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GORAN MUIĆ AND GORDAN SAVIN

ABSTRACT. Let $G$ be a hermitian quaternionic group. We determine complementary series for representations of $G$ induced from super-cuspidal representations of a Levi factor of the Siegel maximal parabolic subgroup of $G$.

INTRODUCTION

Let $F$ be a non-archimedean field of characteristic zero. Let $F'$ be a finite dimensional division algebra over $F$ with an anti-involution $\tau$, such that the set of fixed points of $\tau$ is $F$. We have three cases:

(I) $F' = F$ and $\tau$ is the identity map on $F$.
(II) $F'$ is a quadratic extension of $F$ and $\tau$ is the non-trivial element of the Galois group $\text{Gal}(F'/F)$.
(III) $F' = D$ is the unique (up to an isomorphism) quaternion algebra, central over $F$ and $\tau$ is the usual involution, fixing the center of $D$.

Every such algebra $F'$ defines a reductive group $G$ over $F$ as follows. Let

$$V_n = e_1 F' \oplus \cdots \oplus e_n F' \oplus e_{n+1} F' \oplus \cdots \oplus e_{2n} F',$$

be a right vector space over $F'$. If we fix $\varepsilon \in \{\pm 1\}$, then $(e_i, e_{2n-j+1}) = \delta_{ij}$ defines an $\varepsilon$-hermitian form on $V_n$:

$$\begin{cases}
(v, v') = \varepsilon \cdot \tau((v, v'))), & v, v' \in V_n, \\
(vx, v'x') = \tau(x)(v, v')x', & x, x' \in F'.
\end{cases}$$

Let $G = G_n(F', \varepsilon)$ be the group of isometries of the form $(\cdot, \cdot)$, and let $P$ be the parabolic subgroup of $G$, which stabilizes the isotropic space

$$V_n' = e_1 F' \oplus \cdots \oplus e_n F'.$$

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The group $P$ has a Levi decomposition $P = MN$, where $M \cong \text{Aut}_{F'}(V_n'')$. We fix an isomorphism $M \cong GL(n, F')$ using the above fixed basis of $V_n''$.

Let $\rho \in \text{Irr}(M)$ be a unitary representation and let $s$ be a real number. Define a generalized principal series representation by
\[
I(\rho, s) = \text{Ind}_P^G([\det_{F'}]^s \otimes \rho),
\]
where $\det_{F'}$ is the reduced norm of the simple algebra $M(n, F')$ of all $n \times n$ matrices with coefficients in $F'$, and $||$ is the normalized absolute value of $F$ in the cases (I) and (III) and the normalized absolute value of $F'$ in the case (II). Let $\overline{P} = MN$ be the opposite parabolic subgroup. Analogously, we define the induced representation $\overline{I}(\rho, s)$ for $\overline{P}$. For $f \in I(\rho, s)$, let
\[
A(s, \rho, N, \overline{N})f(g) = \int_{\overline{N}} f(ng)d\overline{n} \quad (g \in G)
\]
be the standard intertwining operator from $I(\rho, s)$ to $\overline{I}(\rho, s)$ (meromorphically continued from the domain of convergence of the integral). Let $\mu(s, \rho)$ be the Plancherel measure defined by
\[
A(s, \rho, \overline{N}, N)A(s, \rho, N, \overline{N}) = \mu(s, \rho)^{-1}.
\]
It follows from the work of Harish-Chandra [Si] that the Plancherel measure $\mu(s, \rho)$ determines points of reducibility and complementary series of $I(\rho, s)$ if $\rho$ is supercuspidal.

In the cases (I) and (II) the group $G$ is quasi-split. Thus, if $\rho$ is supercuspidal, the reducibilities and complementary series of $I(\rho, s)$ are part of a general theory of Shahidi [Sh2] for generic representations. For more details and for a nice interpretation in terms of conjectural twisted endoscopy theory, see [Sh1] (case (I)) and [G] (case (II)). In this paper we study the remaining case (III). Then $G$ is no longer quasi-split and our induced representations do not have Whittaker models.

Let us describe the main results of this paper in more details. First, note that $G$ is an inner form of the group
\[
G' = G_{2n}(F, -\epsilon) = \begin{cases} 
    Sp(4n, F) & \text{if } \epsilon = +1 \\
    SO(4n, F) & \text{if } \epsilon = -1.
\end{cases}
\]

Let $P' = M'N'$ be the Siegel maximal parabolic subgroup of $G'$ as above. Note that $M' = GL(2n, F)$. Furthermore, there is a natural 1-1 correspondence between regular elliptic conjugacy classes of $GL(n, D)$ and $GL(2n, F)$ . For each $\pi \in \text{Irr}(G)$, we write $\chi_\pi$ for its character, which is, by a well-known result of Harish-Chandra, a locally integrable function, locally constant on the set of all regular conjugacy classes. By [DKV], there exists a 1-1 correspondence $\rho \leftrightarrow \rho'$ between the sets of all classes irreducible essentially square-integrable representations of $GL(n, D)$ and $GL(2n, F)$ characterized by
\[
(-1)^n \chi_\rho = \chi_{\rho'}
\]
on the set of the regular elliptic classes. In Section 2 we prove (Proposition 2.1)
\[ \mu(s, \rho) = \mu(s, \rho'), \]
under certain normalizations of Haar measures on \( N \) and \( N' \); for more details see Section 2. Combining this with the results of [Sh1], we compute reducibility and complementary series of \( I(\rho, s) \) if \( \rho \) is supercuspidal. This can be found in Section 3.

1. Results of DKV

In this section we describe the correspondence \( \rho \leftrightarrow \rho' \), between essentially square integrable representations of \( GL(n, D) \) and \( GL(2n, F) \), in more details.

By a result of Bernstein [Ze], there exists a positive integer \( k \) and a supercuspidal representation \( \rho_0 \) of \( GL(2n/k, F) \) such that \( \rho' \) is the unique irreducible subrepresentation of
\[ \nu^{(k-1)/2}_\rho \times \cdots \times \nu^{-(k-1)/2}_\rho \nu_0. \]
(Here, as usual [Ze], \( \nu = |\det|_F \).) We will write \( \rho' = \delta(\rho_0, k) \). Now, by [DKV; B.2.b] \( \rho \) is supercuspidal if and only if the lowest common multiple of 2 and \( 2n/k \) is \( 2n \). Thus, if \( n \) is even, \( \rho \) is supercuspidal if and only if \( \rho' \) is. If \( n \) is odd, \( \rho \) is supercuspidal if and only if either \( \rho' \) is supercuspidal, or \( \rho' = \delta(\rho_0, 2) \).

Assume now that \( \rho \) is a supercuspidal representation. Define, as in [T; pg. 53], the character \( \nu_\rho \) of \( GL(n, D) \) by
\[ \nu_\rho = \begin{cases} 
|\det_D|_F & \text{if } \rho' \text{ is supercuspidal} \\
|\det_D|_F^2 & \text{if } \rho' = \delta(\rho_0, 2). 
\end{cases} \]
Let \( \delta(\rho, k) \) be the unique irreducible subrepresentation of
\[ \nu^{(k-1)/2}_\rho \times \cdots \times \nu^{-(k-1)/2}_\rho \nu_0. \]
By [DKV; B.2] and [T; Prop. 2.7] this representation is essentially square integrable. Furthermore, its lift to \( GL(2n, F) \) is given by [DKV; B.2.b]
\[ \delta(\rho, k)' = \delta(\rho', k) \text{ if } \rho' \text{ is supercuspidal} \]
\[ \delta(\rho, k)' = \delta(\rho_0, 2k) \text{ if } \rho' = \delta(\rho_0, 2). \]

We will end this section by introducing the natural involution on the set of irreducible representations of \( GL(n, D) \). First, for \( g = (g_{ij}) \in GL(n, D) \) define \( \tau(g) = (\tau(g_{ij})) \in GL(n, D) \). If \( g^t \) denote the transpose matrix (with respect to the main diagonal), we put \( g^\tau = \tau(g^t) \). The map \( g \mapsto (g^\tau)^{-1} \) is a continuous involution on \( GL(n, D) \), for any \( n \). Thus, it acts on representations by \( \pi^\tau(g) = \pi((g^\tau)^{-1}) \). Now, we will prove
Lemma 1.1. Let \( \pi \) be an irreducible representation of \( GL(n, D) \). Let \( \tilde{\pi} \) be the contragredient representation of \( \pi \). Then \( \pi^* \cong \tilde{\pi} \).

Proof. We will prove this result under our assumption that the characteristic of \( F \) is zero. This assumption enables us to consider the characters \( \chi_\pi \) and \( \chi_{\pi^*} \) as locally integrable functions, locally constant on the set of all regular semisimple conjugacy classes. Hence, to prove the lemma, it is enough to check
\[
\chi_{\pi^*}(g) = \chi_\pi(g),
\]
for all regular semisimple elements \( g \in GL(n, D) \). This is equivalent to
\[
\chi_\pi(g^*) = \chi_\pi(g).
\]
Hence, it is enough to check that \( g^* \) and \( g \) are conjugate for all regular semisimple \( g \).

Note that \( g^* \) and \( g \) have the same characteristic polynomial. In particular, they are conjugate over the algebraic closure \( \bar{F} \) of \( F \). Let \( \mathcal{A} \) be the centralizer of \( g \) in \( M(n, D) \). Then
\[
\mathcal{A} = \bigoplus_j F_j,
\]
where for any \( j \), \( F_j \) is an extension of \( F \) (of degree \( [F_j : F] \)), and \( \sum_j [F_j : F] = n \). Thus, the centralizer of \( g \) in \( GL(n, D) \) is
\[
GL(\mathcal{A}) = \times_j F_j^e.
\]
By the Hilbert Theorem 90, the first Galois cohomology group \( H^1(\text{Gal}(\bar{F}/F), GL(\mathcal{A})) \) is trivial. In particular, \( g^* \) and \( g \) are conjugated over \( F \). \( \square \)

2. Plancherel measures

In this section we will prove the equality of Plancherel measures. Abusing our notation, let \( G = G_n(F', \epsilon) \) and let \( P = MN \) be the Siegel maximal parabolic subgroup as in the Introduction. First, we need to normalize Haar measures on \( N \) and \( \overline{N} \). We shall fix a non-trivial additive character \( \psi_F \) of \( F \). Let \( M(n, F') \) be the vector space over \( F' \) of \( n \times n \)-matrices with coefficients in \( F' \). Then
\[
M(n, F') = M(n, F')^+ \oplus M(n, F')^-,
\]
where \( M(n, F')^+ \) and \( M(n, F')^- \) are the sets of \( \tau \)-hermitian symmetric and \( \tau \)-hermitian skew-symmetric matrices. Then, using the basis \( e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n} \) of \( V_n \) we can identify both \( N \) and \( \overline{N} \) with
\[
\left\{ \begin{array}{ll} M(n, F')^+ & \text{if } \epsilon = -1 \\ M(n, F')^- & \text{if } \epsilon = +1. \end{array} \right.
\]
Let \( \mu_n(F') \) be the Haar measure on \( M(n, F') \) self-dual with respect to \( \psi_F \) and the bilinear form \( \text{Tr}_{F'}(xy) \), where \( \text{Tr}_{F'} \) is the reduced trace on \( M(n, F') \). Let \( \mu_n^\pm(F') \) be the self-dual Haar measure on \( M(n, F')^\pm \) such that
\[
\mu_n(F') = \mu_n^+(F') \cdot \mu_n^-(F').
\]
Specifying \( F' = D \) and \( F' = F \) we obtain normalizations of the Haar measures for \( G = G_n(D, \epsilon) \) and \( G' = G_{2n}(F, -\epsilon) \), respectively.
Proposition 2.1. Assume that \( \rho \) is a square integrable representation of \( GL(n, D) \), and \( \rho' \) its lift to \( GL(2n, F) \). Then, under above normalization of Haar measures on \( N \) and \( N' \), we have
\[
\mu(s, \rho) = \mu(s, \rho').
\]

Proof. We will prove the proposition using global means. Let \( k \) be an algebraic number field. For each place \( v \) of \( k \) let \( k_v \) denote its completion at \( v \). Let \( \mathcal{A} \) be the ring of adeles of \( k \). We may assume that \( k \) has two places \( v_1 \) and \( v_2 \), such that
\[
\begin{cases}
  k_{v_1} \cong F \\
  k_{v_2} \cong F.
\end{cases}
\]

Let \( D \) be a quaternion algebra over \( k \), ramified at \( v_1 \) and \( v_2 \) only. Let \( G = G_n(D, \epsilon) \). It is a form of \( G \) defined over \( k \). Note that
\[
G(k_{v_1}) \cong G(k_{v_2}) \cong G
\]
\[
G(k_v) \text{ is split if } v \notin \{v_1, v_2\}.
\]

Let \( P = \text{MN} \) be the Siegel parabolic subgroup of \( G \). Note that \( M = GL(n, D) \). Take a nontrivial additive character \( \psi = \otimes_v \psi_v \) on \( \mathcal{A} \), trivial on \( k \), such that \( \psi_{v_1} = \psi_F \) and \( \psi_{v_2} = \psi_F \). For each place \( v \), we fix the Haar measures on \( N(k_v) \), and \( \overline{N}(k_v) \) self-dual with respect to \( \psi_v \), as above. In this way we have fixed Tamagawa measures (see [We; pg. 113]) on \( N(\mathcal{A}) \) and \( \overline{N}(\mathcal{A}) \). This means that
\[
\begin{cases}
  v\text{d}(N(k) \backslash N(\mathcal{A})) = 1 \\
  v\text{d}(\overline{N}(k) \backslash \overline{N}(\mathcal{A})) = 1.
\end{cases}
\]

Let \( G' = G_{2n}(k, -\epsilon) \). It is a split form of \( G \). Let \( P' = M'N' \) be the Siegel parabolic subgroup of \( G' \). Note that \( M' = GL(2n, k) \). As in the case of \( G \), we fix Tamagawa measures on \( N'(\mathcal{A}) \) and \( \overline{N}'(\mathcal{A}) \) using \( \psi = \otimes_v \psi_v \).

Now, we will fix a unitary character \( \omega \) of \( \mathcal{A}^\times \), trivial on \( k^\times \), such that \( \omega_{v_1} \) and \( \omega_{v_2} \) are equal to the central character of \( \rho \).

Lemma 2.1. Fix a finite place \( u \), different from \( v_1 \) and \( v_2 \), and choose any supercuspidal unitary representation \( \delta \) of \( GL(2n, k_u) \), whose central character is \( \omega_u \). Then there exists an automorphic cuspidal representation \( \pi' = \otimes_v \pi'_v \) of \( GL(2n, \mathcal{A}) = M'(\mathcal{A}) \), whose central character is \( \omega \), such that
\[
\pi'_{v_1} \cong \pi'_{v_2} \cong \rho' \text{ and } \pi'_u \cong \delta.
\]

Proof. This lemma is an application of the trace formula. For example, the proof of [F; Proposition III.3] can be adapted to this situation. We leave details to the reader. \( \square \)
The automorphic cuspidal representation, described in Lemma 2.1, can be lifted [FK; Theorem 3] to the automorphic cuspidal representation \( \pi = \otimes_v \pi_v \) of \( M(A) \), defined as follows:

\[
\pi_{v_1} \cong \pi_{v_2} \cong \rho\]

\( \pi_v \cong \pi_v' \), for any \( v, v \not\in \{v_1, v_2\} \).

Let \( S \) be a finite set of places of \( k \) containing \( \{v_1, v_2\} \) and all places of residual characteristic 2, such that if \( v \not\in S \) then \( \psi_v \) and \( \pi_v \) are unramified. For every \( v \not\in S \), we denote by \( f_v^s \) (resp. \( f_v^{s'} \)) the unique unramified vector in \( I(\pi_v, s) \) (resp. \( I(\pi_v, s') \)) normalized as in [Sh1; pg. 6]. Since for \( v \not\in S \) our choice of Haar measures coincides with the usual one (where on each root subgroup one takes a self dual measure with respect to \( \psi_v \)), we can apply a result of Langlands (see for example [Sh1; pg. 6]):

\[
\left\{ \begin{array}{l}
A(s, \pi_v, N(k_v), \overline{N(k_v)}) f_v^s(g) = c_v(s, \psi_v) f_v^{s'}(g) \\
A(s, \pi_v, \overline{N(k_v)}, N(k_v)) f_v^{s'}(g) = c_v(-s, \psi_v) f_v^s(g),
\end{array} \right.
\]

where \( c_v(s, \pi_v) \) is a quotient of certain \( L \)-functions. The explicit formula for \( c_v(s, \pi) \) can be found in [Sh1; pg 6]. For our purpose, it is important that the Euler product

\[
c_S(s, \pi) = \prod_{v \not\in S} c_v(s, \pi_v)
\]

converges for \( \Re(s) > 0 \), and it continues to a meromorphic function on \( \mathbb{C} \).

Take \( f^s = \otimes_v f_v^s \) in \( I(\pi, s) \) such that \( f_v^s \) is the unramified vector as above, for all \( v \not\in S \). In view of (2.1) we can apply [MW; Theorem IV.1.10] and obtain

\[
A(s, \pi, N(A), \overline{N(A)}) A(s, \pi, N(A), \overline{N(A)}) f^s = f^s.
\]

Now, using (2.2), it follows from (2.3) that

\[
\prod_{v \in S} \mu(s, \pi_v) \cdot c_S(s, \pi) \cdot c_S(-s, \overline{\pi}) = 1.
\]

Analogously, we can prove

\[
\prod_{v \in S} \mu(s, \pi_v') \cdot c_S(s, \pi') \cdot c_S(-s, \pi') = 1.
\]

Next, if \( v \not\in \{v_1, v_2\} \), then \( D_v \cong M(2, k_v) \). This induces an isomorphism of \( G(k_v) \) and \( G'(k_v) \) restricting to isomorphisms

\[
\left\{ \begin{array}{l}
N(k_v) \cong N'(k_v) \\
\overline{N}(k_v) \cong \overline{N}'(k_v).
\end{array} \right.
\]
It is easy to check that these isomorphisms preserve the self-dual measures. In particular it follows that

\[(2.7) \quad \mu(s, \pi'_v) = \mu(s, \pi_v)\]

if \(v \not\in \{v_1, v_2\}\). Now, (2.4), (2.5) and (2.7) imply

\[(\mu(s, \rho))^2 = (\mu(s, \rho'))^2.\]

Since both Plancherel measures are non-negative along the imaginary axis \(\text{Re}(s) = 0\) [Si, Chapter 5], we obtain the proposition. \(\square\)

3. Applications

Let \(\rho\) be a unitarizable supercuspidal representation of \(GL(n, D)\). In this section we determine the reducibility points of \(I(\rho, s)\), where \(s\) is a real number. First, let us write \(w_0\) for the non-trivial element of the group \(N_G(M)\). Clearly, \(w_0\) acts on representations of \(M = GL(n, D)\). More precisely, the action is given by \(w_0(\rho)(g) = \rho((g^r)^{-1})\). Hence, by Lemma 1.1 we have

\[(3.1) \quad w_0(\rho) \cong \bar{\rho}.\]

Now, we have

**Proposition 3.1.** If \(\rho \not\cong \bar{\rho}\), then \(I(\rho, s)\) is irreducible for all real numbers \(s\). Moreover, \(I(\rho, s)\) is unitarizable only for \(s = 0\).

**Proof.** It follows from [Be; Theorem 28] and (3.1) (and also from Harish-Chandra [Si]) that \(\rho \cong \bar{\rho}\) is a necessary condition for reducibility of \(I(\rho, s)\). Hence \(I(\rho, s)\) is irreducible, for real numbers \(s\). Since, this representation is not Hermitian for \(s \neq 0\), the lemma follows. \(\square\)

In what follows we shall assume that \(\rho \cong \bar{\rho}\). Then, by a result of Silberger, there exists the unique \(s_0 \geq 0\) such that \(I(\rho, \pm s_0)\) reduces, and \(I(\rho, s)\) is irreducible for \(|s| \neq s_0\) [Si; Lemma 1.2]. Moreover, by the general theory of Harish-Chandra [Si; Chapter 5], we have

\[(3.2) \quad \left\{\begin{array}{ll}
s_0 = 0 & \text{if and only if } \mu(s_0, \rho) \neq 0 \\
s_0 > 0 & \text{if and only if } \mu(s_0, \rho) = \infty
\end{array}\right.\]

In the remainder of this section we will calculate \(s_0\), using Proposition 2.1 and (3.2) Thus, let \(\rho'\) be the corresponding square integrable representation of \(GL(2n, F)\). Note that \(\rho'\) is also self-contragredient.

First, we shall assume that \(\rho'\) is supercuspidal. Then, the work of Shahidi [Sh1] implies that there is \(s'_0 \in \{0, 1/2\}\), such that \(I(\rho', \pm s'_0)\) is reducible and \(I(\rho', s)\) is irreducible for \(|s| \neq s'_0\). As in [Sh1], we call \(\rho'\) a representation of symplectic type if \(I(\rho', 1/2)\) is reducible, and a representation of orthogonal type if \(I(\rho', 0)\) is reducible. Also, [Sh1; Lemma 3.6] implies that every self-contragredient supercuspidal representation of \(GL(2n, F)\) is exactly
of one of the above types. Moreover, these definitions do not depend on the choice of the group $G'$ (that is, $G'$ can be either $SO(4n, F)$ or $Sp(4n, F)$).

Furthermore, the dual group of $GL(2n)$ is $GL(2n, \mathbb{C})$. Let $\rho_{2n}$ be the standard representation of $GL(2n, \mathbb{C})$. Let $\wedge^2 \rho_{2n}$ and $\text{Sym}^2 \rho_{2n}$ be the exterior square and symmetric square representation of $GL(2n, \mathbb{C})$, respectively. Shahidi has defined local $L$-functions $L(s, \rho', \wedge^2 \rho_{2n})$ and $L(s, \rho', \text{Sym}^2 \rho_{2n})$ ([Sh1], [Sh2]), and has proved that a self-contragredient representation $\rho'$ has symplectic (resp. orthogonal) type if and only if $L(s, \rho', \wedge^2 \rho_{2n})$ (resp. $L(s, \rho', \text{Sym}^2 \rho_{2n})$) has a pole at $s = 0$.

**Example 3.1.** Let $\rho'$ be a self-contragredient supercuspidal representation of $GL(2, F)$, and let $\omega'$ be its central character. If $\omega' = 1$ then $\rho'$ is of symplectic type, and if $\omega' \neq 1$ then $\rho'$ is of orthogonal type.

Our first result is

**Theorem 3.1.** Assume that $\rho$ is a self-contragredient unitarizable supercuspidal representation of $GL(n, D)$, being the lift of a supercuspidal representation $\rho'$ of $GL(2n, F)$. Then we have

(i) If $\rho'$ has symplectic type, then $I(\rho, \pm 1/2)$ is reducible, and $I(\rho, s)$ is irreducible for $|s| \neq 1/2$. Moreover, $I(\rho, s)$ is in the complementary series if and only if $|s| < 1/2$.

(ii) If $\rho'$ has orthogonal type, then $I(\rho', 0)$ is reducible, and $I(\rho, s)$ is an irreducible non-unitarizable representation for $s \neq 0$.

**Proof.** As explained before, to find the point of reducibility $s_0$, we need to study the poles and zeroes of $\mu(s, \rho)$. Proposition 2.1 implies $\mu(s, \rho) = \mu(s, \rho')$, and the theorem follows. $\square$

In other words, Theorem 3.1 says that $I(\rho, s)$ reduces if and only if $I(\rho', s)$ reduces. On the other hand, reducibility of $I(\rho', 1/2)$ can be checked as follows. Let $w$ be a non-singular skew-symmetric matrix in $GL(2n, F)$. Put

$$Sp(2n, F) = \{g \in GL(2n, F); g^t wg = w\}.$$ 

We have the following result of Shahidi [Sh1; Theorem 5.3].

**Proposition 3.2.** Assume that $\omega'$ is the central character of $\rho'$. For each function $f \in C_c^\infty (GL(2n, F))$, such that

$$f_{\rho'}(g) = \int_Z f(zg)\omega^{-1}(z)dz$$

defines a non-trivial matrix coefficient of $\rho'$, we put

$$I(f) = \int_{Sp(2n, F) \setminus GL(2n, F)} f(g^t \cdot wgw^{-1}) dg.$$ 

Then, $I(\rho', 1/2)$ is reducible if and only if there exists $f$ as above, such that $I(f) \neq 0$.

Finally, we note that Murnaghan and Repka [MR] have computed this integral for a large family of tamely ramified supercuspidal representations.
Now, we will assume that the lift $\rho'$ is not supercuspidal. Hence, by (1.1), $\rho' \cong \delta(\rho_0, 2)$, where $\rho_0$ is an irreducible supercuspidal representation of $GL(n, F)$ and $n$ is odd. Since $\rho$ is self-contragredient, $\rho_0$ must also be self-contragredient. Now, we have

**Theorem 3.2.** Assume that $G' = SO(4n, F) \ (n \text{ is odd}).$ Let $\rho$ be a self-contragredient unitarizable supercuspidal representation of $GL(n, D)$. Assume that $\rho$ corresponds to a discrete series representation $\rho' = \delta(\rho_0, 2)$ of $GL(2n, F)$. Let $I(\rho, s)$ be the induced representation of $G = G_n(D, -1)$. Then $I(\rho, \pm 1/2)$ is reducible, and $I(\rho, s)$ is irreducible for $|s| \neq 1/2$. Moreover, $I(\rho, s)$ is in the complementary series if and only if $|s| < 1/2$.

**Proof.** As explained before, to find the point of reducibility $s_0$, we need to find the poles and zeroes of $\mu(s, \rho) = \mu(s, \rho')$. Let $q$ be the order of the residue field of $F$. Combining (3.16) and (7.4) of [Sh2], the Plancherel measure $\mu(s, \rho')$ is, up to a monomial in $q^s$, equal to

\[
\frac{L(1 + 2s, \rho', \Lambda^2 \rho_2n) L(1 - 2s, \rho', \Lambda^2 \rho_2n)}{L(-2s, \rho', \Lambda^2 \rho_2n) L(2s, \rho', \Lambda^2 \rho_2n)}.
\]

In fact, since both $\mu(s, \rho')$ and the function given by the formula (3.3) are even, they are equal up to a non-zero constant. Since $\rho' = \delta(\rho_0, 2)$, by [Sh1; Proposition 8.1],

\[
L(s, \rho', \Lambda^2 \rho_2n) = L(s + 1, \rho_0, \Lambda^2 \rho_n) L(s, \rho_0, \text{Sym}^2 \rho_n).
\]

Since $n$ is odd, we have that $L(s, \rho_0, \text{Sym}^2 \rho_n)$ has a pole at $s = 0$, while $L(s, \rho_0, \Lambda^2 \rho_n)$ is holomorphic there [Sh1; Proposition 3.5]. Since the $L$-functions of supercuspidal representations have all poles on the imaginary axis $Re(s) = 0$ [Sh2; Proposition 7.3], we see that the only real pole of the $L$-function on the left hand-side of (3.4) is $s = 0$. Now, since local $L$-functions never vanish, (3.3) implies that $s_0 = 1/2$. The theorem is proved. \( \square \)

**Theorem 3.3.** Assume that $G' = Sp(4n, F) \ (n \text{ is odd}).$ Let $\rho$ be a self-contragredient unitarizable supercuspidal representation of $GL(n, D)$. Assume that $\rho$ corresponds to a discrete series representation $\rho' = \delta(\rho_0, 2)$ of $GL(2n, F)$. Let $I(\rho, s)$ be the induced representation of $G = G_n(D, +1)$. Then:

(i) If $n = 1$ and $\rho_0 = 1_{F \times}$, then $I(\rho, \pm 3/2)$ is reducible, and $I(\rho, s)$ is irreducible for $|s| \neq 3/2$. Moreover, $I(\rho, s)$ is in the complementary series if and only if $|s| < 3/2$ (note that the unique irreducible subrepresentation of $I(3/2, \rho)$ is the Steinberg representation of $G$).

(ii) If $n = 1$ and $\rho_0^2 = 1_{F \times}$, $\rho_0 \neq 1_{F \times}$, then $I(\rho, 0)$ is reducible, and $I(\rho, s)$ is an irreducible non-unitarizable representation for $s \neq 0$.

(iii) If $n > 1$ then $I(\rho, \pm 1/2)$ is reducible, and $I(\rho, s)$ is irreducible for $|s| \neq 1/2$. Moreover, $I(\rho, s)$ is in the complementary series if and only if $|s| < 1/2$.

**Proof.** The proof is similar to the proof of Theorem 3.2. This time note that, up to a non-zero constant, the Plancherel measure $\mu(s, \rho')$ is equal to

\[
\frac{L(1 + 2s, \rho', \Lambda^2 \rho_2n) L(1 - 2s, \rho', \Lambda^2 \rho_2n)}{L(-2s, \rho', \Lambda^2 \rho_2n) L(2s, \rho', \Lambda^2 \rho_2n)} \cdot \frac{L(1 + s, \rho') L(1 - s, \rho')}{{L(-s, \rho') L(s, \rho')}}.
\]
where $L(s, \rho')$ is the principal $L$-function [J]. Since $\rho' = \delta(\rho_0, 2)$, by [J; Proposition 3.1.3]
\[
L(s, \rho') = L(s + 1/2, \rho_0).
\]
Note that $L(s, \rho_0)$ has a real pole if and only if $n = 1$ and $\rho_0 = 1_{F^\times}$. Moreover, $s = 0$ is the only real pole of $L(s, 1_{F^\times})$. The theorem follows from (3.5) and a case by case discussion. □

**Remark 3.1.** Note that Theorem 3.1 (combined with Example 3.1) and Theorem 3.3 give a classification of the non-cuspidal part of the unitary dual of the rank one, non-split form of $\text{Sp}(4, F)$.

**Remark 3.2.** We could also calculate the reducibility of $I(\rho, 0)$, where $\rho$ is a unitarizable discrete series representation of $\text{GL}(n, D)$ using the theory of $R$-groups and the results of Shahidi. Note that Shahidi has determined all of the reducibilities $I(\rho', 0)$, where $\rho'$ is a discrete series representation of $M' = \text{GL}(2n, F)$ (see Section 9 and Theorem 9.1 in [Sh1]).

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**References**


Department of Mathematics, University of Utah, Salt Lake City, UT 84112
E-mail address: gmui@math.utah.edu

Department of Mathematics, University of Utah, Salt Lake City, UT 84112
E-mail address: savin@math.utah.edu