segments $\Delta_1$ and $\Delta_i$ are linked. Then the kernel is isomorphic to $\langle \Delta_2 \rangle \times \cdots \times \langle \Delta_{i-1} \rangle \times \langle \Delta_i \cup \Delta_i \times \langle \Delta_{i+1} \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg}$

which, by Corollary 2-5, has the same composition series as

$$(9-6) \quad \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_{i-1} \rangle \times \langle \Delta_1 \cup \Delta_i \rangle \times \langle \Delta_1 \cap \Delta_i \rangle \times \langle \Delta_{i+1} \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg}.$$
To compute Jacquet modules, we write segment as in the paragraph before the statement of Lemma 7-4. Since \( \Delta_1 \) and \( \Delta_i \) are linked and \( e(\Delta_1) \geq e(\Delta_i) \) we must have \( \rho_1 \simeq \rho_i \), \( -\alpha_i < -\alpha_1 \), \( \beta_i < \beta_1 \), and

\[
\begin{aligned}
\Delta_1 \cup \Delta_i &= [\nu^{-\alpha_1} \rho_1, \nu^{\beta_1} \rho_1] \\
\Delta_1 \cap \Delta_i &= [\nu^{-\alpha_1} \rho_1, \nu^{\beta_i} \rho_1].
\end{aligned}
\]

This implies the following:

\[ e(\Delta_i), e(\Delta_1 \cup \Delta_i), e(\Delta_1 \cap \Delta_i) < e(\Delta_1). \]

In particular, we see \( i > i_1 \) (see (7-1)). Now, if (7-2) is a subquotient of the appropriate Jacquet module of the induced representation in (9-6), then

\[
\mu^*(\langle \Delta_2 \rangle \times \cdots \times \langle \Delta_{i-1} \rangle \times \langle \Delta_1 \cup \Delta_i \rangle \times \langle \Delta_1 \cap \Delta_i \rangle \times \langle \Delta_{i+1} \rangle \times \cdots \times \langle \Delta_k \rangle \times \sigma_{neg}) \geq \\
\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_{i-1} \rangle \otimes \langle \Delta_{i+1}, \ldots, \Delta_k; \sigma_{neg} \rangle.
\]

Now, we apply Theorem 1-4 to (9-7). There are indices \( 0 \leq b_j \leq a_j \leq a_j + \beta_j + 1, j = 2, \ldots, i-1, i+1, \ldots, k \), and \( 0 \leq b' \leq a' \leq a' \beta_i + 1, 0 \leq b'' \leq a'' \leq \alpha_1 + \beta_i + 1 \) (corresponding to \( \Delta_1 \cap \Delta_i \) and \( \Delta_1 \cup \Delta_i \), respectively) such that

\[
\prod_{j=2, j \neq i}^k \langle [\rho^{-\beta_j} \tilde{\rho}_j, \nu^{-a_j + \alpha_j} \tilde{\rho}_j] \rangle \times \langle [\nu^{-\alpha_1} \rho_j, \nu^{b_j-a_j-1} \rho_j] \rangle \times \\
\langle [\rho^{-\beta_i} \tilde{\rho}_i, \nu^{-a_i + \alpha_i} \tilde{\rho}_i] \rangle \times \langle [\nu^{-\alpha_1} \rho_i, \nu^{b''-a_i-1} \rho_i] \rangle \times \langle [\nu^{-\alpha_1} \rho_i, \nu^{b''-a_i-1} \rho_i] \rangle \times \delta_2 \geq \prod_{j=1}^{i_1} \langle [\nu^{-\alpha_1} \rho_j, \nu^{\beta_j} \rho_j] \rangle.
\]

where \( \delta_2 \otimes \tau_2 \leq \mu^*(\sigma_{neg}) \) is an irreducible constituent. Now, we repeat the analysis done after (6-6) in Section 6 first arranging \( \beta_1 \geq \beta_2 \geq \ldots \geq \beta_i \) and \( \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_i \), as in the proof of Lemma 7-4. But the highest exponent \( \beta_1 \) on the right hand side does not appear on the left hand side. The last kernel can be considered analogously.

Theorem 9-4 is an analogue of the well-known factorization of the long intertwining operator (see [25]).

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**References**


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