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GENERALIZATIONS OF HLAWKA'S INEQUALITY

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1. Introduction

Let E be complex pre-Hilbert (unitary) space. We shall denote the norm of $a \leqslant E$ by $|a|$. The following identity is due to E. Hlawka [1]:

$$\begin{aligned}&(|a_1|+|a_2|+|a_3|-|a_2+a_3|-|a_3+a_1|-|a_1+a_2|+\\&+|a_1+a_2+a_3|)\times(|a_1|+|a_2|+|a_3|+|a_1+a_2+a_3|)=\\&=(|a_2|+|a_3|-|a_2+a_3|)(|a_1|-|a_2+a_3|+|a_1+a_2+a_3|)+\\&+(|a_3|+|a_1|-|a_3+a_1|)(|a_2|-|a_3+a_1|+|a_1+a_2+a_3|)+\\&+(|a_1|+|a_2|-|a_1+a_2|)(|a_3|-|a_1+a_2|+|a_1+a_2+a_3|),\end{aligned} \tag{1}$$

where a_1, a_2, a_3 are arbitrary elements of E. From this it follows that

$$\begin{aligned}&|a_1|+|a_2|+|a_3|-|a_2+a_3|-|a_3+a_1|-|a_1+a_2|+\\&\qquad+|a_1+a_2+a_3|\geq 0.\end{aligned} \tag{2}$$

This is Hlawka's inequality.

D. Adamović [2] has proved the more general inequality

$$\sum_{1\leq i<j\leq n} |a_i+a_j| \leq (n-2)\sum_{i=1}^{n}|a_i| + \left|\sum_{i=1}^{n} a_i\right| \quad (n\geq 2), \tag{3}$$

which contains (2) as a special case.

In this paper we prove a new inequality

$$\begin{aligned}&\sum_{1\leq i_1<i_2<\ldots<i_k\leq n} |a_{i_1}+a_{i_2}+\ldots+a_{i_k}| \leq\\&\leq \binom{n-2}{k-2}\left(\frac{n-k}{k-1}\sum_{i=1}^{n}|a_i| + \left|\sum_{i=1}^{n} a_i\right|\right)\end{aligned} \tag{4}$$

$$(n=3,4,\ldots;\ k=2,3,\ldots,n-1),$$

which contains (3) as a special case and we establish the necessary and sufficient conditions that the equality holds.

2. Proof of (4)

We start from the identity

$$\binom{n-2}{k-2}\left[\left(\sum_{i=1}^{n}|a_i|\right)^2-\left|\sum_{i=1}^{n}a_i\right|^2\right]=\sum_{i_1<i_2<\ldots<i_k}(|a_{i_1}|+|a_{i_2}|+\ldots+|a_{i_k}|)^2-\sum_{i_1<i_2<\ldots<i_k}|a_{i_1}+a_{i_2}+\ldots+a_{i_k}|^2. \tag{5}$$

Both sides of (5) are equal to

$$\binom{n-2}{k-2}\sum_{i<j}\left[2|a_i|\,|a_j|-(a_i|a_j)-(a_j|a_i)\right],$$

where $(a|b)$ denotes the scalar product of a and b $(a, b \in E)$.

From (5) we obtain

$$\binom{n-2}{k-2}\left(\left|\sum_{i=1}^{n}a_i\right|-\sum_{i=1}^{n}|a_i|\right)\left(\left|\sum_{i=1}^{n}a_i\right|+\sum_{i=1}^{n}|a_i|\right)=$$

$$=-\sum_{i_1<\ldots<i_k}(|a_{i_1}|+\ldots+|a_{i_k}|-|a_{i_1}+\ldots+a_{i_k}|)\times \tag{6}$$

$$\times(|a_{i_1}|+\ldots+|a_{i_k}|+|a_{i_1}+\ldots+a_{i_k}|).$$

On the other hand we have

$$\binom{n-1}{k-1}\sum_{i=1}^{n}|a_i|-\sum_{i_1<i_2<\ldots<i_k}|a_{i_1}+a_{i_2}+\ldots+a_{i_k}|=$$

$$=\sum_{i_1<i_2<\ldots<i_k}(|a_{i_1}|+|a_{i_2}|+\ldots+|a_{i_k}|-|a_{i_1}+a_{i_2}+\ldots+a_{i_k}|). \tag{7}$$

Multiplying (7) by $\sum_{i=1}^{n}|a_i|+\left|\sum_{i=1}^{n}a_i\right|$ and adding to (6), we get

$$\left[\binom{n-2}{k-2}\left(\frac{n-k}{k-1}\sum_{i=1}^{n}|a_i|+\left|\sum_{i=1}^{n}a_i\right|\right)-\right.$$

$$\left.-\sum_{i_1<i_2<\ldots<i_k}|a_{i_1}+a_{i_2}+\ldots+a_{i_k}|\right]\times\left(\sum_{i=1}^{n}|a_i|+\left|\sum_{i=1}^{n}a_i\right|\right)=$$

$$=\sum_{i_1<i_2<\ldots<i_k}(|a_{i_1}|+\ldots+|a_{i_k}|-|a_{i_1}+\ldots+a_{i_k}|)\times \tag{8}$$

$$\times\left(\sum_{i=1}^{n}|a_i|+\left|\sum_{i=1}^{n}a_i\right|-|a_{i_1}|-\ldots-|a_{i_k}|-|a_{i_1}+\ldots+a_{i_k}|\right).$$

Let $j_1, j_2, \ldots, j_{n-k}$ be complementary indices, i. e. those indices $1, 2, \ldots, n$ which are different from one another and from $i_1, i_2, \ldots, i_k$. We can write

$$\sum_{i=1}^{n} |a_i| + |\sum_{i=1}^{n} a_i| - |a_{i_1}| - \ldots - |a_{i_k}| - |a_{i_1} + \ldots + a_{i_k}| =$$

$$= |-a_{j_1}| + |-a_{j_2}| + \ldots + |-a_{j_{n-k}}| +$$

$$+ |\sum_{i=1}^{n} a_i| - |a_{i_1} + a_{i_2} + \ldots + a_{i_k}| \geq 0,$$

since

$$-a_{j_1} - a_{j_2} - \ldots - a_{j_{n-k}} + \sum_{i=1}^{n} a_i = a_{i_1} + a_{i_2} + \ldots + a_{i_k}.$$

Therefore, the right-hand side of (8) is always non-negative. It follows that the left-hand side of (8) is also non-negative which is equivalent to (4). The proof is completed.

3. Conditions for equality

We shall prove that in (4) equality holds if and only if some of the following conditions is satisfied:

1° there is a vector $a \in E$ such that $a_i = \lambda_i a$ $(i = 1, 2, \ldots, n)$ and $\lambda_i \geq 0$ $(i = 1, 2, \ldots, n)$;

2° all vectors a_i but two are zero-vectors;

3° all vectors a_i but three, say a_1, a_2, a_3, are zero-vectors and $a_1 + a_2 + a_3 = 0$;

4° there is a vector $a \in E$ such that $a_i = \lambda_i a$ $(i = 1, 2, \ldots, n)$, where all λ_i but one, say λ_n, are non-negative, $\lambda_n < 0$, and

$$|\lambda_n| \geq \sum_{i=1}^{n-1} \lambda_i ;$$

5° $k = n - 1$ and $\sum_{i=1}^{n} a_i = 0$.

Proof. Let us suppose that $a_i \in E$ $(i = 1, 2, \ldots, n)$ are such that

$$\sum_{i_1 < \ldots < i_k} |a_{i_1} + a_{i_2} + \ldots + a_{i_k}| = \binom{n-2}{k-2}\left(\frac{n-k}{k-1} \sum_{i=1}^{n} |a_i| + |\sum_{i=1}^{n} a_i|\right) \tag{9}$$

and none of the conditions 1°—5° is satisfied.

According to (8) we obtain

$$\sum_{i_1<\ldots<i_k} (|a_{i_1}|+\ldots+|a_{i_k}|-|a_{i_1}+\ldots+a_{i_k}|)\times$$
$$\times(\sum_{i=1}^{n}|a_i|+|\sum_{i=1}^{n}a_i|-|a_{i_1}|-\ldots-|a_{i_k}|-|a_{i_1}+\ldots+a_{i_k}|)=0. \tag{10}$$

In other words, for each combination $(i_1, i_2, \ldots, i_k)$ of indices $1, 2, \ldots, n$ we must have

$$|a_{i_1}|+|a_{i_2}|+\ldots+|a_{i_k}|=|a_{i_1}+a_{i_2}+\ldots+a_{i_k}| \tag{11}$$

or

$$|-a_{j_1}|+\ldots+|-a_{j_{n-k}}|+|\sum_{i=1}^{n}a_i|=|a_{i_1}+\ldots+a_{i_k}|, \tag{12}$$

where $j_1, j_2, \ldots, j_{n-k}$ are complementary indices.

The equation (11) is equivalent (see [3], p. 32.) to the statement that vectors $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ are *non-negative scalar multiples* (*nnsm*) of a vector a.

Hence, we have the following

L e m m a. *For each combination* $(i_1, i_2, \ldots, i_k)$

(I) $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ *are nnsm of a vector, or*

(II) $-a_{j_1}, -a_{j_2}, \ldots, -a_{j_{n-k}}, \sum_{i=1}^{n} a_i$ *are nnsm of some vector.*

a) Now, we suppose that for a certain combination, say $(1, 2, \ldots, k)$ the alternative (I) applies and that at least two vectors $a_1, a_2, \ldots, a_k$ are non-zero, say a_1 and a_2. Vectors $a_1, a_2, \ldots, a_k$ are *nnsm* of a certain vector a. By 1° and by our assumption a vector between $a_{k+1}, a_{k+2}, \ldots, a_n$, say a_{k+1}, is not a *nnsm* of a. Taking into consideration the combination $(2, 3, \ldots, k+1)$, we conclude that the alternative (II) must occur. So, $-a_1, -a_{k+2}, \ldots, -a_n, \sum_{i=1}^{n} a_i$ are *nnsm* of $-a$. Hence, all vectors a_i $(i = 1, 2, \ldots, k, k+2, \ldots, n)$ and $-\sum_{i=1}^{n} a_i$ are *nnsm* of a. The condition 4° is satisfied which is in contradiction with our assumption.

b) We next suppose that for a combination, say $(1, 2, \ldots, k)$ the alternative (I) applies again, but only one vector between $a_1, a_2, \ldots, a_k$, say a_1, is non-zero. If there are two non-zero vectors between $a_1, a_{k+1}, a_{k+2}, \ldots, a_n$ which are *nnsm*, we can apply the argument of a). By 2° we conclude that between $a_{k+1}, a_{k+2}, \ldots, a_n$ there are at least two non-zero vectors, say a_{n-1} and a_n. Take the combination $(1, 2, \ldots, k-1, n)$. Since a_1 and a_n are not *nnsm* the alternative (II) applies. We conclude that $-a_k, -a_{k+1}, \ldots, -a_{n-1}$, $\sum_{i=1}^{n} a_i$ are *nnsm*. Taking the combination $(2, 3, \ldots, k-1, n-1, n)$ we find that $-a_1, -a_k, -a_{k+1}, \ldots, -a_{n-2}, \sum_{i=1}^{n} a_i$ are *nnsm*. But a_1 and a_{n-1} are not *nnsm*, so it must be $a_k = a_{k+1} = a_{k+2} = \ldots = a_{n-2} = 0$ and $\sum_{i=1}^{n} a_i = 0$. The condition 3° is satisfied which is in contradiction with our assumption.

c) Finally, we suppose that for each combination the alternative (II) applies. By 5° we can suppose that $k \leq n-2$. By 2° we conclude that at least three vectors, say a_1, a_2, a_3 are not zero. Taking the combination $(n-k+1, n-k+2, \ldots, n)$ we obtain that $-a_1, -a_2, \ldots, -a_{n-k}, \sum_{i=1}^{n} a_i$ are *nnsm* of a certain vector a. Changing the combination we can prove that all vectors a_i are *nnsm* of $-a$. The condition 1° is satisfied which is in contradiction with our assumption.

Conversely, if some condition 1°—5° is satisfied it can be easily checked that (9) holds.

The proof is finished.

4. Some remarks

A) Inequality (4) can be written as follows

$$\sum_{i_1<\ldots<i_k} |a_{i_1} + \ldots + a_{i_k}| \leq \binom{n-1}{k-1}\left(\frac{n-k}{n-1}\sum_{i=1}^{n}|a_i| + \frac{k-1}{n-1}\left|\sum_{i=1}^{n} a_i\right|\right). \tag{13}$$

On the other hand, we have obviously

$$\sum_{i_1<\ldots<i_k} |a_{i_1} + \ldots + a_{i_k}| \leq \binom{n-1}{k-1}\sum_{i=1}^{n}|a_i|. \tag{14}$$

Let $\lambda, \mu \geq 0, \lambda + \mu = 1$. Multiplying (13) by λ, (14) by μ and adding together, we get

$$\sum_{i_1<\ldots<i_k} |a_{i_1} + \ldots + a_{i_k}| \leq \binom{n-1}{k-1}\left[\left(\frac{n-k}{n-1}\lambda + \mu\right)\sum_{i=1}^{n}|a_i| + \frac{k-1}{n-1}\lambda\left|\sum_{i=1}^{n} a_i\right|\right].$$

Putting $\frac{n-k}{n-1}\lambda+\mu=\alpha,\ \frac{k-1}{n-1}\lambda=\beta,$ we obtain

$$\sum_{i_1<\ldots<i_k}|a_{i_1}+\ldots+a_{i_k}|\leq\binom{n-1}{k-1}\left(\alpha\sum_{i=1}^{n}|a_i|+\beta\left|\sum_{i=1}^{n}a_i\right|\right) \tag{15}$$

$$\alpha,\beta\geq 0,\ \alpha+\beta=1,\ \alpha\geq\frac{n-k}{k-1}\beta.$$

We shall show that the constant $\frac{n-k}{k-1}$ in the inequality

$$\alpha\geq\frac{n-k}{k-1}\beta$$

is the best possible. Indeed, let us suppose that inequality (15) is true for some α,β such that

$$\alpha,\beta\geq 0,\ \alpha+\beta=1,\ \alpha<\frac{n-k}{k-1}\beta. \tag{16}$$

We take $a_1=a_2=\ldots=a_{n-1}=a\neq 0,\ a_n=-(n-1)a$. Then we obtain

$$\sum_{i_1<\ldots<i_n}|a_{i_1}+\ldots+a_{i_k}|=\binom{n-1}{k}k|a|+\binom{n-1}{k-1}(n-k)|a|,$$

$$\sum_{i=1}^{n}|a_i|=2(n-1)|a|,\ \left|\sum_{i=1}^{n}a_i\right|=0,$$

and inequality (15) gives $\alpha\geq\frac{n-k}{n-1}$ which is in contradiction with (16).

In the optimal case $\alpha=\frac{n-k}{k-1}\beta$ we obtain the inequality (4).

B) Let us consider the following inequalities

$$|a|+|b|-|a+b|\geq 0, \tag{17}$$

$$|a|+|b|+|c|-|b+c|-|c+a|-|a+b|+|a+b+c|\geq 0, \tag{18}$$

which are true for any $a, b, c\in E$.

It is very attractive to suppose that inequality

$$|a|+|b|+|c|+|d|-|a+b|-|a+c|-|a+d|-$$
$$-|b+c|-|b+d|-|c+d|+|b+c+d|+|c+d+a|+$$
$$+|d+a+b|+|a+b+c|-|a+b+c+d|\geq 0$$

is also true. But it is easy to show that it is false. It is sufficient to put $a = b = c \neq 0, d = -2a$.

C) D. Adamović [2] has shown by a counterexample that the inequality (3) does not hold in all Banach spaces. The same argument shows that in general Banach spaces (4) holds for no n and k.

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GENERALIZACIJE HLAWKINE NEJEDNAKOSTI

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Sadržaj

D. Adamović [2] je dokazao nejednakost (3) koja generališe Hlawkinu nejednakost (2). Sa a_i odnosno $|a_i|$ su označeni vektori nekog unitarnog prostora odnosno njihove norme.

U ovom radu se dokazuje nejednakost (4) koja generališe nejednakost (3). Takođe su nađeni potrebni i dovoljni uslovi da bi u (4) važio znak jednakosti.

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