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CONTINUOUS IMAGES OF ORDERED CONTINUA

Sibe Mardešić and Pavle Papić, Zagreb

All spaces in this paper are Hausdorff spaces. By a *continuum* we mean any Hausdorff compact and connected space. A continuum is non-degenerate if it consists of more than one point. An *ordered continuum (ordered compact)* is a continuum C (a compact space K) provided with a total ordering $<$ such that the topology of C (of K) is the order topology induced by $<$ (see e. g. [1], p. 57). The only metric ordered continuum is the arc, i. e. the homeomorph of the real line unit segment $I = [0, 1]$. Examples of nonmetric ordered continua are given by various »transfinite lines« (see e. g. [1], L, p. 164 and also [3]).

The object of this paper is to study the class $\mathfrak{X}$ consisting of all spaces X, where X is obtainable as the image of at least one ordered continuum C under a map $f: C \to X$ onto X. To spaces of class $\mathfrak{X}$ we shall refer merely as to *continuous images of ordered continua.*

Since each C is a locally connected continuum, the same is true of spaces of class $\mathfrak{X}$. By a classical theorem of H. Hahn and S. Mazurkiewicz (see e. g. [5], Chapter III) in the case of metric continua also the converse is true, for all metric locally connected continua are continuous images of I and thus belong to $\mathfrak{X}$. In this paper we exhibit a further necessary condition for a space to belong to the class $\mathfrak{X}$, which in the general nonmetric case is independent of the previous ones. It involves the notion of *weight* $\mathrm{w}(X)$ of a space X. $\mathrm{w}(X)$ is the least cardinal which occurs as the cardinal of a basis for the topology of X. Thus metric non-degenerate continua are characterized by $\mathrm{w}(X) = \aleph_0$. For spaces of class $\mathfrak{X}$ we prove that the weight is invariant under mappings $p: X \to Y$, $p(X) = Y$, which have the property that $p^{-1}(y)$ is *nowhere dense*[1] in X, for each $y \in Y$ (see Theorem 1).

This result is applied to give a complete characterization of those product spaces $\Pi_a X_a$ which belong to $\mathfrak{X}$ (see Theorem 3). Finally, we prove that for spaces X of class $\mathfrak{X}$ the weight $\mathrm{w}(X)$ coincides with the *degree of separability* $\mathrm{s}(X)$, which is the least cardinal which occurs as the cardinal of a subset of X dense in X.

[1] A set $A \subset X$ is said to be nowhere dense in X if every open set $U \subset X$, $U \neq 0$, contains an open subset $V \subset U$ which is non-empty and is disjoint with A. For closed A this amounts to having an empty interior.

The proofs of these assertions are based on several lemmas.

Lemma 1. Let $f: C \to X$ be a mapping of the ordered continuum C onto X and let K be a closed subset of C having the property that f maps each component U_α of $C \setminus K$ into a nowhere dense set of X. Then $f(K) = X$.

Proof. First observe that for any finite collection $\{U_{\alpha 1}, \ldots, U_{\alpha n}\}$ of components of $C \setminus K$ we have

$$f(C \setminus (U_{\alpha 1} \cup \ldots \cup U_{\alpha n})) = X. \tag{1}$$

Indeed, the set $f(U_{\alpha 1} \cup \ldots \cup U_{\alpha n})$ is nowhere dense in X and thus the set

$$f(C \setminus (U_{\alpha 1} \cup \ldots \cup U_{\alpha n})) \supset X \setminus f(U_{\alpha 1} \cup \ldots \cup U_{\alpha n}) \tag{2}$$

is dense in X. However, this set is closed and must contain all of X.

Now assume (conversely to Lemma 1) that there is an $x \in X$ with

$$f^{-1}(x) \subset C \setminus K = \bigcup_\alpha U_\alpha. \tag{3}$$

Then $\{U_\alpha\}$ is an open covering of the compact space $f^{-1}(x)$ and thus reduces to a finite subcovering. Hence, $f^{-1}(x) \subset U_{\alpha 1} \cup \ldots \cup U_{\alpha n}$, which contradicts (1).

Lemma 2. Let $f: C \to X$ and $p: X \to Y$ be mappings onto. If p has the property that $p^{-1}(y)$ is nowhere dense in X, for each $y \in Y$, then there exists a closed subset $K \subset C$, which is $w(Y)$-separable[2] and such that $f(K) = X$.

Proof. Let $\{U_\alpha\}$, $\alpha \in A$, be the family of all maximal connected open subsets of C for which the restriction map $pf \mid U_\alpha$ is constant. $\{U_\alpha\}$, $\alpha \in A$, is a well- defined family of disjoint open sets U_α (it may be empty) and is obtained as follows. For each $y \in Y$, consider the interior of the set $(pf)^{-1}(y)$. The sets U_α are obtained as components of these interiors, when y runs through Y. Since $p^{-1}(y)$ is nowhere dense, it follows that, for each U_α, $f(U_\alpha)$ is nowhere dense in X. K is defined as $C \setminus \bigcup_\alpha U_\alpha$, $\alpha \in A$. It follows from Lemma 1 that $f(K) = X$.

In order to show that K is $w(Y)$-separable, consider the decomposition of C into (closed) segments $\mathrm{Cl}\, U_\alpha$, $\alpha \in A$, and all remaining points. Let C_1 denote the corresponding quotient space and $m: C \to C_1$ the natural projection map. m being monotone, the ordering in C induces an ordering in C_1 and it is readily seen that the induced order topology of C_1 coincides with the quotient topology. Thus C_1 is another ordered continuum. Since, for each $\alpha \in A$, $pf(\mathrm{Cl}\, U_\alpha)$ is a single point, we obtain a unique map $g: C_1 \to Y$ verifying $gm = pf$. By the maximality property of sets U_α it follows that g is a light mapping (i. e. $g^{-1}(y)$ is totally disconnected, for each $y \in Y$). According to a theorem of one of

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[2] Here $w(Y)$ denotes the weight of Y. We say that a space X is k-separable if it admits a dense subset of power $\leq k$, i. e. if its degree of separability $s(X) \leq k$.

the authors, the weight is an invariant of light maps for locally connected continua ([2]). Thus $w(C_1) = w(Y)$. It follows that C_1 contains a dense subset R_1 of power $w(Y)$.

In order to define a dense subset R in K observe that, for each $t_1 \in C_1$, $m^{-1}(t_1)$ is either a closed segment $\mathrm{Cl}\, U_a$ of C with the two end-points belonging to K or it is a point of $C \setminus \bigcup_a (\mathrm{Cl}\, U_a) \subset K$, which can also be considered as a segment of C with identical end-points from K (if U_a is an end-section of C then it may happen that $(\mathrm{Cl}\, U_a) \setminus U_a$ consists of only one point). We define R as the set consisting of all the end-points of $m^{-1}(t_1)$, when t_1 runs through R_1. R is clearly a dense subset of K and of the same cardinality as R_1 (Notice that $w(Y) \geqq \aleph_0$, for otherwise Y would be a point, contradicting the assumptions on p). This completes the proof of Lemma 2.

L e m m a 3.[3] *Let* X *be a locally connected space,* K *an ordered compact and* $f: K \to X$ *a mapping onto* X. *Then* $w(X) \leqq s(K)$, *where* $w(X)$ *is the weight of* X *and* $s(K)$ *is the degree of separability of* K.

It is clear that $s(X) \leqq s(K)$ and that $w(X) \leqq w(K)$, but notice that for ordered compact spaces K one can have $s(K) < w(K)$. An example is obtained by replacing uncountably many points $t \in I = [0,1]$ by pairs of points (t', t''), where one considers that $t' < t''$. The newly obtained ordered compact is still $\aleph_0$ -separable but fails to be metric. Also notice that for locally connected continua X one can have $s(X) < w(X)$. An example is given by the product of $\aleph_1$ copies of I. This continuum is $\aleph_0$ -separable but fails to be metric (see [1], N, p. 103).

P r o o f o f L e m m a 3. It suffices to consider the case when both spaces K and X are infinite. Let R be a subset dense in K and of power $\aleph_\tau$. Since $\mathrm{Cl}\, R = K$ and $f(K) = X$ it follows that $f(R)$ is dense in X and thus $s(X) \leqq \aleph_\tau$. Denote by $\mathfrak{R}$ the family of all closed segments of K of the form $[r_\xi, s_\xi]_K$, where $r_\xi, s_\xi \in R, r_\xi < s_\xi$.[4] Clearly, the cardinal $k(\mathfrak{R}) = k(R) = \aleph_\tau$. Let $\mathfrak{Q}$ denote the family of all sets Q which are unions of finitely many segments $[r_{\xi i}, s_{\xi i}]_K \in \mathfrak{R} \cdot k(\mathfrak{Q}) = k(\mathfrak{R}) = \aleph_\tau$. For each $Q \in \mathfrak{Q}$ consider the interior $\mathrm{Int}\, f(Q)$ of the set $f(Q)$. Due to local connectedness of X all components of $\mathrm{Int}\, f(Q)$ are disjoint open sets. Furthermore, $s(X) \leqq \aleph_\tau$ implies that, for each $Q \in \mathfrak{Q}$, the set $\mathrm{Int}\, f(Q)$ has at most $\aleph_\tau$ components. Let $\mathfrak{B}$ denote the family of all components of $\mathrm{Int}\, f(Q)$, when Q runs through $\mathfrak{Q}$. $\mathfrak{B}$ is a family of open sets and $k(\mathfrak{B}) = \aleph_\tau$ To complete the proof it suffices to show that $\mathfrak{B}$ is a basis for the topology of X.

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[3] Instead of Lemma 3 the first draft of this paper contained a slightly weaker statement. The authors are indebted to Dr. A. J. W a r d for the observation that their proof of the original lemma essentially established the neater and stronger Lemma 3 given at present.

[4] Here $[a, b]_K, a < b$, denotes the set of all points $t \in K$ with $a \leqq t \leqq b$.

Let $x \in X$ and let U be an arbitrary open set, $x \in U \subset X$. We have to find a set $V \in \mathfrak{B}$ with $x \in V \subset U$. Since $f^{-1}(U)$ is an open set of K, it is the union of a family of disjoint open intervals $(a_i, b_i)_K$ of K[5]. The set of those $(a_i, b_i)_K$ which intersect $f^{-1}(x)$ is finite. Let these intervals be $(a_1, b_1)_K, \ldots, (a_n, b_n)_K$. Their union contains $f^{-1}(x)$ and is contained in $f^{-1}(U)$. Denote by a_i', b_i' the first and the last point of $(a_i, b_i)_K \cap f^{-1}(x) \cdot a_i', b_i'$ exist and $a_i < a_i' \leqq b_i' < b_i$. Now we shall define, for each $i \in \{1, \ldots, n\}$, a closed segment $J_i = [c_i, d_i]_K$ such that

$$[a_i', b_i']_K \subset \operatorname{Int} J_i \subset J_i \subset (a_i, b_i)_K \tag{4}$$

as follows. If $(a_i, a_i')_K \cap R \neq 0$, choose $c_i \in (a_i, a_i')_K \cap R$; otherwise, let $c_i = a_i'$. Notice that in the second case $(a_i, a_i')_K$ must be empty, which means that a_i' is an interior point of $[a_i', b_i']_K$ with respect to $K \cdot d_i$ is defined in the same way so that (4) follows.

Denote by J the union $J = J_1 \cup \ldots \cup J_n$. (4) implies

$$f^{-1}(x) \subset \operatorname{Int} J \subset J \subset f^{-1}(U). \tag{5}$$

Int J being an open set containing $f^{-1}(x)$, it follows from the continuity of f that there is an open set $U', x \in U'$, such that $f^{-1}(U') \subset \operatorname{Int} J$ (this is easily proved by contradiction). Consequently, $x \in U' \subset f(J)$, which proves

$$x \in \operatorname{Int} f(J). \tag{6}$$

On the other hand (5) implies

$$\operatorname{Int} f(J) \subset f(J) \subset U. \tag{7}$$

J is not necessarily a member of $\mathfrak{Q}$, because the end-points of some J_i may belong to $K \setminus R$. Therefore, we replace all J_i by new closed segments J_i' with end-points from R and such that $J_i \subset J_i'$, that $J_i' \setminus J_i$ is a closed set (which is empty or consists of one or two segments of K) and that

$$f(J_i' \setminus J_i) \cap f(J) = 0\,. \tag{8}$$

If both end-points of J_i are from R we set $J_i' = J_i$. If for a given $J_i = [c_i, d_i]_K$ we have $c_i = a_i'$ and $d_i \in R$, then $(a_i, a_i')_K = 0$, $a_i < a_i'$, and a_i is either a cluster point of points from R which precede a_i or $a_i \in R$. Since $f(a_i) \in X \setminus U$ and $f(J) \subset U$, there exists a point $c_i' \in R$, $c_i' \leqq a_i$, close enough to a_i or equal to a_i and such that

$$f([c_i', a_i]_K) \cap f(J) = 0. \tag{9}$$

In this case we set

$$J_i' = [c_i', a_i]_K \cup [a_i', d_i]_K = [c_i', d_i]_K \tag{10}$$

and the above conditions are verified. We proceed in exactly the same way with the right end-point in cases when $d_i = b_i'$.

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[5] Here $(a, b)_K, a < b$, denotes the set of all points $t \in K$ with $a < t < b$. If the first (the last) point of K belongs to $f^{-1}(U)$, then the corresponding interval of $f^{-1}(U)$ consists of all points $t < b_i$ ($t > a_i$).

Now denote by J' the union $J_1' \cup \ldots \cup J_n'$ and observe that $J' \setminus J = (J_1' \setminus J_1) \cup \ldots \cup (J_n' \setminus J_n)$ is a closed set. Clearly, $J' \in \mathfrak{Q}$ and $J \subset J'$, which implies (by (6)) that $x \in \operatorname{Int} f(J')$. Furthermore, (8) implies

$$f(J' \setminus J) \cap f(J) = 0. \tag{11}$$

This means that

$$f(J') = f(J' \setminus J) \cup f(J) \tag{12}$$

is a decomposition of $f(J')$ in disjoint closed sets and therefore the component V of $\operatorname{Int} f(J')$ which contains x coincides with that component of $\operatorname{Int} f(J)$ which contains x. Thus (7) implies that $x \in V \subset \subset U$, $V \in \mathfrak{B}$, and the proof is completed.

Theorem 1. *Let* X *be a continuous image of an ordered continuum,* $\mathrm{p} : \mathrm{X} \to \mathrm{Y}$ *a mapping onto another continuum* Y. *If* p *has the property that, for each* $\mathrm{y} \in \mathrm{Y}$, $\mathrm{p}^{-1}(\mathrm{y})$ *is a nowhere dense set in* X, *then* $\mathrm{w}(\mathrm{X}) = \mathrm{w}(\mathrm{Y})$, *i. e.* X *and* Y *have equal weights.*

Proof. Y being an image of X under a continuous map p it follows that $\mathrm{w}(Y) \leqq \mathrm{w}(X)$ (see e. g. Lemma 3 in [2]) $\cdot$ $\mathrm{w}(X) \leqq \mathrm{w}(Y)$ is an immediate consequence of Lemmas 2 and 3.

This theorem can be compared to the theorem on invariance of weight for locally connected continua under light mappings [2]. Now the assumptions on maps are weaker, but the assumptions on spaces are stronger.

Theorem 2. *Let* X *be a continuum and* I *the real line segment. If* $\mathrm{X} \times \mathrm{I}$ *is the continuous image of an ordered continuum, then* X *is metric and thus a Peano continuum.*

Proof. $X \times I$ is a locally connected continuum and so is X. Let $p : X \times I \to I$ be the natural projection given by $p(x, u) = u$, $x \in X$, $u \in I$. Clearly, $p^{-1}(u) = X \times u$ is nowhere dense in $X \times I$, for any $u \in I$. Applying Theorem 1 we obtain $\mathrm{w}(X \times I) = \mathrm{w}(I) = \aleph_0$, which means that $X \times I$ and thus also X is a metric continuum.

Now consider products $X \times Y$ of two non-degenerate continua. By the Urysohn lemma there exists a map $g : Y \to I$ sending two distinct points of Y into the two end-points of I. Connectedness of Y implies that g is onto and so is $(1 \times g) : X \times Y \to X \times I$. Consequently, if $X \times Y$ is the image of an ordered continuum, then so is $X \times I$ and Theorem 1 implies that X is a Peano continuum. The same is true of Y. In general we have.

Theorem 3. *In order that a product space* $\Pi_a \mathrm{X}_a$, $a \in \mathrm{A}$ *(cardinal* $\mathrm{k}(\mathrm{A}) > 1$*), of non-degenerate continua be the continuous image of an ordered continuum it is necessary and sufficient that all* X_a *be metric Peano continua and that* $\mathrm{k}(\mathrm{A}) \leqq \aleph_0$. *In this case* $\Pi_a \mathrm{X}_a$ *is itself a Peano continuum and thus a continuous image of* I.

Proof. For an arbitrary $a' \in A$ let $Y_{a'} = \Pi'_a X_a$, where Π'_a denotes the product taken over all $a \in A$ except a'. Clearly, $\Pi_a X_a = X_{a'} \times Y_{a'}$. Consequently, if $\Pi_a X_a$ is the image of an ordered continuum then both $X_{a'}$ and $Y_{a'}$ are Peano continua. Hence $\Pi_a X_a$

is itself a Peano continuum and an image of I (the Hahn-Mazurkiewicz theorem). Clearly, $k(A) \leqq \aleph_0$; for $\Pi_\alpha X_\alpha$, $\alpha \in A$, can be mapped (the Urysohn lemma) onto $\Pi_\alpha I_\alpha$, $\alpha \in A$, where $I_\alpha = I$ and it is well-known that, for infinite A, $w(\Pi_\alpha I_\alpha) = k(A)$ (this follows from [4]). Thus $\aleph_0 \geqq w(\Pi_\alpha I_\alpha) = k(A)$, because $\Pi_\alpha I_\alpha$, $\alpha \in A$, is a metric continuum.

Theorem 3 presents a considerable strengthening of a former result of one of the authors, according to which all the X_α had to possess the Suslin property (see [3]).

C o r o l l a r y 1. If all X_α, $k(A) > 1$, are ordered continua and $\Pi_\alpha X_\alpha$, $\alpha \in A$, is a continuous image of an ordered continuum, then all $X_\alpha = I$ and $k(A) \leqq \aleph_0$. Hence $\Pi_\alpha X_\alpha$ is either an n-dimensional cube or the Hilbert cube.*)

A product $\Pi_\alpha I_\alpha$, $\alpha \in A$, of uncountably many copies I_α of the line segment I is a locally connected continuum, which by Corollary 1 fails to be the image of an ordered continuum. Moreover, observe that it also fails to be the image of an ordered compact K. Indeed, each K is easily imbedded in an ordered continuum C by an order preserving imbedding, and a map of K onto $\Pi_\alpha I_\alpha$, $\alpha \in A$, could always be extended to all of C by the Tietze extension theorem.

C o r o l l a r y 2. Let C_1, C_2, C_3 be three ordered continua and f a map of C_1 onto $C_2 \times C_3$. Then $C_2 = C_3 = I$ and f can be factored into a (monotone) map $g : C_1 \to I$ and a »Peano map« $h : I \to I \times I$ onto $I \times I$.

Proof. $C_2 = C_3 = I$ is a consequence of Corollary 1. Applying the monotone-light factorization theorem we obtain g and h (see e. g. [6], p. 141). g being monotone, $g(C_1)$ is also an ordered continum. Since $h : g(C_1) \to I \times I$ is light it follows (by Theorem 1) that $w(g(C_1)) = w(I \times I) = \aleph_0$; hence $g(C_1) = I$.

A consequence of this corollary is the fact that I is the only ordered continuum which admits a map onto its square $I \times I$ (the Peano phenomenon). This has been established already in [3] by other methods.

It is clear that for ordered continua C the degree of separability $s(C)$ and the weight $w(C)$ coincide. Now we shall establish the same fact for images of ordered continua.

T h e o r e m 4. *Let* X *be a continuous image of an ordered continuum. Then the weight* $w(X)$ *and the degree of separability* $s(X)$ *are equal.*

The proof is an immediate consequence of Theorem 1 and this

L e m m a 4. Let X be a non-degenerate $\aleph_\tau$- separable continuum. Then there exists a continuum Y of weight $w(Y) \leqq \aleph_\tau$ and a map $p : X \to Y$ onto Y such that, for each $y \in Y$, the set $p^{-1}(y)$ is nowhere dense in X.

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*) Added in proof: Prof. Đ. Kurepa has informed the authors that he also obtained this result.

Proof. Let $R \subset X$ be a dense subset of power $\leqq \aleph_\tau$, $\tau \geqq 0$. The set $R \times R$ of all pairs (r, r'), $r, r' \in R$, is also of power $\aleph_\tau$. For each pair (r, r') define (by the Urysohn lemma) a map $p_{rr'}: X \to I = [0, 1]$ such that

$$p_{rr'}(r) = 0, \; p_{rr'}(r') = 1. \tag{13}$$

The maps $p_{rr'}$, $(r, r') \in R \times R$, define a mapping p of X into the product space $\Pi_{rr'} I_{rr'}$, $(r, r') \in R \times R$, $I_{rr'} = I$. This product space is of weight $k(R \times R) \leqq \aleph_\tau$. Let $Y = p(X) \subset \Pi_{rr'} I_{rr'}$. Y is a continuum of weight $w(Y) \leqq \aleph_\tau$. It remains to show that, for each $y \in Y$, $p^{-1}(y)$ is nowhere dense in X. If it were not so, we would have a $y \in Y$ and an open set $U \subset X$ contained in $p^{-1}(y)$ $(U \neq 0)$. Since X has no isolated points, we could find two distinct points $r \neq r'$, $r \in R \cap U$, $r' \in R \cap U$. Now $U \subset p^{-1}(y)$ would imply $p(r) = p(r') = y$ and therefore also $p_{rr'}(r) = p_{rr'}(r')$, which contradicts (13). This establishes the lemma.

Finally, let us point out that the question of a topological characterization of continuous images of ordered continua remains open.

Question. Can one characterize images of ordered continua as connected and locally connected spaces which are continuous images of ordered compacta? In the metric case the answer is affirmative, because metric compact spaces and images of ordered metric compacta coincide.

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NEPREKIDNE SLIKE UREĐENIH KONTINUUMA

Sibe Mardešić i Pavle Papić, Zagreb

Sadržaj

Pod kontinuumom razumijevamo u ovom članku svaki Hausdorffov kompaktan povezan prostor. Kontinuum je nedegeneriran ako se sastoji od više nego jedne točke. Uređeni kontinuum (ure-

đeni kompakt) je svaki kontinuum C (kompaktan prostor K), koji posjeduje takovo potpuno uređenje $<$, da se topologija prostora C (prostora K) podudara s uređajnom topologijom koju inducira $<$. Jedini metrički uređeni kontinuum je luk, t. j. homeomorfna slika jediničnog segmenta $I = [0, 1]$ realnog pravca. Primjeri nemetričkih uređenih kontinuuma dani su raznim »transfinitnim linijama« (vidi na pr. [1], str. 164. i [3]).

Predmet ovog rada je ispitivanje razreda $\mathfrak{X}$ svih Hausdorffovih prostora X, koji se mogu dobiti kao slike barem jednog uređenog kontinuuma C pri neprekidnom preslikavanju $f : C \to X$ na čitavi X. O ovim prostorima govorimo kraće kao o *neprekidnim slikama uređenih kontinuuma.*

Kako je svaki C lokalno povezan kontinuum, to su svi prostori razreda $\mathfrak{X}$ također lokalno povezani kontinuumi. Prema klasičnom teoremu H. Hahna i S. Mazurkiewicza u metričkom slučaju vrijedi i obrat, jer je svaki metrički lokalno povezani kontinuum neprekidna slika segmenta I. U ovom članku iznosi se jedan daljnji nuždan uvjet, da bi neki prostor pripadao razredu $\mathfrak{X}$. Ovaj uvjet je u općem nemetričkom slučaju nezavisan od gore spomenutih uvjeta. Formulacija iziskuje pojam težine $\mathrm{w}(X)$ prostora X. $\mathrm{w}(X)$ je najmanji kardinalni broj, koji se javlja kao kardinalni broj neke baze okolina prostora X.

Teorem 1. Neka je X neprekidna slika uređenog kontinuuma, a $p : X \to Y$ preslikavanje na čitavi kontinuum Y. Ako p ima svojstvo da je za svaki $y \in Y$ skup $p^{-1}(y)$ nigdje gust u X, tada je $\mathrm{w}(X) = \mathrm{w}(Y)$.

Uz pomoć ovog rezultata dobiven je

Teorem 3. Da bi produkt $\Pi_\alpha X_\alpha$, $\alpha \in A$, potencija $\mathrm{k}(A) > 1$, nedegeneriranih kontinuuma X_α bio neprekidna slika uređenog kontinuuma nužno je i dovoljno da svi X_α budu metrički Peanovi kontinuumi i da bude $\mathrm{k}(A) \leq \aleph_0$. U ovom slučaju je sam $\Pi_\alpha X_\alpha$ Peanov kontinuum, te je neprekidna slika od I.

Na kraju radnje se uspoređuje težina $\mathrm{w}(X)$ prostora $X \in \mathfrak{X}$ sa stepenom separabilnosti $\mathrm{s}(X) \cdot \mathrm{s}(X)$ je definiran kao minimum potencija skupova gustih na X.

Teorem 4. Neka je X neprekidna slika uređenog kontinuuma. Tada je $\mathrm{w}(X) = \mathrm{s}(X)$.

Ključnu ulogu u dokazivanju navedenih teorema igra ova

Lema 3. Neka je X lokalno povezan kompaktan prostor, K uređen kompakt, a $f : K \to X$ preslikavanje na čitavi X. Tada je $\mathrm{w}(X) \leq \mathrm{s}(K)$.

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