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ON THE CARDINAL NUMBER OF ORDERED SETS AND OF SYMMETRICAL STRUCTURES IN DEPENDENCE ON THE CARDINAL NUMBERS OF ITS CHAINS AND ANTICHAINS

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Introduction[1)]

The cardinal number kS (resp. kG) of an ordered set S (of a graph G) depends on the cardinal numbers of its *chains* and *antichains.* A particular kind of ordered sets — *trees or ramified tables* T — was considered in our *Thesis* (Kurepa [1]) in connexion with the Suslin problem, when I was lead to the *hypothesis that* $\mathrm{k}\,\mathrm{T}$ *is the supremum of* $\mathrm{k}_c\mathrm{T}$ *and* $\mathrm{k}_{\bar{c}}\mathrm{T}$[2)] (for definitions see the glossary). Another kind of this problematics is related to the question whether the numbers $\mathrm{k}_c\,\mathrm{S}$, $\mathrm{k}_{\bar{c}}\,\mathrm{S}$ *are reached.* In this connexion we proved that every infinite *»narrow« tree* T *contains a chain of the same cardinality as the set* T *itself* (cf. Kurepa [1] p. 80 Th. 5 bis, also in Kurepa [8] where the same theorem with its proof are reproduced). As far as we know both kinds of these problems were considered for the first time in our Thesis.

The next step was the same problematics for general *ordered sets.* The question was resolved in 1937 (cf. *»relation fondamentale«* (1) in Kurepa [3]; and [4]). In particular, kS depends exponentially upon $k_c S$ and one has $k\,S \leq (2\,k_{\bar{c}}\,S)^{k_c S}$ (cf. § 5). The proof of this item is extensionable to binary symmetrical relations as I found in 1950. I thought that the same result should hold for *n-ary symmetrical relations* (Kurepa [6], [7]); therefore, I postponed the complete publication of this paper which was promised to the *Journal*

1) We delivered these lectures on the matter:
1. Über binäre symmetrische Relation, Munich, 06. 9. 1952 (Congress of the »Deutsche Mathematikervereinigung«).
2. O simetričnim relacijama i grafovima, Zagreb, 03. 12. 1952 (Colloquium, Društvo matematičara i fizičara NR Hrvatske).
3. Sur les relations binaires, Paris, 24. 02. 1953 (Faculté des Sciences).
4. O kombinacijama, Zagreb 17. 3. 1954 (Colloquium, Društvo matematičara i fizičara NR Hrvatske).

2) The Suslin problem is equivalent to the problem whether every tree is countable if each of its chains and antichains is countable (Kurepa [1] pp. 106 (passage b), 124, 132).

für die reine und angewandte Mathematik after my lecture on the 09. 9. 1952 in Munich and sent for publicationly a part of it (Kurepa [6]).

In 1952 I lectured on the same subject in Paris, at the Faculty of Sciences; then I was informed by G. Riguet about the work of Ramsey [1], P. Erdös [1] and R. Rado. Anyway I delayed the publication of my paper hoping to extend the theorem 7.2 (resp. 9.2) in the same form writing Ir instead of $I2$ (resp. writing r instead of 2), for any integer $r > 1$ (cf. Kurepa [7]) and not only to have evaluation of kS contained in the theorem 8.3 (resp. 9.4) I was stopped, too, by the problem we announce in § 10, in particular as to the existence of the number $R(r, n, \aleph_\alpha)$, for finite r and n. Moreover, I had the idea to gather all the results and publish them in a particular work.

Now, that I was informed by P. Erdös that Hajnal proved that the evolution of kS in theorem 8.3 (resp. 9.4) is the best one, I decided to publish this paper jointly with reimpression of my original paper [4].

Obviously, there are connexions between my papers and those of Erdös — Rado.

It is instructive to notify how the *tree considerations* are playing an important role in the theory of *general symmetrical relations* (cf. §§ 3.2; 6.2, 8.4.5). We stress also the idea of *product of relations:* this idea played an essential role in our proofs[3].

1. Definitions and Notations.

1.1 Definition. For an ordered set $(S; <)$ let ΓS denote the first ordinal number which is not representable in $(S; <)$.

In other words, $I\Gamma S$ denotes the system of all ordered types of well ordered subsets of the ordering $(S; <)$.

1.2. ΓS^* denotes $\Gamma(S; >)$. Consequently, $I\Gamma S^*$ denotes the order types of inversely well ordered subsets of $(S; <)$.

1.3. $K_c S$ resp. $K_{\bar c} S$ denotes the first cardinal number non-representable as chain resp. as antichain in $(S; <)$.

1.4. In particular, let $W_c S$ denote the first cardinal non representable as a well-ordered subset of $(S; <)$; $W_c S^* = W_c\ (S; >)$.

1.5. Definition.

$$w_c S = (W_c S)^-, \; w_d S = (W_c S^*)^-;$$

$$k_a S = k_c S = (K_{\bar c} S)^-.$$

1.6. Analogously, for every *binary graph* $(G; \varrho)$ let

$$K_c G \text{ resp. } K_{\bar c} G = K_a G$$

[3]) The idea of intersection of relations is the very basis of dimension theory of ordered sets in the sense Dushnik-Miller.

denote the first cardinal number which is not representable as a chain resp. as an antichain of the graph $(G; \varrho)$[4].

Let $k_c G = (K_c G)^-$, $k^a G = (K^a G)^-$.

1.6. The R-operator. Let X be a part of an ordered set $(S; <)$. We denote by RX any *maximal antichain* of X such that $RX \supseteq R_0 X$; $R_0 X$ denotes the set of all the *initial points* of X i. e. $R_0 X = \{x \mid x \in X, X(., x) = \emptyset\}$.

Analogously, for any graph $(G; \varrho)$ and $X \subseteq G$ we denote by RX any maximal antichain of X.

1.7. Numbers NS, nS.

For a chain C of $(S; <)$ let $S(C, .)$ denote the set of all the points x of S satisfying $C < x$ i. e. $y < x$ for each $y \in C$. In particular, $S(\emptyset, .) = R_0 S$. We denote by $N(S; <)$ or NS the first cardinal number $> kRS(C, .)$ for each chain $C \subseteq S$. We denote by $n(s, <)$ or nS the number $(NS)^-$.

2. Trees or Ramified Tables.

2.1. Definition. An ordered set T is said to, be a *tree* or a *ramified table*, provided for every point $x \in T$ the set $T(., x)$ is a *well ordered* subset of T. The void set $\emptyset$ is a tree too.

2.2. Definition. Let $R_\alpha T$ be the set of all the points $x \in T$ such that the set $T(., x)$ be of order type α.

2.3. Definition. The first ordinal such that $R_\alpha T = \emptyset$ is called the height or the rank γT of T.

2.4.1. *One has this disjointed partition of* T *in rows* $R_\alpha T$ *of* T:

$$T = \bigcup_\alpha R_\alpha T, \quad (\alpha < \gamma T)$$

$$kT = \sum_\alpha k R_\alpha T, \quad (\alpha < \gamma T),$$

from which it follows that

$$kT \leq mT \cdot k\gamma T \text{ with}$$

$$mT = \sup_\alpha k R_\alpha T.$$

and the more

$$kT \leq k_a T \cdot k\gamma T$$

each row being an antichain.

2.4.2. *According to our hypothesis we have*

$$kT \leq k_a T \cdot k_c T$$

[4]) ϱ is a binary symmetric relation in the set G. If a subset of G has no pair of *ϱ-comparable* (resp. *ϱ-incomparable)* distinct elements, this subset is referred to as an *antichain* (resp. *chain*) of the graph $(G; \varrho)$.

for every tree T. This hypothesis is equivalent to the positive answer to the *general Suslin problem: each ordered chain* C *contains a set of cardinality* s C *which is everywhere dense in* C[5]

2.5. Lemma. *For every* $\gamma' < \gamma \mathrm{T}$ *the set* $\mathrm{R}_{\gamma'}\mathrm{T}$ *is non empty; there exists at least on point* x *such that the order-type of* T(. , x) *be* γ'.

2.5.1. $\gamma T = \Gamma T$.

2.6. A *node* of a tree T is each maximal subset in which the mapping $x \to T(.,x)$ is constant. In particular, $R_0 T$ is a node of T.

2.6.1. Lemma. *For any tree* T *the cardinal* NT *or* N *is the first cardinal number* $> \mathrm{k}\,\mathrm{X}$, X *being any node of* T.

2.7.1. Theorem. *For each* $\alpha < \gamma \mathrm{T}$ *one has* $\mathrm{k}\mathrm{R}_\alpha \mathrm{T} \leq k_\alpha{}^{a_\alpha}$, *where* k_α, a_α *are cardinals satisfying* $\mathrm{k}_\alpha \leq \mathrm{N}^-$, $\mathrm{a}_\alpha \leq \mathrm{k}(1+\alpha)$. T *is similar to a subset of the set* $\mathrm{T}(\mathrm{n};\gamma)$ *of all the sequences of length* $<\gamma \mathrm{T}$ *of ordinals* $<\omega(\mathrm{n})$ *ordered by* $\dashv$ *relation.*[6] *One has* $\mathrm{k}\,\mathrm{T}(\mathrm{n},\gamma) = \Sigma\, \mathrm{n}^{\mathrm{k}\alpha}$, $(\alpha \leq \gamma)$.

The proof is carried out by induction.

2.7.2. Theorem. *There are two mappings* $\alpha \to \mathrm{k}_\alpha$, $\alpha \to \mathrm{a}_\alpha$ of $\mathrm{I}\gamma\mathrm{T}$ *into cardinals such that* $\mathrm{k}_\alpha \leq \mathrm{n}$, $\mathrm{a}_\alpha \leq \mathrm{k}(1+\alpha)$ *and* $\mathrm{k}\mathrm{T} \leq \sum_\alpha \mathrm{k}_\alpha{}^{\mathrm{a}\alpha}$ $(\alpha < \gamma \mathrm{T})$.

The theorem is a consequence of the disjointed partition $T = \bigcup_\alpha R_\alpha T\ (\alpha < \gamma T)$ and of the 2.7.1.

2.7.3. Theorem.

$$\mathrm{k}\,\mathrm{R}_\alpha \mathrm{T} \leq (\mathrm{n}\,\mathrm{T})^{\mathrm{w_c T}}\ (\alpha < \gamma \mathrm{T}) \text{ and } \mathrm{k}\,\mathrm{T} \leq (\mathrm{n}\,\mathrm{T})^{\mathrm{w_c T}}\, \mathrm{w_c T}\,;$$

in particular $\mathrm{k}\mathrm{T} \leq (\mathrm{n}\,\mathrm{T})^{\mathrm{w_c}}\,\mathrm{T}$ *provided* $\mathrm{n}\mathrm{T} > 1$.

The theorem is a consequence of 2.7.2.

2.8. Theorem (a) *Let* T *be a tree and* $\mathrm{N} = \mathrm{N}\mathrm{T}$ *the first cardinal number greater than any node of* T. *There exist two* $\gamma\mathrm{T}$-*sequences of cardinals*

$$k_\alpha \leq N^-,\ a_\alpha \leq k(1+\alpha),\ a_\alpha < K_c \quad (\alpha < \gamma T) \tag{1}$$

such that

$$kT \leq \sum_\alpha k_\alpha{}^{a\alpha} \quad (\alpha < \gamma T). \tag{2}$$

If N *is regular one could request, moreover, that*

$$k_\alpha < N \text{ for every } \alpha < \gamma T\,; \tag{3}$$

(b) *The general continuum hypothesis implies for any regular* N *the existence of two mappings of* $\mathrm{I}\,\mathrm{K}_c$:

$$x \to k_x \text{ into } IN$$
$$\text{and } x \to a_x \text{ into } IK_c$$

such that

$$\mathrm{k}\mathrm{T} \leq \sum_x \mathrm{k_x}{}^{\mathrm{a_x}} \quad (\mathrm{x} \in \mathrm{I}\mathrm{K_c}). \tag{4}$$

[5]) s^C denotes the supremum of the cardinals kF, F being a disjointed *system* of open non empty intervals of **C**.

[6]) $x \dashv y$ meaus: x is a proper initral part of y.

P r o o f. 2.8.1. The coexistence of the relations (1) and (2) was proved in § 2.7.1; only it remains to prove that the condition $a_\alpha < K_c$ might be required. Now, $\sup k\gamma' \leq k_c \leq k\gamma T$. If γ is not initial, then $k_c = k\gamma$ and one could suppose $a_\alpha < K_c$. Let, therefore, γ bi initial. If $k_c < k\gamma$, then $k\gamma' \leq k_c$ thus $a_\alpha < K_c$. There remains the case $k_c = (k\gamma)^- = k\gamma$; then $k\gamma' < k_c$. Thus in any case we could demand that $a_\alpha < K_c$.

2.8.2. Let us prove (2) under assumptions (4) for any regular N.

Let $n = N^-$. We have $n \leq N$. If $n < N$, it is sufficiant to put $k_\alpha = n$, $a_\alpha = k(1+\alpha)$. Therefore, let us consider the case $n = N$. One has either $kT = k\gamma T$ or $kT > k\gamma T$. In the first case it is sufficient to put $k_\alpha = 1$ and $a_\alpha = 1$ for every $\alpha < \gamma T$: the relations (1), (2) hold good. In the second case $kT > k\gamma T$ we have $kT = mT$ with $mT = \sup k R_\alpha T$, $(\alpha < \gamma T)$. Now, either $n \geq k\gamma T$ or $n < k\gamma T$. If $n \geq k\gamma T$ we put $k_0 = k R_0 T$, $k_1 = \sum_{x \varepsilon R_0 T} k R(x, .)$. Let $0 < \alpha < \gamma T$ such that the cardinals $k_{\alpha'}, a_{\alpha'}$, be determined and that

$$k R_{\alpha'} T \leq \Sigma k_{\alpha'}{}^{a\alpha'} \quad (\alpha' < \alpha). \tag{5}$$

If α is of the second kind we put

$$k_\alpha = \sup k_{\alpha'}, \quad a_\alpha = k(1+\alpha);$$

the number $n (= N)$ being regular and $> kT$ one has $k_\alpha < n$ and obviously $k R_\alpha T \leq k_\alpha{}^{k(1+\alpha)}$. Let now α be of the first kind. Then the cardinal $k R_{\alpha-1} T$ is either $< n$ or $\geq n$. In the first case we put $k_\alpha = \Sigma\, k\, R\, T(x, .)$, $(x \in R_{\alpha-1} T)$, $a_\alpha = k(1+\alpha)$; in the second case put $k_\alpha = k_{\alpha-1}$. In both cases the numbers k_α, a_α are determined and one sees that the equation obtained from (5) by the substitution $\alpha \to \alpha + 1$ holds.

2.8.3. Now, we shall prove that the domain of the mappings $x \to k_x$, $x \to a_x$ might be the set $I K_c$ of cardinals $< K_c$ (instead the set $I\gamma T$ of ordinals $< \gamma T$), provided both N be regular and the general continuum hypothesis is holding.

We have to consider two alternatives, according as the number $k_c = K_c^-$ is reached or not reached.

2.8.3.1. k_c is reached i. e. there is a chain of cardinality k_c. Then $k_c = k\gamma$ or $k_c < k\gamma$. Let $k_c = k\gamma$. If $N \leq k_c$ then $k_x{}^{k_c} \leq k_c{}^{k_c} = 2^{k_c}$ and $\sum_{<\gamma} 2^{k_c} = 2^{k_c} \cdot k_c = 2^{k_c}$ i. e. (4) holds. If $N > k_c$, then $k_x < N \Rightarrow$ $\Rightarrow k_x{}^{k_c} \leq N$ and $N^{k_c} = N$ (N is regular, and the continuum hypothesis is assumed!) and $kT \leq N$ — again (4) is satisfied.

In $N^- < N$ then $N^- \geq k_c$ and $(N^-)^{k_c} \leq N$; if N^- is regular, then $(N^-)^{k_c} = N^-$ and $kT \leq N^-$ i. e. $kT = N^-$ — all right!

If N^- is singular, then $(N^-)^{k_c}$ is either N^- or N; in both cases $kT = (N^-)^{k_c} = \sum_{Tk_c} (N^-)^{k_c}$.

2.8.3.2. k_c is reached and $k_c < k\gamma$; then $\operatorname{ind} k_c + 1 = \operatorname{ind} k\gamma$, where $\alpha = \operatorname{ind} \aleph_\alpha$. The preceding reasoning applies in this case too.

2.8.3.3. k_c is not reached: $k_c = K_c = K_c^{-} = k\,\gamma T$ being initial. We have these alternatives: $N \leq k_c$ and $N > k_c$.

2.8.3.3.1. If $N \leq k_c$, then $k_x < k_c$ and one could take $a_{\gamma'} \geq k_{\gamma'}$, thus $k_{\gamma'}{}^{a_{\gamma'}} = 2^{a_{\gamma'}}$. Therefore, $kT \leq \sum_{\gamma'} 2^{a_{\gamma'}}$. Now, $a_{\gamma'} < k_c$ and consequently $2^{a_{\gamma'}} \leq k\gamma = \aleph_\lambda$. On the other hand one proves readily this

L e m m a. To every γ-sequence of cardinals $b_{\gamma'} < \aleph_\lambda$ corresponds a λ-sequence of cardinals $d_{\lambda'} < \aleph_\lambda$ such that $\sum_{\gamma'} b_{\gamma'} \leq \sum_{\lambda'} d_{\lambda'}$. (*)

As a matter of fact $cf\,\omega_\lambda = cf\,\lambda$. On the other hand the number $(*)_1$ is $\leq \aleph_\lambda$; now, the number $(*)_2$ might be $\aleph_\lambda$, although $d_{\lambda'} < \aleph_\lambda$.

2.8.3.3.2. Let us now consider the case $N > k_c$. We might suppose $k_{\gamma'} > a_{\gamma'}$ and $k_{\gamma'}$ to be regular and therefore $k_{\gamma'}{}^{a_{\gamma'}} = k_{\gamma'}$ (continuum hypothesis!) and finally $kT \leq N^{-}$.

Now, $N^{-} \leq N$. The relation $N^{-} = N$ is not possible: otherwise one would have $kT = N$ and the number $kT\,(= N)$ would be the sum of a γ-sequence of numbers $< N$ — absurdity, N being regular. Therefore, necessarily $N^{-} < N$; and in this case it is sufficient to put $k = N^{-}$, $a_x = 1$ for any $x \in I K_c$, to convince us that the relation (4) holds.

The theorem is completely proved.

2.8.4. R e m a r k. If N is not regular, the relation (2) might be false under the assumption (3). This is shown by a tree T satisfying $\gamma T = 2$, $R_0 T = \{a_0, a_1, \ldots a_n, \ldots\}$, $k\,R\,T(a_{\omega'}, \cdot) = \aleph_{\omega'}$.

3. **Ranged Sets.**

3.1. D e f i n i t i o n. An ordered set B *is ranged* provided each of its *chains* is *well ordered.*

3.1.1. L e m m a. *The set* $R_0 B$ *of all the initial points of a ranged set is a maximal antichain of* B *i. e.* $R_0 B = RB$ (cf. § 1.6).

3.2. *A tree* TB *associated to* B (cf. Kurepa [4] § 2). Let us consider the sets

$$B(.\,, x] \qquad (x \in B).$$

The maximal chains of any of these sets form a well defined family of chains of $(B; <)$; we shall denote it by

TB or more explicitly $(TB; =\!|)$ where the relation $=\!|$ means *»to be an initial segment of«;* in other words if X, Y are sequences or well ordered sets then $X =\!| Y$ means that X is a beginning part of Y; in particular $X -\!| Y$ means $X =\!| Y$ and $X \neq Y$ i. e. X is a proper initial portion[7] of Y.

By induction argument one sees that

[7]) The symbols $=\!|$, $-\!|$ replace the symbols $\leqq\cdot$, $\leqq_k$, $<\cdot$ of some of my previous papers.

3.2.1. L e m m a. $R_0 T B = \{a_0\}$, $(a_0 \leqslant R_0 B)$

$$R_1 T B = (a_0, a_1), (a_0 \leqslant R_0 B, (a_1 \leqslant R_0 T(a_0;))$$

and for every $\alpha < \Gamma$ B

$$R_\alpha T B = \{X \pm (a) \mid X \leqslant R_{\alpha-1}\ T B, a \leqslant R_0 B(X, \cdot)\} \text{ provided } \alpha \leqslant I$$

$$R_\lambda T B = \{\sup C \mid C$$

being any maximal λ-chain in the set $(\bigcup_{\alpha'} R_{\alpha'}\, T B; \text{—|})\}$.

By definition, *sup* C means the least sequence s such that $x =\!| s$ for every $x \leqslant C$.

3.2.2. $\Gamma\, T\, B = \Gamma\, B$

3.2.3. $k_c T B = k_c B$

3.2.4. $N T B = N B, \quad n T B = n B$.

3.3. T h e o r e m o n r a n g e d s e t s. *For each ranged set* (B; <) *we have*

$$k B \leq (n B)^{k_c B} \cdot k_c B.$$

In particular

$$\mathrm{k\,B} \leq (\mathrm{n\,B})^{\mathrm{kc\,B}} \textit{ provided } \mathrm{n\,B} > 1.$$

The theorem is an immediate consequence of the obvious relation $k B \leq k(TB)$ and of the 2.7.3, 3.2.2, 3.2.3 and 3.2.4.
Since $n B \leq k_c B$ the theorem 3.3. implies this corollary:
3.3.1. C o r o l l a r y. *For any ranged set* B *one has* $\mathrm{k\,B} \leq (2\,\mathrm{k_a\,B})^{\mathrm{k_c\,B}}$ (cf. Kurepa [4] Lemma p. 63).

3.3.2. T h e o r e m (a) *For any ranged set* B *there are two* ΓB-*sequences of cardinal numbers*

$$k_\alpha \leq (N B)^-, a_\alpha \leq k(1 + \alpha), \underset{\alpha}{a} < K_c, (\alpha < \Gamma B)$$

such that

$$k B \leq \sum_\alpha k_\alpha a^\alpha \, (\alpha < \Gamma B).$$

If N *is regular, one could require that, moreover,*

$$k_\alpha < N B, (\alpha < \Gamma B).$$

(b) The general continuum hypothesis implies for any regular N *the existence of two mappings of* $\mathrm{I\,K_c} : \mathrm{x} \to \mathrm{k_x}$ *into* I N *and* $\mathrm{x} \to \mathrm{a_x}$ into $\mathrm{I\,K_c}$ *such that* $\mathrm{k\,B} \leq \Sigma\, \mathrm{k_x}^{\mathrm{a_x}}$, (x $\mathrm{I\,K_c}$).

The theorem is implied by Th. 2.8. and the lemmas 3.2.2, 3.2.4.

3'. G-ranged Sets.

Let $\rightarrow$ be a binary antisymmetrical relation; this means that for distinct points a, b the relations $a \rightarrow b$ and $b \rightarrow a$ are not possible (the relation $a \rightarrow a$ is not excluded; the transitive property of $\rightarrow$ is not excluded either).

3'.1. Def. *An oriented graph* is any ordered pair $(S; \rightarrow)$ of a set S and an antisymmetrical binary relation $\rightarrow$ in S.

3'.2. A *g-ranged set* is any oriented graph $(G; \rightarrow)$ in which every non void chain C has an initial element i. e. an element e such that $e \rightarrow x$ for every $x \in C$.

3'.3. For any $X \subseteq G$ let $R\,X$ be a maximal antichain containing every initial point of X.

For any ranged set $(G; \rightarrow)$ and any $X \subseteq G$ the antichain $R\,X$ is well determined just like for ranged sets $(B; <)$.

3'.4. *Dual g-ranged set* of $(S, \rightarrow)$ is the structure $(S, \leftarrow)$ where $a \leftarrow b$ means $b \rightarrow a$.

3'.5. *The preceding considerations on ranged sets hold for g-ranged sets too.*

4. **Ordered Chains.**

4.1. Let E be a chain and ω a normal well — ordering of E. Let B be the set E ordered by *superposition* of the given order in E and the well-order w. B is a ranged set and

$$K_c\,B \leq W_c\,E; \tag{1}$$

$$K_a\,B \leq W_d\,E, \tag{2}$$

$$\Gamma\,G \leq \Gamma\,E: \tag{3}$$

Let us prove (2). Let A be any antichain in B. Now in the wellorder w, the set A is well-ordered; the same set A in the given chain E is inversely well - ordered, - otherwise A would be no antichain in B: any couple of distinct points of A are distinctly ordered in E and w.

Since $k\,E = k\,B$ on applying the theorem 3.3 we conclude that

$$k\,E \leq (w_d\,E)^{w_c\,E}.$$

Analogously on considering the order $(S; >)$ instead of the order $(S; <)$ we see that $w_d(S, >) = w_c(S, <)$, $w_c(S; >) = w_d(S, <)$ and the preceding relation yields

$$k\,E \leq (w_c\,E)^{w_d\,E}.$$

Thus we have the following result.

4.2. T h e o r e m. *For every totally ordered set* E *we have*

$$k\,E \leq a^b$$

where

$$a = \sup\{w_c\,E, w_d\,E\},\ b = \inf\{w_c\,E, w_d\,E\}.$$

4.2.1. C o r o l l a r y. *For every ordered chain* E *we have*

$$k\,E \leq 2^a,\ a = \sup\{w_c\,E, w_d\,E\}. \text{ (Hausdorff).}$$

The theorem 4.2. is a strengthening of the preceding corollary. E. g. if for a chain $w_c\,E = 2^{\aleph_0}$, $w_d\,E = \aleph_0$, then the theorem yields $k\,E \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$; by the corollary one has the weaker majoration $k\,E \leq 2^{\aleph_0}$.

4.3. Remark. As application of the theorem 3.3.2. one gets a corresponding statement for ordered chains.

4.4. *The s-number of a chain* E. For a family F of sets let SF be the first cardinal $> kD$, D being any disjointed sistem of sets which are elements of F. We put $sF = (SF)^{-}$. Thus sF is the supremum of kD, D having the same meaning. For an ordered set E we denote by SE, sE respectively the numbers SF, sF, F meaning the family of all the open intervals of E.

4.4.1. Lemma. *For any ordered chain* E $\mathrm{sE} \geq \mathrm{w_c E}$, $\mathrm{sE} \geq \mathrm{w_d E}$.

4.4.2. Theorem. $kE \leq 2^{sE}$

The theorem is a corrolary of 4.2.1. and 4.4.1.

5. Ordered Sets.

5.1. Let $(E; <)$ be any ordered set (partially or totaly ordered); let w mean a *normal well-order* of E. Let (B, ϱ) mean the ordering of E obtained as the product of the orderings $(E, <)$ and w i. e. $x \varrho y$ means that x precedes y in $(E; <)$ and in w. The set $(B; \varrho)$ is ranged. By theorem 3.3 we have

$$kB \leq (nB)^{k_c B}\, k_c B. \tag{1}$$

Now, $k_c B \leq w_c (E, <)$; therefore

$$kB \leq (nB)^{w_c E}\, w_c E. \tag{2}$$

On the other hand

$$nB \leq k_a (B, \varrho). \tag{3}$$

Now, let A be any antichain in $(B; \varrho)$; let $(A; \sigma)$ be the order of A obtained as the product of the orders of A in $(E, >)$ and in w. The set $(A; \sigma)$ in ranged and obviously

$$K_a (A, \sigma) \leq K_a (E, <) \text{ i. e. } k_a (A, \sigma) \leq k_a E$$
$$k_c (A, \sigma) \leq k_c (E, >) = k_d (E, <) = k_d E. \tag{4}$$

By the theorem 3.3. we have, therefore,

$$kA \leq (k_a A)^{w_c A} \cdot w_c A \leq (k_a E)^{w_d E}\, w_c A. \tag{5}$$

Taking here the supremum with respect to the antichain A in (A, ϱ) one gets

$$k_a B = \sup kA \leq (k_a E)^{w_d E}\, w_d A \tag{6}$$

and the formula (2) yields

$$kB \leq [(k_a E)^{w_d E}\, w_d A]^{k_c B}\, w_c B = (k_a E)^{w_d E \cdot w_c E}\, (w_d E)^{w_c B}$$

i. e. (since $kB = kE$):

$$kE \leq (k_a E)^{w_c E\ w_d E}\, (w_d E)^{w_c E}. \tag{7}$$

By permuting the indices c and d, one gets another similar formula. Therefore, we have the following theorem.

5.2. *Theorem on ordered sets. Putting for any ordered set* E

$$x = \sup\{w_c E, w_d E\}, \quad y = \inf\{w_c E, w_d E\}$$

we have

$$k E \leq (k_a E)^x \cdot x^y. \tag{8}$$

5.2.1. Corollary. *For any chain E we have* $k E \leq x^y$ (put in (8) $k_a E = 1$; cf. Theorem 4.2).

5.2.2. Corollary. *For any ordered set E we have* $k E \leq$ $\leq (2 k_a E)^x$, $x = \sup\{w_c E, w_d E\}$ (cf. relation (4) in Kurepa [4]).

As a matter of fact $x^y \leq x^x = 2^x$, and the relation (8) yields $k E \leq (k_a E)^x 2^x = (2 k_a E)^x$. Q. E. D.

6. Binary Symmetrical Relations. Graphs.

6.1. As an immediate generalization of preceding considerations on *ordered sets* one has the corresponding results for *binary graphs* $(G; \varrho)$. The role of the comparability) (cesf. incomparability) relation in ordered sets is played now by any binary *symmetrical* relation ϱ. Obviously, in this case the numbers W_c, W_d are to be replaced by the number K_c defined as the first cardinal number $> k C$, C being any *ϱ-chain* of the graph $(G; \varrho)$. Analogously, K_a or $K_{\bar{c}}$ is the least cardinal $> k A$, A being any *antichain* of the graph.

6.1.1. *Dual graph* $(G; \varrho^*)$ of a graph $(G; \varrho)$ is obtained from $(G; \varrho)$ by permuting the connexion and the disconnection relation:

$$a \varrho^* b \langle=\rangle a \text{ non } \varrho b.$$

Consequently,

$$k_c(G, \varrho) = k_{\bar{c}}(G; \varrho^*)$$

$$k_{\bar{c}}(G, \varrho) = k_c(G; \varrho).$$

6.2. To every graph $(G; \varrho)$ we associate a tree $(T G; =|)$ in the following way. (cf. § 3.2). Let w a normal well-order of the set G; let the relation $\rightarrow$ mean the product of the ϱ-relation and of the well-order relation w i. e. $a \rightarrow b \langle=\rangle a \varrho b$ and $w a < w b$. Then for any $\rightarrow$ chain C we have the set $G(C, .) =$

$$\{x \mid x \in G \setminus C, c \rightarrow x \text{ for every } c \in C\}$$

as well as the set $R G(C, .)$ of the first points of $G(C, .)$. Then to every $a \in G$ one associates a $\rightarrow$-chain $C(a) = C_0(a), C_1(a), \ldots$ such that $a \in C(a)$ and $C'(a) \cdot \rightarrow a$ where $C'(a) = C(a) - \{a\}$ and that $C_\xi(a) \in R G(\{C_0(a); . C_\xi(a) . .\}, .)$. The set $C(a)$ is a maximal $\rightarrow$ chain of the set $G(, . a]$. The length $\gamma C(a)$ of $C(a)$ is $\leq \omega(a)$, $C(a)$ being also a ϱ-chain, one has necessarily $\gamma C(a) \leq \omega_{(kc)}$. The tree $T G$ will be formed of the chains $C(a)$, $(a \in G)$ and ordered by the relation $=|$.

6.2.1. Lemma. *The sets* (G, ϱ) *and* $(T G; -|)$ *are connected by the relations:*

$$k\,G \leq k\,T\,G \tag{1}$$

$$\gamma\,T\,G \leq \omega_{(kc)} \tag{2}$$

$$N\,T\,G \leq K_a\,G \tag{3}$$

Let us prove for instance the last relation. Let C be a chain in $(T\,G\,;\,=\!|)$; then $\bigcup_{X \varepsilon C} X = X'$ is a chain in $(G\,;\rightarrow)$; the set $G(X',.)$ and its initial row R are well determined; R is a ϱ-antichain and one sees that the elements $X' \cup \{x\}$, $(x \in R)$ form the node $R\,(T\,G\,(C,.))$ of $(T\,G\,;\,-\!|)$.

6.2.2. Theorem. *For any graph* $(G\,;\varrho)$ *one has*

$$k\,G \leq x^y$$

where

$$x = \sup\,\{k_a\,G, k_c\,G\}, \quad y = \inf\,\{k_a\,G, k_c\,G\}.$$

As a matter of fact, the n^{os} 2.7.3. and 6.2.1. imply

$$k\,G \leq (k_a\,G)^{k_c G} \,.\, k_c\,G.$$

Now, for dual graph (G, ϱ^*) the analogous relation yields $k\,G \leq \leq (k_c\,G)^{k_a G} \,.\, k_a\,G$. And the last two formulas yield the required formula of the theorem.

6.2.3. Theorem. *Let* $(G\,;\varrho)$ *be a graph of cardinality* $> 2^{\aleph_\alpha}$; *then* G *contains a* ϱ*-chain or a* ϱ*-antichain of cardinality* $> \aleph_\alpha$. This is a direct consequence of 6.2.2. T.

7. On Symmetrical Mappings with 2 Variables.

7.1. Definition. Let $I2 = \{0,1\}$; for any set S let $S_{11}{}^{I2}$ be the set of all the ordered pairs (x, y) such that $x \in S$, $y \in S$ and $x \neq y$.

7.2. Theorem. *Let* S *be any set and* f *a symmetrical mapping of* $S_{11}{}^{I2}$ *into* In[8]*, where for a given number* n *we denote by* In *the* set of numbers $< n$.

If $k\,S > 2^{\aleph_\alpha}$ and $n < \omega$, then there exists a subset X of S such that $k\,X > \aleph_\alpha$ and that f be constant in $X_{11}{}^{I2}$. The conclusion need not hold provided $k\,S \leq 2^{\aleph_\alpha}$ or provided f be non symmetrical, regardless of the number $k\,S$.

Proof 7.2.1. The proof will be carried out by the induction argument on n.

First step: $n = 2$. Let us denote the relation $f(a.\,b) = 0$ by $a\,\varrho\,b$; then we have the graph $(S\,;\varrho)$ and the wording »X *is a chain (antichain) in* $(S\,;\varrho)$« is equivalent to the wording $X_{11}{}^{I2} \subset \{f^{-1}\,0\}$, (resp. $\{f^{-1}\,1\}$). Therefore, the theorem 6.2.3. implies the theorem 7.2 for $n = 2$.

[8]) i. e. $f(x, y) = f(y, x)$.

Second step: let $2 < n < \omega$ and suppose that the theorem 7.2 holds for every $n < l$. We shall prove that it holds also for $n = l$. For this, let $a \sigma b$ mean $f(a, b) = l-1$. On gets the graph $(S; \sigma)$. We have these alternatives: First case: S contains $a \sigma$-chain X of cardinality $> \aleph_\alpha$; this means that the theorem holds for $n = l$. Second case: every σ-chain in $(S; \sigma)$ is $\leq \aleph_\alpha$. In this case, S contains necessarily an σ-antichain A of cardinality $> 2^{\aleph_\alpha}$; in the opposite case, one would have $k A \leq 2^{\aleph_\alpha}$ for every σ-antichain. In virtue of the theorem 6.2.2. one would have $k S \leq (2^{\aleph_\alpha})^{\aleph_\alpha} = 2^{\aleph_\alpha}$, contrarily to the hypothesis. Consequently, there exists a σ-antichain A of cardinality $> 2^{\aleph_\alpha}$; this means that the restriction of f on A is a mapping of A^{I2} into $I(l-1)$; by induction hypothesis, A contains a subset X of cardinality $> \aleph_\alpha$ such that f be constant in $X_{11}{}^{I2}$. Finally, the theorem holds for $n = l$ too. Thus it holds for each $n < \omega$.

7.2.2. On the other hand, let M_α be the set, ordered alphabetically, of ω_α-sequences of rational numbers and ω a normal well-order of M_α. If we put $f(a, b) = 0$ if and only if a precedes b and and $w\, a < w\, b$, and $f(a, b) \neq 0 \langle = \rangle f(a, b) = 1$, then f is a mapping of $M_\alpha{}^{I2}$ into $I2$ which is non constant in every square of cardinality $> \aleph_\alpha$; the cardinal number of M_α is $\aleph_0{}^{\aleph_\alpha}$ i. e. $2^{\aleph_\alpha}$. Thus the condition $k S > 2^{\aleph_\alpha}$ of the theorem is necessary.

7.2.3. On the other hand, let S be any set and f a mapping of S^{I2} onto $I2$ such that $f(a, b) \neq f(b, a)$ for $a \neq b$. Then f is non-constant on the square X^{I2} for each $X \subseteq S, k X > 1$. Thus the symmetry condition of f in the theorem is necessary.

7.2.4. Remark. I thought that by induction argument the theorem 7.2. holds for Ir instead of $I2$ for any integer $r > 1$ (cf. Kurepa [6], [7]); cf. also the theorems 8.3 and 8.4, 9.4, 9.5 and the remark. 9.6).

8. On Symmetrical Mappings.

8.1. Definition. Let (A, B) be any ordered pair of sets and $A_{11}{}^B$ or $A_{11}(B)$ the set of all the one-to-one mappings of B into A. In particular, r being any ordinal number, $A_{11}{}^{Ir}$ denotes all the one-to-one r-sequences of elements of A.

By definition we put $A = A_{11}{}^{I1}$.

8.2. Definition. Let (m, n) be any ordered fair of numbers and r any ordinal $< \omega$; we define $m_r\, n$ in the following way:

$$m_0\, n = n, \quad m_1\, n = m^n, \quad m_{x+1} = m^{m_x\, n}$$

$$\text{E. g. } 3_2\, 4 = 3^{3^4}$$

8.3. Main theorem. *Let* S *be a set and* r *a positive integer and* $\aleph_\alpha$ *any aleph. If there exists a symmetrical mapping* f *of* $S_{11}{}^{Ir}$ *into a set* M *of cardinality* m *such that the relations*

$$X \subseteq S, k f X_{11}{}^{Ir} = 1 \text{ imply } k X \leq \aleph_\alpha,$$

then

$$k S \leq m_{r-1} \aleph_a.$$

The theorem is equivalent to the following theorem.

8.4. Theorem. *For any positive integer* r *and set* S *let* f *be a symmetrical mapping of* $S_{11}{}^{Ir}$ *into a set of cardinality* m. *If* $m \leq \aleph_a$ *and* $kS > m_{r-1} \aleph_a$, *there exists a subset* X *of* S *such that* $kX > \aleph_a$ *and that Let* f *be constant in* $X_{11}{}^{Ir}$.

Therefore, let us prove the theorem 8.4. The proof will be carried out by induction on r.

8.4.1. The theorem holds for $r = 1$: if a set of cardinality $> m_0 \aleph$ $(= \aleph_a)$ is mapped by f into M with $k M \leq \aleph_a$, then f is constant on a subset of S of a cardinality $> \aleph_a$. In the opposite case, there would be $k \{ {}^{-}f a \} \leq \aleph_a$ for each $a \in M$ and the relation $\bigcup\limits_{a \varepsilon M} \{ {}^{-}f a \} = S$ would imply $k M \cdot \aleph_a \geq k S$ i. e. $\aleph_a \cdot \aleph_a \geq k S$, contrary to the hypothesis $k S > \aleph_a$.

Let now e be any integer > 1 and suppose that the theorem 8.3 holds for each $r < e$; we shall prove that it holds for $r = e$ too.

8.4.2. Let

$$(w) \ldots w_0, w_1, w_2, \ldots$$

be a 1—1 mapping of $I \omega_{(ks)}$ onto S. For every $a \in S$ we define a subsequence $C(a)$ of points $x \in S$ satisfying ${}^{-}x \leq {}^{-} w a$[9]. We put

$$C(w_0) = (w_0), C(w_1) = (w_0 w_1), \ldots, C(w_{e-1}) = (w_0, w_1, \ldots, w_{e-2}).$$

For any other point $a \in S$ we put $a_0 = w_0, a_1 = w_1, \ldots, a_{e-2} = w_{e-2}$ and define a_{e-1} as the first element x in the well-order (w) such that $f(s_{e-1} x) = f(s_{e-1} a)$, where $s_{e-1} = a_0 a_1 \ldots a_{e-2}$. Let α be an ordinal such that the α-sequence $a^\alpha = (a_{\alpha'})_{\alpha'}$ be defined and that each of its e-subsequences s satisfies $f(s) = f(s' a)$, where s' means the sequence s without it last term. We define then a_α as the first element $x \in S, {}^{-}x \leq {}^{-} w a$ in (w) such that $f(y x) = f(y a)$ for each $(e-1)$-subsequence y of a^α.

The formation of $C(a)$ is finished when the point a becomes an element of $C(a)$.

8.4.3. Obviously $\gamma C(a) \leq \gamma w a$.

8.4.4. The mapping $a \rightarrow C(a)$ $(a \in S)$ is one-to-one. First of all the mapping is uniform. Secondly, the inverse mapping is uniform too.

As a matter of fact if x, y are distinct elements of S then either ${}^{-}w x < {}^{-} w y$ or ${}^{-} w x > {}^{-} w y$. In the first case one has $x \in C(x)$, $y \in' C(x)$; in the second case, $x \in' C(y)$, $y \in C(y)$; thus $C(x) \neq (y)$.

8.4.5. Let $T = T S$ be the tree whose elements are all the initial portions of the sequences $C(a)$, $(a \in S)$; we order $T S$ by $-|$.

8.4.6. *Every node of* T *is* $\leq$ m, i. e. for every sequence $s = a_0 a_1 \ldots a_{\alpha'}, \ldots$ the number of the sequences of the form $s x$ satisfying $s x \in T$, $x \in S$ is $\leq m$.

[9]) The relations $w_\alpha = x$, $\alpha = {}^{-}w x$ are equivalent.

In fact, for every subsequence y of $(e-1)$ terms of s and every value $v \in M$ let $\{^{-}f(y)\,v\}$ mean the set of all the x satisfying $x \in S$, $f(y\,x) = v$. For a given $v \in M$ the intersection of all these sets is well determined as well as its first element u; u depends upon s and v i. e. $u = u\,(s\,;\,v)$. It might happen that for some $v \in M$ the point $u\,(s\,;\,v)$ does not exist; anyway the immediate followers of s in T are of the form $s, u\,(s\,;\,v)$, v running through M; therefore, the number of these followers is $\leq m$. Q. E. D.

8.4.7. *One has*

$$k\,T \leq \sum_{a} m^{k\,a}, \; (a < \gamma, \gamma < \sup \gamma\, C\,(a),\, a \in S).$$

This is an immediate consequence of 4. and § 2. 7. 1.

8.4.8. *There exists an* a $\in$ S *such that*

$$k\,C\,(a) > m_{e-2}\,\aleph_a\,(= b). \tag{1}$$

In the opposite case, for every $a \in S$ one would have $k\,C\,(a) \leq b$ and $\gamma\,C\,(a) < \omega_{(b)+1}$ thus

$$k\,T \leq \sum_{a<\omega_{(b)+1}} m^{k\,a} = \aleph_{(b)+1} \cdot m^b = (m_{e-1}\,\aleph_a)^+ \cdot m_{e-1}\,\aleph_a = m_{e-1}\,\aleph_a$$

Hence $k\,T \leq m_{e+1}\aleph_a$ which is in contradiction with $k\,S \leq k\,T$ and $k\,S > m_{e-1}\aleph_a$. This proves the relation (1).

8.4.9. Now, the definition of $C\,(a)$ implies that

$$f\,(s) = f\,(s\,a) \text{ for every e-subsequence } s \text{ of } C\,(a). \tag{2}$$

In this way we get a determined symmetrical mapping

$$x \rightarrow f\,(x\,a) \quad (x \in C\,(a)_{11}{}^{I(e-1)}) \tag{3}$$

The relation (1) enables us to apply the induction hypothesis: the set $C\,(a)$ contains a subset X of cardinality $> \aleph_a$ such that the mapping (3) be constant in $X_{11}{}^{I(e-1)}$. This means, in virtue of (2), that also the mapping f is constant in $X_{11}{}^{I\,e}$. Q. E. D.

8.5. *Remark on the symmetry condition.* The symmetry condition in theorem 8.3, 8.4 is needed.

In fact, let S be any set and f a mapping of S onto $I\,2$ such that $f\,(x, y) \neq f\,(y, x)$ for every $x, y \in S, x \neq y$. Then f is non-constant on the set $X_{11}{}^{I\,r}$ for each $X \subseteq S$, $k\,X > 1$.

8.6. Remark. *For* $\mathrm{r} = 2$ *the condition* $\mathrm{k\,S} > \mathrm{m}_1\,\aleph_a$ *is needed: there exists a set* S *such that* $\mathrm{k\,S} = 2^{\aleph_a}$ *and a symmetrical mapping of* $\mathrm{S}_{11}{}^{\mathrm{I}2}$ *into* I 2 *which is non-constant in* $\mathrm{X}_{11}\,(\mathrm{I}\,2)$ *for each subset* X *of cardinality* $> \aleph_a$.

As a matter of fact let

$$S = Q\,(\omega_a)$$

be a system of all the ω_a-sequences of rational numbers ordered by the principle of the first differences; S is a chain, each interval of S ha $k\,S = 2^{\aleph_a}$ points and every strictly increasing (decreasing) sequence in S is of a cardinality $\leq \aleph_a$. Now, let w be a normal well-ordering of S. Let then the order relation $\prec$ be defined in S as the superposition (product) of the preceding two orderings of $S : a \prec b$ means that a precedes b in $(S ; <)$ and in $(S ; w)$. Then each chain (antichain) in $(S ; \prec)$ as a well-ordered (resp. a dually well-ordered) subset of $(S ; \prec)$ is $\aleph_\alpha$, although $k\,S = 2^{\aleph_\alpha}$(cf. Kurepa 18). If then $f(a, b) = 0$ means that a, b are in $(S ; \prec)$ comparable relative to $\prec$ and if $f(a, b) = 1$ means that the points a, b are incomparable relative to $\prec$, then we are dealing with a symmetrical mapping f of $S_{11}{}^{I2}$ into $I2$ and which is non-constant in $X_{11}{}^{I2}$ for each $X \subseteq S$ with $k\,S > \aleph_\alpha$.

9. **On Combinations.**

9.1. Definition. For any set S and any cardinal number n let $\binom{S}{n}$ denote the system of all subsets of S, of cardinality n each.

If S, M are sets, then $\binom{S}{M}$ denotes the set of all the subsets of S, of cardinality $k\,M$ each.

If $n > k\,S$ and if $k\,M > k\,S$, one puts $\binom{S}{n} = \emptyset = \binom{S}{M}$. Any mapping f of $\binom{S}{n}$ is a symmetrical mapping of $S_{11}(B)$, where $k\,B = n$.[10)]

Therefore the results of §§ 7 and 8 imply the following statements.

9.2. Theorem. *Let $\aleph_{\alpha}$ be given. In order that for each mapping* f *of* $\binom{S}{2}$ *into* M *of cardinality* $\leq \aleph_\alpha$ *there exists a subset* X *of* S *such that* $k\,X > \aleph_\alpha$ *and that* f *be constant in* $\binom{X}{2}$ *it in necessary and sufficient that* $k\,S > 2^{\aleph_\alpha}$ *(cf § 7).*

9.3. Theorem. *Let $\aleph_\alpha$ be given. In order that for each partition* P *of* $\binom{S}{2}$ *into* m *classes there exists a subset* X *of* S *of cardinality* $> \aleph_\alpha$ *and such that* $\binom{X}{2}$ *be entirely contained in one class of the partition, it is necessary and sufficient that* $k\,S > 2^{\aleph_\alpha}$.

The statement 9.3. is equivalent to the statement 9.2 as it is visible by the correspondence $f x = A \Leftrightarrow x \in A \in P$, $(x \in \binom{S}{2})$.

One gets in this way a mapping of $\binom{S}{2}$ into the set P which takes now the role of the set M in statement 9.2.

10) A mapping f of $S_{11}(B)$ is symmetrical provided $fg = fbg$ $(g \in S_{11}(B)$, $b \in B!)$; $B!$ denotes the set of all the permutations of B.

9.4. T h e o r e m. *Let* S, r, $\aleph_\alpha$ *be any set, any positive integer and any aleph respectively; il there exists a mapping* f *of* $\binom{S}{r}$ *into a set* M *of cardinality* m *such that the relations*

$$X \subseteq S, kf\binom{X}{r} = 1 \text{ imply } kX \leq \aleph_\alpha$$

then

$$kS \leq m_{r-1} S.$$

The theorem 9.4 is a special case of the theorem 8.3.

9.5. T h e o r e m. *For any set* S, *any cardinal number* $\aleph_\alpha$ *and any positive integer* r *let* f *be a mapping of* $\binom{S}{r}$ *into a set* M *of cardinality* $m \leq \aleph_\alpha$; *if* $kS > m_{r-1}\aleph_\alpha$, *where* $m_0 \aleph_\alpha = \aleph_\alpha$, $m_1 \aleph_\alpha = m^{\aleph_\alpha}$, $m_{x+1} = m^{m_x \aleph_\alpha}$ *then there exists a subset* X *of* S *such that* $kX > \aleph_\alpha$ *and that* f *be constant in* $\binom{X}{r}$.

The theorem is an immediate consequence of the § 7.5.

9.6. R e m a r k. *The converse of the theorem 9.4 holds too for* r = 1,2 (cf. § 8.6)[11])

10. **Problem.**

10.1. P r o b l e m. *Let* $\aleph_\alpha$ *be given. Does there exist a cardinal number* $R(\aleph_\alpha)$ *such that for every set* S *and for every mapping* f *of* $\binom{S}{\aleph_\alpha}$ *into* I2 *the relation* $kS > R(\aleph_\alpha)$ *implies the existence of a set* X *in* S *such that* f *be constant in* $\binom{X}{\aleph_\alpha}$ *and that* $kX > \aleph_\alpha$.

10.2. P r o b l e m. *Given cardinals* m, $\aleph_\alpha$. *Let* S *be any set and* f *any mapping of* $\binom{S}{\aleph_\alpha}$ *into* I2; *if every set* $X \subseteq S$ *such that* f *be constant in* $\binom{S}{\aleph_\alpha}$ *is of a cardinality* $\leq m$, *determine sup* kS.

10.3. G e n e r a l p r o b l e m. *Let* a, m, c *be given numbers (each finite or infinite); let us consider any set* S, *the set* $\binom{S}{a}$ *and any mapping* f *of* $\binom{S}{a}$ *into a set* M *of a cardinality* $\leq m$. *Does there exist — and determine — a number* R = R (a, m, c) *sucht that the relation* $kS > R$ *implies for any mapping* f *of* $\binom{S}{a}$ *into* M *the existence of a subset* X *of* S *of a cardinality* > c *and such that* f *be constant in* $\binom{X}{a}$? *E. g.* $R(2, 2, \aleph_\alpha) = 2^{\aleph_\alpha}$ *and* $R(2, n, \aleph_\alpha) = 2^{\aleph_\alpha}$ *for each* $1 < n \leq \aleph_0$. We thought that $R(a, n, \aleph_\alpha) = 2^{\aleph_\alpha}$ for any finite $a > 1$ (cf. Kurepa [6], [7]); therefore, the publication of this paper stopped since 1952.

[11]) As I was told by P. E r d ö s, the converse was recently proved in 1959 by H a j n a l for each positive integer *r*.

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Glossary and Notations

Antichain	1.6 (Note)
Chain	1.6 (Note)
γ	2.2
Γ	1.1
G	6.1
Combination, $\binom{S}{M}$, $\binom{S}{M}$	9.1
Graph	6.1
$G(C,.)$	6.2
G-ranged	3.2
Ix (x any number): the set of all the numbers $< x$	
11, $A_{11}(B)$, $A_{11}{}^B$	8.1
kS ... the cardinality of S	
k_c, $k_{\bar{c}} = k_a$	1.6; 6.1
K_c, $K_{\bar{c}} = K_a$	1.3; 6.1
$m_r n$	8.2

$m\,T$ 2.4.1
N, n 1.8
Node 2.6
Oriented graph 3.1
R; R_0 . . . . 1.7; 3'.3
Ranged 3.1
Rank 2.2
R_a 2.3
s, S 4.4
Tree 2.1
Tree associated to G 6.2
$x^- = \sup Ix$; x' means $x' < x$ . . .
$\omega_{(x)}$ is the first ordinal of cardinality x
$W_c^{(x)}, W_d$ 1.4
w_c, w_d 1.5
$=\!|$ means »to be an initial part of« . (3.2)
$-\!|$ means »to be a proper initial part of« (3.2)
${}^{-}w\,x$ 8.4.2
! . . . . 9.1 (Note)

O KARDINALNOM BROJU UREĐENIH SKUPOVA I SIMETRIČNIH STRUKTURA U ZAVISNOSTI OD KARDINALNIH BROJEVA NJIHOVIH LANACA I ANTILANACA[1]

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Sadržaj

Neka je S (potpuno ili nepotpuno) uređen skup, T stablo, a G graf (t. j. skup u kojem je definirana simetrična binarna relacija, ϱ) Tad se mogu promatrati brojevi $K_c\,G$, $K_{\bar{c}}\,G$ kao oni najmanji kardinalni brojevi koji su veći od kardinalnog broja svakog lanca odnosno antilanaca iz G. Već se u Tezi pojavio problem da li je $k\,T$ supremum brojeva $k_c\,T = (K_c\,T)^-$, $k_{\bar{c}}\,T = K_{\bar{c}}\,T)^-$ te problem da li su brojevi $K_c\,T$, $k_{\bar{c}}\,T$ dostignuti u pojedinom T. Specijalno za »uske« T-ove dokazano je da svako beskonačno T ima isti kardinalni broj kao nekoji lanac iz T (Kurepa [1] str. 80. T. 5^{bis}; u [8] je taj teorem reproduciran). Za uređene skupove S problem je riješen »osnovnom relacijom« (1) u Kurepa [3] te [4]: vrijedi $k\,S \leq (2\,k_{\bar{c}}\,S)\,k_c\,S$ (isp. § 5). Rezultat i dokaz se prenose na binarne simetrične relacije. Mislio sam da će isti rezultat vrijediti i za n-arne simetrične relacije ([6], [7]) pa je zato potpuno objavljivanje sve dalje odgađano sve dok nedavno nisam doznao od P. Erdösa da je procjena za $k\,S$ u T. 8.3 (odn. 9.4) najbolja kao što je Hajnal dokazao.

[1]) O tom predmetu govorio sam: 06. 9. 1952 u Münchenu na kongresu njemačkih matematičara, 03. 12. 1952 na kolokviju u Zagrebu, 24. 02. 1953. na Faculté des Sciences u Parizu te 17. 3. 1954 na kolokviju u Zagrebu.

Od interesa je imati u vidu kako razmatranja o stablima ulaze u razmatranja o općim simetričnim relacijama (cf. §§ 3.2, 6.2, 8.4.5).

§ 1. *Definicije*

1.1. ΓS označuje prvi redni broj koji je veći od svakog rednog broja koji se može predstaviti unutar S.

1.4. $W_c S$ je prvi glavni broj koji nije pretstavljiv kao glavni broj nekog dobro uređenog podskupa iz S; $W_c S^* = W_c (S, >)$.

1.6. Ako je $X \subseteq S$, tad RX znači bilo koji maksimalni antilanac iz S koji obuhvata $R_0 X$; $R_0 X$ označuje množinu svih početnih elemenata iz X. Slično, RX za $X \subseteq G$ znači bilo koji maksimalni antilanac iz grafa G.

1.7. Za lanac C iz S neka $S(C,.)$ znači skup svih $x \in S$ za koje je $y < x$ za svako $y \in C$. Neka NS znači prvi kardinalni broj $> kRS(C,.)$ za svako $C \leq S$. Stavljamo $mS = n = (NS)^-$.

§ 2. *Stabla.*

2.7.1. Teorem.[2] Za svako $\alpha < \gamma T$ vrijedi $k R_\alpha T \leq k_\alpha^{a_\alpha}$ gdje su k_α, a_α kardinalni brojevi za koje je $k_\alpha \leq N^-$, $a_\alpha \leq k(1 + \alpha)$. T je slično s nekim dijelom stabla $T(n;\gamma)$ sastavljenog od svih nizova dužine $< \gamma T$ rednih brojeva $< \omega_{(n)}$ i uređenog relacijom $\dashv$. Vrijedi $kT(n, \gamma) = \sum_{\alpha<\gamma} n^{k_\alpha}$. Teoremi 2.7.2 2.7.3 i 2.8 mogu se razabrati iz engleskog teksta.

§ 3. *Razvrstani skupovi.*

To su uređeni skupovi B kod kojih je svaki lanac dobro uređen. Svakom B može se pridružiti drvo TB (Kurepa [4] § 2) promatrajući skupove $B(.\,, x]$ ($x \in B$), njihove maksimalne lance i uvodeći međusobni poredak pomoću relacije $\dashv$. Tad se može pokazati teorem 3.3 kao i teorem 3.3.2.

§ 3'. *G-razvrstani skupovi.*

Tu se radi o binarnoj antisimetričnoj relaciji; označujemo je sa $\rightarrow$ pa se može govoriti o orijentiranim grafovima $(S;\rightarrow)$ pri čemu je S proizvoljan skup. Teoremi o razvrstanim skupovima prenose se na t. zv. g-razvrstane skupove t. j. na orijentirane grafove kod kojih svaki lanac ima početan član.

§ 4. *Uređeni lanci.*

4.1. Ako je E uređen lanac, a w normalno dobro-uređenje od E, tad se superpozicijom tih dvaju uređenja dobije razvrstan skup B za koje vrijede obrasci (1), (2), (3) kao i teorem 4.2.

[2]) O oznaci vidi alfabetski popis na kraju članka.

4.4. s-broj lanca E. Pridružimo svakoj obitelji F skupova broj SF kao prvi kardinalni broj $> kD$, gdje je D proizvoljan disjunktivan podsistem od F; stavljamo $sF = (SF)^{-}$. Tad vrijede iskazi 4.4.1 i 4.4.2.

§ 5. *Uređeni skupovi.*

Glavni iskaz o uređenim skupovima nalazi se u § 5.2.

§ 6. *Binarne simetrične relacije. Grafovi.*

Gornja razmatranja o uređenim skupovima prenose se na skupove snabdjevene proizvoljnom simetričnom binarnom relacijom ϱ koja zamijenjuje relaciju uporedljivosti kod uređenih skupova. Specijalno važe teoremi 6.2.2. i 6.2.3.

§ 7. *Simetrične funkcije s 2 varijable.*

7.2. Neka $S_{11}{}^{I2}$ označuje skup svih uređenih pari (x, y) za koje je $x \in S$, $y \in S$, $x \neq y$. Ako je f simetrično preslikavanje os $S_{11}{}^{I2}$ na In (n prirodan broj), pa ako je $kS > 2^{\aleph_\alpha}$, tad S sadrži dio X od $> \aleph_\alpha$ članova i to tako da f bude konstantno u $X_{11}{}^{I2}$.

7.2.4. Primjedba. Držao sam ([6], [7]) da gornji iskaz važi i onda kad se mjesto $I2$ čita Ir (za bilo koji prirodni broj $r > 1$) pa je nastojanje da se prvobitni pogrešni »dokaz« toga popravi oteglo objavljivanje samog članka.

§ 8. *O simetričnim preslikavanjima.*

8.3. Ako za neki skup S, prirodni broj r i alef $\aleph_\alpha$ postoji simetrično preslikavanje f od $S_{11}{}^{Ir}$ na skup M od m elemenata tako da vrijedi relacija (1) tad je $kS \leq m_{r-1}\, \aleph_\alpha$. Pritom se posljednji simbol definira rekurzijom u § 8.2. Drugim riječima:

8.4. Ako je $kS > m_{r-1}\, \aleph_\alpha$ tad S sadrži skup k od $> \aleph_\alpha$ članova tako da f bude konstanta u X_{11}^{Ir}.

§ 9. *O kombinacijama.*

9.1. Za skup S i broj m neka $\binom{S}{m}$ označuje množinu svih dijelova od S po m članova. Ako je M skup od m članova, stavljamo $\binom{S}{m} = \binom{S}{M}$. Kako je svaki član iz S_{11}^{Ir} u vezi određenom r-kombinacijom mogu se teoremi iz §§ 7 i 8 iskazati i pomoću kombinacija.

9.2. Teorem. Neka je $\aleph_\alpha$ zadano. Da bi za svako preslikavanje od $\binom{S}{2}$ na skup M od $\leq \aleph_\alpha$ elemenata postojao podskup $X \leq S$ tako da f bude u $\binom{X}{2}$ konstantno i $kX > \aleph_\alpha$, treba a i dosta je da vrijedi $kS > 2^{\aleph_\alpha}$ (isp. § 7.).

9.3. Teorem. Zadano je $\aleph_\alpha$; da bi za svaku podjelu P od $\binom{S}{2}$ na m razreda postojalo neko $X \subseteq S$ od $> \aleph_\alpha$ članova sa svojstvom da čitavo $\binom{X}{2}$ leži u jednom članu od P potrebno je i dovoljno da bude $kS > 2^{\aleph_\alpha}$.

9.4. Neka su $S, r, \aleph_\alpha$ skup, prirodan broj i alef; postoji li simetrično preslikavanje f od $\binom{S}{r}$ na M od m članova tako da iz $X \subseteq S$, $kf\binom{X}{r} = 1$ slijedi $kX \leq \aleph_\alpha$, tad je $kS \leq m_{r-1}\,\aleph_\alpha$.

9.5. Uz prepostavke kao u 9.4 relacija $kS > m_{r-1}\,\aleph_\alpha$ uslovljava postojanje množine X iz S sa svojstvima $kX > \aleph_\alpha$ i da f bude konstantno u $\binom{X}{r}$.

9.6. Primjedba. Za $r = 1,2$ procjena za kS u iskazu 9.4 je najbolja (isp. 8.6); prema obavijesti P. Erdösa procjena 9.4 za kS je najbolja za svako r kao što je to Hajnal dokazao u 1959.

§ 10. *Problem.*

10.1. Problem. Neka je zadano $\aleph_\alpha$; postoji li kardinalan broj $R = R(\aleph_\alpha)$ sa svojstvom da za svaki skup S i svako preslikavanje f od $\binom{S}{\aleph_\alpha}$ na $I2$ relacija $ks > R$ ima za posljedicu da S sadrži dio X tako da f bude konstanta u $\binom{X}{\aleph_\alpha}$?

10.3. Opći problem. Neka su a, m, c brojevi (svaki od njih konačan ili beskonačan). Neka je S skup, a f preslikavanje od $\binom{S}{a}$ na M potencije $\leq m$. Postoji li broj $R = R(a, m, c)$ sa svojstvom da iz $kS > R$ slijedi da S obuhvata dio X od $> c$ članova i da f u $\binom{X}{a}$ bude konstantno?

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