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CHAINABLE CONTINUA AND INVERSE LIMITS

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1. *Introduction*

A *chain* $(U_1, \ldots, U_n)$ is a finite collection of sets U_i such that $U_i \cap U_j \neq 0$ if and only if $|i-j| \leq 1$. The sets U_i are called *links* of the chain $(U_1, \ldots, U_n)$; U_1 and U_n are the two *end-links*. A topological Hausdorff space[1)] X is said to be *chainable* if each open covering of X can be refined by an open covering $u = \{U_1, \ldots, U_n\}$ such that $(U_1, \ldots, U_n)$ is a chain (U_i need not be connected). It is clear that each chainable X is compact and connected, i. e. a continuum. Moreover, X has *covering dimension* $\dim X \leq 1$[2)].

The arc, i. e. the homeomorph of the real line segment $I = [0,1]$, is an obvious example of a chainable continuum. Furthermore, if $\{I_a; \pi_{aa'}\}$ is an inverse system[3)] of arcs I_a, then the inverse limit[3)] $\lim I_a$ is also a chainable continuum (see Lemma 1). From a paper by J. R. Isbell [3] one derives that each metrizable chainable continuum is the inverse limit of a sequence of arcs. It is natural to ask whether metrizability is actually needed, in other words one has the question: *Is every chainable continuum the inverse limit of an inverse system of arcs?* This question has been raised in a recent paper by R. H. Rosen ([8], p. 170).

In this paper we answer Rosen's question negatively (see Theorem 6), by constructing in Section 4 a chainable continuum C with *inductive dimension* ind $C > 1$. By a previous result of ours, a compact space X with ind $X > 1$ cannot be obtained as the inverse limit of 1-dimensional and 0-dimensional polyhedra[4)] and a fortiori cannot be the limit of an inverse system of arcs (see [7], Theorem 4).

The problem of expanding chainable continua into inverse systems of arcs is in many respects analogous to the problem of expanding compact spaces X of (covering) dimension $\dim X \leq n$ into inverse systems of polyhedra of dimension not greater than n (see

[1)] All spaces in this paper are assumed to be topological Hausdorff spaces.

[2)] dim X can be zero only in the case when X consists of a single point.

[3)] α ranges through an arbitrary directed set $(A, \leq)$, $\pi_{\alpha\alpha'}$: $I_{\alpha'} \to I_\alpha$ are mappings onto, defined whenever $\alpha \leq \alpha'$. For basic definitions and facts concerning inverse systems and their limits, see e. g. [2] and [4].

[4)] By a polyhedron we always mean a compact polyhedron.

[7], especially the introduction). In [7] we established the existence of an expansion of an arbitrary compact space X with $\dim X \leq n$ into an inverse system of metrizable compacta X_a with $\dim X_a \leq n$. In this paper we show that an arbitrary chainable continuum is the inverse limit of an inverse system of metrizable chainable continua (see Theorem 2′). Both situations can be treated in the same way so that it appears convenient to state and prove a unique set of theorems, which would contain both, the theorems on dimension and those on chainable continua, as special cases. This is carried through in Section 2.

Section 3. starts with a few elementary facts about chainable continua. In particular, we verify certain conditions from Section 2 in order to make sure that general results from that section are applicable to chainable continua. Some of the results thus obtained for chainable continua are then stated explicitly.

2. *General Expansion Theorems for Compact Spaces*

Let $\mathfrak{P}$ be a property well-defined on compact spaces, so that each compact space X either has property $\mathfrak{P}$ or does not. We shall be especially interested in properties $\mathfrak{P}$, which satisfy the following three conditions:

(A) *Approximation condition. Let* X *be any compact space having property* $\mathfrak{P}$, *let* P *be a polyhedron with a given metric* d, $\mathrm{r} > 0$ *a real number and* $\mathrm{f}: \mathrm{X} \to \mathrm{P}$ *a mapping. Then there exists a polyhedron* Q *having property* $\mathfrak{P}$, *and there exist a map* $\mathrm{g}: \mathrm{X} \to \mathrm{Q}$ *and a map* $\mathrm{p}: \mathrm{Q} \to \mathrm{P}$ *such that*

$$g(X) = Q, \tag{1}$$

$$d(f, pg) \leq r; \tag{2}$$

pg denotes as usual the composite mapping, while $d(f, pg) = \operatorname{Sup} d(f(x), pg(x)), x \in X$.

(B) *Continuity condition. Whenever* $\{\mathrm{X}_a; \pi_{aa'}\}$ *is an inverse system of compact spaces* X_a *all of which have property* $\mathfrak{P}$, *and* $\pi_{aa'}: \mathrm{X}_{a'} \to \mathrm{X}_a$ *are mappings onto, then the limit* $\mathrm{X} = \lim \mathrm{X}_a$ *also has property* $\mathfrak{P}$.

(C) *If* X *is a compact space,* Q *a polyhedron,* $\mathrm{f}: \mathrm{X} \to \mathrm{Q}$ *a mapping, and both* X *and* Q *have property* $\mathfrak{P}$, *then* f(X) *also has property* $\mathfrak{P}$.

Example. Let n be a fixed non-negative integer and let $\mathfrak{P}$ be the property of a compact space X to have covering dimension $\dim X \leq n$. Then $\mathfrak{P}$ verifies conditions (A), (B) and (C) (see [7], Lemma 1 and Section 1.4).

We can state now the main expansion and factorization theorems; $\mathfrak{P}$ shall denote a property defined for compact spaces.

Theorem 1. *If* $\mathfrak{P}$ *verifies the condition* (A), *then each metrizable compact space* X *having property* $\mathfrak{P}$ *is homeomorphic with* *the inverse limit of an inverse sequence*[5] $\{Q_i; q_{ij}\}$ *of polyhedra* Q_i *all of which have property* $\mathfrak{P}$.

Theorem 2. *If* $\mathfrak{P}$ *verifies the conditions* (A), (B) *and* (C), *then each compact space* X *having property* $\mathfrak{P}$ *is homeomorphic with the inverse limit of an inverse system* $\{Q_b; p_{bb'}\}$ *of metrizable compacta* Q_b *all of which have property* $\mathfrak{P}$; b *ranges through a directed set* B *of cardinality* $k(B) \leq$ *weight* $w(X)$[6]

An immediate consequence of these two theorems is

Corollary 1. *If* $\mathfrak{P}$ *verifies the conditions* (A), (B) *and* (C), *then each compact space* X *having property* $\mathfrak{P}$ *is homeomorphic with a double iterated inverse limit* $\lim_b (\lim_i P_{bi})$ *of polyhedra* P_{bi} *all of which have property* $\mathfrak{P}$; i *ranges through positive integers.*

Theorem 3. *Let* $\mathfrak{P}$ *verify the conditions* (A), (B) *and* (C), *let* X *and* P[7] *be two compact spaces, let* X *have property* $\mathfrak{P}$ *and let* $f: X \to P$ *be a mapping. Then there exists a compact space* Q *having property* $\mathfrak{P}$, *and there are mappings* $g: X \to Q$, $p: Q \to P$ *such that* g *is onto and*

$$w(Q) \leq w(P)^{6)}, \tag{3}$$

$$f = pg. \tag{4}$$

Denote by $\omega_{\tau(X)}$ the initial ordinal whose cardinality is equal to the weight $w(X)$. Then we have

Theorem 4. *Let* $\mathfrak{P}$ *verify the conditions* (A), (B) *and* (C). *Then each non-metrizable compact space* X *having property* $\mathfrak{P}$ *is homeomorphic with the inverse limit of an inverse system* $\{X_\beta; p_{\beta\beta'}\}$; β *ranges through the set of all ordinals* $\beta < \omega_{\tau(X)}$. X_β *are compact spaces all of which have property* $\mathfrak{P}$ *and verify*

$$w(X_\beta) < w(X). \tag{5}$$

Furthermore, if $k(\beta)$ *denotes the cardinality of the ordinal* β, *then*

$$w(X_\beta) \leq k(\beta), \quad \omega_0 \leq \beta < \omega_{\tau(X)}, \tag{6}$$

$$w(X_\beta) \leq \aleph_0, \quad \beta < \omega_0. \tag{7}$$

If β is a limit ordinal, then

$$X_\beta = \lim \{X_\alpha; p_{\alpha\alpha'}\}, \quad \alpha < \beta, \tag{8}$$

$p_{\alpha\beta}: X_\beta \to X_\alpha$ *being the corresponding projections.*

A proof of Theorems 1, 2, 3, and 4 will be obtained from the corresponding considerations in [7], by replacing the specific property $\mathfrak{P}$ of having dimension $\leq n$ by an arbitrary property $\mathfrak{P}$ which verifies the conditions (A), (B) and (C). We shall briefly indicate the main lines of argument, emphasizing only deviations from the proofs in [7].

[5]) An inverse sequence is an inverse system $\{X_\alpha; \pi_{\alpha\alpha'}\}$, $\alpha \leq A$, for which A is the set of positive integers.

[6]) The weight $w(X)$ of a space X is the least cardinal which is the cardinal number of a basis for the topology of X.

[7]) We exclude the case when P is a finite set of points.

All proofs are based on analogues of Lemmas 1 to 5 of [7]. The exact statements of these analogues are obtained by adding to the statements of the lemmas in [7] the requirement that X has property $\mathfrak{P}$, which verifies the conditions (A), (B) and (C); furthermore, one has to replace the assertion, that the spaces Q, Q_i and Q_b respectively have dimension not greater than dim X, by the assertion that these spaces have property $\mathfrak{P}$.

The analogue of Lemma 1 is true according to (A). The analogue of Lemma 2 follows in the same way as in [7]. This enables us to prove the analogue of Lemma 3 as well as Theorem 1 of this paper, by following the proofs given in [7] (in particular in Sections 2.2 and 2.4).

In order to prove the analogue of Lemma 4, especially in the case $n = 1$, we have to consider a mapping $f_1 : X \to Q$, where X is a compact space having property $\mathfrak{P}$, f_1 is onto, and Q is the inverse limit of an inverse sequence $\{Q_i ; q_{ij}\}$ of polyhedra Q_i all of which have property $\mathfrak{P}$. We need the conclusion that Q also has property $\mathfrak{P}$ (compare with Section 2.3 of [7]). First, we replace $\{Q_i ; q_{ij}\}$ by the inverse sequence $\{q^i(Q) ; q_{ij}\}$, where $q^i : Q \to Q_i$ are the natural projections[8]. This sequence has the same limit Q and q_{ij} now become mappings onto. According to (B) it suffices to show that all $q^i(Q)$ have property $\mathfrak{P}$. This follows from (C) and the fact that $q^i(Q) = q^i g(X) \subset Q_i$.

Next we proceed to the analogue of Lemma 5 and hereafter to Theorems 2 and 3 (which are the analogues of Theorems 1 and 2 of [7]). Observe that the projections $p_{bb'}$ of the inverse system $\{Q_b ; p_{bb'}\}$ in Lemma 5 of [7] are mappings onto. Furthermore, all Q_b have property $\mathfrak{P}$, so that $Q = \lim Q_b$ also has property $\mathfrak{P}$ according to (B). This observation is needed in the proof of the above Theorem 3.

Finally, we proceed to the proof of Theorem 4 (the analogue of Theorem 3 of [7]). Notice that the mappings $p_{\alpha\alpha'} : X_{\alpha'} \to X_\alpha$, which appear in the system $\{X_\alpha ; p_{\alpha\alpha'}\}$, $\alpha < \beta$ (see [7], Section 4), are mappings onto. Hence, we can conclude that $X_\beta = \lim X_\alpha$ has property $\mathfrak{P}$ provided that all X_α have property $\mathfrak{P}$.

3. *Theorems on Chainable Continua*

1. Let $\mathfrak{C}$ denote the property of a compact space to be chainable. We shall show that the property $\mathfrak{C}$ verifies the three conditions (A), (B) and (C) of Section 2.

Here is a standard proposition that we shall utilize in verifying the condition (A).

Let X be a compact space and let Ω be a family of finite open coverings of X, which is cofinal in the set of all open coverings of X (the ordering is given by the notion of refinement). Let P be a

[8]) Notations are those of [7], Section 2.3.

compact polyhedron with a given metric d, *let* $r > 0$ *be a real number and* $f: X \to P$ *a mapping. Then there exist a covering* $v \in \Omega$, *a mapping* $g: X \to N(v)$ *of* X *into the nerve* N(v) *of* v, *and a mapping* $p: N(v) \to P$ *such that*

$$d(f, pg) \leq r. \tag{1}$$

A proof of this proposition can be obtained e. g. from the first part of the proof of Lemma 1 of [7]. One has to replace the family of coverings, whose order is not greater than $1 + \dim X$, by the family Ω.

Now take for Ω the family of all open coverings of X, which are chains. If X is a chainable continuum, then Ω is cofinal and the proposition of above is applicable. It yields an approximate factorization of f through the nerve of a chain-covering of X, i. e. through an arc I. Since X is connected, so is $g(X) \subset I$, and thus $g(X)$ is either an arc or a single point. Hence, in both cases $g(X)$ is a chainable polyhedron. This establishes the assertion (A) for $\mathfrak{C}$.

Remark 1. The only chainable polyhedra are: the arc and a single point. Indeed, chainable continua have (covering) dimension 0 (only in the case of a single point) or 1. Hence, a chainable polyhedron is either a single point or a connected graph. Furthermore, it is clear that a chainable continuum cannot contain a triod; this rules out all the graphs except the arc and the circle. Finally, a chainable continuum must be acyclic, which leaves the arc as the only 1-dimensional chainable polyhedron.

2. Now, we shall prove a slightly stronger statement (Lemma 1) than the assertion that $\mathfrak{C}$ verifies (B); it involves the notion of end-points of a chainable continuum and will be needed in Section 4.

Let X be a chainable continuum. A point $a \in X$ is said to be an *end-point* of X, if each open covering v of X admits a refinement $u = \{U_1, \ldots, U_n\}$, such that $(U_1, \ldots, U_n)$ is a chain and a belongs to one of the two end-links U_1 and U_n. Similarly, a pair of points a, b of X is said to be a *pair of opposite end-points* of X, if each open covering v of X admits a refinement $u = \{U_1, \ldots, U_n\}$, such that $(U_1, \ldots, U_n)$ is a chain and $a \in U_1$, $b \in U_n$. Observe that one can always assume that $a \in U_1 \setminus U_2$, $b \in U_n \setminus U_{n-1}$ (otherwise take $U_1 \cup U_2$ and $U_{n-1} \cup U_n$ for end-links and assume that u is a star refinement of v).

Remark 2. There exist (metrizable) chainable continua without any end-points, with only one end-point, and such having every point for an end-point (see [1], p. 662).

L e m m a 1. *Let* $\{X_\alpha; p_{\alpha\alpha'}\}$, $\alpha \in A$, *be an inverse system of chainable continua* X_α *and let* $p_{\alpha\alpha'}$ *be mappings onto. Then* $X = \lim X_\alpha$ *is also a chainable continuum. Furthermore, let* (a, b) *be a pair of points of* X *and let* $a_\alpha = p^\alpha(a) \in X_\alpha$, $b_\alpha = p^\alpha(b) \in X_\alpha$, *where* $p^\alpha: X \to X_\alpha$ *denotes the natural projection. If* (a_α, b_α) *is a pair of opposite end-points of* X_α, *for all* $\alpha \in A$, *then* (a, b) *is a pair of opposite end-points of* X.

P r o o f. Let v be an arbitrary open covering of X. We can assume that v is finite (X is compact) and that the elements of v are sets of the form $(p^a)^{-1}(V_i)$, where V_i are open sets of certain X_a, $a \in A$. Furthermore, there is no loss of generality in assuming that all V_i belong to the same X_a. Then the sets V_i cover $p^a(X) = X_a$. Let $(U_1, \ldots, U_n)$ be a chain-refinement of the covering $\{V_i\}$. Then $\{(p^a)^{-1}(U_j)\}$ refines $\{(p^a)^{-1}(V_i)\}$. In order to show that X is chainable, it remains only to verify that $((p^a)^{-1}(U_1), \ldots, (p^a)^{-1}(U_n))$ is a chain. This is immediate, because $(p^a)^{-1}(U_j) \cap (p^a)^{-1}(U_k) \neq 0$ if and only if $U_j \cap U_k \neq 0$, and $(U_1, \ldots, U_n)$ is a chain.

Since (a_a, b_a) is a pair of opposite end-points of X_a, one can choose the chain $(U_1, \ldots, U_n)$ on X_a in such a way that $a_a \in U_1$ and $b_a \in U_n$. Then a and b belong to the end-links $(p^a)^{-1}(U_1)$ and $(p^a)^{-1}(U_n)$ respectively of the chain $((p^a)^{-1}(U_1), \ldots, (p^a)^{-1}(U_n))$, which refines the arbitrarily chosen covering v of X. This concludes the proof of Lemma 1.

3. Finally, let us verify (C). By Remark 1, Q is either a single point or an arc. Since X is chainable and thus connected, it follows that $f(X)$ is also connected and thus again either a point or an arc. This completes the proof of the following

T h e o r e m 5. *The property $\mathfrak{C}$ of a compact space to be chainable verifies the conditions (A), (B) and (C). Therefore, the Theorems 1, 2, 3 and 4 are valid for $\mathfrak{P} = \mathfrak{C}$.*

For example, Theorem 1 becomes a proposition asserting that metrizable chainable continua are inverse limits of arcs.

Remark 3. There is another class of chainable continua, all of which are inverse limits of arcs. These are *ordered continua*[9].

Theorem 2 for $\mathfrak{P} = \mathfrak{C}$ yields

T h e o r e m 2'. *Every chainable continuum* X *is homeomorphic with the inverse limit of an inverse system* $\{\mathrm{Q_b}; \mathrm{p_{bb'}}\}$ *of metrizable chainable continua* $\mathrm{Q_b}$; b *ranges through a directed set* B *of cardinality* $\mathrm{k(B)} \leq$ *weight* $\mathrm{w(X)}$.

Corollary 1 goes over into

C o r o l l a r y 1'. *Every chainable continuum* X *is homeomorphic with a double iterated inverse limit* $\lim_{\mathrm{b}}(\lim_{\mathrm{i}} \mathrm{I_{bi}})$ *of arcs* $\mathrm{I_{bi}}$; i *ranges through positive integers.*

4. We conclude this Section with a simple lemma which shall be needed in the next Section.

L e m m a 2. *Let* $\mathrm{X'}$ *and* $\mathrm{X''}$ *be two chainable continua (considered as disjoints sets) and let* $(\mathrm{a'}, \mathrm{b'})$ *and* $(\mathrm{a''}, \mathrm{b''})$ *be pairs of opposite end-points for* $\mathrm{X'}$ *and* $\mathrm{X''}$ *respectively. Let* X *be the continuum obtained by taking* $\mathrm{X'} \cup \mathrm{X''}$ *and identifying the two points* $\mathrm{b'}$ *and* $\mathrm{a''}$. *Then* X *is again a chainable continuum and* $(\mathrm{a'}, \mathrm{b''})$ *is a pair of opposite end-points of* X.

The proof is straightforward and is omitted.

[9]) A proof of this assertion will appear elsewhere.

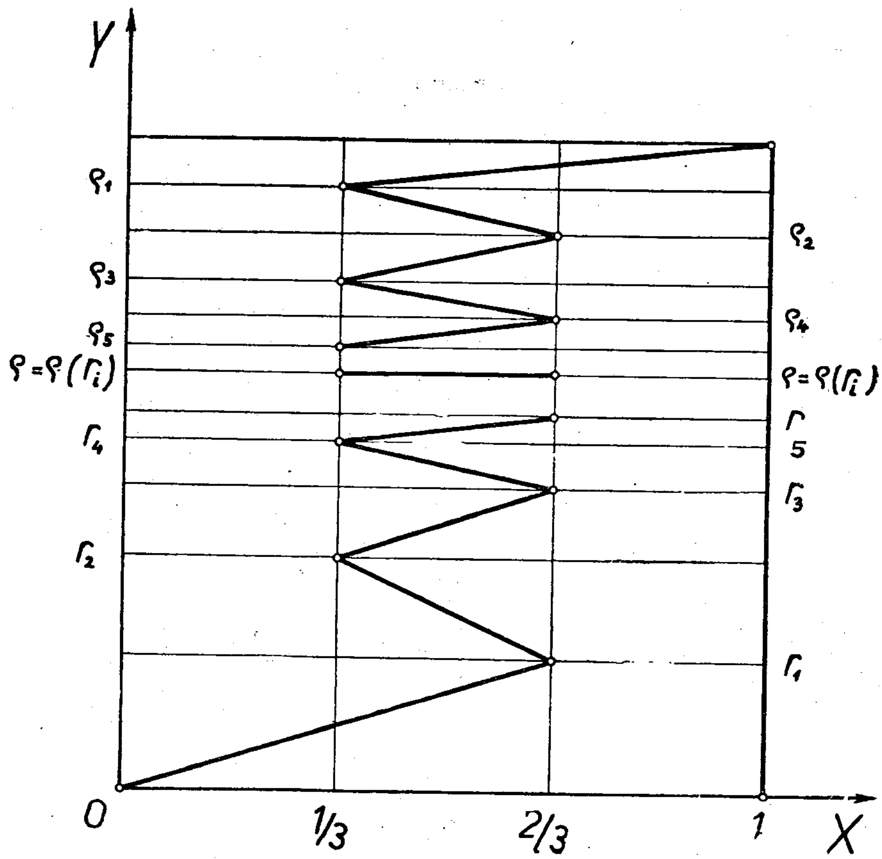
4. *A Chainable Continuum of Inductive Dimension Two*

1. Theorem 6. *There exist chainable continua X with inductive dimension ind X > 1. These continua are not obtainable as inverse limits of inverse systems of polyhedra of dimension ≤ 1, a fortiori they are not obtainable as inverse limits of arcs.*

The second assertion is a consequence of Theorem 4 of [7]. In order to prove the first assertion, we shall define a certain chainable continuum C which satisfies

$$\operatorname{ind} C = 2. \tag{1}$$

Since the covering dimension of a chainable continuum is ≤ 1, we shall have $\dim C = 1$. Therefore, the existence of C presents a strengthening of previous results of A. Lunc [6] and O. V. Lokucievskiĭ [5], who have also constructed compact spaces X with $\dim X = 1$, $\operatorname{ind} X = 2$. It is readily verified that the spaces of Lunc and Lokucievskiĭ are not chainable (they contain triods).



Now, we proceed to the construction of C, which shall be performed in several steps.

2. For each real number ϱ, $0 < \varrho < 1$, choose a fixed sequence of reals $1 > \varrho_1 > \varrho_2 > \ldots$, converging towards ϱ. Next, consider sequences $0 < r_1 < r_2 < \ldots < 1$ of *rational* numbers r_i, for which the limit $\lim r_i < 1$; the limit will be denoted by $\varrho = \varrho(r_i) = \lim r_i$. With every such sequence (r_i) we associate a certain continuum $K = K(r_i)$, contained in the unit square $[0,1] \times [0,1]$ of the XY coordinate plane. K is defined as indicated in the adjoint figure; it consists of countably many straight line segments joining $(0,0)$ with $(2/3, r_1)$, $(2/3, r_1)$ with $(1/3, r_2)$, $(1/3, r_2)$ with $(2/3, r_3), \ldots, (1/3, \varrho)$ with $(2/3, \varrho), \ldots, (2/3, \varrho_2)$ with $(1/3, \varrho_1)$, $(1/3, \varrho_1)$ with $(1,1)$ and $(1,1)$ with $(1,0)$ respectively.

It is readily seen that K can be obtained as the inverse limit of a sequence of broken lines[10] with fixed end-points $(0,0)$ and $(1,0)$. It follows (Lemma 1) that K is a (metrizable) chainable continuum and that the points $(0,0)$ and $(1,0)$ form a pair of opposite end-points of K.

3. Let ω_σ be the initial ordinal of cardinality $2^{\aleph_0}$ and let A denote the set of all ordinals $\alpha \leq \omega_\sigma$. Consider the union M' of a family of disjoint copies I_α, $\alpha \in A \setminus \{\omega_\sigma\}$, of the real line segment $I = [0,1]$. We introduce a total ordering in M' by setting $x < y$, $x \in I_\alpha$, $y \in I_{\alpha'}$, $\alpha, \alpha' \in A \setminus \{\omega_\sigma\}$, whenever $\alpha < \alpha'$ or whenever $\alpha = \alpha'$ and $x < y$ in the natural ordering of $I_\alpha = [0,1]$. Adjoin a new point b to M' and denote $M' \cup \{b\}$ by M. Consider M as ordered with b as its maximal element. M also has a minimal element $a = 0 \in I_0 \subset M$. Consider A as a subset of M by identifying $\alpha \in A \setminus \{\omega_\sigma\}$ with $0 \in I_\alpha \subset M$ and $\omega_\sigma \in A$ with $b \in M$. It is readily seen that M is an ordered continuum under the order topology.

Now, we shall define a subset L of the Cartesian product $M \times I$, $I = [0,1]$. First, consider all sequences of rationals $0 < r_1 < r_2 < \ldots$ with $\lim r_i = \varrho(r_i) < 1$. There are $2^{\aleph_0}$ of these sequences. Therefore, one can establish a fixed one-to-one correspondence between all of these sequences and all the ordinals $\alpha < \omega_\sigma$. Let $(r_i)_\alpha$ denote the sequence thus assigned to the ordinal α.

For each $\alpha < \omega_\sigma$, consider $I_\alpha \times I \subset M \times I$ as the unit square $I \times I$ of Section 2. Let $K_\alpha \subset I_\alpha \times I \subset M \times I$ be the set corresponding to $K((r_i)_\alpha)$ of Section 2. Now, define $L \subset M \times I$ by

$$L = (\bigcup_\alpha K_\alpha) \cup (\bigcup_\alpha \alpha \times I), \tag{2}$$

where α in the first term ranges through $A \setminus \{\omega_\sigma\}$ and in the second term ranges through $A \setminus \{0\}$[11].

Lemma 3. *L is a chainable continuum with the points* $a \times 0$ *and* $b \times 0$ *as a pair of opposite end-points.*

Proof. Introduce the folowing notations

$$[\alpha, \alpha']_M = \{x \mid x \in M, \alpha \leq x \leq \alpha'\}, \; \alpha, \alpha' \in A, \tag{3}$$

$$L(\alpha) = L \cap ([0, \alpha]_M \times I), \; \alpha \in A. \tag{4}$$

[10]) Such a broken line can be obtained from the part of K exhibited in the figure, by adding the line segments joining $(2/3, r_5)$ with $(1/3, \varrho)$ and $(1/3, \varrho_5)$ with $(2/3, \varrho)$.

[11]) Notice that in the expression $\alpha \times I$, α denotes $0 \in I_\alpha$.

We shall prove, by transfinite induction on α, the following proposition:

Each $\mathrm{L}(\alpha), \alpha \leq \omega_\sigma$, *is a chainable continuum and the points* $a \times 0$ *and* $\alpha \times 0$ *form a pair of opposite end-points of* $\mathrm{L}(\alpha)$.

Since $L = L(\omega_\sigma)$, this will prove Lemma 3.

The proposition is true for $\alpha = 1$, because $L(1) = K_0 = K((r_i)_0)$ is chainable and has the points 0×0 and 1×0 for a pair of opposite end-points (see 2.). Now, assume the proposition true for all the ordinals $< \alpha, 1 < \alpha \leq \omega_\sigma$. One has to distinguish two cases: case (a) when $\alpha - 1$ exists and case (b) when α is a limit ordinal.

Case (a). By assumption $L(\alpha - 1)$ is a chainable continuum with the points $a \times 0$ and $(\alpha - 1) \times 0$ as a pair of opposite end-points. Furthermore, $L(\alpha) = L(\alpha - 1) \cup K_{\alpha-1}$ and $L(\alpha - 1) \cap K_{\alpha-1} = \{(\alpha - 1) \times 0\}$. Since both $L(\alpha - 1)$ and $K_{\alpha-1}$ are chainable continua with pairs of opposite end-points $(a \times 0, (\alpha - 1) \times 0)$ and $((\alpha - 1) \times 0, \alpha \times 0)$ respectively, it follows (Lemma 2) that $L(\alpha)$ is a chainable continuum and $(a \times 0, \alpha \times 0)$ a pair of its opposite end-points.

Case (b). First, observe that $L(\alpha)$ is a closed subset of the compact space $[0, \alpha]_M \times I$ and thus is compact itself. Indeed, if $x \in ([0, \alpha]_M \times I) \setminus L(\alpha)$, then there is a $\beta < \alpha$, such that x belongs to the open set $(I_\beta \setminus \{\beta, \beta + 1\}) \times I$ of $[0, \alpha]_M \times I$. Since $L(\alpha) \cap (I_\beta \times I) = K_\beta \cup (\beta \times I)$ is closed in $I_\beta \times I$, one can easily find a neighborhood $U(x)$ of x in $(I_\beta \setminus \{\beta, \beta + 1\}) \times I$, such that $U(x) \cap L(\alpha) = 0$. Clearly, $U(x)$ is open in $M \times I$.

Now, assume that α is a limit ordinal, $\alpha \leq \omega_\sigma$. Observe that for each pair $\beta < \beta' \leq \omega_\sigma$ we have

$$L(\beta') = L(\beta) \cup (L \cap ([\beta, \beta']_M \times I)). \tag{5}$$

Define a mapping $\pi_{\beta\beta'}: L(\beta') \to L(\beta)$ by taking for $\pi_{\beta\beta'} \mid L(\beta)$ the identity mapping and for $\pi_{\beta\beta'} \mid L \cap ([\beta, \beta']_M \times I)$ the mapping, which sends $x \times y$ into $\beta \times y \in \beta \times I \subset L(\beta) \cdot \pi_{\beta\beta'}$ is clearly continuous and onto; it verifies

$$\pi_{\beta\beta'} \pi_{\beta'\beta''} = \pi_{\beta\beta''}, \tag{6}$$

for $\beta \leq \beta' \leq \beta''$ ($\pi_{\beta\beta}$ is defined as the identity map).

It is immediate that $\{L(\beta); \pi_{\beta\beta'}\}, \beta, \beta' < \alpha$, is an inverse system and that $\pi_{\beta\alpha}: L(\alpha) \to L(\beta)$ is a system of mappings onto, inducing a mapping $\pi^\beta: L(\alpha) \to \lim \{L(\beta); \pi_{\beta\beta'}\}$. Due to the compactness of $L(\alpha)$, π^β is a mapping onto. On the other hand, it is readily seen that π^β is a one-to-one mapping, which proves that

$$L(\alpha) = \lim \{L(\beta); \pi_{\beta\beta'}\}, \beta, \beta' < \alpha; \tag{7}$$

$\pi_{\beta\alpha}: L(\alpha) \to L(\beta)$ are the corresponding projections.

Notice that $\pi_{\beta\alpha}(a \times 0) = a \times 0$ and $\pi_{\beta\alpha}(\alpha \times 0) = \beta \times 0$. Our assumptions and Lemma 1 yield thus the assertion of the above proposition. This concludes the proof of Lemma 3.

4. Let ω_1 be the first uncountable ordinal and B the set of all ordinals $\beta \leq \omega_1$. Consider the union N' of family of disjoint copies M_β, $\beta \in B \setminus \{\omega_1\}$, of ordered continua M (defined in Section 3). The first and the last element of M_β will be denoted hereafter by a_β and b_β respectively and the segments of M_β, corresponding to I_α in M, will be denoted by $I_{\beta\alpha}$. We introduce a total ordering in N' by setting $x < y$, $x \in M_\beta$, $y \in M_{\beta'}$, $\beta, \beta' \in B \setminus \{\omega_1\}$, whenever $\beta < \beta'$ or whenever $\beta = \beta'$ and $x < y$ in the ordering of $M_\beta = M$. Adjoin a new point b to N' and denote $N' \cup \{b\}$ by N. Consider N as ordered with b as its maximal element. N also has a minimal element $a = a_0 \in M_0 \subset N$. Consider B as a subset of N, by identifying $\beta \in B \setminus \{\omega_1\}$ with $a_\beta \in M_\beta \subset N$ and $\omega_1 \in B$ with $b \in N$. It is readily seen that N is an ordered continuum under the order topology.

Now, we shall define a continuum $C \subset N \times I$ as follows. For each $\beta < \omega_1$, consider $M_\beta \times I \subset N \times I$ as the Cartesian product $M \times I$ of Section 3. Let $L_\beta \subset M_\beta \times I \subset N \times I$ be the set corresponding to $L \subset M \times I$ of Section 3 and let $K_{\beta\alpha} \subset L_\beta$ be the sets corresponding to $K_\alpha \subset L$. Now, define $C \subset N \times I$ by

$$C = (\bigcup_\beta L_\beta) \cup (\bigcup_\beta \beta \times I), \tag{8}$$

where β in the first term ranges through $B \setminus \{\omega_1\}$ and in the second term ranges through $B \setminus \{0\}$[12].

Lemma 4. *C is a chainable continuum.*

The proof follows closely the proof of Lemma 3. Let

$$[\beta, \beta']_N = \{x \mid x \in N, \beta \leq x \leq \beta'\}, \beta, \beta' \in B, \tag{9}$$

$$C(\beta) = C \cap ([0, \beta]_N \times I), \beta \in B. \tag{10}$$

Notice that $C(\omega_1) = C$.

One proves, by transfinite induction on β, the following proposition

Each $C(\beta)$, $\beta \leq \omega_1$, *is a chainable continuum and the points* $a \times 0$ *and* $\beta \times 0$ *form a pair of opposite end-points of* $C(\beta)$.

For $\beta = 1$ the proposition reduces to Lemma 3. Assuming the proposition true for ordinals $< \beta \leq \omega_1$, we conclude that it is true for β. Again one has to distinguish the case when $\beta - 1$ exists and the case of a limit ordinal β. The first case is treated by recurring to Lemma 2. In the second case, one has to prove that $C(\beta)$ is closed in $[0, \beta]_N \times I$. Then one defines mappings $\pi_{\beta\beta'}: C(\beta') \to C(\beta)$, $\beta \leq \beta'$, taking for $\pi_{\beta\beta'} \mid C(\beta)$ the identity map and by sending $x \times y \in C \cap$ $\cap ([\beta, \beta']_N \times I)$ into $\beta \times y \in \beta \times I \subset C(\beta)$. Next, one proves that $\pi_{\gamma\beta}: C(\beta) \to C(\gamma)$, $\gamma \leq \beta$, induce a homeomorphism between $C(\beta)$ and the limit of the inverse system $\{C(\gamma); \pi_{\gamma\gamma'}\}$, $\gamma, \gamma' < \beta$. We conclude the argument by applying Lemma 1 to this system.

5. Since N and I are ordered continua, we have

[12]) Notice that in the expression $\beta \times I$, β denotes $a_\beta \in M_\beta$.

$$\operatorname{ind}(N \times I) = 2, \tag{11}$$

so that $C \subset N \times I$ implies

$$\operatorname{ind} C \leq 2. \tag{12}$$

Now, we wish to establish

$$\operatorname{ind} C \geq 2. \tag{13}$$

It suffices to find a point $c \in C$ and a neighborhood V of c (in C) such that each open set U of C, $c \in U \subset V$, has the property that its frontier $\operatorname{Fr} U = \overline{U} \setminus U$ contains a nondegenerate real line segment; this will imply $\operatorname{ind} \operatorname{Fr}(U) \geq 1$ and thus (13).

Take for V e.g. the open set $(N \times (1/3, 2/3)) \cap C$ and for c the point $b \times 1/2$[13]. Let U be any open subset of C such that $c = b \times 1/2 \in U \subset V = (N \times (1/3, 2/3)) \cap C$. Let ϱ be the least upper bound of reals t, for which $b \times t \in U$. Then clearly, $0 < \varrho < 1$ and

$$(b \times [\varrho, 1]) \cap U = 0^{14)}. \tag{14}$$

Choose a sequence $t_1 < t_3 < t_5 < \ldots$ of reals, $0 < t_{2i+1} < \varrho$, such that $\varrho = \lim t_{2i+1}$ and

$$b \times t_{2i+1} \in U. \tag{15}$$

Next, choose a sequence of rationals $0 < r_1 < r_2 < \ldots < r_i < \ldots$, $r_{2i+1} < t_{2i+1} < r_{2i+2}$, and choose a sequence of ordinals $\beta_1, \beta_3, \ldots, \beta_{2i+1}, \ldots \in B \setminus \{\omega_1\}$ in such a way that

$$([\beta_{2i+1}, b]_N \times [r_{2i+1}, r_{2i+2}]) \cap C \subset U, \; i = 0,1, \ldots \tag{16}$$

This is possible due to (15). Let β be the least upper bound of the sequence $\beta_{2i+1} \in B \setminus \{\omega\}_1$, $i = 0,1, \ldots$. It follows that $\beta < \omega_1$. We conclude from (16) that

$$\bigcup_{i=0}^{\infty} ([\beta, b]_N \times [r_{2i+1}, r_{2i+2}]) \cap C \subset U. \tag{17}$$

For each $\beta' \in B$, $\beta \leq \beta' < \omega_1$, one has $M_{\beta'} = [\beta', \beta' + 1]_N \subset [\beta, b]_N$ and $L_{\beta'} \subset C$. Therefore, (17) yields

$$\bigcup_{i=0}^{\infty} (M_{\beta'} \times [r_{2i+1}, r_{2i+2}]) \cap L_{\beta'} \subset U, \tag{18}$$

for each β', $\beta \leq \beta' < \omega_1$.

Now, consider the sequence $0 < r_1 < r_2 < \ldots$ chosen above. There is a certain ordinal $\alpha < \omega_\sigma$ such that (r_i) is precisely the sequence of rationals $(r_i)_\alpha$ assigned to α in Section 3. Since $I_{\beta'\alpha} \subset M_{\beta'}$, $K_{\beta'\alpha} \subset L_{\beta'}$[15], we obtain from (18)

[13]) $(1/3, 2/3)$ denotes the open interval $\{t \mid t \in I, 1/3 < t < 2/3\}$; $b = \omega_1 \in B$ is the last point of N.

[14]) $[\varrho, 1]$ denotes the closed segment $\{t \mid t \in I, \varrho \leq t \leq 1\}$.

[15]) Recall that $I_{\beta\alpha}$ $(K_{\beta\alpha})$ are the subsets of M_β (of L_β), which correspond to the sets I_α (K_α) of M (of L).

$$\bigcup_{i=0}^{\infty} (I_{\beta'\alpha} \times [r_{2i+1},\ r_{2i+2}]) \cap K_{\beta'\alpha} \subset U. \tag{19}$$

Observing that $K_{\beta'\alpha}$ corresponds to $K((r_i)_\alpha) = K(r_i)$ and inspecting the figure, we conclude that $(I_{\beta'\alpha} \times [r_{2i+1}, r_{2i+2}]) \cap K_{\beta'\alpha}$ is the straight line segment l_i, joining the points $(2/3,\ r_{2i+1})$ and $(1/3,\ r_{2i+2})$ in $I_{\beta'\alpha} \times I$. It is evident that the line segment $[1/3,\ 2/3] \times \varrho \subset I_{\beta'\alpha} \times \times I$, $\varrho = \lim r_i$, belongs to the closure of the union of the segments l_i. By (19), it belongs a fortiori to $\overline{U}$, so that we have

$$[1/3, 2/3]_{\beta'\alpha} \times \varrho \subset \overline{U}, \tag{20}$$

for each β', $\beta \leq \beta' < \omega_1$; here $[1/3, 2/3]_{\beta'\alpha}$ denotes the segment $[1/3, 2/3]$ of the copy $I_{\beta'\alpha}$ of $I = [0, 1]$ and $\overline{U}$ denotes the closure of U with respect to C.

If we assume the existence of at least one β', $\beta \leq \beta' < \omega_1$, having the property that the line segment $[1/3, 2/3]_{\beta'\alpha}$ contains a non-degenerate subsegment which is disjoint with U, then it follows from (20) that Fr U contains that subsegment. On the other hand, if there is no such β', $\beta \leq \beta' < \omega_1$, then for each β', one can certainly find a point $c_{\beta'\alpha} \in [1/3, 2/3]_{\beta'\alpha}$ contained in U. U being open, it also contains the intersection of $K_{\beta'\alpha} \subset C$ with a set of the form $[x_1, x_2] \times [\varrho - \varepsilon,\ \varrho + \varepsilon]$, where $[x_1, x_2]$ is a non-degenerate subsegment of $[1/3, 2/3]_{\beta'\alpha}$ and $\varepsilon > 0$. Inspecting again the figure, it is evident that one can choose, for each β', a line segment contained in this intersection (and a fortiori in U), whose end-points have distinct rational coordinates $s_{\beta'}$ and $t_{\beta'}$, such that $\varrho < s_{\beta'} < t_{\beta'}$. The set $B' = \{\beta' \mid \beta \leq \beta' < \omega_1\}$ being uncountable, there is a subset B'' of B', which is cofinal in the set B' and has the property that both $s_{\beta'}$ and $t_{\beta'}$ have constant (rational) values s and t respectively, for all $\beta' \in B''$. It follows immediately, that the non-degenerate line segment $b \times [s, t] \subset b \times I \subset C$ belongs to $\overline{U}$ ($b = \omega_1 \in B$ is the last point of N). On the other hand, $\varrho < s < t$, so that (14) implies $(b \times \times [s, t]) \cap U = 0$. Hence, $b \times [s, t]$ is a non-degenerate segment of Fr U. This proves that Fr U always contains a non-degenerate line segment, which implies (13) and completes the proof of Theorem 6.

Remark 4. We believe that there exist chainable continua X with arbitrarily large inductive dimension.

REFERENCES:

[1] *R. H. Bing*, Snake-like continua, Duke Math. J. **18** (1951), 653—663.
[2] *S. Eilenberg* and *N. Steenrod*, Foundations of algebraic topology, Princeton, 1952.
[3] *J. R. Isbell*, Embeddings of inverse limits, Ann. of Math. **70** (1959), 73—84.
[4] *S. Lefschetz*, Algebraic topology, Colloquium Publ. Amer. Math. Soc., New York, 1942.
]5] *О. В. Локуциевский*, О размерности бикомпактов, Докл. Акад. Наук СССР (Н. С.) **67** (1949), 217—219.

[6] *А. Лунц*, Бикомпакт, индуктивная размерность которого больше, чем размерность, определенная при помощи покрытий, Докл. Акад. Наук СССР (Н. С.) **66** (1949), 801—803.
[7] *S. Mardešić*, On covering dimension and inverse limits of compact spaces, to appear in Illinois J. Math.
[8] *R. H. Rosen*, Fixed points for multi-valued functions on snake-like continua, Proc. Amer. Math. Soc. **10** (1959), 167—173.

LANČASTI KONTINUUMI I INVERZNI LIMESI

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Sadržaj

Konačna uređena familija $(U_1, \ldots, U_n)$ otvorenih skupova U_i zove se *lanac*, ako je $U_i \cap U_j \neq 0$ onda i samo onda, kada je $|i - j| \leq 1$. Kaže se da je topološki Hausdorffov prostor[1]) X *lančast*, ako svaki otvoreni pokrivač prostora X dopušta jedno profinjenje $\{U_1, \ldots, U_n\}$, takovo da je $(U_1, \ldots, U_n)$ lanac. Svaki lančasti X je nužno kompaktan i povezan, dakle kontinuum.

Osnovni primjer lančastog kontinuuma je lûk, t. j. homeomorfna slika segmenta $I = [0,1]$ realnih brojeva. R. H. Rosen je nedavno postavio ovaj problem ([8], str. 170.): *Da li je moguće dobiti svaki lančasti kontinuum kao inverzni limes*[2]) *nekog inverznog sistema*[2]) *lukova?*

U ovom članku se gornji problem dovodi u vezu s ovim problemom o dimenziji prekrivanja:

Da li se svaki kompaktni prostor X, za kojeg je dimenzija prekrivanja $\dim X \leq n$, može dobiti kao inverzni limes nekog inverznog sistema poliedara P_α, za koje je $\dim P_\alpha \leq n$?

U radu [7] bio je dan negativni odgovor na drugi od gornjih problema. Točnije, bilo je pokazano da kompaktni prostori X, za koje je $\dim X = 1$, dok im je induktivna dimenzija $\operatorname{ind} X > 1$, ne mogu biti inverzni limesi poliedara dimenzije ≤ 1 ([7], Teorem 4.). Egzistenciju kompaktnih prostora s gornjim svojstvima utvrdili su ranije A. Lunc [6] i O. V. Lokucievskiĭ [5].

U ovom članku (u točki 4.) se konstruktivno definira jedan *lančasti kontinuum* C *sa svojstvom* ind $C = 2$. Kako je dimenzija prekrivanja za svaki lančasti kontinuum očito ≤ 1, to postojanje kontinuuma C predstavlja pooštrenje rezultata Lunca i Lokucievskog. Iz [7], Teorem 4. slijedi da C ne može biti inverzni limes lukova, te je tako dobiven negativni odgovor na problem Rosen-a.

1) Za sve prostore u ovom članku pretpostavlja se da su topološki Hausdorffovi prostori.

2) Osnovne definicije i činjenice u vezi s inverznim sistemima i njihovim limesima opisane su na primjer u [2] i [4].

Nadalje je u [7] bilo pokazano, da je svaki n-dimenzionalni kompaktni prostor X inverzni limes jednog inverznog sistema n-dimenzionalnih metrizabilnih kompakata. U ovom radu (točka 3.) se dokazuje analogni teorem (Teorem 2'.): *Svaki lančasti kontinuum je inverzni limes jednog inverznog sistema metrizabilnih lančastih kontinuuma.* Dokazi obaju teorema teku paralelno, pa je stoga poželjno dobiti jedan općeniti teorem, koji bi oba gornja teorema obuhvatao kao specijalne slučajeve. To je učinjeno u točki 2. ovog rada, gdje se promatra neko opće svojstvo $\mathfrak{P}$, koje je definirano za sve kompaktne prostore, tako da svaki kompaktni X ili ima svojstvo $\mathfrak{P}$ ili nema to svojstvo. Na svojstvo $\mathfrak{P}$ se zatim postavljaju ova tri uvjeta:

(A) *Uvjet aproksimacije.* Neka je X bilo koji kompaktni prostor sa svojstvom $\mathfrak{P}$, neka je P neki poliedar sa zadanom metrikom d, $r > 0$ neka je realni broj, a $f: X \to P$ neko preslikavanje (neprekidno). Tada postoji poliedar Q sa svojstvom $\mathfrak{P}$ i postoje preslikavanja $g: X \to Q$, $p: Q \to P$, za koja vrijedi $g(X) = Q$ i $d(f, pg) \leq r$.

(B) *Uvjet neprekidnosti.* Neka je $\{X_a; \pi_{aa'}\}$ neki inverzni sistem kompaktnih prostora X_a, koji svi imaju svojstvo $\mathfrak{P}$, i neka su $\pi_{aa'}: X_{a'} \to X_a$ preslikavanja na čitavi X_a. Tada i $X = \lim X_a$ ima svojstvo $\mathfrak{P}$.

(C) Ako je X kompaktni prostor, Q neki poliedar, a $f: X \to Q$ preslikavanje, te ako X i Q imaju svojstvo $\mathfrak{P}$, tada i $f(X)$ ima svojstvo $\mathfrak{P}$.

Primjeri. (1) Neka tvrdnja »X ima svojstvo $\mathfrak{P}$« znači da je $\dim X \leq n$, gdje je n neki čvrsti cijeli broj, $n \geq 0$.

(2) Neka tvrdnja »X ima svojstvo $\mathfrak{P}$« znači da je X lančast.

Ova oba svojstva $\mathfrak{P}$ zadovoljavaju (A), (B) i (C).

Sada možemo izreći ovaj općeniti teorem:

Ako svojstvo $\mathfrak{P}$ zadovoljava (A), (B) *i* (C), *tada se svaki kompaktni prostor* X *sa svojstvom $\mathfrak{P}$ dade prikazati kao inverzni limes jednog inverznog sistema metrizabilnih kompakata sa svojstvom $\mathfrak{P}$* (Teorem 2.).

U radu su dokazana još dva opća teorema o faktorizaciji (Teorem 3.), odnosno o razvijanju u inverzne sisteme (Teorem 4.).

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