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REAL SYMMETRIC BANACH $*$-ALGEBRAS

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Abstract. In this paper new results in the theory of real symmetric Banach $*$-algebras are presented. The main results characterize such algebras among all real Banach $*$-algebras.

Introduction

All algebras in this paper will be supposed to possess identity elements. The identity element will be denoted by e and we shall assume that $\|e\| = 1$. A real Banach algebra A is called real Banach $*$-algebra if there exists an involution (linear anti-isomorphism of period two) $x \mapsto x^*$ on A. An element $h \in A$ is said to be hermitian if $h^* = h$ and skew-hermitian if $h^* = -h$. An element $u \in A$ is said to be unitary if $u^*u = uu^* = e$. An element $x \in A$ is said to be normal if $x^*x = xx^*$. The sets of all hermitian, skew-hermitian, unitary and normal elements of A will be denoted by $H(A)$, $SH(A)$, $U(A)$ and $N(A)$, respectively. It is easy to see that each element $x \in A$ has a unique decomposition $x = h + k$ with $h \in H(A)$, $k \in SH(A)$. An element $x \in A$ is normal if and only if h and k commute. In the study of real Banach $*$-algebras it is very useful that any real Banach $*$-algebra can be isometrically and isomorphically embedded in a certain complex Banach $*$-algebra. This can be done as follows. Let A be a real Banach $*$-algebra. Denote by A_c the cartesian product $A \times A$. Then A_c becomes a complex $*$-algebra if we define operations and the involution as follows

$$(x, y) + (u, v) = (x + u, y + v), \ (\alpha + \beta i)(x, y) =$$

$$= (\alpha x - \beta y, \alpha y + \beta x), \ (x, y)(u, v) = (xu - yv, xv + yu),$$

$$(x, y)^* = (x^*, -y^*).$$

The mapping $x \mapsto (x, 0)$ is an isomorphism of A into A_c. It is possible to introduce a norm in A_c, such that A_c becomes a complex Banach $*$-algebra and the isomorphism $x \mapsto (x, 0)$ an isometry (see [6, p. 5]

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for details). The Banach $*$-algebra A_c is called the complexification of A. We shall write $x + iy$ for an element $(x, y) \in A_c$. The spectrum $\mathrm{sp}_A(x)$ of $x \in A$ is defined to be equal to the spectrum of x as an element of the complexification A_c of A. We shall usually write $\mathrm{sp}(x)$ for $\mathrm{sp}_A(x)$. We shall write $r(x)$ for the spectral radius of $x \in A$, and $p(x)$ for $r(x^*x)^{1/2}$. An element $h \in H(A)$ is said to be positive, $h > 0$ if $\mathrm{sp}(h) > 0$, and nonnegative, $h \geqslant 0$ if $\mathrm{sp}(h) \geqslant 0$. The radical of A will be denoted by $\mathrm{rad}(A)$.

In the first section of this paper we are investigating real symmetric Banach $*$-algebras. The main attention is devoted to positive hermitian functionals on such algebras, and to characterizations of symmetric Banach $*$-algebras. V. Ptak proved in [5] that the inequality $r(x) \leqslant p(x)$ characterizes complex symmetric Banach $*$-algebras among all complex Banach $*$-algebras. In the real case the inequality $r(x) \leqslant p(x)$ does not guarantee the symmetry of a Banach $*$-algebra, although this inequality holds also in real symmetric Banach $*$-algebras. For characterizations of real symmetric Banach $*$-algebras in terms of spectral radius stronger assumptions are necessary. We prove, for example, that a real Banach $*$-algebra A is symmetric if and only if $r(x)^2 \leqslant r(x^*x + y^*y)$ holds for all pairs $x, y \in A$. All results are proved without the assumption that involution in continuous or locally continuous.

In the second section we prove some characterizations of real hermitian B^*algebras and some characterizations of real hermitian Banach $*$-algebras with an equivalent B^* norm. Some of those results were proved by L. Ingelstam [2] and T. W. Palmer [4] by different methods.

In this paper we use methods from the complex case, especially those presented in V. Ptak's paper [5].

1. Real symmetric Banach $*$-algebras

A lemma of J. W. M. Ford [1] makes it possible to develop much of the theory of complex Banach $*$-algebras without the assumption that the involution is continuous or locally continuous. Ford's square root lemma can be proved also for real Banach $*$-algebras (see the proof of Lemma 1 in [4]).

LEMMA 1 (J. W. M. Ford, T. W. Palmer). *Let A be a real Banach $*$-algebra. Let $h \in H(A)$, and suppose that $r(e - h) < 1$. Then there exists $u \in H(A)$, such that u commutes with h and $u^2 = h$. Moreover, if* $\mathrm{sp}(h)$ *is positive, then so is* $\mathrm{sp}(u)$.

DEFINITION 2. *Let A be a real Banach $*$-algebra. We say that A is hermitian if* $\mathrm{sp}(h)$ *is real for each $h \in H(A)$, skew-hermitian if* $\mathrm{sp}(k)$ *is imaginary for each $k \in SH(A)$, and symmetric if $(e + x^*x)^{-1}$ exists for each $x \in A$.*

It is obvious that a complex Banach $*$-algebra is hermitian if and only if it is skew-hermitian. It is routine to prove that any complex symmetric Banach $*$-algebra is hermitian. Conversely, any complex hermitian Banach $*$-algebra is symmetric. This result was first proved by S. Shirali and J. W. M. Ford [7]. T. W. Palmer (Lemma 1 in [4]) showed that the methods developed in [7] can be used also for the real case and proved the following

LEMMA 3 (S. Shirali, J. W. M. Ford, T. W. Palmer). *A real hermitian and skew-hermitian Banach $*$-algebra is symmetric.*

In the theorem below results presented by V. Ptak [5] concerning complex symmetric Banach $*$-algebras are extended to the real case.

THEOREM 4. *Let A be a real symmetric Banach $*$-algebra. Then the following statements are valid.*

1° *To each positive $h \in H(A)$ there corresponds positive $u \in H(A)$, such that $h = u^2$;*

2° $r(x) \leq p(x)$ *for each* $x \in A$;

3° $r(uv) \leq r(u)\, r(v)$ *for each pair* $u, v \in H(A)$;

4° $p(xy) \leq p(x)\, p(y)$ *for each pair* $x, y \in A$;

5° $x \in \operatorname{rad}(A) \Leftrightarrow p(x) = 0$;

6° *If $u \in H(A)$ and $v \in H(A)$ are nonnegative, then so is $u + v$;*

7° $r(u + v) \leq r(u) + r(v)$ *for each pair* $u, v \in H(A)$;

8° $r(\frac{1}{2}(x \pm x^*)) \leq p(x)$ *for each* $x \in A$;

9° $p(x + y) \leq p(x) + p(y)$ *for each pair* $x, y \in A$.

Proof. 1° A simple consequence of Lemma 1. 2° It is possible to use the proof from the complex case (see the proof of Theorem (5,2) in [5] for details). 3° The proof is based on the inequality $r(x) \leq p(x)$ and the fact that in any Banach algebra $r(xy) = r(yx)$ (see the proof of (5,3) in [5]). 4° A simple consequence of 3°. 5° If $x \in \operatorname{rad}(A)$ then $x^*x \in \operatorname{rad}(A)$, and by Theorem (2.3.4.) in [6, p. 56] $r(x^*x) = 0$. Conversely, if $p(x) = 0$ then for each $y \in A$ we have $r(yx) \leq p(yx) \leq p(y)\, p(x) = 0$. Since $r(yx) = 0$ for each $y \in A$, we have $y \in \operatorname{rad}(A)$. 6° Let u and v be nonnegative. We have to prove that $u + v$ is also nonnegative. Let t be an arbitrary positive real number. Then $\frac{t}{2} e + u$, $\frac{t}{2} e + v$ are positive. It follows from the first statement of this theorem, that $\frac{t}{2} e + u = h^2$, $\frac{t}{2} e + v = k^2$ for some positive h and k. We have $u + v + te = h^2 + k^2 = h(e + h^{-1}k^2h^{-1})h = h(e + (kh^{-1})^*(kh^{-1}))h.$

Here h is regular, and so is $e + (kh^{-1})^* (kh^{-1})$, since A is symmetric. Therefore $-t \notin \mathrm{sp}(u + v)$. 7° A simple consequence of 6°. 8° Let $x = u + v$, $u \in H(A)$, $v \in SH(A)$ be an arbitrary vector. Then $x^*x + xx^* = 2(u^2 - v^2)$. Obviously $r(u^2 - v^2)e - (u^2 - v^2) \geqslant 0$. Since $-v^2 \geqslant 0$, $u^2 \geqslant 0$, and the sum of nonnegative elements is nonnegative, we obtain $r(u^2 - v^2)e - u^2 \geqslant 0$, $r(u^2 - v^2)e + v^2 \geqslant 0$. Therefore $r(u^2) \leqslant r(u^2 - v^2)$, $r(v^2) \leqslant r(u^2 - v^2)$. Hence $r(\frac{1}{2}(x + x^*))^2 = r(u)^2 \leqslant \frac{1}{2} r(x^*x + xx^*)$, $r(\frac{1}{2}(x - x^*))^2 = r(v)^2 \leqslant \frac{1}{2} r(x^*x + xx^*)$. Using the subaditivity of the spectral radius on $H(A)$, we obtain $r(\frac{1}{2}(x \pm x^*))^2 \leqslant \frac{1}{2} r(x^*x + xx^*) \leqslant \frac{1}{2}(r(x^*x) + r(xx^*)) = p(x)^2$. 9° The subaditivity of the function p follows from the subaditivity of the spectral radius on $H(A)$, the statement 8°, and the submultiplicativity of the function p. (See the proof of (5,8) in [5] for details.) The proof of the theorem is complete.

V. Ptak proved that the inequality $r(x) \leqslant p(x)$ characterizes complex symmetric Banach $*$-algebras among all complex Banach $*$-algebras. We proved in the theorem above that this inequality holds also in real symmetric Banach $*$-algebras. The converse is not true. There exist real Banach $*$-algebras which are not symmetric, although the inequality $r(x) \leqslant p(x)$ holds. In the theorem below we shall prove that the inequality $r(x) \leqslant p(x)$ implies symmetry of real Banach $*$-algebra if we assume that the algebra is hermitian. Similarly, other characterizations of complex symmetric Banach $*$-algebras, included in Theorem (5,10) [5], are extended to the real case.

THEOREM 5. *Let A be a real Banach $*$-algebra. Then the following statements are equivalent.*

1° *A is symmetric.*

2° *A is hermitian and $r(x) \leqslant p(x)$ for all $x \in A$.*

3° *A is hermitian and $r(x) \leqslant p(x)$ for all $x \in N(A)$.*

4° *A is hermitian and $r(x) = p(x)$ for all $x \in N(A)$.*

5° *A is hermitian and $r(\frac{1}{2}(x \pm x^*)) \leqslant p(x)$ for all $x \in A$.*

6° *A is hermitian and $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in A$.*

7° *A is hermitian and $r(u) = 1$ for all $u \in U(A)$.*

8° *A is hermitian and $r(u) \leqslant 1$ for all $u \in U(A)$.*

9° *A is hermitian and $r(u) \leqslant a$ for all $u \in U(A)$ and some $a > 0$.*

10° *A is hermitian and skew-hermitian.*

Proof. The implication 10° $\Rightarrow$ 1° is contained in Lemma 3. The implication 1° $\Rightarrow$ 2° follows from the fact that each real symmetric Banach $*$-algebra is hermitian and from Theorem 4. The implications 2° $\Rightarrow$ 3° and 3° $\Rightarrow$ 4° are trivial. Assume 4° and prove 10°. We have to prove that the spectrum of each skew-hermitian element is imaginary. Suppose, on the contrary, there exists a $k \in SH(A)$ such that $\alpha + \beta i \in \mathrm{sp}(k)$, $\alpha \neq 0$. Then for each real number t the element $te + k$ is normal and $t + \alpha + \beta i \in \mathrm{sp}(te + k)$. Using condition 4° we obtain $(t + \alpha)^2 + \beta^2 \leqslant r(te + k)^2 = r((te + k)^*(te + k)) = r(t^2 e - k^2) \leqslant t^2 + r(k)^2$. Therefore $2\alpha t + \alpha^2 + \beta^2 \leqslant r(k)^2$. Since t is an arbitrary real number, the contradiction is obvious. The implication 1° $\Rightarrow$ 5° is contained in Theorem 4. Assume 6° and prove 5°. Since $x + x^*$ is hermitian and $x - x^*$ skew-hermitian, we have $r(x \pm x^*) = p(x \pm x^*)$. Therefore $r(\frac{1}{2}(x \pm x^*)) = p(\frac{1}{2}(x \pm x^*)) \leqslant \frac{1}{2} p(x) + \frac{1}{2} p(x^*) = p(x)$. Assume 5° and prove 3°. Let $x = h + k$, $h \in H(A)$, $k \in SH(A)$ be a normal element. Then, since h and k commute, we have $r(x) \leqslant r(h) + r(k)$. Using 5° we obtain $r(x) \leqslant r(h) + r(k) \leqslant p(x) + p(x) = 2p(x)$. Therefore $r(x) \leqslant 2p(x)$ for each $x \in N(A)$. Since $x \in N(A)$ implies $x^n \in N(A)$ it follows that

$$r(x^n) \leqslant 2p(x^n), \quad x \in N(A). \tag{5.1.}$$

It is easy to obtain $p(x^n) = p(x)^n$. Substituting this in (5.1.) we obtain $r(x)^n = r(x^n) \leqslant 2p(x)^n$, hence $r(x) \leqslant 2^{1/n} p(x)$ for each integer n. The proof of the implication 5° $\Rightarrow$ 3° is complete. The implications 7° $\Rightarrow$ 8° $\Rightarrow$ 9° are trivial. Let us prove the implication 9° $\Rightarrow$ 8°. If $u \in U(A)$ then also $u^n \in U(A)$ for each integer n, so that $r(u) = r(u^n)^{1/n} \leqslant \alpha^{1/n}$. Therefore $r(u) \leqslant 1$. The implication 8° $\Rightarrow$ 7° is a consequence of the fact that $u \in U(A)$ implies $u^{-1} \in U(A)$. Let us prove that 7° implies 10°. We have to prove that A is skew-hermitian. Let k, $r(k) < 1$ be a skew-hermitian element. We shall consider k as an element of A_c. Let C be a maximal commutative $*$-subalgebra of A_c containing k. Then $\mathrm{sp}_C(k) = \mathrm{sp}_{A_c}(k)$. Since $r(e - (e + k^2)) = r(k^2) < 1$, there exists $h \in H(A) \cap C$ such that $e + k^2 = h^2$. The element $u = h + k$ is unitary. Let f be an arbitrary multiplicative linear functional on C. Since $r(u) = 1$ we have

$$\begin{aligned} f(h) + f(k) &= f(u) = \exp(\alpha i), \\ f(h) - f(k) &= f(u^*) = \exp(\beta i). \end{aligned} \tag{5.2.}$$

Since A is hermitian $f(h)$ is real. It follows from (5.2.) that

$$0 \leqslant f(h)^2 \leqslant 1. \tag{5.3.}$$

At the same time $f(h)^2 - f(k)^2 = f(h^2 - k^2) = f(u^* u) = f(e) = 1$. Therefore $f(k)^2 = f(h)^2 - 1$ and (5.3.) implies that $f(k)$ is imaginary. Since f was an arbitrary multiplicative linear functional on C it follows that $\mathrm{sp}_C(k) = \mathrm{sp}_{A_c}(k)$ is imaginary. The proof of the implication 7° $\Rightarrow$ 10° is complete. Since the implication 1° $\Rightarrow$ 6° is included in Theorem 4 and the implication 4° $\Rightarrow$ 7° is obvious the proof of the theorem is complete.

Now we shall list some elementary results concerning positive functionals on real Banach $*$-algebras.

DEFINITION 6. *Let A be a real Banach $*$-algebra. Let f be a linear functional on A. We call a functional f hermitian if $f(x^*) = f(x)$ for each $x \in A$, skew-hermitian if $f(x^*) = -f(x)$ for each $x \in A$, positive if $f(x^*x) \geqslant 0$ for each $x \in A$, and weakly positive if $f(h^2) \geqslant 0$ for each $h \in H(A)$.*

LEMMA 7. *Let A be a real Banach $*$-algebra and f a positive hermitian functional. Then the following statements are valid.*

1° $|f(y^*x)|^2 \leqslant f(x^*x) f(y^*y)$ *for all pairs* $x, y \in A$;

2° $|f(x)|^2 \leqslant f(e) f(x^*x)$ *for all* $x \in A$.

Proof. Obviously 2° follows from 1°. The proof of 1° is similar to the proof in the complex case. (See [6, p. 212] for details.)

LEMMA 8. *Let A be a real Banach $*$-algebra. For arbitrary weakly positive linear functional f the inequality $|f(h)| \leqslant f(e)\, r(h)$ is valid for each $h \in H(A)$. If f is positive and hermitian we have*

1° $|f(x)|^2 \leqslant f(e)^2\, r(x^*x)$ *for each* $x \in A$;

2° $f(x^*xx^*x) \leqslant r(x^*x) f(x^*x)$ *for each* $x \in A$.

Proof. Let h be in $H(A)$ and ε an arbitrary positive real number. Denote $(r(h) + \varepsilon)^{-1} h$ by k. Since $r(e - (e - k)) < 1$ there exists $u \in H(A)$ such that $e - k = u^2$ by Ford's square root lemma. Hence $f(e - k) = f(u^2) \geqslant 0$. This implies $f(h) \leqslant f(e)(r(h) + \varepsilon)$. Therefore $f(h) \leqslant f(e)\, r(h)$. If we replace h by $-h$, we obtain $|f(h)| \leqslant \leqslant f(e)\, r(h)$.

Let f be positive and hermitian. Then using Lemma 7 and $f(x^*x) \leqslant \leqslant f(e)\, r(x^*x)$ we obtain $|f(x)|^2 \leqslant f(e)^2\, r(x^*x)$. It remains to prove 2°. If f is positive and hermitian, then a routine calculation shows that so is g defined by $g(x) = f(a^*xa)$. Since g is positive and hermitian, we have $g(xx^*) \leqslant g(e)\, r(xx^*)$. If we replace a by x, we obtain $f(x^*xx^*x) \leqslant f(x^*x)\, r(x^*x)$.

It is well known that each positive functional on a unital complex Banach $*$-algebra is bounded (see (2,6) in [5]). The same is true for positive hermitian functionals on unital real Banach $*$-algebras. This can be proved in the following way. Denote by f a positive hermitian functional acting on a real Banach $*$-algebra A, and define the linear functional F on A_c as follows $F(x + iy) = f(x) + if(y)$, $x + + iy \in A_c$. Then routine calculation shows that F is positive functional on complex Banach $*$-algebra A_c and therefore bounded. It follows that the functional f is bounded too.

We can say more about positive functionals on a real Banach $*$-algebra if algebra is symmetric. The complex version of the following results concerning positive functionals can be found in [5].

PROPOSITION 9. *Each weakly positive functional on a real symmetric Banach $*$-algebra is positive.*

Proof. Let $x \in A$ be an arbitrary element and ε arbitrary positive real number. Then $\varepsilon e + x^*x$ is positive since the algebra is symmetric. By Theorem 4 there exists $u \in H(A)$ such that $\varepsilon e + x^*x = u^2$. Therefore $f(\varepsilon e + x^*x) = f(u^2) \geqslant 0$ since f is weakly positive. This implies $f(x^*x) \geqslant 0$.

The theorem below is similar to (6,4) in [5] proved for the complex case.

THEOREM 10. *Let A be a real symmetric Banach $*$-algebra. Let f be a linear functional on A such that $f(e) > 0$. Then the following conditions are equivalent.*

1° *f is hermitian and $|f(h)| < f(e)\, r(h)$ for each $h \in H(A)$.*

2° *f is positive and hermitian.*

3° *f is weakly positive and hermitian.*

4° *$|f(x)| \leqslant f(e)\, p(x)$ for each $x \in A$.*

5° *$|f(x)| \leqslant f(e)\, p(x)$ for each $x \in N(A)$.*

Proof. Assume 1° and prove 2°. The proof is borrowed from the proof of (6,3) in [5]. We may assume that $f(e) = 1$. Let us take $h \in H(A)$ and write

$$\alpha = \inf_{\lambda \in \mathrm{sp}(h)} \{\lambda\}, \qquad \beta = \sup_{\lambda \in \mathrm{sp}(h)} \{\lambda\},$$

$$\varrho = \tfrac{1}{2}(\alpha + \beta), \qquad \delta = \tfrac{1}{2}(\beta - \alpha).$$

Since $\alpha \leqslant \mathrm{sp}(h) \leqslant \beta$ we have $r(h - \varrho e) \leqslant \delta$. By the hypotheses of the theorem we have $|f(h) - \varrho| = |f(h - \varrho e)| \leqslant r(h - \varrho e) \leqslant \delta$. Therefore

$$\alpha = \varrho - \delta \leqslant f(h) \leqslant \varrho + \delta = \beta. \tag{10.1.}$$

Let x be an arbitrary element in A. Since A is symmetric, we have $\mathrm{sp}(x^*x) \geqslant 0$. Using (10.1.) we obtain $f(x^*x) \geqslant 0$. The implications 2° $\Rightarrow$ 3° and 4° $\Rightarrow$ 5° are trivial. The implication 3° $\Rightarrow$ 4° follows from Proposition 9 and Lemma 8. It remains to prove the implication 5° $\Rightarrow$ 1°. Let h be hermitian. Then $|f(h)| \leqslant f(e)\, p(h) = f(e)\, r(h)$. Therefore we have only to prove that the functional f is hermitian. For this purpose let us write f in the form $f = f_1 + f_2$. Here f_1 is hermitian and f_2 a skew-hermitian functional. We shall prove that $f_2 = 0$. It suffices to prove that $f_2(k) = 0$ for each $k \in SH(A)$. Let k be in $SH(A)$ and let t be an arbitrary real number. The element $x = te + k$ is normal. Using $p(x)^2 = p(x^*x)$ and the subaditivity of p we obtain

$$p(x)^2 \leqslant t^2 + p(k)^2. \tag{10.2.}$$

Using condition 5° and (10.2.) we obtain

$$|f(x)|^2 = (tf_1(e) + f_2(k))^2 \leqslant f(e)^2 p(x)^2 \leqslant$$
$$\leqslant t^2 f(e)^2 + f(e)^2 p(k)^2.$$

Since $f(e) = f_1(e)$ we have finaly

$$2tf(e) f_2(k) + f_2(k)^2 \leqslant f(e)^2 p(k)^2$$

for each real number t. This implies $f_2(k) = 0$. The proof of the theorem is complete.

The theorem above allows us to prove two characterizations of real symmetric Banach $*$-algebras in terms of positive functionals.

THEOREM 11. *Let A be a real Banach $*$-algebra. The following statements are equivalent.*

1° *A is symmetric.*

2° *To each proper left ideal $I \subset A$ there corresponds a positive hermitian functional f, $f(e) = 1$, such that $f(I) = 0$.*

3° *For each $x \in A$ $\sup f(x^*x) = p(x)^2$, f a positive hermitian functional such that $f(e) = 1$.*

Proof. Let us assume 1° and prove 2°. We can define a linear functional f on the linear subspace $X = \{x + te;\ x \in I,\ t \text{ real}\}$ as follows

$$f(x + te) = t.$$

It is evident that $f(e) = 1$, $f(I) = 0$ and that $t \in \operatorname{sp}(x + te)$. Since A is symmetric, we have $r(x + te) \leqslant p(x + te)$. Using that inequality and the fact that $t \in \operatorname{sp}(x + te)$, we obtain $|f(x + te)| = |t| \leqslant r(x + te) \leqslant p(x + te)$. The function p is subaditive since A is symmetric. Hence by Hahn-Banach theorem f can be extended to the whole of A satisfying the condition $|f(x)| \leqslant p(x)$ for each $x \in A$. By Theorem 10 f is positive and hermitian. The implication 1° $\Rightarrow$ 2° is so proved. Let us prove the implication 2° $\Rightarrow$ 3°. Since $\sup\limits_{f(e)=1} f(x^*x) \leqslant p(x)^2$ by Lemma 8, it suffices to prove that there exists a hermitian positive functional f, $f(e) = 1$, such that $f(x^*x) = p(x)^2$. The element $r(x^*x)\,e - x^*x$ is without left inverse. Hence $r(x^*x)\,e - x^*x$ is contained in some proper left ideal I. By assumption there exists a positive hermitian functional f, $f(e) = 1$, such that $f(I) = 0$ and therefore $f(r(x^*x)\,e -$

$- x^*x) = 0$. The implication $2° \Rightarrow 3°$ is so proved. It remains to prove the implication $3° \Rightarrow 1°$. The proof of this implication is borrowed from the proof of Theorem (4.7.21.) in [6]. Let x be given, let us write ϱ for $r(x^*x)$ and set $u = \varrho e - x^*x$. For arbitrary positive hermitian functional f, $f(e) = 1$, we obtain $f(u^2) = \varrho^2 - 2\varrho f(x^*x) + f(x^*xx^*x)$. Combining this equation with $f(x^*xx^*x) \leqslant \varrho f(x^*x)$ (see Lemma 8), we obtain $f(u^2) \leqslant \varrho^2 - \varrho f(x^*x) \leqslant \varrho^2$. Since f was an arbitrary positive hermitian functional such that $f(e) = 1$ it follows from assumption that $r(u)^2 = r(u^2) = \sup_{f(e)=1} f(u^2) \leqslant \varrho^2$. Hence $r(u) \leqslant \varrho$ or $r(r(x^*x)e - x^*x) \leqslant r(x^*x)$. This inequality implies that A is symmetric. The proof of the theorem is complete.

The complex version of the equivalence $1° \Leftrightarrow 2°$ was proved by N. Namsraj [3]. The equivalence $1° \Leftrightarrow 3°$ in the complex case is a well known result, first proved by D. A. Raikov (see Theorem 4.7.21. in [6] and Theorem (6,5) in [5]).

We mentioned that in the real case the inequality $r(x) \leqslant p(x)$ does not guarantee the symmetry of a real Banach $*$-algebra. For characterizations of real symmetric Banach $*$-algebras in terms of spectral radius stronger assumptions are necesary. In the theorem below some results of this kind are presented.

THEOREM 12. *Let A be a real Banach $*$-algebra. The following conditions are equivalent.*

$1°$ *A is symmetric.*

$2°$ *$r(x)^2 \leqslant r(x^*x)$ for all $x \in A$ and $r(x^*x) \leqslant r(x^*x + y^*y)$ for all pairs $x, y \in A$.*

$3°$ *$r(x)^2 \leqslant r(x^*x)$ for all $x \in N(A)$ and $r(x^*x) \leqslant r(x^*x + y^*y)$ for all commuting pairs $x, y \in N(A)$.*

$4°$ *$r(x)^2 \leqslant r(x^*x + y^*y)$ for all $x, y \in A$.*

$5°$ *$r(x)^2 \leqslant r(x^*x + y^*y)$ for all commuting pairs $x, y \in N(A)$.*

Proof. Assume $1°$ and prove $2°$. Since A is symmetric we have $r(x)^2 \leqslant r(x^*x)$ for each $x \in A$ by Theorem 4. Using the result in Theorem 11 we obtain $r(x^*x) = \sup_{f(e)=1} f(x^*x) \leqslant \sup_{f(e)=1} f(x^*x + y^*y) = r(x^*x + y^*y)$. The implications $2° \Rightarrow 4° \Rightarrow 5°$, $2° \Rightarrow 3° \Rightarrow 5°$ are obvious. It remains to prove the implication $5° \Rightarrow 1°$. Let B be an arbitrary closed commutative $*$-subalgebra of A. Let $x + iy \in B_c$ be unitary. Then $x^*x + y^*y = e$. Since x and y commute we have $r(x + iy) \leqslant r(x) + r(y)$. Using the assumption we obtain $r(x + iy) \leqslant r(x) + r(y) \leqslant 2r(x^*x + y^*y)^{1/2} = 2r(e) = 2$. Since we proved that $r(x + iy) \leqslant 2$ for arbitrary $x + iy \in U(B_c)$ it follows that B_c is symmetric (see Theorem (5,10) in [5]). Therefore B is also symmetric. We proved that each closed commutative $*$-subalgebra of algebra A is symmetric. This implies that A is symmetric. The proof of the theorem is complete.

2. Real B^* algebras

DEFINITION 13. *A real Banach $*$-algebra is called a B^*algebra if* $\|x^*x\| = \|x\|^2$ *for each* $x \in A$.

The well known result of I. M. Gelfand and M. A. Naimark that every complex B^*algebra is isometrically $*$-isomorphic to some C^* algebra is not true in real case (see [2, p. 265]). L. Ingelstam [2] proved that the Gelfand-Naimark theorem holds for real B^*algebras if we assume that real B^* algebra is also symmetric. Similar result was obtained by T. Palmer [4]. T. Palmer considered real Banach $*$-algebras with generalized involution. Symmetry of real B^* algebra follows from weaker assumption that B^* algebra is hermitian. Using suitable characterizations of real symmetric Banach $*$-algebras from the first section, this can be proved in a very simple way.

PROPOSITION 14. *A real hermitian B^*algebra is symmetric.*

Proof. From $\|x^*x\| = \|x\|^2$ it follows that $r(h) = \|h\|$ for each $h \in H(A)$. Then $p(x)^2 = \|x^*x\| = \|x\|^2$. Hence $p(x)$ is a norm and since A is hermitian it follows from Theorem 5 that A is symmetric.

The characterizations of real hermitian B^* algebras and characterizations of real hermitian Banach $*$-algebras with equivalent B^* norm, presented in the theorems below, are similar to some results obtained by L. Ingelstam [2] and T. Palmer [4]. The proofs presented here seem to be simpler.

THEOREM 15. *Let A be a real Banach $*$-algebra. Then the following conditions are equivalent.*

1° *A is a hermitian B^* algebra.*

2° $\|x\|^2 = \|x^*x\|$ *for each* $x \in A$ *and* $\|x\|^2 \leqslant \|x^*x + y^*y\|$ *for each pair* $x, y \in A$.

3° $\|x\|^2 = \|x^*x\|$ *for each* $x \in A$ *and* $\|x\|^2 \leqslant \|x^*x + y^*y\|$ *for all commuting pairs* $x, y \in N(A)$.

4° $\|x\|^2 \leqslant \|x^*x + y^*y\|$ *for each pair* $x, y \in A$.

Proof. Assume 1° and prove 2°. Since in B^* algebra $r(h) = \|h\|$, $h \in H(A)$ and since by Proposition 14 hermitian B^* algebra is symmetric, we have by Theorem 12 $\|x^*x\| = r(x^*x) \leqslant r(x^*x + y^*y) = \|x^*x + y^*y\|$. The implication 1° $\Rightarrow$ 2° is proved. The implications 2° $\Rightarrow$ 3° and 2° $\Rightarrow$ 4° are obvious. Assume 3° and prove 1°. We have to prove that A is hermitian. Let commuting pair $x, y \in N(A)$ be given. Since $r(h) = \|h\|$ for each $h \in H(A)$, we have $r(x)^2 \leqslant \|x\|^2 = \|x^*x + y^*y\| = r(x^*x + y^*y)$ for all commuting pairs $x, y \in N(A)$. By Theorem 12 A is symmetric. The implication 4° $\Rightarrow$ 1° can be proved in a similar way. The proof of the theorem is complete.

THEOREM 16. *Let A be a real Banach $*$-algebra. Then the following conditions are equivalent.*

1° *A is hermitian with an equivalent B^*norm.*

2° *A is hermitian and $\alpha \|x\|^2 \leqslant \|x^*x\|$ for each $x \in A$ and some $\alpha > 0$.*

3° *A is hermitian and $\alpha \|x\|^2 \leqslant \|x^*x\|$ for each $x \in N(A)$ and some $\alpha > 0$.*

4° *A is hermitian and $\|u\| \leqslant \alpha$ for each $u \in U(A)$ and some $\alpha > 0$, $\beta \|h\| \leqslant r(h)$ for each $h \in H(A)$ and some $\beta > 0$.*

5° *A is symmetric and $\alpha \|h\| \leqslant r(h)$ for all hermitian and skew-hermitian elements h and some $\alpha > 0$.*

6° *$\alpha \|x\|^2 \leqslant \|x^*x + y^*y\|$ for all pairs $x, y \in A$ and some $\alpha > 0$.*

7° *$\alpha \|x\|^2 \leqslant \|x^*x + y^*y\|$ for all commuting pairs $x, y \in N(A)$ and some $\alpha > 0$.*

Proof. The implications $1° \Rightarrow 2° \Rightarrow 3°$ are trivial. Assume 3° and prove 4°. Let $h \in H(A)$ be given. Then by assumptions $\alpha \|h\|^2 \leqslant \|h^2\|$. This inequality implies $\alpha^2 \|h\|^2 \leqslant \alpha \|h^2\|$ whence $\alpha \|h\| \leqslant (\alpha \|h^2\|)^{1/2}$. Inductively, $\alpha \|h\| \leqslant (\alpha \|h^{1/2^n}\|)^{1/2^n}$ for each integer n. This implies $\alpha \|h\| \leqslant r(h)$. Obviously $\alpha \|u\|^2 \leqslant 1$ for all $u \in U(A)$. The implication $3° \Rightarrow 4°$ is so proved. Let us prove that 4° implies 5°. Since $\|u\| \leqslant \alpha$ for all $u \in U(A)$, A is symmetric by condition 9° in Theorem 5. Let $k \in SH(A)$, $r(k) < 1$ be given. Since $r(e - (e + k^2)) < 1$, there exists $h \in H(A)$ commuting with k such that $e + k^2 = h^2$. Therefore the element $u = h + k$ is unitary. Since $k = \frac{1}{2}(u - u^*)$, we obtain using the assumption $\|k\| \leqslant \frac{1}{2}(\|u\| + \|u^*\|) \leqslant \alpha$. Hence $\|k\| \leqslant \alpha r(h)$ for each $k \in SH(A)$ and the implication $4° \Rightarrow 5°$ is proved. Let us prove that 5° implies 6°. Let $x = h + k$, $h \in H(A)$, $k \in SH(A)$ be given. Since A is symmetric we have $r(h) \leqslant p(x)$, $r(k) \leqslant p(x)$ by condition 8° in Theorem 4. Using that and the assumption we obtain $\alpha \|x\| \leqslant \alpha(\|h\| + \|k\|) \leqslant r(h) + r(k) \leqslant 2p(x)$. By condition 2° in Theorem 12 $p(x)^2 \leqslant r(x^*x + y^*y)$ for all pairs $x, y \in A$. Hence $\alpha^2 \|x\|^2 \leqslant 4p(x)^2 \leqslant 4r(x^*x + y^*y) \leqslant 4\|x^*x + y^*y\|$ for all pairs $x, y \in A$. The implication $5° \Rightarrow 6°$ is proved. Since the implication $6° \Rightarrow 7°$ is trivial it suffices to prove that 7° implies 1°. It is possible to prove that

$$\alpha \|h\| \leqslant r(h) \qquad (16.1.)$$

for all hermitian and skew-hermitian elements h since $\alpha \|x\|^2 \leqslant \|x^*x\|$ for all $x \in N(A)$ (see the proof of the implication $3° \Rightarrow 4°$). Let us first prove that A is symmetric. Let $x = h + k \in N(A)$ be given. Then using the inequality (16.1.) we obtain $\alpha^2 r(x)^2 \leqslant \alpha^2 \|x\|^2 \leqslant \alpha \|x^*x + y^*y\| \leqslant r(x^*x + y^*y)$. Hence $\alpha^2 r(x)^2 \leqslant r(x^*x + y^*y)$ for all commuting pairs $x, y \in N(A)$ and some $\alpha > 0$. This implies the symmetry of A (see the proof of the implication $5° \Rightarrow 1°$ in Theorem 12). It remains to prove that there exists a B^* norm equivalent to the given norm. Let an arbitrary $x = h + k$ be given. Since A is symmetric we have $r(h) \leq p(x)$, $r(k) \leq p(x)$. Using that and the inequality (16.1.) we obtain $\alpha \|x\| \leq \alpha \|h\| + \alpha \|k\| \leq r(h) + r(k) \leq 2p(x)$. Therefore

$$\alpha \|x\| \leq 2p(x) \tag{16.2.}$$

for all $x \in A$. This inequality implies that A is semisimple (see condition 5° in Theorem 4). We shall prove that the involution is continuous. It suffices to prove that the involution is closed. Assume that $x_n \to 0$ and $x_n^* \to y$. Then using the subaditivity of the function p we obtain $p(y) \leq p(y - x_n^* + x_n^*) \leq p(y - x_n^*) + p(x_n^*) \leq \|y^* - x_n\|^{1/2} \|y - x_n^*\|^{1/2} + \|x_n\|^{1/2} \|x_n^*\|^{1/2}$. In each of the summands on the right side one factor is bounded and one tends to zero. It follows that $p(y) = 0$ so that y is contained in the radical whence $y = 0$. Let us prove that there exists a constant $\beta > 0$ such that for each $x \in A$

$$p(x) \leq \beta \|x\|. \tag{16.3.}$$

Since the involution is bounded we have $p(x)^2 = r(x^*x) \leq \|x^*x\| \leq \|x^*\| \|x\| \leq \beta^2 \|x\|^2$. The function p is a norm since A is symmetric and semisimple. Therefore since $p(x)^2 = p(x^*x)$ we have a B^* norm on A which is by inequalities (16.2.) and (16.3.) equivalent with the given norm. The proof of the theorem is complete.

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REALNE SIMETRIČNE BANACHOVE ALGEBRE Z INVOLUCIJO

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Povzetek

Realna ali kompleksna Banachova algebra z involucijo je simetrična, če za vsak x iz algebre obstaja $(e + x^*x)^{-1}$, kjer je e enota algebre. V članku so obravnavane realne simetrične Banachove algebre z involucijo. Glavna pozornost je posvečena pozitivnim funkcionalom nad temi algebrami in karakterizacijam teh algeber.