GLASNIK MAT. - FIZ. I ASTR.
Tom 12. — No. 1-2 — 1957.

CALCULATION OF THE CORRECTIONS FOR THE ANALYSIS OF THE β- SPECTRA AND THEIR APPLICATION TO Ho^{166}

G. Alaga and *B. Jakšić*

CONTENTS

Summary

We have submitted a short article to The Physical Review [1] containing the main results of our present paper, i. e., that it is possible to account qualitatively for the large $\log ft = 8.1$ value and approximately allowed shape of the $Ho^{166} \rightarrow Er^{166}$ ground state $(0 \longrightarrow 0+)$ transition [2], by a tensor pseudoscalar mixture. The amount of the pseudoscalar admixture can not be determined without new experimental data or calculation of the matrix elements involved.

Two possibilities have been investigated. The case of destructive interference [3] of pseudoscalar and tensor coupling in which case the pseudoscalar plays a dominant rôle and the case of a singular tensor matrix element $\langle \beta \vec{\sigma} \cdot \mathbf{r} \rangle$. In the first case all the corrections like those of finite size of the nucleus [4] and finite wave length are important and have been calculated. The dominant correction in the second case is the variation of the lepton fields over the nuclear region [5].

After a short introduction on the present status of the β-decay theory [6] we present the calculation of the above mentioned corrections. First we write the β-interactions in tensor form [7], because that is the most convenient and simplest way of calculating the finite wave length effects and the missing cross term between the two part of the pseudoscalar interaction. Beside the tensor and pseudoscalar interaction we have written the other interactions in the tensor form for the sake of completeness and references. The extraction of the major part of the pseudoscalar interaction and non relativistic form of the relativistic matrix elements [8] has been done with the aid of the Foldy-Wouthuysen transformation. The final result for the $Ho^{166} \rightarrow Er^{166}$ correction factor is presented in the last section.

For the calculation of the effects of the finite size of the nucleus we used the method of Rose [4], assuming a charge density distribution slightly different from a constant. The variation of the electron wave function inside the nucleus, which is essential in the second case, was calculated, for the same nuclear charge distribution, by expanding the lepton wave function in powers of r/r_0. Destructive interference even corrected for all the mentioned corrections failed to reproduce the experimental spectrum. So we examined in some details the second case and were able to obtain agreement with the spectrum. The numerical results are given in the figures.

I. *Introduction*

The theory of β-decay was formulated originally by E. Fermi [9] in close analogy with the field theory of electromagnetic radiation [10]. The direct lepton (electron, neutrino*) nucleon (proton, neutron) interaction was assumed to be of the form of a scalar product of a four vector made of lepton emission operators and the nucleon exchange current, multiplied by a constant which measures the strengths of the lepton nucleon coupling.

* Prof. Pauli announced at the Geneva Conference in June, 1956, that the existence of the neutrino was definitely established by direct experiments by Cowan and Reines.

The theory was successful in explaining some general features of this phenomenon, like spectrum shape and half life (in general). These are essentially given by the apriori probability for the case if some amount of energy is shared among 3 particles (statistical factor) and the magnitude of nuclear matrix elements or nuclear moments. The last measures the overlap of the initial and final state wave functions. The original four vector formulation and the close analogy with electromagnetic radiation turned out to be too narrow to explain all the variety of the β-radioactivity [11]. Scalar, tensor, pseudovector and pseudoscalar couplings have been introduced beside the original four vector. In this way beside the interaction, which has classical analogy in the theory of electromagnetic radiation, new couplings have been introduced. The five covariants just mentioned are all the possible covariants which can be formed from lepton operators without including the derivatives, and the »type« of spinor field [12]. So one can say that the theory of β-decay is in this respect complete.

The problem is now to find the coupling constants of the right linear combinations of the mentioned couplings (by analysing the β-spectra, half lives, β^+/K ratios and electron-neutrino angular correlations) [6].

The great advance in the field of experimental technique was actually made in 1949 with the aid of high flux neutron reactors and electromagnetic separation methods, sources of high specific activity, have been prepared and precise measurements have been made on some isotopes of long half life. In this way the earlier measurements have been completed and so we have now available valuable data on half-lives, spectra, β^+/K ratios and angular correlations of He^6, Ne^{19}, Na^{22}, P^{32}, S^{35}, Cl^{36}, Y^{91}, Tc^{99}, Cs^{137}, Pm^{147}, Tm^{170}, Tl^{204}, and many others.

The scalar and vector, tensor and axial vector coupling do interfere even in the allowed transitions. The interference terms are called the F i e r z terms [13]. They cause deviations of the spectra, etc. from the statistical shape which is called the allowed shape.

It seams that the best available values of coupling constants compatible with all the experimental data are [13] $|g_1/g_3|^2 = \frac{0.75}{1.15}$ and $|g_4|$ and $|g_2|$ are 1—3% and 15% of the g_3. With $|g_1| = |g_3|$ one would get $g_\beta = 1{\cdot}7 \times 10^{-49}$ erg cm^3. With g_1, g_2, g_3, g_4 and g_5 we have denoted the relative scalar, vector, tensor, pseudovector and pseudoscalar coupling constant. g_β measures the coupling strengths of the lepton field to the nucleons.

For the determination of the relative sign between the scalar and tensor coupling constants it is necessary to investigate such transitions where the two couplings interfere. Tc^{99}, Cs^{137}, Rb^{87}, Cl^{36}, RaE seem to be beta emiters of this type. Different investigators [14] report different relative signs so that the problem of sign is not properly settled as yet. Anyhow all these conclusions about the signs are based on the calculation of nuclear matrix elements involved and therefore are not very reliable.

About the pseudoscalar coupling there is only some indirect evidence from the analysis of the ft-values of the presumably $\Delta I = 0$ yes transitions. The ft-value of these transitions (Hg^{205}, Tl^{207} and Pb^{209}) and (Tl^{206}, Pb^{210} and Pb^{212}) ($0 \longrightarrow 0+$) seem to be somewhat smaller ($\log ft \approx 5{\cdot}5$) than expected what is interpreted as a substantial contribution of the pseudoscalar coupling to this transitions [15]. Very little is known about the shapes of these transitions. Beside these examples there are recent measurements of the Canadian group on the β-spectrum and branching of the $Ho^{166} \rightarrow Er^{166}$ transition [2]. They found that the ground-ground state transition is a ($0 \longrightarrow 0+$) transition with an approximatly allowed shape with the maximum energy $W_0 = 4{\cdot}63$ and a rather large ft-value ($\log ft = 8{\cdot}1$).

If one excludes the axial vector coupling (which also gives an allowed shape) for obvious reasons [13] already discussed, we are left with the tensor and pseudoscalar coupling or a mixture of both to explain the experimental results on ($0 \longrightarrow 0+$) transitions.

The purpose of this paper is to show how it is possible to understand qualitatively the large ft-value and the approximately allowed shape of this ($0 \longrightarrow 0+$) transition. Two possibilities have been considered:

a) destructive interference of the pseudoscalar and the tensor coupling,

b) small value of the $\langle \beta \vec{\sigma} \cdot r \rangle$ tensor matrix element.

II. *General Formulation of the Theory*

The interaction Hamiltonian density responsible for β-decay may be written in the form [16]:

$$\mathcal{H}_\beta = g_\beta \sum_{i=1}^{5} g_i (\mathcal{H}_i + \mathcal{H}_i^*) \tag{1}$$

with

$$\mathcal{H}_1 = (\Psi_P^* \beta \Psi_N)(\psi^* \beta \varphi) \tag{2a}$$

$$\mathcal{H}_2 = (\Psi_P^* \Psi_N)(\psi^* \varphi) - (\Psi_P^* \vec{\alpha} \Psi_N) \cdot (\psi^* \vec{\alpha} \varphi) \tag{2b}$$

$$\mathcal{H}_3 = (\Psi_P^* \beta \vec{\sigma} \Psi_N) \cdot (\psi^* \beta \vec{\sigma} \varphi) + (\Psi_P^* \beta \vec{\alpha} \Psi_N) \cdot (\psi^* \beta \vec{\alpha} \varphi) \tag{2c}$$

$$\mathcal{H}_4 = (\Psi_P^* \vec{\sigma} \Psi_N) \cdot (\psi^* \vec{\sigma} \varphi) - (\Psi_P^* \gamma_5 \Psi_N)(\psi^* \gamma_5 \varphi) \tag{2d}$$

$$\mathcal{H}_5 = (\Psi_P^* \beta \gamma_5 \Psi_N)(\psi^* \beta \gamma_5 \varphi) \tag{2e}$$

Ψ_P and Ψ_N are the proton respectively neutron and ψ and φ are the lepton field operators. They figure as emission and absorption operators. Nucleon and lepton fields are taken at the same space time point. $\beta, \vec{\sigma}, \vec{\alpha}$ and γ_5 are the usual Dirac matrices [17]. We can note that in the form of interaction (2a)—(2e) we have adopted a definite sequence of the operators. That is also an additional assumption corresponding to the decay sheme

$$N + \nu \gtrless P + e, \tag{3}$$

where a neutron and a neutrino is absorbed and an electron and proton emitted. The Hermitian conjugate operator corresponding to the inverse process is included in (1) to make the expression for $\mathcal{H}_\beta$ Hermitian.

One usually treats in β - decay the nucleons in the configuration space [18] because the nucleon pair creation state can be neglected.

The interaction Hamiltonian density in the configuration space of A nucleons is represented by

$$\mathcal{H}_i = \sum_{k=1}^{A} O_k{}^i \psi^*(\mathbf{r}) O^i \varphi(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}_k), \tag{4}$$

O^i is the obvious abreviation for the Dirac operator in (2a—2e) including the proton-neutron exchange operator for nucleons.

To obtain the interaction energy one has to integrate the expression (4) over the whole space $\int \mathcal{H}_i \, d\tau = H_i$ and write the matrix elements between the initial and final lepton states. We can denote that by

$$\langle H_i \rangle = \sum_{k=1}^{A} O_k{}^i \langle e \bar{\nu} \mid \psi^*(x_k) O_i \varphi(x_k) \mid 0 \rangle. \tag{4a}$$

In that way we obtain as interaction energy operator the expression

$$H_\beta = g_\beta \sum_{i=1}^{5} g_i [\langle H_i \rangle + \langle H_i^* \rangle] \tag{5}$$

wich can be used in the ordinary perturbation treatement in the configuration space, so that the probability of the β - emission is given by the known formula [17]

$$N = \frac{2\pi}{\hbar} \Sigma \left| \int \Psi_f^* H_\beta \Psi_i \, d\tau_1 \ldots d\tau_A \right|^2 \varrho \tag{6}$$

where ϱ is the density of the final state for a definite total energy. It is actually the density which normalizes the final lepton wave functions to the total energy. Summation runs over all not observed quantum numbers of the final states and includes averaging over all experimentally not specified initial states. Ψ_i and Ψ_f are the initial, respectively final, wave function of the nucleus.

III. *Solution of Dirac Equation for Leptons*

Since leptons obey the Dirac equation, we have to solve this equation both for electrons and for neutrinos. In the presence of an external electromagnetic field $A_\mu = (\mathbf{A}, i\Phi)$, the Dirac equation in relativistic units ($m = \hbar = c = 1$) is

$$\left[\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - i e A_\mu\right) + 1\right]\psi = 0\,,\ \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}. \tag{7}$$

With

$$\gamma_k = -i\beta\alpha_k,\ \gamma_4 = -\beta,\ \mathbf{p} = -i\nabla,\ H = i\frac{\partial}{\partial t}$$

we get the Dirac equation in an external spherically symmetric electrostatic field ($\mathbf{A} = 0$, $\Phi = \Phi(r)$) as

$$H\psi = i\frac{\partial\psi}{\partial t},\ H = -\vec{\alpha}\cdot\mathbf{p} - \beta + V(r),\ V(r) = e\,\Phi(r). \tag{8}$$

The matrices $\vec{\sigma}$, β will be used in the Dirac representation [17] throughout the paper:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix},\ \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8a}$$

Since the Hamiltonian (8) is spherically symmetric, it will commute with the operator of the total angular momentum $\mathbf{J}$

$$\mathbf{J} = \mathbf{L} + \mathbf{S} = \begin{pmatrix} \mathbf{L} + {}^1/_2\vec{\sigma} & 0 \\ 0 & \mathbf{L} + {}^1/_2\vec{\sigma} \end{pmatrix},\ \mathbf{S} = {}^1/_2\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix},\ [H, \mathbf{J}] = 0. \tag{9}$$

It also commutes with the inversion operator I

$$\begin{aligned} &I = \beta I_p,\ I_p\,\mathbf{r}\,I_p = -\mathbf{r},\ I_p\vec{\alpha}\,I_p = -\vec{\alpha}, \\ &[H, I] = 0,\ I_p\,\mathbf{p}\,I_p = -\mathbf{p},\ I_p\beta I_p = \beta. \end{aligned} \tag{10}$$

Before we introduce spherical coordinates we shall transform the Hamiltonian into a more convenient form. We define first the quantities

$$p_r = \frac{1}{r}\mathbf{r}\cdot\mathbf{p} = \frac{1}{i}\frac{\partial}{\partial r},\ \alpha_r = \frac{\vec{\alpha}\cdot\mathbf{r}}{r},\quad \sigma_r = \frac{\vec{\sigma}\cdot\mathbf{r}}{r},\ \alpha_r^2 = 1,\ \sigma_r^2 = 1. \tag{11}$$

One can show now that we can write

$$\vec{\alpha} \cdot \mathbf{p} = \alpha_r \left(p_r - \frac{\mathbf{i}}{r} + i \beta \frac{K}{r} \right). \tag{12}$$

where K is the constant of motion given by

$$K = \beta (2 \mathbf{S} \cdot \mathbf{L} + 1) = \begin{pmatrix} \vec{\sigma} \cdot \mathbf{L} + 1 & 0 \\ 0 & -(\vec{\sigma} \cdot \mathbf{L} + 1) \end{pmatrix}, \quad [H, K] = 0. \tag{13}$$

This operator satisfies the two following relations

$$\mathrm{K}^2 = \mathbf{J}^2 + 1/4, \quad \mathrm{K}(\mathrm{K} - \beta) = \mathbf{L}^2. \tag{14}$$

Since it neither depends nor operates on r, it will commute with p_r, as well as α_r. It commutes also with α_r and β. With the help of (12) the Hamiltonian (8) takes the form [17]

$$H = -\alpha_r \left(p_r - \frac{i}{r} \right) - i \alpha_r \beta \frac{K}{r} - \beta + V(r). \tag{15}$$

The problem is to find now the simultaneous eigenfunctions of the complete set of commuting operators H, $\mathbf{J}^2$, J_z, I, or of the set H, K, J_z. We will take the second set so that we have to solve the following eigenvalue problem

$$H \psi_{w \varkappa \mu} = W \psi_{w \varkappa \mu} \tag{16a}$$

$$K \psi_{w \varkappa \mu} = \varkappa \psi_{w \varkappa \mu} \tag{16b}$$

$$J_z \psi_{w \varkappa \mu} = \mu \psi_{w \varkappa \mu} \tag{16c}$$

For the continuous spectrum, in which we are interested, we have $W > 1$. Writing $\psi_{w \varkappa \mu}$ by means of two two-component wave functions $\varphi^{(1)}_{w \varkappa \mu}$ and $\varphi^{(2)}_{w \varkappa \mu}$

$$\psi_{w \varkappa \mu} = \begin{pmatrix} \varphi^{(1)}_{w \varkappa \mu} \\ \varphi^{(2)}_{w \varkappa \mu} \end{pmatrix} \tag{16d}$$

one can easily find that the angular part of the solution can be expressed by means of the functions $\chi_\varkappa^\mu$ defined by

$$(\vec{\sigma} \cdot \mathbf{L} + 1) \chi_\varkappa^\mu = -\varkappa \chi_\varkappa^\mu \tag{17a}$$

$$(L_z + 1/2 \sigma_z) \chi_\varkappa^\mu = \mu \chi_\varkappa^\mu \tag{17b}$$

in the following form

$$\psi_{w \varkappa \mu} = \begin{pmatrix} -i e^{i \varphi_\varkappa} f_{w \varkappa}(r) \chi^\mu_{-\varkappa} \\ g_{w \varkappa}(r) \chi^\mu_\varkappa \end{pmatrix} \tag{18}$$

where the phase factor $\exp(i \varphi_\varkappa)$ defined by (17c), was added so that real $f_{w \varkappa}$ and $g_{w \varkappa}$ can be chosen. The angular equations (16b, c) are obviously satisfied by (18). Because of (14) one finds immediately that $\chi_\varkappa^\mu$ is a function corresponding to the eigenvalues j and $l_\varkappa$ of the operators $(\mathbf{L} + 1/2 \vec{\sigma})^2$, respectively $\mathbf{L}^2$

$$j = k - 1/2 = l_\varkappa - 1/2\, S_\varkappa,\ l_\varkappa = k + 1/2\,(S_\varkappa - 1), \tag{19}$$

$$l_\varkappa - l_{-\varkappa} = S_\varkappa,\ \varkappa = \pm 1, \pm 2, \ldots, k = |\varkappa|, S_\varkappa = \frac{\varkappa}{|\varkappa|}\ ,\ -j \leq \mu \leq j.$$

The functions $\chi_{\varkappa}{}^{\mu}$ can be therefore constructed by means of the vector addition coefficients [19]

$$\chi_{\varkappa}{}^{\mu} = \sum_{m'm} (l_\varkappa\, 1/2\, m\, m' \,|\, l_\varkappa\, 1/2\, j\, \mu)\ Y_{l\varkappa}{}^{m}\, \chi^{m'}{}_{1/2}, \tag{20}$$

where $Y_l{}^m$ and $\chi_{1/2}{}^m$ are the usual spherical harmonics and spin functions, respectively. In general, owing to the unitarity and the anticommutativity with $(\overset{\rightarrow}{\sigma} \cdot \mathbf{L} + 1)$ of the operator σ_r, we shall obtain as the result of the application of σ_r upon $\chi_{\varkappa}^{\mu}$ the function $\chi_{-\varkappa}^{\mu}$, multiplied by the phase-factor used already in (18)

$$\sigma_r\, \chi_{\varkappa}{}^{\mu} = -\exp(i\varphi_\varkappa)\ \chi_{-\varkappa}{}^{\mu} \tag{17c}$$

With the special phases for (20) as in T A S [20], the phase factor in (17c) $\exp(i\varphi_\varkappa) = 1$. Using (15), (16a, b), (11), (18) and (17c) we obtain for the radial wave functions the system

$$\frac{d f_\varkappa}{d r} = \frac{\varkappa - 1}{r} f_\varkappa - (W - 1 - V)\, g_\varkappa \tag{21a}$$

$$\frac{d g_\varkappa}{d r} = (W + 1 - V)\, f_\varkappa - \frac{\varkappa + 1}{r}\ g_\varkappa\ . \tag{21b}$$

They are invariant against the substitution $f \to g$, $\varkappa \to -\varkappa$, $W \to -W$, $V \to -V$. From these equations follows the nonlinear equation for the ratio $\left(\frac{f_\varkappa}{g_\varkappa}\right)$

$$\frac{d}{dr}\left(\frac{f_\varkappa}{g_\varkappa}\right) = -(W - 1 - V) + \frac{2\varkappa}{r}\left(\frac{f_\varkappa}{g_\varkappa}\right) - (W + 1 - V)\left(\frac{f_\varkappa}{g_\varkappa}\right)^2. \tag{21c}$$

The index denoting the energy eigenvalue W was dropped for convenience. One can see from (21) that the radial wave functions $f_\varkappa$, $g_\varkappa$ can be taken as real. We must solve now the radial equation (21) under some assumptions on the potential energy $V(r)$.

a) Solution in Coulomb Field

In this case the potential energy of an electron moving in the field of a positive point charge Ze is given by

$$V = V_c = -\frac{\alpha Z}{r},\ \alpha = e^2. \tag{22}$$

The case of a positron moving in the same field can be obtained formally by changing the sign of α.

Since $(f_\varkappa, g_\varkappa)$ can be chosen as real functions, we can express this explicitly by writing

$$rf_\varkappa = \sqrt{1-W}\,(\sigma_\varkappa - \sigma_\varkappa^*) = i\sqrt{W-1}\,(\sigma_\varkappa - \sigma_\varkappa^*),\ W > 1, \tag{23a}$$

$$rg_\varkappa = \sqrt{1+W}\,(\sigma_\varkappa + \sigma_\varkappa^*) \tag{23b}$$

where $\sigma_\varkappa$ is a complex function. The factors in front are introduced to simplify the equation for $\sigma_\varkappa$ and they can be guessed from the assymptotic form of (21). Inserting (23) into (21) we obtain

$$\frac{d\sigma_\varkappa}{dr} = i\left(p + \frac{y}{r}\right)\sigma_\varkappa - \frac{\varkappa - iy/W}{r}\,\sigma_\varkappa^*, \tag{24a}$$

$$\frac{d\sigma_\varkappa^*}{dr} = -i\left(p + \frac{y}{r}\right)\sigma_\varkappa^* - \frac{\varkappa + iy/W}{r}\,\sigma_\varkappa, \tag{24b}$$

where y is the quantity peculiar to the Coulomb field

$$y = \frac{\alpha ZW}{p} = \frac{\alpha Z}{v}. \tag{25}$$

The fine-structure constant α enters the theory only through y, so that the case of a positron is obtained by changig only the sign of $y \to -y$. By elimination, one obtains from (24) the differential equation for $\sigma_\varkappa$

$$\frac{d^2\sigma_\varkappa}{dr^2} + \frac{1}{r}\frac{d\sigma_\varkappa}{dr} + \left(p^2 + \frac{2py - ip}{r} - \frac{\gamma_\varkappa^2}{r^2}\right)\sigma_\varkappa = 0, \tag{26}$$

where $\gamma_\varkappa$ is defined by

$$\gamma_\varkappa = +\sqrt{\varkappa^2 - (\alpha Z)^2}. \tag{27}$$

The regular solution of (26) can be found in a straight — forward way in terms of a hypergeometric function

$$\sigma_\varkappa = C_\varkappa\, e^{-ipr}\,(2pr)^{\gamma_\varkappa} F(\gamma_\varkappa + 1 + iy,\ 2\gamma_\varkappa + 1,\ 2ipr) \tag{28}$$

where the complex constant $C_\varkappa$ must be determined from the condition that $\sigma_\varkappa$ satisfies (24) (it is sufficient to apply this to the first term in the expansion of $\sigma_\varkappa$) and from the normalization of the wave function. If one normalizes the asymptotic form of the wave functions in such a way that it represents on the average one particle in a sphere of unit radius, we obtain

$$\begin{aligned} C_\varkappa &= |C_\varkappa|\, e^{i\eta_\varkappa}\ \ |C_\varkappa| = \frac{e^{\pi y/2}}{2\sqrt{W}}\,\frac{|\Gamma(\gamma_\varkappa + iy)|}{\Gamma(2\gamma_\varkappa + 1)}, \\ \eta_\varkappa &= {}^1\!/_2\, arg\left[\frac{\varkappa - \dfrac{iy}{W}}{\gamma_\varkappa + iy}\right]. \end{aligned} \tag{29}$$

In this way we finally arrive to the solution of the radial wave functions for the continuous energy spectrum $W > 1$ [21],

$$r\begin{pmatrix} f_\varkappa \\ g_\varkappa \end{pmatrix} = \frac{\sqrt{1 \mp W}}{2\sqrt{W}}\, e^{\pi y/2}\, \frac{|\Gamma(\gamma_\varkappa + iy)|}{\Gamma(2\gamma_\varkappa + 1)} \cdot \tag{30}$$

$$\cdot (2pr)^{\gamma_\varkappa}[e^{-ipr+i\eta_\varkappa}\,(\gamma_\varkappa + iy)\; F(\gamma_\varkappa + 1 + iy,\; 2\gamma_\varkappa + 1, 2ipr) \mp c.c.].$$

By means of the formula for the assymptotic expansion of hypergeometric functions we can determine the assymptotic behaviour of the radial wave functions

$$r\begin{pmatrix} f_\varkappa \\ g_\varkappa \end{pmatrix} \sim \sqrt{\frac{W \mp 1}{W}}\begin{pmatrix} \cos \\ \sin \end{pmatrix}\left(pr + y\ln 2pr - \frac{l_\varkappa \pi}{2} + \delta_\varkappa\right), \tag{30a}$$

$$\delta_\varkappa = \eta_\varkappa - \zeta_\varkappa + [k - \gamma_\varkappa + \tfrac{1}{2}(S_\varkappa + 1)]\,\frac{\pi}{2},\quad \zeta = arg\;\Gamma(\gamma_\varkappa + iy).$$

One can see from (30a) that the normalization was chosen properly:

$$\overline{\sum_s \int |\psi_{w\varkappa\mu}|^2 r^2\, d\Omega} = r^2\,\overline{(f_\varkappa^2 + g_\varkappa^2)} = 1\,, \tag{30b}$$

where the sum denotes summation over spin variables and the bar averaging over r.

The second linearly independent solution can be obtained from (30) if one changes $\gamma_\varkappa$ into $-\gamma_\varkappa$. Such solution will be denoted by $\bar{f}_\varkappa$, $\bar{g}_\varkappa$. That this is a solution follows from the form of the differential equation (26) which depends only on $\gamma_\varkappa^2$; that it is linearly independent follows from the fact that $r\bar{f}_\varkappa$, $r\bar{g}_\varkappa$ are singular at the origin.

For many applications one needs the values of regular and irregular Dirac functions for such a small radius as is the nuclear radius. The natural approach in this case would be to expand the functions appearing in (30) in a power series, which converges rapidly owing to the small value of the radius. However, if one does not insist on normalized functions, it is more practical for numerical calculations to solve the radial differential equations (21) directly by assuming power series expansions for $f_\varkappa$ and $g_\varkappa$

$$rf_\varkappa = A_\varkappa\,(2pr)^{s_1}\sum_{n=0}^{\infty} a_{\varkappa n}\, r^n,\quad a_{\varkappa 0} = 1, \tag{31a}$$

$$rg_\varkappa = B_\varkappa\,(2pr)^{s_1}\sum_{n=0}^{\infty} b_{\varkappa n}\, r^n,\quad b_{\varkappa 0} = 1.$$

Inserting (31) into (21) we obtain

$$s_1 = s_2 = \gamma_\varkappa,\; \varrho_\varkappa = \frac{A_\varkappa}{B_\varkappa} = \frac{\varkappa + \gamma_\varkappa}{\alpha Z} = \frac{\alpha Z}{\varkappa - \gamma_\varkappa},\; \bar{\varrho}_\varkappa = \frac{\alpha Z}{\varkappa + \gamma_\varkappa} = \frac{\varkappa - \gamma_\varkappa}{\alpha Z} \tag{32a}$$

and the recurrence formulae for the coefficients

$$a_{\varkappa n} = -\frac{\alpha Z}{n(2\gamma_\varkappa + n)}\left[\frac{(W-1)(\varkappa+\gamma_\varkappa+n)}{\varkappa+\gamma_\varkappa} b_{\varkappa,n-1} + (W+1)\, a_{\varkappa,n-1}\right]$$

$$a_{\varkappa 0} = 1, \tag{32b}$$

$$b_{\varkappa n} = \frac{\alpha Z}{n(2\gamma_\varkappa + n)}\left[\frac{(W+1)(-\varkappa+\gamma_\varkappa+n)}{\varkappa-\gamma_\varkappa} a_{\varkappa,n-1} - (W-1)\, b_{\varkappa,n-1}\right],$$

$$b_{\varkappa 0} = 1. \tag{32c}$$

To obtain the normalized functions represented by (31), we must compare them with (30). In this way one gets the following expressions for $A_\varkappa$ and $B_\varkappa$

$$A_\varkappa = \varrho_\varkappa B_\varkappa = S_\varkappa\, \alpha Z\, e^{\pi y/2} \frac{|\Gamma(\gamma_\varkappa + iy)|}{\Gamma(2\gamma_\varkappa+1)} \sqrt{\frac{\varkappa W - \gamma_\varkappa}{2W(\varkappa-\gamma_\varkappa)}} \tag{33a}$$

$$B_\varkappa = e^{\pi y/2} \frac{|\Gamma(\gamma_\varkappa + iy)|}{\Gamma(2\gamma_\varkappa+1)} \sqrt{\frac{(\varkappa-\gamma_\varkappa)(\varkappa W - \gamma_\varkappa)}{2W}}. \tag{33b}$$

In evaluating (33a, b) we have chosen the solution for η, given by (29), which satisfies $\cos(\eta+\varphi) \geq 0$, φ being $\arg(\gamma_\varkappa + iy)$. This corresponds to the choice $-\pi \leq \eta \leq 0$ for attractive Coulomb interaction (electrons) and $0 \leq \eta \leq \pi$ for repulsive interaction (positrons).

The corresponding solution $\bar{f}_\varkappa$, $\bar{g}_\varkappa$ are obtained by substituting $(-\gamma_\varkappa)$ for $\gamma_\varkappa$, so that the ratios of Dirac radial wave functions are given by

$$\left(\frac{f_\varkappa}{g_\varkappa}\right) = \varrho_\varkappa \frac{\sum_{n=0}^{\infty} a_{\varkappa n} r^n}{\sum_{n=0}^{\infty} b_{\varkappa n} r^n}, \quad \left(\frac{\bar{f}_\varkappa}{\bar{g}_\varkappa}\right) = \bar{\varrho}_\varkappa \frac{\sum_{n=0}^{\infty} \bar{a}_{\varkappa n} r^n}{\sum_{n=0}^{\infty} \bar{b}_{\varkappa n} r^n} \tag{34}$$

$$\left(\frac{f_\varkappa}{\bar{f}_\varkappa}\right) = \frac{\varrho_\varkappa}{\bar{\varrho}_\varkappa}\left(\frac{g_\varkappa}{\bar{g}_\varkappa}\right), \quad \left(\frac{g_\varkappa}{\bar{g}_\varkappa}\right) = \tau_\varkappa (2pr)^{2\gamma_\varkappa} \frac{\sum_{n=0}^{\infty} b_{\varkappa n} r^n}{\sum_{n=0}^{\infty} \bar{b}_{\varkappa n} r^n}$$

where

$$\tau_\varkappa = \frac{B_\varkappa}{\bar{B}_\varkappa} = \frac{\alpha Z}{\varkappa+\gamma_\varkappa} \frac{\Gamma(-2\gamma_\varkappa+1)}{\Gamma(2\gamma_\varkappa+1)} \left|\frac{\Gamma(\gamma_\varkappa+1+iy)}{\Gamma(-\gamma_\varkappa+iy)}\right| \frac{p}{\varkappa W+\gamma_\varkappa}. \tag{34a}$$

b) Solution for Free Particles

This case is obtained if we put $V = 0$ in the radial differential equation (21a, b). The solution can be obtained either directly from (30) by putting $y = 0$, or by solving the equations separatly for this case. The result is expressed in terms of spherical Bessel functions

$$j_l(z) = \sqrt{\frac{\pi}{2z}}\, J_{l+1/2}(z) = \frac{z^l}{(2l+1)!!} \sum_{n=0}^{\infty} \frac{(-1)^n (2l+1)!!\, z^{2n}}{(2n)!!\,(2l+2n+1)!!} \tag{35}$$

in the following way

$$F_\varkappa = S_\varkappa\, p \sqrt{\frac{W-1}{W}}\, j_{l_{-\varkappa}} \tag{36a}$$

$$G_\varkappa = p \sqrt{\frac{W+1}{W}}\, j_{l_\varkappa}\ (\mathrm{pr}). \tag{36b}$$

The asymptotic behaviour is given by (30a) taken for $y = 0$, $\delta_\varkappa = 0$, so that the functions (34) are also normalized to one particle in a sphere of unit radius.

If the particle has zero mass we must omit 1 following W in (36) and put $p = W$, so that in this case we have

$$F_\varkappa = S_\varkappa\, G_{-\varkappa} = S_\varkappa\, W\, j_{l_{-\varkappa}}\ (\mathrm{Wr}) \tag{37a}$$

$$G_\varkappa = W\, j_{l_\varkappa}\ (\mathrm{Wr}). \tag{37b}$$

c) Solution in a Modified Coulomb Field

Since Coulomb field is not a good approximation inside the nuclear charge distribution, we must solve the Dirac equation for a field which is approximately Coulomb field only outside the radius r_0 of the nuclear charge distribution. Inside the nucleus the field is given by the charge density distribution by means of the Poisson equation. We will assume a spherically symmetric charge density distribution, since the deviation from such a distribution should not be very important. In this case the field outside the nucleus will be exactly the Coulomb field

$$V = V_c = -\frac{\alpha Z}{r},\ r \geq r_0 \tag{38a}$$

and inside the nucleus

$$\triangle V^{(i)} = 4\pi e \varrho,\ r \leq r_0. \tag{38b}$$

Both solutions must fulfill the continuity conditions at the boundary

$$V^{(i)} = V_c,\ \frac{dV^{(i)}}{dr} = \frac{dV_c}{dr},\ r = r_0. \tag{38c}$$

The method of solving Dirac equation in such field is to solve it separately for $r < r_0$ and $r > r_0$, and to apply the continuity conditions on the solutions at the boundary. If the not normalized solution inside the nucleus is denoted by $f_\varkappa^{(i)}$, $g_\varkappa^{(i)}$, and the normalized solution of the problem by $f'_\varkappa$, $g'_\varkappa$, we have

$$\left.\begin{aligned} f'_\varkappa &= a_\varkappa\, f_\varkappa^{(i)} = b_\varkappa\, f_\varkappa + c_\varkappa\, \bar{f}_\varkappa \\ g'_\varkappa &= a_\varkappa\, g_\varkappa^{(i)} = b_\varkappa\, g_\varkappa + c_\varkappa\, \bar{g}_\varkappa \end{aligned}\right\},\quad r = r_0. \tag{39}$$

If we introduce, according to Rose [4], the following notation

$$\mu'_\varkappa = \frac{\dfrac{f_\varkappa}{g_\varkappa} - \dfrac{f_\varkappa^{(i)}}{g_\varkappa^{(i)}}}{\dfrac{f_\varkappa^{(i)}}{g_\varkappa^{(i)}} - \dfrac{\bar{f}_\varkappa}{\bar{g}_\varkappa}},\quad \mu_\varkappa = \frac{\bar{f}_\varkappa/\bar{g}_\varkappa}{f_\varkappa/g_\varkappa}\,\mu'_\varkappa,\quad \lambda_\varkappa = \mu'_\varkappa\,\frac{g_\varkappa}{\bar{g}_\varkappa} = \mu_\varkappa\,\frac{f_\varkappa}{\bar{f}_\varkappa} \tag{40}$$

in which all quantities are taken at the radius r_0, we can write

$$b_\varkappa = \frac{1}{\sqrt{1 + 2\,\lambda_\varkappa \cos(\delta_\varkappa - \delta_{-\varkappa}) + \lambda_\varkappa^2}},\quad c_\varkappa = \lambda_\varkappa\, b_\varkappa, \tag{41}$$

where $\delta_\varkappa$ is the phase shift given by (30a). Because of the factor $g_\varkappa/\bar{g}_\varkappa \sim (2pr)^{2\gamma_\varkappa}$, $\lambda_\varkappa$ will be small if $pr \ll 1$, so that $b_\varkappa \approx 1$. For more precise calculation, particularly in the case of destructive interference, it is necessary to use the exact formula (41). By means of (41) and (42) we can write

$$\left.\begin{aligned} f'_\varkappa &= b_\varkappa\,(1 + \lambda_\varkappa\,\mu_\varkappa)\, f_\varkappa \\ g'_\varkappa &= b_\varkappa\,(1 + \lambda_\varkappa\,\mu'_\varkappa)\, g_\varkappa \end{aligned}\right\},\quad a_\varkappa = \frac{f'_\varkappa}{f_\varkappa^{(i)}} = \frac{g'_\varkappa}{g_\varkappa^{(i)}},\quad r = r_0. \tag{39a}$$

If we assume the potential energy $V^{(i)}(r)$ inside the nucleons in the form of a power series in r

$$V^{(i)}(r) = \sum_{n=0}^{\infty} V_n\, x^n,\quad x = \frac{r}{r_0}, \tag{38d}$$

the regular solution of the radial Dirac equation (21) can be easily written down by means of the Rose method [4.] According to him it is possible to write the radial functions in the form

$$u_{1\varkappa}(r) = r\, f_\varkappa^{(i)} = r^\varkappa\left[\frac{1 + S_\varkappa}{2}\, D_\varkappa + \int_0^r r^{-\varkappa}\, a_{12}(r)\, u_{2\varkappa}(r)\, dr\right] \tag{41a}$$

$$u_{2\varkappa}(r) = r\, g_\varkappa^{(i)} = r^{-\varkappa}\left[\frac{1 - S_\varkappa}{2}\, D_\varkappa + \int_0^r r^{\varkappa}\, a_{21}(r)\, u_{1\varkappa}(r)\, dr\right] \tag{41b}$$

where $D_\varkappa$ is a constant, and

$$a_{12} = \sum_{n=0}^{\infty} a_{12}^{(n)}\, x^n,\quad a_{12}^{(0)} = -(W - 1 - V_0),\quad a_{12}^{(n)} = V_n,\ n \neq 0,$$

$$a_{21} = \sum_{n=0}^{\infty} a_{21}^{(n)}\, x^n,\quad a_{21}^{(0)} = W + 1 - V_0,\quad a_{21}^{(n)} = -V_n,\ n \neq 0. \tag{41c}$$

By successive substitutions one obtains the power series expansion in r for the functions $u_{1\varkappa}$ and $u_{2\varkappa}$, where the coefficients are polynomials in x, i. e. still depending on r.

$$f_\varkappa^{(i)}(r) = \frac{1}{r} u_{1\varkappa}(r) = D_\varkappa r^{l-\varkappa} \sum_{n=0}^{\infty} u_{1\varkappa}^{(n)}(x)\, r^{2n}, \tag{42a}$$

$$g_\varkappa^{(i)}(r) = \frac{1}{r} u_{2\varkappa}(r) = D_\varkappa r^{l_\varkappa} \sum_{n=0}^{\infty} u_{2\varkappa}^{(n)}(x)\, r^{2n}. \tag{42b}$$

The polynomials $u_{1\varkappa}^{(n)}(x)$ and $u_{2\varkappa}^{(n)}(x)$ are given by the formulas

$$u_{1k}^{(n)}(x) = \tag{43a}$$

$$= \sum_{p_1, \ldots, p_{2n}} \prod_{r=1}^{n} \frac{a_{21}^{(p_{2r-1})}}{\sum_{i=1}^{2r-1} p_i + 2(k+r) - 1} \prod_{s=1}^{n} \frac{a_{12}^{(p_{2s})}}{\sum_{j=1}^{2s} p_j + 2s}\, x^{p_1+p_2+\ldots+p_{2n}}$$

$$u_{2k}^{(n)}(x) = x^{-2(k+n)-1} \int_0^x x_1^{2(k+n)}\, a_{21}(x_1)\, u_{1k}^{(n)}(x_1)\, dx_1 = \tag{43b}$$

$$= \sum_{p_1, \ldots, p_{2n+1}} \prod_{r=1}^{n+1} \frac{a_{21}^{(p_{2r-1})}}{\sum_{i=1}^{2r-1} p_i + 2(k+r) - 1} \prod_{s=1}^{n} \frac{a_{12}^{(p_{2s})}}{\sum_{j=1}^{2s} p_j + 2s}\, x^{p_1+p_2+\ldots+p_{2n+1}}$$

The corresponding formulas for $\varkappa < 0$ follow from (43) if one changes u_{1k} into $u_{2,-k}$, u_{2k} into $u_{1,-k}$ and $a_{12} \rightleftarrows a_{21}$. The normalization factor $a_\varkappa$ can be obtained from (39a).

Although we have got the solution in a closed form, it is still not so simple to perform all operations indicated in (43) and to obtain the expression ready for numerical evalution. This is particularly pronounced for higher — order terms. Fortunately, the series converges rapidly if the energy is not too high, owing to the very small nuclear charge radius r_0, so that it is sufficient to retain only the first few terms.

IV. *Foldy-Wouthuysen Transformation for Nucleons*

The general formulation of the β-decay theory is relativistic, and in the expression (6) for the transition probability Ψ_f and Ψ_i are relativistic wave functions of the nucleus.

Furthermore, the nucleon operators occuring in β-decay α and γ_5 mix the large and the small components of the wave function Ψ_f and Ψ_i. On the other hand it is well known that all the proposed

nuclear models (α particle model, shell model of M. Goepert-Mayer and H. Jensen; collective model of A. Bohr and B. R. Mottelson; crystal cloudy ball model, etc. [22]) are non-relativistic, thus we have so far only non-relativistic wave functions. Because the models had some success in explaining some qualitative features of nuclear structure (spins, parities, magnetic and quadrupole moments... etc.) one may hope that the relativistic effects are small corrections, and that the calculation of matrix elements can be performed to some extent with the nonrelativistic wave functions. In order to be able to handle the nuclear matrix elements containing $\vec{\alpha}$ and γ_5 we shall describe here a method for obtaining the non relativistic form of such operators by means of a unitary transformation as a series expansion in $(v/c)^n$ [23].

If one writes the Dirac equation of a quantum mechanical system in the form

$$H\Psi = i\dot{\Psi} \tag{8}$$

and if one performs a unitary time independent transformation upon the wave function

$$\Psi' = e^S \Psi, \tag{44}$$

the new wave equation will be

$$H'\Psi' = i\dot{\Psi}', \tag{45}$$

with the new Hamiltonian

$$H' = e^S H e^{-S} = \sum_{n=0}^{\infty} \frac{1}{n} [S, H]_{(n)} = H + \frac{1}{1!}[S, H] + \frac{1}{2!}[S, [S, H]] + \ldots \tag{46}$$

Since little is known about the dynamics of the nucleons in the nucleus, we shall describe it in terms of a model in which we assume that each nucleon is moving in some relativistically covariant average static field. This model leads to the following Dirac Hamiltonian for each nucleon.

$$H = -\vec{\alpha}\mathbf{P} - \beta M - \beta V_1 + V_2 + \vec{i\alpha}\,\mathbf{V}_2 + \vec{\beta\sigma}\,\mathbf{V}_3 - \vec{\beta\alpha}\,\mathbf{V}'_3 - \vec{\sigma}_4\,\mathbf{V}_4 - i\gamma_5 V_4 - \beta\gamma_5 V_5 \tag{47}$$

or

$$H = -\beta M + \mathcal{E} + \mathcal{O}, \tag{47a}$$

where $\mathcal{E}$ and $\mathcal{O}$ are the parts of the Hamiltonian containing only even, respectively, odd operators. An operator is called odd (even) if it mixes (or not) the large and the small components of the Dirac wave function. An alternative definition of odd and even operators is $[\beta, \mathcal{O}]_+ = 0$, respectively $[\beta, \mathcal{E}]_- = 0$. According to that definition $\vec{\alpha}$ and γ_5 are odd, while β and $\vec{\sigma}$ are even operators. In the Foldy-Wouthuysen transformation S is chosen to satisfy

$$[S, H] = -\mathcal{O} + R\,(v/c), \tag{48}$$

where the odd part of $R\ (v/c)$ should be at least of the order $(v/c)^3$. One can easily see that

$$S = -\frac{\beta}{2M}\mathcal{O} \tag{48a}$$

is an obvious choice which satisfies that condition and gives the new Hamiltonian

$$H' = -\beta M + \tag{47b}$$

$$+\left\{\mathcal{E} - \frac{\beta\mathcal{O}^2}{2M} - \frac{1}{2}\left(\frac{1}{2M}\right)^2[\mathcal{O},[\mathcal{O},\mathcal{E}]] + \frac{7}{6}\left(\frac{1}{2M}\right)^3\beta\mathcal{O}^4 + \ldots\right\} +$$

$$+\left\{-\frac{1}{2M}\beta[\mathcal{O},\mathcal{E}] - \frac{4}{3}\left(\frac{1}{2M}\right)^2\mathcal{O}^3 + \frac{1}{6}\left(\frac{1}{2M}\right)^3[\beta\mathcal{O},[\mathcal{O},[\mathcal{O},\mathcal{E}]]] + \ldots.\right\}$$

$$= -\beta M + \mathcal{E}' + \mathcal{O}'.$$

A rough estimate of the quantities $(\mathcal{E}/M)$ and $(\mathcal{O}/M)^2$ is of the order $(v/c)^2$ so that the requirement (48) is fulfilled. Therefore, the odd part $\mathcal{O}'$ of the transformed Hamiltonian H' is only of the order of magnitude (v/c).[3] The small components in the transformed wave function are of that order of magnitude too.

We can perform now the next transformation with

$$S' = -\frac{\beta}{2M}\mathcal{O}' \tag{48b}$$

and proceed in this way until we succeed in eliminating the odd operators up to the desired power of (v/c). The resulting transformation can then be written in the form

$$S = S + S' + S'' + \ldots \tag{48c}$$

With the help of [8]

$$S = -\frac{\beta}{2M}\left\{\mathcal{O} - \frac{1}{2M}\beta[\mathcal{O},\mathcal{E}] - \frac{4}{3}\left(\frac{1}{2M}\right)^2\mathcal{O}^3\right\} \tag{49}$$

all the odd terms up to $(v/c)^4$ are eliminated from the Hamiltonian and the small components vanish identically in this approximation.

V. *Calculation of the β-Spectra*

The β-spectrum is given by the perturbation theory in the first order approximation in the coupling constant g_β with the aid of formula (6). Because the lepton wave functions are normalized (30b) to one particle in a unit sphere, we have for ϱ (in c. g. s. units)

$$\varrho(E_0) = \frac{1}{\pi^2} \frac{m^5 c^5}{\hbar^6} \frac{W}{p} . \tag{50}$$

The β - spectrum is than [7]

$$N(W) = \frac{2}{\pi} \frac{W}{p} \sum_{\substack{\varkappa,\varkappa_\nu \\ \mu,\mu_\nu}} \sum_f S_i \left| \Psi_f^* H_\beta \Psi_i \, d\tau_1 \ldots d\tau_A \right|^2, \tag{51}$$

where W and p are the energy and the momentum of the electron. The summation is to be taken over the electron and the neutrino quantum numbres $\varkappa$, $\varkappa_\nu$, μ and μ_ν and the final states of the nucleus consistent with the conservation of energy. The symbol S denotes the averaging over the initial states of the nucleus.

It is very useful to define the β - correction factor in the form

$$C_\beta = \frac{4\pi^2}{p^2 q^2 F} \sum_{\substack{\varkappa,\varkappa_\nu \\ \mu,\mu_\nu}} \sum_f S_i \left| \Psi_f^* H_\beta \Psi_i \, d\tau_1 \ldots d\tau_A \right|^2 \tag{52}$$

which is connected with the transition probability by means of

$$N(W) = \frac{1}{2\pi^3} W p q^2 F C_\beta. \tag{51a}$$

Wpq^2 is the statistical factor in the absence of Coulomb field or $Z = 0$ approximation. Statistical factor multiplied by the function F, the so called F e r m i function [9]

$$F(Z, W) = 4 (2pr_0)^{2(\gamma-1)} e^{\pi y} \left| \frac{\Gamma(\gamma + iy)}{\Gamma(2\gamma + 1)} \right|^2, \gamma = \sqrt{1 - (\alpha Z)^2}$$

$$F(0, W) = 1 \tag{53}$$

gives the allowed spectrum.* C_β is a measure of the departure of the spectrum from the allowed shape.

We proceed in calculating C_β by using the solution (18) with (30) for the electron and (37) for the neutrino. The solution (37) refers to the positive energy states of the neutrino, but accordinog to our sheme (3) it is the antineutrino which is emitted together with the electron. It is emited in the state — $C K_0 \psi_{w \varkappa \mu}$ [24] where C is the charge conjugation operator connecting the corresponding solutions of the Dirac equation with the opposite sign of energy. Therefore the lepton matrix elements for the emission of an electron and an antineutrino, for the different interaction (2a) — (2c), can be written in the following very useful form

* That is actually the definition of the allowed shape based on the assumption that $C_\beta = L_0 = (1 + \gamma)/2$ for the allowed transition is energy independent. It is true only for low maximum energy and for low Z. The variation of L_0 for heavy elements over the energy range may be about 6% what is inside the experimental error of the present day technique.

$$
\begin{aligned}
\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} &= i\left[f_\varkappa G_{\varkappa_\nu}\left(\chi^\mu_{-\varkappa} T \chi^{\mu_\nu}_{\varkappa_\nu}\right) \pm g_\varkappa F_{\varkappa_\nu}\left(\chi^\mu_\varkappa T \chi^{\mu_\nu}_{-\varkappa_\nu}\right)\right], \\
\begin{pmatrix} \mathbf{L}_2 \\ \mathbf{L}_3' \end{pmatrix} &= f_\varkappa F_{\varkappa_\nu}\left(\chi^\mu_{-\varkappa} \vec{\sigma} T \chi^{\mu_\nu}_{-\varkappa_\nu}\right) \pm g_\varkappa G_{\varkappa_\nu}\left(\chi^\mu_\varkappa \vec{\sigma} T \chi^{\mu_\nu}_{\varkappa_\nu}\right), \\
\begin{pmatrix} \mathbf{L}_3 \\ \mathbf{L}_4 \end{pmatrix} &= i\left[f_\varkappa G_{\varkappa_\nu}\left(\chi^\mu_{-\varkappa} \vec{\sigma} T \chi^{\mu_\nu}_{\varkappa_\nu}\right) \pm g_\varkappa F_{\varkappa_\nu}\left(\chi^\mu_\varkappa \vec{\sigma} T \chi^{\mu_\nu}_{-\varkappa_\nu}\right)\right], \\
\begin{pmatrix} L_4 \\ L_5 \end{pmatrix} &= f_\varkappa F_{\varkappa_\nu}\left(\chi^\mu_{-\varkappa} T \chi^{\mu_\nu}_{-\varkappa_\nu}\right) \pm g_\varkappa G_{\varkappa_\nu}\left(\chi^\mu_\varkappa T \chi^{\mu_\nu}_{\varkappa_\nu}\right).
\end{aligned}
\tag{54}
$$

The expression in paranthesis are the scalar products. The used abbreviations for the lepton matrix elements are as follows

$$
\begin{aligned}
L_1 &= \langle e\bar{\nu} \mid \psi^* \beta \varphi \mid 0 \rangle, \\
\begin{pmatrix} L_2 \\ \mathbf{L}_2 \end{pmatrix} &= \langle e\bar{\nu} \mid \psi^* \begin{pmatrix} 1 \\ \vec{\alpha} \end{pmatrix} \varphi \mid 0 \rangle, \\
\begin{pmatrix} \mathbf{L}_3 \\ \mathbf{L}_3' \end{pmatrix} &= \langle e\bar{\nu} \mid \psi^* \begin{pmatrix} \beta\vec{\sigma} \\ \beta\vec{\alpha} \end{pmatrix} \varphi \mid 0 \rangle, \\
\begin{pmatrix} \mathbf{L}_4 \\ L_4 \end{pmatrix} &= \langle e\bar{\nu} \mid \psi^* \begin{pmatrix} \vec{\sigma} \\ \gamma_5 \end{pmatrix} \varphi \mid 0 \rangle, \\
L_5 &= \langle e\bar{\nu} \mid \psi^* \beta \gamma_5 \varphi \mid 0 \rangle.
\end{aligned}
\tag{54a}
$$

T is defined as

$$T = \beta \gamma_5 C K_0 = \sigma_2 K_0$$

where C is the charge conjugation matrix given by

$$C = B \gamma_4 \gamma_5 = -i \gamma_2 = -\beta \alpha_2.$$

$$B = i \gamma_2 \gamma_4 \gamma_5 = i \gamma_1 \gamma_3 = -i \alpha_1 \alpha_3 = -\sigma_2 ;$$

K_0 is the operator of complex conjugation. T operates on the angular part of the wave function and gives

$$T \chi_{\varkappa^\mu} = (-1)^{l_\varkappa + 1/2 - j + \mu} \chi_{\varkappa^{-\mu}} \tag{55}$$

With the aid of (55) and some straightforward R a c a h-algebra [25] we get for the $(\chi_{\varkappa^\mu} T \chi_{\varkappa',\mu'})$

$$(\chi_{\varkappa^\mu} T \chi_{\varkappa',\mu'}) = (-)^{1/2 + \mu + \mu'} \frac{1}{\sqrt{4\pi}} \sum_{\Lambda M} R_\Lambda(\varkappa, \varkappa') \tag{56}$$

$$(j j' \Lambda - M \mid j - \mu\, j' - \mu')\, Y^{-M}_\Lambda.$$

The abbreviation $R_\Lambda(\varkappa, \varkappa')$ stands for

$$R_\Lambda(\varkappa, \varkappa') = (-)^{l_\varkappa - \Lambda + j' - 1/2} \left[\frac{(2j+1)(2j'+1)(2l_\varkappa+1)(2l_{\varkappa'}+1)}{2\Lambda+1}\right]^{1/2}$$

$$W(j j' l_\varkappa l_{\varkappa'} \mid \Lambda\, 1/2)(l_\varkappa l_{\varkappa'}, \Lambda\, 0 \mid l_\varkappa\, 0\, l_{\varkappa'}\, 0). \tag{57}$$

The expression for $(\chi_{\varkappa}{}^{\mu} \vec{\sigma} T \chi_{\varkappa'}{}^{\mu'})$ can be obtained in the same way as the expression (56). Therefore we have

$$(\chi_{\varkappa}{}^{\mu} \vec{\sigma} T \chi_{\varkappa'}{}^{\mu'}) = \sum_{\nu} \mathbf{e}_1^{-\nu} (-) (\chi_{\varkappa}{}^{\mu} \sigma_1^{\nu} T \chi_{\varkappa'}{}^{\mu'}) \tag{58}$$

$$= (-)^{1/2 + \mu + \mu'} \frac{1}{\sqrt{4\pi}} \sum_{\Lambda J} R_{\Lambda J}(\varkappa, \varkappa') (j\, j'\, J\, -M' \mid j\, -\mu\, j'\, -\mu') \mathbf{Y}_J^{-M'},$$

$R_{\Lambda J}(\varkappa, \varkappa')$ being

$$R_{\Lambda J}(\varkappa, \varkappa') = (-)^{j-j'-J} \sqrt{6}\, [(2l_{\varkappa}+1)(2l_{\varkappa'}+1)(2j+1)(2j'+1)]^{1/2}$$

$$(l_{\varkappa}\, 0\, l_{\varkappa'}\, 0 \mid l_{\varkappa}\, l_{\varkappa'}\, \Lambda\, 0) \begin{Bmatrix} j & j' & J \\ l_{\varkappa} & l_{\varkappa'} & \Lambda \\ 1/2 & 1/2 & 1 \end{Bmatrix} \tag{59}$$

where

$$\vec{Y}_J^{-M'} = \sum_{\nu M} (1\, -\nu\, \Lambda\, -M \mid 1\, \Lambda\, J\, -M')\, \mathbf{e}_1^{-\nu}\, Y_{\Lambda}^{-M} \tag{60}$$

is the vector spherical harmonics [26] $\mathbf{e}_1^{\nu}$ are the spherical component of the unit vector, and $W(\,|\,)$ is the Racah coeficient defined by [25]

$$\sum_{\alpha,\beta,\gamma,\delta,\varepsilon,\varphi} (a\, \alpha\, b\, \beta \mid a\, b\, c\, \gamma)\, (d\, \delta\, b\, \beta \mid d\, b\, e\, \varepsilon)\, (a\, \alpha\, f\, \varphi \mid a\, f\, d\, \delta) \tag{61}$$

$$(c\, \gamma\, f\, \varphi \mid c\, f\, e\, \varepsilon) = \sqrt{(2c+1)(2d+1)}\, (2e+1)\, W(c\, b\, f\, d \mid a\, e);$$

The nine-j symbol is that defined by Wigner and Edmonds [19]

$$\begin{Bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{Bmatrix} = \sum_{\lambda} (-)^{2\lambda} (2\lambda+1) \begin{Bmatrix} j_{11} & j_{21} & j_{31} \\ j_{21} & \lambda & j_{23} \end{Bmatrix} \begin{Bmatrix} j_{12} & j_{22} & j_{32} \\ j_{21} & \lambda & j_{23} \end{Bmatrix} \begin{Bmatrix} j_{13} & j_{23} & j_{33} \\ \lambda & j_{11} & j_{12} \end{Bmatrix} \tag{62}$$

in terms of the six-j symbols $\{\ \}$. Between the six-j symbols and the Racah coefficients there exists the relation [19]

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = (-)^{j_1+j_2+l_1+l_2} W(j_1\, j_2\, l_2\, l_1 \mid j_3\, l_3). \tag{63}$$

Using these abbreviations one can write the scalar and pseudoscalar parts of the lepton covariants in the following form:

$$\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = \frac{1}{\sqrt{4\pi}} (-)^{\mu+\mu_{\nu}+1} \sum_{\Lambda} [f_{\varkappa}\, G_{\varkappa_{\nu}}\, R_{\Lambda}(-\varkappa, \varkappa_{\nu}) \pm g_{\varkappa}\, F_{\varkappa_{\nu}}\, R_{\Lambda}(\varkappa, -\varkappa_{\nu})]$$

$$(j\, -\mu\, j_{\nu}\, -\mu_{\nu} \mid j\, j_{\nu}\, \Lambda\, -M)\, Y_{\Lambda}^{-M}, \tag{64a}$$

$$\begin{pmatrix} L_4 \\ L_5 \end{pmatrix} = i\, (-)^{\mu+\mu_{\nu}} \frac{1}{\sqrt{4\pi}} \sum [f_{\varkappa}\, F_{\varkappa_{\nu}}\, R_{\Lambda}(-\varkappa, -\varkappa_{\nu}) \pm g_{\varkappa}\, G_{\varkappa_{\nu}}\, R_{\Lambda}(\varkappa, \varkappa_{\nu})]$$

$$(j\, -\mu\, j_{\nu}\, -\mu_{\nu} \mid j\, j_{\nu}\, \Lambda\, -M)\, Y_{\Lambda}^{-M}, \tag{64b}$$

Since Racah coefficients are invariants, the tensor character of interaction is contained only in Clebsch-Gordan coefficients and spherical harmonics.

Making use of the vector spherical harmonics it is possible to write the vector and pseudovector parts of the lepton covariants in a similar form:

$$\begin{pmatrix} \mathbf{L}_3 \\ \mathbf{L}_4 \end{pmatrix} = (-)^{\mu+\mu_\nu+1} \frac{1}{\sqrt{4\pi}} \sum_{\Lambda J} \left[f_\varkappa G_{\varkappa_\nu} R_{\Lambda J}(-\varkappa, \varkappa_\nu) \pm g_\varkappa F_{\varkappa_\nu} R_{\Lambda J}(\varkappa, -\varkappa_\nu) \right]$$
$$(j - \mu\, j_\nu - \mu_\nu \mid j\, j_\nu\, J - M')\, \mathbf{Y}_J^{-M} \tag{65a}$$

$$\begin{pmatrix} \mathbf{L}_2 \\ \mathbf{L}'_3 \end{pmatrix} = i\,(-)^{\mu+\mu_\nu} \frac{1}{\sqrt{4\pi}} \sum_{\Lambda J} \left[f_\varkappa F_{\varkappa_\nu} R_{\Lambda J}(-\varkappa, -\varkappa_\nu) \pm g_\varkappa G_{\varkappa_\nu} R_{\Lambda J}(\varkappa, \varkappa_\nu) \right]$$
$$(j - \mu\, j_\nu - \mu_\nu \mid j\, j_\nu\, J - M')\, \mathbf{Y}_J^{-M'} \tag{65b}$$

The lepton covariants are expanded in series of ordinary and vector spherical harmonics. Such expansions are usually called the multipole expansions. Owing to the expressions (64) and (65) the transition probability is given as a series expansion in angular momentum of leptons, and it converges rather rapidly. The first term in the development ($\Lambda = 0$ and $J = 1$, no change of parity), i. e. the lowest approximation, are the so called allowed transitions. Then follow the first, second and higher forbidden transitions.

We can further note the relations like

$$\vec{\sigma}_K \cdot \mathbf{Y}_J^M = \sum_{\nu M'} (1\, \nu\, \Lambda\, M' \mid 1\, \Lambda\, J\, M)\, \sigma_{K\backslash}^{\nu}\, \mathbf{Y}_\Lambda^{M'} = T_{KJ\Lambda}^M \tag{66}$$

with abbreviation

$$\sum_{K=1}^{A} \vec{\sigma}_K \cdot \mathbf{Y}_J^M = \sum_{K=1}^{A} T_{KJ\Lambda}^M = T_{J\Lambda}^M . \tag{67}$$

If we introduce — for the labeling of the states ot the initial and the final nucleus — the total angular momenta J' and J'' and its projections along the z-axis M' and M'', τ' and τ'' for the radial quantum numbers, we can further take advantage of Racah algebra in order to simplify the expression (52). It is especially useful in the non-relativistic treatment. Let us define the part of the matrix element which does not depend on the magnetic quantum numbers as [25]

$$\sum_f \sum_i S \left| \int \Psi_f^*\, O^i\, \Psi_i\, d\tau_1 \ldots d\tau_A \right|^2 = \frac{1}{2J'+1} \sum_{M''M'M} \tag{68}$$
$$\left| < J''\, M''\, \tau'' \mid T_J^M\, f(r) \mid J'\, M'\, \tau' > \right|^2 =$$
$$\left| < J'' \parallel T_J \parallel J' > \right|^2 \left| < \tau'' \mid f(r) \mid \tau' > \right|^2 .$$

This definition of the reduced or double-bar matrix element $<\| \ \| >_\beta$ is the one used in β- decay. There are several other definitions of the double-bar matrices. We would like to mention and to use occasionally the more symmetric one of Wigner [19]

$$<J'' M'' \mid T_J^M \mid J' M' > = (-)^{J''-M''} \begin{pmatrix} J'' & J & J' \\ -M'' & M & M' \end{pmatrix} <J'' \| T_J \| J' >_W \tag{69}$$

$\begin{pmatrix} J'' & J & J' \\ -M'' & M & M' \end{pmatrix}$ is the se called three -$j$symbol which is connected with the Clebsch-Gordan coefficients by the relation [19]

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-)^{j_1-j_2-m_3} \frac{1}{\sqrt{2j_3+1}} (j_1 j_2 j_3 - m_3 \mid j_1 m_1 j_2 m_2) \tag{70}$$

J and M are the maximum amount of angular momentum and its projection along the z - axes which can be taken away by the electron and the neutrino. Because of the unitary character of the three - j transformation coefficients we can choose the phases so that the following relation between the angular parts of double-bar matrices holds

$$\sqrt{2J'+1} <J'' \| T_J \| J' >_\beta = <J'' \| T_J \| J' >_W. \tag{71}$$

It is now very simple to write down the exact correction factors for the non-relativistic interactions (64a) and (65a). For the sake of simplicity we limit our consideration to the tensor correction factor.

$$C_\beta^T = \frac{\pi G^2 g_3}{p^2 q^2 F} \sum_{\varkappa,\varkappa_\nu} \left| \sum_{J\Lambda} <J'' \| T_{J\Lambda} \| J' >_\beta <\tau'' \mid [f_\varkappa G_{\varkappa_\nu} R_{\Lambda J}(-\varkappa, \varkappa_\nu) + g_\varkappa F_{\varkappa_\nu} R_{\Lambda J}(\varkappa, -\varkappa_\nu)] \mid \tau' > \right|^2. \tag{72}$$

In the general case instead of a single term we have the linear combination of the β- couplings.

The first factor together with $R_{\Lambda J}$ contains only the dependence on angular properties of the radiation and the nucleus and their coupling while the second factor depends on the radial wave functions of the nucleus and the leptons, i. e., upon a more detailed knowledge of the situation inside the nucleus

The series for (72) converges very rapidly. The order of magnitude of the leading term of the nonrelativistic interactions for the given transition is $|<r^\Lambda>|^2$ with the lowest possible Λ. This term is obtained if the summation over $\varkappa, \varkappa_\nu$, is carried out only over those $\varkappa, \varkappa_\nu$ for which the algebraic sum of the orbital momenta of the electron and the neutrino is equal to this lowest Λ. Owing to the very small nuclear radius, the higher order terms are completely negligible. With the relativistic interactions the situation is not so simple, but the leading term can be found in a similar way.

As for the lepton wave functions, the usual treatment is to make the series expansion (35) for the neutrino wave function and to stop after the first term. For most purposes in the analysis of the experimental data, unless the released energy is too big, this is a sufficient approximation. The wave function for the electron is taken either as for pure Coulomb field (30), or for modified Coulomb field (39), (42). Then, both lepton functions, each multiplied by r^{-l} are taken out of the nuclear matrix element and evaluated at the boundary $r = r_0$. With the help of the definitions [16], [27]:

$$\begin{aligned}
L_{k-1} &= (2p^2F)^{-1}\, r_0^{2-2k}\,(g^2_{-k} + f^2_k),\\
M_{k-1} &= (2p^2F)^{-1}\, r_0^{-2k}\,(g^2_k + f^2_{-k}),\\
N_{k-1} &= (2p^2F)^{-1}\, r_0^{1-2k}\,(f_{-k}\,g_{-k} - f_k\,g_k),\\
P_{k-1} &= (2p^2F)^{-1}\, r_0^{2-2k}\,(g^2_{-k} - f^2_k),\\
Q_{k-1} &= (2p^2F)^{-1}\, r_0^{-2k}\,(g^2_k - f^2_{-k}),\\
R_{k-1} &= (2p^2F)^{-1}\, r_0^{1-2k}\,(f_{-k}\,g_{-k} + f_k\,g_k),
\end{aligned} \tag{73}$$

it is possible to write the correction factors in terms of L, M, N, P, Q and R. For pure Coulomb functions these quantities have been tabulated and they are very useful in the analysis of the β-spectra and the ft-values. If one uses modified Coulomb functions $f_\varkappa'$, $g_\varkappa'$ in (73) one obtains the quantities (73) corrected for finite charge distribution of the nucleus. The corrections $\Delta L = \dfrac{L' - L}{L}$ etc., can be easily calculated by means of (39 a) [4]

$$\begin{aligned}
\begin{pmatrix}\Delta L_{k-1}\\ \Delta P_{k-1}\end{pmatrix} &= \frac{g^2_{-k}\,(\mu'^2_{-k} + 2\mu'_{-k}) \pm f^2_k\,(\mu_k^2 + 2\mu_k)}{g^2_{-k} \pm f^2_k}\\
\begin{pmatrix}\Delta M_{k-1}\\ \Delta Q_{k-1}\end{pmatrix} &= \frac{g^2_k\,(\mu'^2_k + 2\mu'_k) \pm f^2_{-k}\,(\mu^2_{-k} + 2\mu_{-k})}{g^2_k \pm f^2_{-k}}\\
\begin{pmatrix}\Delta N_{k-1}\\ \Delta R_{k-1}\end{pmatrix} &= \frac{f_{-k}\,g_{-k}\,(\mu_{-k}\,\mu'_{-k} + \mu_{-k} + \mu'_{-k}) \mp f_k\,g_k\,(\mu_k\,\mu'_k + \mu_k + \mu'_k)}{f_{-k}\,g_{-k} \mp f_k\,g_k}
\end{aligned} \tag{74}$$

where $\mu_\varkappa$, $\mu'_\varkappa$ are defined as in (40) and all functions are pure Coulomb functions taken at $r = r_0$.

(a) Non-relativistic Form of the Relativistic β-interaction

The pseudoscalar operator $\beta\gamma_5$ mixes the small and the large component of the nuclear wave function and therefore has not a classical analogy. One would think therefore, that the series expansions like (72) (according to multipoles), which is a non relativistic expansion, would not be very appropriate. So one might try

to find a suitable method of extracting the major parts from the relativistic series of (52). Such a method is provided by the Foldy-Wouthuysen transformation [23].

Beside the pseudoscalar operator there are also some other operators which mix large and small components. These are the relativistic parts of the vector, tensor and pseudovector interaction. Using the transformation (49) we can try to obtain the corresponding non-relativistic form of these interactions. In order to save space we can immediately write down the matrix elements, or the nuclear moments, occuring in β-decay theory in the non relativistic form. If we write any of these interactions as $K_i = \vec{\alpha} \cdot \mathbf{L}_2$, $\beta\vec{\alpha} \cdot \mathbf{L}_3$, $\gamma_5 L_4$, $\beta\gamma_5 L_5$, the nuclear moment becomes

$$\langle f | K_i | i \rangle = \langle f' | e^{-S} H_i e^{S} | i' \rangle = \langle f' | K_i | i' \rangle + \langle f' | [S, K_i] | i' \rangle + \\ + \frac{1}{2} \langle f' | [S, [S, K_i]] | i' \rangle + \ldots, \tag{75}$$

$|a'\rangle$ stands for the transformed states. The first term vanishes identically. After evaluating the necessary commutators and using $[V, L] = 0$, we get the desired expressions for all these interactions as follows:

$$\langle f | \vec{\alpha} \cdot \mathbf{L}_2 | i \rangle = \frac{1}{M} \langle f' | \mathbf{L}_2 \cdot (-\mathbf{p} + i\,\mathbf{V}_2 + i\vec{\sigma} \times \mathbf{V}_3 - \vec{\sigma} \cdot \mathbf{V}_4) | i' \rangle \\ - \frac{1}{2M} \langle f' | [(i\vec{\sigma} \times \mathbf{p} + \mathbf{p}), \mathbf{L}_2] | i' \rangle \tag{76}$$

$$\langle f | \beta\vec{\alpha} \cdot \mathbf{L}_3' | i \rangle = -\frac{1}{M} \langle f' | \mathbf{L}_3' \cdot (i\vec{\sigma} \times \mathbf{P} + \vec{\sigma} \times \mathbf{V}_2 - \mathbf{V}_3 - \vec{\sigma} \cdot V_5) | i' \rangle - \\ - \frac{1}{2M} \langle f' | [(i\vec{\sigma} \times \mathbf{P} + \mathbf{P}), \mathbf{L}_3] | i' \rangle, \tag{77}$$

$$\langle f | \gamma_5 L_4 | i \rangle = -\frac{1}{M} \langle f' | L_4 (\vec{\sigma} \cdot \mathbf{P} - i\vec{\sigma} \cdot \mathbf{V}_2 + i\,V_4) | i' \rangle - \\ - \frac{1}{2M} \langle f' | [\vec{\sigma} \cdot \mathbf{P}, L_4] | i' \rangle \tag{78}$$

$$\langle f | \beta\gamma_5 L_5 | i \rangle = \frac{1}{M} \langle f' | L_5 (\vec{\sigma} \cdot \mathbf{V}_3 + V_5 | i' \rangle - \frac{1}{2M} \langle f' | [\sigma \cdot \mathbf{P}, L_5] | i' \rangle \\ + \left(\frac{1}{2M}\right)^2 \langle f' | L_5 (V_1 \vec{\sigma} \cdot \mathbf{P} - V_2 \vec{\sigma} \cdot \mathbf{P} - 2\vec{\sigma} \cdot \mathbf{P}\, V_2 - \mathbf{V}_3 \cdot \mathbf{P} + \\ + \mathbf{V}_4 \cdot \mathbf{P} + 2\mathbf{P} \cdot \mathbf{V}_4 - i\mathbf{V}_3 \cdot \vec{\sigma} \times \mathbf{P} + 5\,i\,\mathbf{V}_4 \cdot \vec{\sigma} \times \mathbf{P} + 2\,i\vec{\sigma} \times \mathbf{P} \cdot \mathbf{V}_4 | i' \rangle + \\ + \left(\frac{1}{2M}\right)^2 \langle f' | [(2\,V_1 \vec{\sigma} \cdot \mathbf{P} - 2\,\mathbf{V}_3 \cdot \mathbf{P} + 2\,i\vec{\sigma} \times \mathbf{P}), L_5] | i' \rangle. \tag{79}$$

The terms in the nonrelativistic form of the matrix elements are divided into those which do not contain the derivatives of the lepton field and those with derivatives. The terms containing derivatives seem to be small and in the case of the vector, tensor and pseudovector coupling can be neglected. In the first bracket the first term which is independent of the potentials will usually dominate, and the other terms measuring the influence of nucleon couplings will make small corrections. As far as this condition is fulfilled we can calculate matrix elements in the usual way but we have to bear in mind that there could be some possible additional terms too [8]. Thus the pseudoscalar coupling plays a special role, because it has only matrix elements containing derivatives of lepton wave functions, nuclear potentials and their derivatives, and it contributes to the β-decay probability only through these terms [27]. It is convenient to divide its matrix element into two parts with, respectively without derivatives of the lepton wave functions. Neglecting the last term in (79) we can write the pseudoscalar matrix element in the form

$$\langle f|\beta\gamma_5 L_5|i\rangle = -\frac{1}{2M}\langle f'|[\vec{\sigma}\cdot\mathbf{P}, L_5]|i'\rangle + i\langle f'|\vec{\sigma}\cdot\mathbf{r}\, L_5 F(\mathbf{r},\mathbf{P})|i'\rangle. \quad (80)$$

b) Calculation of β-spectra for Coupling with Derivatives

Until now we have developed methods for calculating correction factors for the interaction which does not contain derivatives, and now we shall show how the procedure can be extended [7]. We hawe only to apply the operators occuring in (76)—(79) to the expressions (64) and (65). For this purpose we shall use the formula [29].

$$\nabla_1^{\mu}\Phi(r)\,T_L^M(\vartheta,\varphi) =$$

$$\left[\frac{L+1}{2L+3}\right]^{1/2}(LM\,1\,\mu|\,L\,1\,L+1\,M+\mu)\,T_{L+1}^{M+\mu}\left(\frac{d}{dr}-\frac{L}{r}\right)\Phi(r) - \quad (81)$$

$$-\left[\frac{L}{2L-1}\right]^{1/2}(LM\,1\mu\,|\,L1\,L-1\,M+\mu)\,T_{L-1}^{M+\mu}\left(\frac{d}{dr}+\frac{L+1}{r}\right)\Phi(r).$$

As an example we show the calculation for pseudoscalar coupling. Using (81) and (64) we can write the first term in (80) in the form

$$\vec{\sigma}\cdot\nabla L_5 = \sum_{\nu}(-)^{\nu}\sigma_1^{-\nu}\nabla_1^{\nu}L_5 = (-)^{-1/2+\mu+\mu_\nu}\frac{1}{\sqrt{4\pi}}\cdot$$

$$\sum_{\Lambda}\left[\sqrt{\frac{\Lambda+1}{2\Lambda+1}}\,T_{\Lambda\Lambda+1}^{-M}D_- - \sqrt{\frac{\Lambda}{2\Lambda+1}}\,T_{\Lambda\Lambda-1}^{-M}D_+\right]\cdot \quad (82)$$

$$(j-\mu j_\nu-\mu_\nu\,|\,j\,j_\nu\,\Lambda-M)\;[f_\varkappa F_{\varkappa_\nu} R_\Lambda(-\varkappa,-\varkappa_\nu) - g_\varkappa G_{\varkappa_\nu} R_\Lambda(\varkappa,\varkappa_\nu)].$$

The abbreviations used are as follows

$$D_+ = \frac{d}{dr} + \frac{\Lambda+1}{r}, \qquad D_- = \frac{d}{dr} - \frac{\Lambda}{r}, \tag{83}$$

$$T^{M}_{\Lambda\Lambda\pm1} = \sum_{\nu} (\Lambda \pm 1\, \nu - M\, 1 - \nu \,|\, \Lambda \pm 1\, 1\, \Lambda - M)\, Y^{M+\nu}_{\Lambda\pm1}\, \sigma_1^{-\nu} \tag{84}$$

The general expression for the correction factor of the derivative part of the pseudoscalar interaction is

$$C^P_\beta = \frac{\pi G^2 g_5^2}{p^2 q^2 F 4 M^2} \sum_{\varkappa,\varkappa_\nu} \Big| \langle J'' \| \sum_{\Lambda} \Big[\sqrt{\frac{\Lambda+1}{2\Lambda+1}}\, T_{\Lambda\Lambda+1}\, D_- - \sqrt{\frac{\Lambda}{2\Lambda+1}}\, T_{\Lambda\Lambda-1}\, D_+ \Big] \| J' \rangle_\beta \langle \tau' | \left[f_\varkappa F_{\varkappa_\nu}\, R_\Lambda (-\varkappa, -\varkappa_\nu) - g_\varkappa G_{\varkappa_\nu}\, R_\Lambda (\varkappa, \varkappa_\nu) \right] | \tau' \rangle \Big|^2 \tag{85}$$

The operators D_+, respectively D_-, operate only upon the lepton functions. Using formula (21a), (21b) and (73) we can express the correction factor again in terms of L, M, etc. as in the case of the usual treatment.

In the same way we can write the other derivative interactions in the tensor form and calculate the correction factors, but since we do not need them we shall omit them.

c) Calculation of the Tensor-Pseudoscalar Correction Factor for $(0 \longrightarrow 0+)$ Transition

Neglecting the pseudovector interaction for reasons mentioned in the introduction we are left in this case with the tensor pseudoscalar mixture. The matrix element for the tensor-pseudoscalar mixture can be written in the form

$$\langle H_\beta \rangle = -g_3 \langle \vec{\sigma} \cdot \mathbf{L}_3 \rangle + i g_5 \langle \vec{\sigma} \cdot \mathbf{r}\, L_5\, f(\mathbf{r}, \mathbf{P}) \rangle + \frac{i g_5}{2M} \langle \vec{\sigma} \cdot \nabla L_5 \rangle \tag{86}$$

Using the general expressions (64), (65), (72), (85) and the more specified expressions given in the *Appendix* for the various terms in (86) we obtain

$$\langle \vec{\sigma} \cdot \vec{\nabla} L_5 \rangle_{\Lambda=0} = (-)^{1/2+\mu+j+l-\varkappa} \frac{1}{4\pi} \Big\langle \frac{\vec{\sigma} \cdot \mathbf{r}}{r} \frac{d}{dr} [f_\varkappa F_\varkappa + g_\varkappa G_\varkappa] \Big\rangle \tag{87}$$

$$\langle \vec{\sigma} \cdot \mathbf{r}\, L_5\, f(\mathbf{r}, \mathbf{P}) \rangle_{\Lambda=0} = (-)^{1/2+\mu+j+l-\varkappa} \frac{1}{4\pi} \langle \vec{\sigma} \cdot \mathbf{r}\, [f_\varkappa F_\varkappa + g_\varkappa G_\varkappa]\, f(\mathbf{r}, \mathbf{P}) \rangle \tag{87}$$

$$\langle \vec{\sigma} \cdot \mathbf{L}_3 \rangle_{J=0,\Lambda=1} = (-)^{\mu+j} \frac{1}{4\pi \sqrt{2j+1}} R_{10}(-\varkappa, \varkappa) \left\langle \frac{\vec{\sigma} \cdot \mathbf{r}}{r} [f_\varkappa G_\varkappa - g_\varkappa F_\varkappa] \right\rangle$$

The correction factor can be calculated now by means of the expressions (86), (21), (87), (72) and (85). If we introduce the quantities

$$g = g_5 / g_3 \tag{88}$$

and

$$\Gamma = -g \frac{\langle \vec{\sigma} \cdot \mathbf{r}\, f(\mathbf{r}, \mathbf{P}) \rangle}{\langle \vec{\sigma} \cdot \mathbf{r} \rangle} \tag{89}$$

the correction factor, in the approximation given on page (51) generalized to the relativistic interactions and expressed in terms of the quantities defined in (73) can be written:

$$C = g_3^2 |\langle \vec{\sigma} \cdot \mathbf{r} \rangle|^2 \{C_1 + g C_2 + \Gamma C_3 + g \Gamma C_4 + g^2 C_5 + \Gamma^2 C_6\} \tag{90}$$

where

$$C_1 = L_0 \frac{q^2}{9} \left[1 - \frac{(q r_0)^2}{5}\right] + M_0 \left[1 - \frac{(q r_0)^2}{3}\right] + \tag{91}$$

$$+ \frac{2}{3} q N_0 \left[1 - \frac{4}{15} (q r_0)^2\right]$$

$$C_2 = -\frac{1}{M} \left\{ \frac{2}{3} q \left[(W - V_c) N_0 + M_0\right] \left[1 - \frac{4}{15} (q r_0)^2\right] + \frac{q^2}{9} \cdot \right.$$

$$\cdot \left[(W - V_c) L_0 - P_0 + 2 N_0\right] \left[1 - \frac{(q r_0)^2}{5}\right] + \left[(W - V_c) M_0 - Q_0\right]$$

$$\left[1 - \frac{(q r_0)^2}{3}\right] - \frac{q^2}{9} N_0 \left[1 - \frac{2}{5} (q r_0)^2\right] - \frac{q^3}{9} L_0 \left[1 - \frac{(q r_0)^2}{5}\right] -$$

$$\left. - \frac{q}{3} M_0 \left[1 - \frac{19}{30} (q r_0)^2\right] - \frac{q^2}{3} N_0 \left[1 - \frac{13}{30} (q r_0)^2\right] \right\} \tag{92}$$

$$C_3 = 2 \left\{ N_0 \left[1 - \frac{4}{9} (q r_0)^2\right] + \frac{q}{3} L_0 \left[1 - \frac{4}{15} (q r_0)^2\right] - \frac{q}{3} r_0^2 M_0 \right\} \tag{93}$$

$$C_4 = -\frac{1}{M}\left\{\frac{q}{3}\left[(W-V_c)\,L_0 - P_0 + 2N_0\right]\left[1-\frac{4}{15}(q r_0)^2\right] - \frac{q}{3} r_0^2 \right.$$
$$\left[(W-V_c)\,M_0 - Q_0\right] - \frac{(q r_0)^2}{9}\left[2M_0 + (W-V_c)\,N_0 - R_0\right]\left[1-\frac{(q r_0)^2}{5}\right] +$$
$$\left[(W-V_c)\,N_0 + R_0\right]\left[1-\frac{(q r_0)^2}{3}\right] - \frac{q^2}{3} L_0\left[1-\frac{13}{30}(q r_0)^2\right] +$$
$$+\frac{(q r_0)^2}{9} M_0\left[1-\frac{2}{5}(q r_0)^2\right] - \frac{q}{3} N_0\left[1-\frac{19}{30}(q r_0)^2\right] +$$
$$\left. +\frac{(q r_0)^2}{9} q N_0\left[1-\frac{(q r_0)^2}{5}\right]\right\} \qquad (94)$$

$$C_5 = \frac{1}{4M^2}\left\{\left[(W-V_c)^2+1)\,M_0 - 2(W-V_c)\,Q_0\right]\left[1-\frac{(q r_0)^2}{3}\right] -\right.$$
$$-\frac{2q^2}{3}\left[(W-V_c)\,N_0 + R_0\right]\left[1-\frac{13}{30}(q r_0)^2\right] + \frac{q^4}{9} L_0\left[1-\frac{(q r_0)^2}{5}\right] +$$
$$+\frac{q^2}{9}\left[(W-V_c)^2+1)\,L_0 - 2(W-V_c)\,P_0 + 4(W-V_c)\,N_0 - 4R_0 + 4M_0\right]$$
$$\left[1-\frac{(q r_0)^2}{5}\right] + \frac{q^2}{9} M_0\left[1-\frac{3}{5}(q r_0)^2\right] - \frac{2q^2}{9}\left[2M_0 + (W-V_c)\,N_0 - R_0\right]$$
$$\left[1-\frac{2}{5}(q r_0)^2\right] + \frac{2}{3} q\left[\left((W-V_c)^2-1\right)N_0 - 2Q_0 + 2(W-V_c)\,M_0\right]$$
$$\left[1-\frac{4}{15}(q r_0)^2\right] - \frac{2}{3} q\left[(W-V_c)\,M_0 - Q_0\right]\left[1-\frac{19}{30}(q r_0)^2\right] -$$
$$-\frac{2}{9} q^3\left[(W-V_c)\,L_0 - P_0 + 2N_0\right]\cdot\left[1-\frac{(q r_0)^2}{5}\right] +$$
$$\left. -\frac{2}{9} q^3 N_0\left[1-\frac{2}{5}(q r_0)^2\right]\right\} \qquad (95)$$

$$C_6 = L_0\left[1-\frac{(q r_0)^2}{3}\right] - \frac{2}{3} q r_0^2 N_0. \qquad (96)$$

The neutrino wave function was calculated by means of the series expansion (35) retaining only the first two terms. The second term in this expansion is the finite wave length correction. This is usually a very small correction since the series expansion converges very rapidly, if the energy of the neutrino $q \ll 1/r_0$, i. e. up to about 10 MeV.

The expressions (91) — (96) are valid for both pure Coulomb field and modified Coulomb field. In the second case, one must only use the corrected quantities L', M', etc., the correction being given by (74).

d) Correction Due to the Variation of the Lepton Wave Fumctions over the Nucleus

In the usual treatment the variation of the lepton wave functions over the nucleus is only partly taken into account. This is a very good approximation for most purposes, except in the case of a small nuclear matrix element (ft-value much larger than normally). To show how this variation can be exactly taken into account, we will give an exemple. Let us consider the radial nuclear matrix element of the product of the lepton wave functions $g_\varkappa(r)$ and $F_{\varkappa_\nu}(r)$. This matrix element may also contain a certain scalar function, like $f(\mathbf{r}, \mathbf{P})$ in the non-derivative part of the pseudoscalar interaction (80), which can be omitted here

$$\langle \tau'' \mid g'_\varkappa(r)\, F_{\varkappa_\nu}(r) \mid \tau' \rangle = \langle \tau'' \mid r^{l_\varkappa + l - \varkappa_\nu} \frac{g'_\varkappa(r)\, r^{-l_\varkappa}}{g'_\varkappa(r_0)\, r_0^{-\varkappa}} \frac{F_{\varkappa_\nu}(r)\, r^{-l-\varkappa_\nu}}{F_{\varkappa_\nu}(r_0)\, r^{-l-\varkappa_\nu}} \mid \tau' \rangle \cdot$$

$$\cdot \left(\frac{g'_\varkappa(r_0)}{r_0^{l_\varkappa}}\right)\left(\frac{F_{\varkappa_\nu}(r_0)}{r_0^{l-\varkappa_\nu}}\right). \tag{97}$$

The ratios of the lepton wave functions which appear in (97), can be written in the form of a power series expansion [5] by means of (39), (42), (37) and (35). If the potential energy (38d) contains only even powers of r the ratio for the electron wave function will also contain only even powers of r, so that we can write

$$\frac{g'_\varkappa(r)\, r^{-\varkappa}}{g'_\varkappa(r_0)\, r_0^{-\varkappa}} \frac{F_{\varkappa_\nu}(r)\, r^{-l-\varkappa_\nu}}{F_{\varkappa_\nu}(r_0)\, r_0^{-l-\varkappa_\nu}} = \sum_{n=0}^{\infty} a_{\varkappa\varkappa_\nu n}(W)\, x^{2n}$$

$$\sum_{n=0}^{\infty} a_{\varkappa\varkappa_\nu n}(W) = 1. \tag{98}$$

In all cases the influence of the neutrino wave function on the expansion (98) is negligible and consequently it has been neglected in our calculations. Inserting (98) into (97) and defining the new parameters

$$\Gamma_{2n+1} = \frac{\langle \tau'' \mid r^{l_\varkappa + l - \varkappa_\nu}\, x^{2n} \mid \tau' \rangle}{\langle \tau'' \mid r^{l_\varkappa + l - \varkappa_\nu} \mid \tau' \rangle}, \tag{99}$$

we obtain

$$\langle \tau'' \mid g'_\varkappa(r)\, F_{\varkappa_\nu}(r) \mid \tau' \rangle = \tag{100}$$

$$= \langle \tau'' \mid r^{l_\varkappa + l - \varkappa_\nu} \mid \tau' \rangle \left(\frac{g'_\varkappa(r_0)}{r_0^{l_\varkappa}}\right)\left(\frac{F_{\varkappa_\nu}(r_0)}{r_0^{l-\varkappa_\nu}}\right)\left[\sum_n a_{\varkappa\varkappa_\nu n}(W)\, \Gamma_{2n+1}\right].$$

The last factor in (100) represents the correction which is usually neglected. This case is obtained if we put $\Gamma_{2n+1} = 1$ This correction can be also written in a form which exhibits its energy dependence in a more explicit way

$$\sum_{n=0}^{\infty} a_{\varkappa\varkappa_\nu n}(W)\, \Gamma_{2n+1} = \sum_{m=0}^{\infty} b_{\varkappa\varkappa_\nu m} W^m \tag{101}$$

with

$$b_{\varkappa\varkappa_\nu m} = \sum_{n=0}^{\infty} a_{\varkappa\varkappa_\nu nm}\, \Gamma_{2n+1}. \tag{101a}$$

The correction factor C_β, corrected for the variation of the lepton wave functions, can be obtained directly from the expression for the correction factor without the correction introduced here, by the substitutions

$$\begin{aligned} L' &\to L'' + P'' & P' &\to P''' + L''' \\ M' &\to M'' + Q'' & Q' &\to Q''' + M''' \\ N' &\to N'' + R'' & R' &\to R''' + N''' \end{aligned} \tag{102}$$

where

$$\begin{aligned} L'' &= C_L L', & M'' &= C_M M', \text{ etc.} \\ L''' &= C_P L', & M''' &= C_Q M', \text{ etc.} \end{aligned} \tag{103}$$

with the factors C_L, C_M etc. in the form

$$C_L = f_L (1 + \beta_L W + \gamma_L W^2 + \ldots), \text{ etc.} \tag{103a}$$

and with f_L, β_L, γ_L, etc. not depending on W. These quantities can be easily calculated by means of (101).

In case of a mixture of different couplings one must bear in mind that there will be as many different Γ_{2n+1} as there are different couplings.

e) Numerical Results for the $Ho^{166} \to Er^{166}$ Transition

We sumarize here the results of our computations for the case of the $Ho^{166} \to Er^{166}$ $(0 \longrightarrow 0+)$ transition.

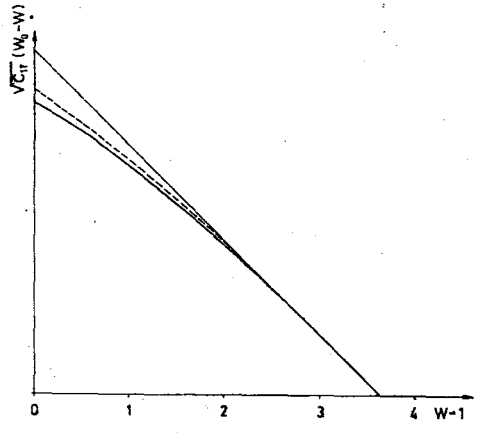

Fig. 1.

Fig. 1. represents the customary Fermi plot for the pure tensor interaction, including finite size correction, with $r_0 = \alpha/2\ A^{1/3}$ (full line), respectively $r'_0 = 0.8\ r_0$ (dashed line). Its deviation from a straight line is very pronounced and outside the experimental error. This means that it is impossible to explain the experimental spectrum without taking into account either additional corrections or other couplings.

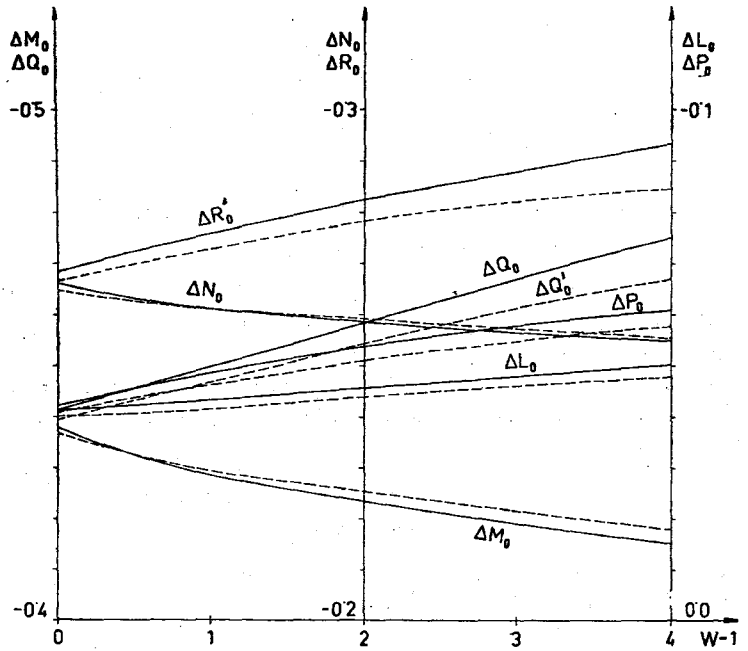

Fig. 2.

Fig. 2. shows the finite size corrections (74) for $r_0 = \alpha/2\ A^{1/3}$ (full lines), respectively $r'_0 = 0.8\ r_0$ (dashed lines), and the parabolic charge distribution with $\frac{\varrho\ (r_0)}{\varrho\ (0)} = 1 + \varepsilon = 1{\cdot}05$. The electron wave functions inside the nucleus were calculated using the expressions (42), (43), (39a) and assuming $b_\varkappa \approx 1$. The wave functions, themselves, are presented in the *Appendix*. To explain the straight spectrum and the large *ft*-value we have introduced the tensor-pseudoscalar mixture (86) which gives the correction factor (90) and leads to the destructive interference for $\Gamma \approx 10$ and $g \approx 1$. We were not able to obtain straight spectrum in this domain of Γ even allowing values of g as large as 15. Larger values of g do not seem very plausible

and would also spoil the spectrum shape of the allowed transitions [8].* The general behaviour of the Fermi plot in this case is more or less pathological because the minimum of the correction factor is extremely small and inside the energy region of the electron. In such cases destructive interference cannot give an allowed

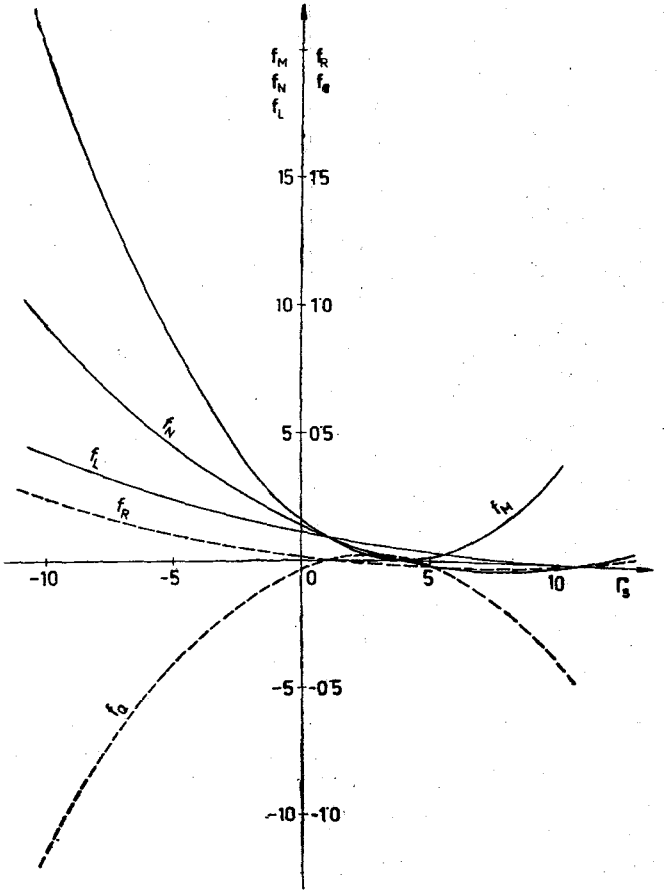

Fig. 3a.

* Note added in proof. Two similar cases, namely, $Pr^{144} \rightarrow Nd^{144}$ and $Tl^{206} \rightarrow Pb^{206}$ transitions have been classified and analyzed as $(0 \rightarrow 0+)$ transition by Zyrjanova. (The Sixth Conference on Nuclear Spectroscopy, Moscow 26th January 1956.) Neglecting the second term in (86) she was able to obtain an approximately allowed shape for the ratio of the coupling constants $g \approx 114$ and 103, respectively (Private communication from Prof. L. A. Sliv). Our corresponding value computed by L. Šips and D. Tadić would be of the same order of magnitude. These values seem to be rather large to be accepted.

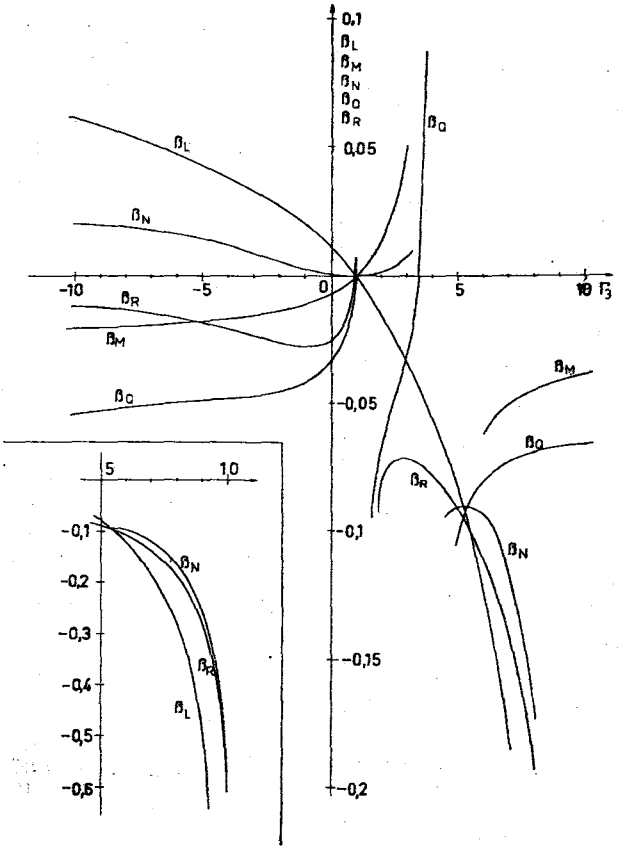

Fig. 3b.

spectrum shape. So we went investigating the second possibility: the case of a small matrix element. Here the correction due to the variation of the lepton wave functions inside the nucleus may be important and we assumed that this is the case. Under this asumption we were able to obtain both the straight spectrum and the large ft-value. The parameters involved here are the Γ'_s as defined in (99), i. e.

$$\Gamma_{2n+1} = \frac{\langle f | \vec{\sigma} \cdot \mathbf{r}\, x^{2n} | i \rangle}{\langle f | \vec{\sigma} \cdot \mathbf{r} | i \rangle} = \frac{\langle \tau'' | r\, x^{2n} | \tau' \rangle}{\langle \tau'' | r | \tau' \rangle}, \tag{104a}$$

$$\Gamma'_{2n+1} = -g \frac{\langle f | r\, x^{2n} f(\mathbf{r}, \mathbf{P}) | i \rangle}{\langle f | \vec{\sigma} \cdot \mathbf{r} | i \rangle} = -g \frac{\langle \tau'' | x^{2n} r f(\mathbf{r}, \mathbf{P}) | \tau' \rangle}{\langle \tau'' | r | \tau' \rangle}. \tag{104b}$$

For $\Gamma_{2n+1} = \Gamma'_{2n+1} = 1$ the correction due to the variation of the lepton wave functions is zero. On the other hand, the series (98) converges so rapidly that only a few first Γ'_s can be important.

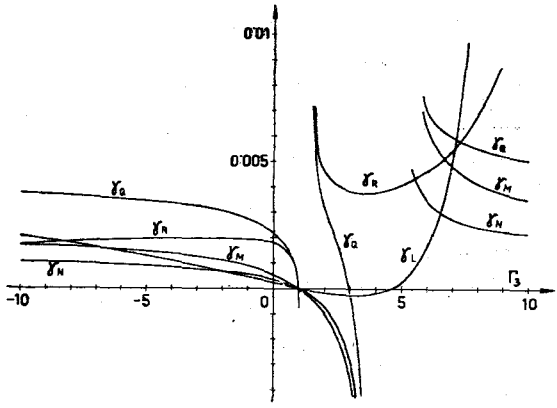

Fig. 3c.

Since by definition $\Gamma_1 = \Gamma'_1 = 1$, we assumed that it is Γ_3 and Γ'_3 which are important, neglecting the influence of higher Γ'_s. The contribution of the derivative part of the pseudoscalar coupling was neglected because it might represent only a very small correction. Therefore, the parameters at our disposal were $\Gamma, \Gamma_3, \Gamma'_3$, and we performed the computation of the quantities which appear in (103a) over a wide range of these parameters. Only terms up to W^2 were taken into account. The results are presented in *Figs.* 3. a, b, c for square-terms (Γ_3 should be read Γ'_3 for the square-pseudoscalar term) and in *Figs.* 4. a, b, c, d, e, f, g for cross-terms.

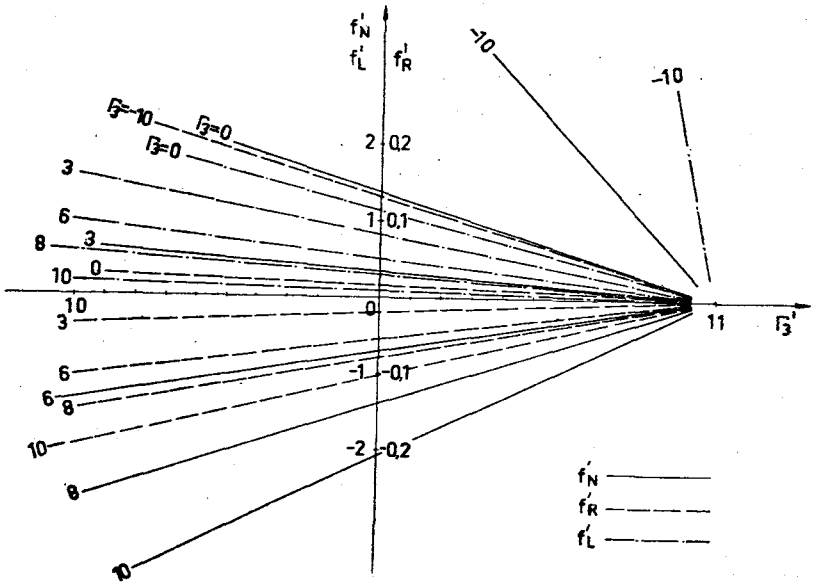

Fig. 4a.

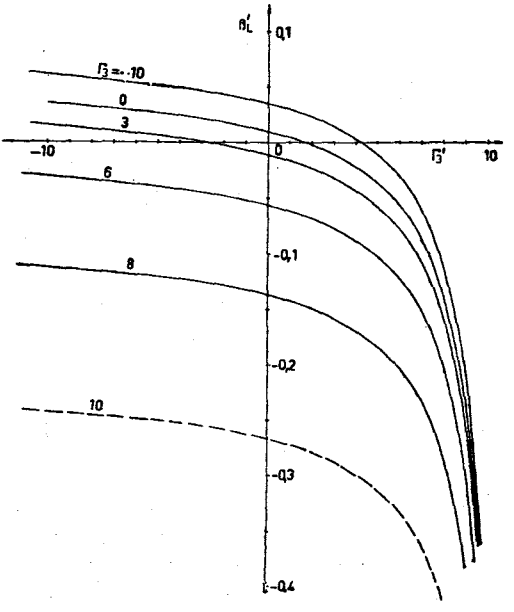
β'_L
0,1
$\Gamma_3=-10$
0
3
-10
0
Γ_3'
10
6
8
10
-0,1
-0,2
-0,3
-0,4

Fig. 4b.

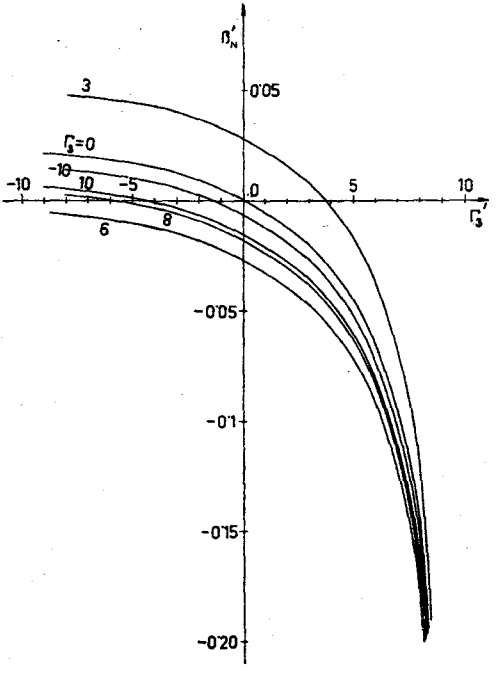
β'_N
3
0·05
$\Gamma_3=0$
-10
-10
10
-5
0
5
10
Γ_3'
8
6
-0·05
-0·1
-0·15
-0·20

Fig. 4c.

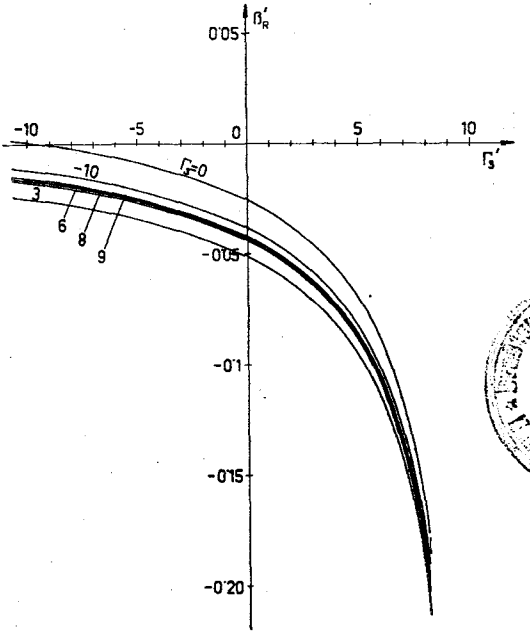
β_R'
0·05
-10
-5
0
5
10
Γ_3'
$\Gamma_3'=0$
-10
3
6
8
9
-0·05
-0·1
-0·15
-0·20

Fig. 4d.

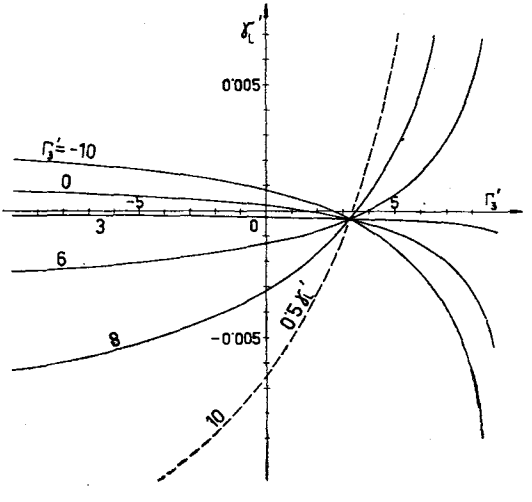
γ_L'
0·005
$\Gamma_3'=-10$
0
-5
3
0
5
Γ_3'
6
8
10
$0{\cdot}5\gamma_L'$
-0·005

Fig. 4e.

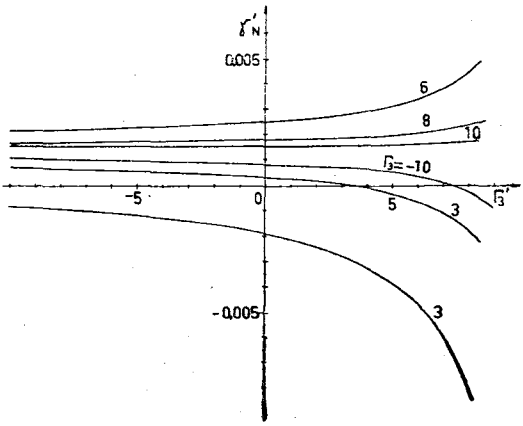

Fig. 4f.

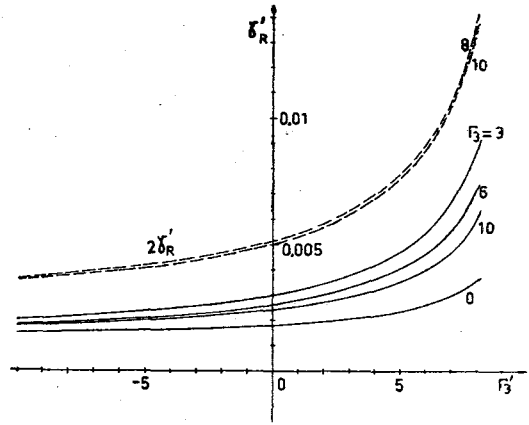

Fig. 4g.

The maximum variation of the correction factor

$$\Delta C = \left[\frac{C(W_0) - C(W)}{C(W_0)}\right]_{\max} \tag{105}$$

is given as a function of our parameters in *Figs.* 5. a, b, c. From these figures one can conclude that the most critical of the parameters is Γ_3, its lowest acceptable value being $\Gamma_3 \approx 6$. The different values of Γ'_3 change slightly the value of Γ_3 for which the variation $\Delta C < 6\%$, but effects the value of Γ rather seriously.

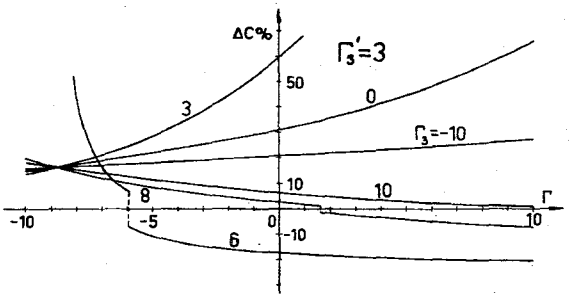

Fig. 5a.

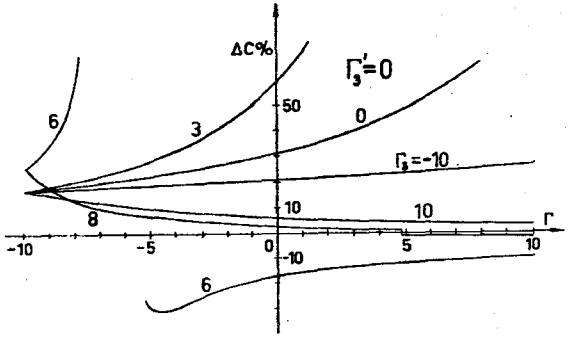

Fig. 5b.

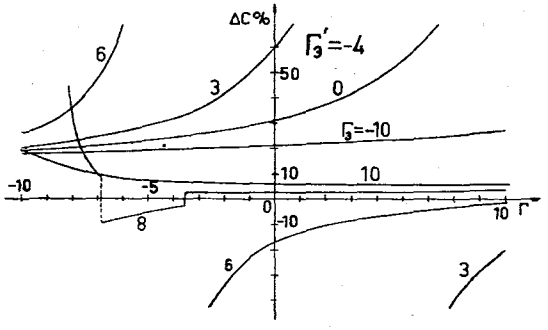

Fig. 5c.

$\Gamma'_3 < 0$ decreases slightly the lowest acceptable Γ_3. The effect of $\Gamma'_3 > 0$ is the opposite. The radius of nuclear charge distribution was taken as $r_0 = a/2\, A^{1/3}$. The alternative value $r'_0 = 0{\cdot}8\, r_0$ was taken to determine the influence of the radius on the correction factor. It has been found that the influence is negligible in the interesting region $\Gamma_3 \geq 6$.

It seems that in this way it is possible to explain the experimental findings on $Ho^{166} \to Er^{166}$ under the assumption of a small matrix element. The small value of the nuclear matrix elements can be obtained when the wave function is a linear combination of several terms. The reductions obtained in this way are rather moderate, i. e. not larger than 10^2. More drastic reduction is obtained in the case of destructive interference inside the matrix element (the case of C^{14}, private communication from Dr. B. H. Flowers). Which of these two possibilities is realized in our case is to be settled by further investigations. The simplest possibility would be to try to set up wave functions for two particles moving in a spheroidal harmonic potential well with some appropriate coupling between the particles. That would lead to a calculation similar to that of Nilsson, Moszkowski and Gottfried [30] but generalized to the case of two particles in a deformed potential.

Appendix

The expressions necessary to evaluate the correction factor for the tensor pseudoscalar mixture for a $(0 \longrightarrow 0 +)$ transition are given here. The numbers refer to the general expressions in the text.

The wave functions for neutrino, for $\varkappa_\nu = \pm 1$, are

$$F_1 = q\left[1 - \frac{(qr)^2}{6} + \ldots.\right], \quad F_1' = -\frac{q^3 r}{3} + \ldots.,$$

$$F_{-1} = -\frac{q^2 r}{3}\left[1 - \frac{(qr)^2}{10} + \ldots.\right], \quad F'_{-1} = -\frac{q^2}{3} + \ldots.,$$

$$G_1 = -F_{-1}, \quad G'_1 = -F'_{-1}, \tag{37A}$$

$$G_{-1} = F_1, \quad G'_{-1} = F'_1.$$

Below are listed the electron wave functions inside the nucleus for $\varkappa = \pm 1$ and the potential energy (40) in the form $V = V_0 + V_2\, x^2 + V_4\, x^4$. The error is less than 0·5‰ for $W < 5$, $x \leq 1$, $r_0 \leq 2 \times 10^{-2}$ and $|\varepsilon| \leq 0{\cdot}05$.

$$f_1^{(i)} = x + \left[\left(-\frac{V_0^2 r_0^2}{6} + \frac{r_0^2}{6}\right) + \left(\frac{V_0 r_0^2}{3}\right) W + \left(-\frac{r_0^2}{6}\right) W^2\right] x^3 +$$

$$+ \left[\left(-\frac{2}{15} V_0 V_2 r_0^2 + \frac{V_0^4 r_0^4}{120} + \frac{V_2 r_0^2}{30}\right) + \left(\frac{2}{15} V_2 r_0^2 - \frac{V_0^3 r_0^4}{30}\right) W + \right.$$

$$\left. + \frac{V_0^2 r_0^4}{20} W^2\right] x^5 + \left[\left(-\frac{V_2^2 r_0^2}{30} + \frac{4}{315} V_0^3 V_2 r_0^4 - \frac{5}{63} V_0 V_4 r_0^2\right) + \right.$$

$$\left. + \left(-\frac{4}{105} V_0^2 V_2 r_0^4 + \frac{5}{63} V_4 r_0^2\right) W + \left(\frac{4}{105} V_0 V_2 r_0^4\right) W^2\right] x^7 +$$

$$+ \left[\left(\frac{29}{3780} V_0^2 V_2^2 r_0^4 - \frac{3}{70} V_2 V_4 r_0^2\right) + \left(-\frac{29}{1890} V_0 V_2^2 r_0^4\right) W\right] x^9 +$$

$$+ \left[\frac{16}{7425} V_0 V_2^3 r_0^4\right] x^{11} + \ldots \qquad (42A)$$

$$g_1^{(i)} = \left[\left(-\frac{V_0 r_0}{3} + \frac{r_0}{3}\right) + \left(\frac{r_0}{3}\right) W\right] x^2 + \left[\left(-\frac{V_2 r_0}{5} + \right.\right.$$

$$\left. + \frac{V_0^3 r_0^3}{30} - \frac{V_0^2 r_0^3}{30} - \frac{V_0 r_0^3}{30}\right) + \left(-\frac{V_0^2 r_0^3}{10} + \frac{V_0 r_0^3}{15}\right) W +$$

$$+ \left(\frac{V_0 r_0^3}{10}\right) W^2 + \left(-\frac{r_0^3}{30}\right) W^3\Big] x^4 + \left[\left(\frac{3}{70} V_0^2 V_2 r_0^3 - \frac{V_4 r_0}{7} - \right.\right.$$

$$\left. - \frac{V_0 V_2 r_0^3}{42} - \frac{V_0^5 r_0^5}{840}\right) + \left(-\frac{3}{35} V_0 V_2 r_0^3 + \frac{V_0^4 r_0^5}{168} + \frac{V_2 r_0^3}{42}\right) W +$$

$$+ \left(\frac{3}{70} V_2 r_0^3 - \frac{V_0^3 r_0^5}{84}\right) W^2\Big] x^6 + \left[\left(\frac{31}{1134} V_0^2 V_4 r_0^3 + \frac{V_0 V_2^2 r_0^3}{54} \right.\right.$$

$$\left. - \frac{V_2^2 r_0^3}{135} - \frac{53}{22480} V_0^4 V_2 r_0^5\right) + \left(-\frac{31}{567} V_0 V_4 r_0^3 - \frac{V_2^2 r_0^3}{54} + \frac{53}{5670}\right.$$

$$\left. V_0^3 V_2 r_0^5\right) W\Big] x^8 + \left[\left(\frac{23}{990} V_0 V_2 V_4 r_0^3 + \frac{V_2^3 r_0^3}{330} - \frac{V_0^3 V_2^2 r_0^5}{540}\right)\right.$$

$$\left. + \left(\frac{V_0^2 V_2^2 r_0^5}{180}\right) W\right] x^{10} + \ldots$$

$f^{(i)}{}_{-1}$ and $g^{(i)}{}_{-1}$ are obtained from $g^{(i)}{}_1$, respectively $f^{(i)}{}_1$, by changing $V_i \to -V_i$ and $W \to -W$.

$$R_0(\pm\varkappa, \pm\varkappa) = (-)^{l \pm \varkappa}\sqrt{2j+1} = -R_0(\mp\varkappa, \mp\varkappa) \qquad (57A)$$

$$\left.\begin{aligned} R_{10}(\pm\varkappa, \mp\varkappa) &= (-)^{-1/2+j+l_{\mp\varkappa}}\sqrt{2}\,[(2l_{\pm\varkappa}+1)(2l_{\mp\varkappa}+1)\cdot \\ &\cdot (2j+1)]^{1/2}(l_{-\varkappa}\,0\,l_{+\varkappa}\,0 \mid l_{\pm\varkappa}\,l_{\mp\varkappa}\,10)\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ l_{\pm\varkappa} & l_{\mp\varkappa} & j \end{Bmatrix} \\ R_{10}(-\varkappa, \varkappa) &= -R_{10}(\varkappa, -\varkappa);\; R_{10}(-1, 1) = -\sqrt{2} \end{aligned}\right\} \qquad (59A)$$

$$\begin{Bmatrix} j & j & 0 \\ l_{-\varkappa} & l_{\varkappa} & 1 \end{Bmatrix} = (-)^{-1/2+l_{\varkappa}+j} \frac{1}{\sqrt{3\,(2j+1)}} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ l_{-\varkappa} & l_{\varkappa} & j \end{Bmatrix} \tag{62A}$$

$$W\left(j\, j\, l_{\pm\varkappa}\, l_{\pm\varkappa} \mid 0\, \frac{1}{2}\right) = -\begin{Bmatrix} \frac{1}{2} & j & l_{\pm\varkappa} \\ 0 & l_{\pm\varkappa} & j \end{Bmatrix} \tag{63A}$$

$$= (-)^{-1/2+l_{\pm\varkappa}+j} \frac{1}{\sqrt{(2j+1)\,(2l_{\pm\varkappa}+1)}}$$

$$T^{0}_{01} = -\frac{1}{4\pi} \frac{\vec{\sigma} \cdot \mathbf{r}}{r} \tag{67A}$$

$$(l\,\mu\, l - \mu \mid l\, l\, 00) = (-)^{l-\mu} \frac{1}{\sqrt{2l+1}} = \begin{pmatrix} l & l & 0 \\ \mu & -\mu & 0 \end{pmatrix} \tag{70A}$$

Faculty of Science, Zagreb
and
Institute »Ruđer Bošković«, Zagreb

REFERENCES:

[1] *G. Alaga — B. Jakšić,* Phys. Rev. **103**, 1441 (1956),
[2] *R. L. Graham — J. L. Wolfson — M. A. Clark,* Bull. Am. Phys. Soc. **30**, № 1., 34. (1955),
[3] *A. G. Petschek — R. E. Marschak,* Phys. Rev. **85**, 698 (1952),
[4] *L. A. Sliv,* ŽETF **17**, 1049 (1947),
M. E. Rose, Phys. Rev. **82**, 389 (1951),
M. E. Rose — D. K. Holmes, ORNL Report, 1022 (1951),
M. E. Rose — C. L. Perry — N. M. Dismuke, ORNL Report, 1459 (1953),
[5] *M. Yamada,* Prog. Theor. Phys. **10**, 245 (1953),
M. Yamada Prog. Theor. Phys. **10**, 241 (1953),
M. R. Nataf, C. R. **238**, 1012 and 1117 (1954),
M. Takebe, Prog. Theor. Phys. **12**, 561 (1954),
[6] *K. Siegbahn:* Beta and Gamma Ray Spectroscopy Amsterdam, **1955**, especially the articles on the theory of β-decay by *M. E. Rose,* Chap. IX., *E. J. Konopinski,* Chap. X, *C. S. Wu,* Chap. XI. and *O. Kofoed-Hansen,* Ch. XII.
[7] *M. E. Rose — R. K. Osborn,* Phys. Rev. **93**, 1315 (1954),
H. Takebe Prog. Theor. Phys. **12**, 561 (1954),
[8] *G. Alaga — O. Kofoed-Hansen — A. Winther,* Dan. Mat. Phys. Medd. **28**, № 3., (1953),
E. J. Konopinski, Phys. Rev. **94**, 492 (1954),
R. W. King — D. C. Peaslee, Phys. Rev. **94**, 1284 (1954),
R. F. Herbst, Phys. Rev. **96**, 372 (1954),
[9] *E. Fermi,* Zs. f. Phys. **88**, 161 (1934) C. f. also reference [6]. The extention of the theory to *K-capture* has been done by *C. Møller,* Phys. Zs. Sowjetunion **11**, 9 (1937), and Phys. Rev. **51**, 84 (1937),
[10] *V. Glaser,* Kovarijantna Kvantna Elektrodinamika, Zagreb, 1955,
[11] *G. Gamow — E. Teller,* Phys. Rev. **49**, 895 (1936),
A. Winther — O. Kofoed-Hansen, Dan. Mat. Phys. Medd. **27**, № **14** (1953),
[12] *C. N. Yang — J. Tiomno,* Phys. Rev. **79**, 495 (1950),
[13] *M. Z. Fierz,* Zs. f. Phys. **104**, 553 (1937),
H. M. Mahmoud — E. J. Konopinski, Phys. Rev. **88**, 1266 (1952),

O. Kofoed-Hansen — A. Winther (private communication from *O. Kofoed-Hansen*),
[14] *M. Morita, J. Fujita, M. Yamada,* Prog. Theor. Phys. **10**, 630 (1953),
G. E. Lee — Whiting, Phys. Rev. **97**, 463 (1955),
M. A. Preston, Phys. Rev. **99**, 1654 (1955),
[15] *R. W. King — D. C. Peaslee,* Phys. Rev. **94**, 1284 (1954),
[16] *H. A. Bethe — R. F. Bacher,* Rev. Mod. Phys. **8**, 193 (1936),
G. E. Konopinski — G. E. Uhlenbeck, Phys. Rev. **60**, 308 (1941),
[17] *P. A. M. Dirac,* The Principles of Quantum Mechanics. Oxford, 1947,
[18] *L. W. Nordheim — F. L. Yost,* Phys. Rev. **51**, 942 (1937),
[19] *E. P. Wigner:* On the Matrices which reduces the Kronecher Products of Representations of S. R. Groups
A. R. Edmonds, Angular Momentum in Quantum Mechanics, CERN 55—26,
[20] TAS.*E. U. Condon — G. H. Shortly:* The Theory of Atomic Spectra, Cambridge, 1953,
[21] *R. H. Hulme,* Proc. Roy Soc. **133** (A), 381 (1931),
Magnus — F. Oberhätinger, Formeln und Sätze für die spezielle Funktionen der Mathematischen Physik, Springer (1943),
A. Erdelji, Higher Transcendental Functions I.
[22] *D. R. Inglis,* Rev. Mod. Phys. **25**, 390, (1953),
M. G. Mayer — H. Jensen, Elementary Theory of Nuclear Shell Structure, New York, 1955,
A. Bohr — B. R. Mottelson, Dan. Mat. Fys. Medd. **27**, № 16 (1953), C. f. also Chap. XVII of reference [6],
H. Feshbach — C. E. Porter — V. P. Weisskopf, Phys. Rev. **96**, 448 (1954),
[23] *L. L. Foldy — S. A. Wouthuysen,* Phys. Rev. **78**, 29 (1950),
[24] *R. H. Good Jr.,* Rev. Mod. Phys. **27**, 187 (1955),
[25] *G. Racah,* Phys. Rev. **62**, 438 (1942),
K. Alder, Helv. Phys. Acta **25**, 235 (1952),
[26] *H. C. Corben — J. S. Schwinger,* Phys. Rev. **58**, 953 (1940),
G. Geortree, Phys. Rev. **70**, 897 (1946),
V. B. Berestetski, ŽETF **18**, 1057 (1948),
[27] *D. L. Pursey,* Phil. Mag. **42**, 1193 (1952),
E. Greuling, Phys. Rev. **61**, 568 (1942),
[28] *G. Alaga,* Dissertation, University of Zagreb, Zagreb (1955). (unpublished),
[29] *E. L. Hill — R. Landshof,* Rev. Mod. Phys. **10**, 87 (1938),
[30] *S. G. Nilsson,* Dan. Mat. Fys. Medd. **29**, № 16 (1955),
S. A. Moszkowski, Phys. Rev. **99**, 803 (1955),
K. Gottfried, Phys. Rev. **103**, 1017 (1956).

IZRAČUNAVANJE KOREKCIJA ZA ANALIZU β-SPEKTARA I NJIHOVA PRIMJENA NA Ho^{166}

G. Alaga i B. Jakšić, Zagreb

Sadržaj

Služeći se analogijom sa teorijom elektromagnetskog zračenja, E. Fermi [9] je prvi formulirao teoriju beta radioaktivnog raspada. On je pretpostavio da je vjerojatnost beta procesa proporcionalna vjerojatnosti prisustva svih čestica na istom mjestu. Za uzajamno djelovanje on uzima skalarni produkt od dva polarna vektora izgrađena od kvantiziranih polja nukleona i leptona. Jačinu uza-

jamnog djelovanja mjeri konstanta g_β koja igra istu ulogu kao naboj u teoriji elektromagnetskog zračenja. Poznato je da su beta procesi sporiji od elektromagnetiskih, pa će konstanta g_β biti manja od naboja »e« mjerena u istom mjerilu. Uzajamno djelovanje polja lakih i teških čestica (1) može se stoga tretirati kao mala smetnja i prijelazna vjerojatnost računati računom smetnje (6).

Ovako formulirana teorija lijepo je objašnjavala opće crte beta procesa, kao što su poluvično trajanje i oblik beta spektra. Te karakteristike su uglavnom dane sa brojem konačnih stanja za slučaj raspada na tri čestice i veličinom matričnog elementa, pa specifičnost uzajamnog djelovanja ne dolazi u obzir.

Detaljnija analiza experimentalnih podataka je pokazala da je teoriju nužno proširiti [11]. Pored vektorskog vezanja uvedena su i ostala vezanja, koja je moguće načiniti od dva spinora. To su: skalarno, vektorsko, tenzorsko, pseudovektorsko i pseudoskalarno. Nazivi su prema transformacionim svojstvima ovih veličina s obzirom na Lorentzovu transformaciju (2a—2e).

Problem je sada naći onu linearnu kombinaciju ovih 5 vezanja, koja će protumačiti sve eksperimentalne nalaze o beta spektrima, o poluvičnom trajanju i angularnim korelacijama elektrona i neutrina.

Godina 1949 bila je ustvari godina velikog napretka u eksperimentalnoj tehnici, tako da je od tada pa do danas sakupljen znatan dio dragocjenog materijala za beta radioaktivno raspadanje [6]. Analizom materijala nađeno je, da je moguće uskladiti eksperimentalne nalaze [13] sa pretpostavkom, da skalarno i tenzorsko vezanje podjednako učestvuju, dok pseudovektorsko i vektorsko učestvuje sa 1—3% odnosno 15%, a o pseudoskalarnom vezanju nije bilo nikakvih direktnih podataka. Možemo možda napomenuti, da se učešće pseudovektorskog vezanja može isključiti dok se, izgleda, vektorsko ne može [13]. Relativni predznak između tenzorskog i skalarnog vezanja, izgleda, da je zasada još neutvrđen [14].

Postoje, naime, izvjesne indikacije, da se veoma brzi prijelazi tipa $(0 \longrightarrow 0 +)$ mogu protumačiti pomoću konstruktivne interference tenzorskog i pseudoskalarnog vezanja. Spektri ovih prijelaza su nepoznati. Ista ta mješavina je odgovorna i za prijelaz $Ho^{166} \rightarrow$ $\rightarrow Er^{166}$, koji, izgleda, da ima po prilici dozvoljen spektar [2] i koji je prema prije pomenutim prijelazima znatno usporen.

Svrha ovog članka je da se pokuša objasniti ova kontradiktorna situacija.

Mi smo pokušali naći rješenje pomoću dvije pretpostavke: a) destruktivne interference [3] između tenzorskog i pseudoskalarnog vezanja i, b) da je uobičajeni tenzorski matrični element veoma mali.

U II je dana opća formulacija beta teorije, a sva moguća vezanja leptonskog i nukleonskog polja dana su sa (2a)—(2e). Prelazeći u konfiguracioni prostor za nukleone i tvoreći matrični element za leptone, dobivamo operator smetnje (5) pomoću koga je dana prijelazna vjerojatnost u jedinici vremena (6).

U III se razmatraju rješenja Diracove jednadžbe za leptone za centralno simetrične potencijale (8). Za matrice $\vec{\alpha}$, β je uzeta Diracova reprezentacija (8a). Definirajući operatore (11) i (13) može se Hamiltonov operator pisati u obliku (15) i tražiti rješenja problema (16a, b, c) za kontinuirani spektar $W > 1$. Definirajući angularni dio rješenja (16d) pomoću (17a, b, c), odnosno (20), dobivamo kao rješenje (18) sa radijalnim jednadžbama (21a, b). U IIIa) je dano rješenje tog sistema za čisto Coulombovo polje (22). Uvodeći novu funkciju $\sigma_\varkappa$ i (23), dolazi se na sistem (24), čije se regularno rješenje, uz definicije (25) i (27), može pisati u obliku (28). Normiranjem rješenja na (30a) dobiva se rješenje za radijalne funkcije u obliku (30). U mnogim slučajevima je zgodnije rješavati sistem (21) direktno pomoću redova (31), čiji su koeficijenti određeni rekurzionim formulama (32) i faktorima normiranja (33). Slučaj slobodnih čestica razmatran je u IIIb). Rješenje je dano pomoću sferičnih Besselovih funkcija (35) u obliku (36), odnosno (37). U IIIc) se razmatraju rješenja za modificirano Coulombovo polje (38). Unutarnje i vanjsko rješenje spojeno je na rubu jezgre $r = r_0$ (39). Normirano rješenje dobiva se pomoću (40), (41), (39a). Unutrašnje rješenje je dano za potencijalnu energiju (40) općenito u obliku (42), (43).

U IV je razmatrana Foldy-Wouthuysenova transformacija, pomoću koje se relativističke valne funkcije za nukleone dobivaju u nerelativističkom obliku kao razvoj po $(v/c)^n$. Definiramo li transformaciju pomoću (44) i (48), dobivamo Hamiltonovu funkciju (47) transformiranu u (49) sa neparnim dijelom višeg reda u (v/c). Uzastopnom primjenom ovakve transformacije može se eliminirati neparni dio Hamiltonove funkcije do bilo kojeg reda $(v/c)^n$. Za naš slučaj je dovoljno uzeti transformaciju (49).

Dio V sadrži sam račun beta spektra. Pomoću definicije korekcionog faktora (52) i Fermijeve funkcije (53) dobivamo za vjerojatnost prelaza (51a). Slijedi račun leptonskih matričnih elemenata (54), (54a), služeći se Racahovom algebrom. Kao rezultat dobivamo (64) i (65). Uvodeći definicije (66), (67), (68) i (69) separirali smo nuklearni matrični element na angularni i radijalni dio i dobili za tenzorski faktor korekcije (72). Uz uobičajenu aproksimaciju korekcioni faktori se mogu izraziti veličinama (73).

U Va) se tretiraju relativistički djelovi vezanja pomoću Foldy-Wouthuysenove transformacije (49) za nukleon u statističkom potencijalu. Transformacija nukleonskih operatora je dana formulom (75) u općem obliku, a za pojedina vezanja formulama (76), (77), (78) i (79). Pseudoskalarno vezanje, za razliku od ostalih, u nerelativističkoj aproksimaciji ne sadržava člana bez derivacije leptonskih funkcija ili bez nuklearnih potencijala ili njihovih derivacija, pa je stoga veoma osjetljivo na karakter potencijala. Izraz (79) može se općenito pisati u obliku (80), ako se zanemari zadnji član u (79). U Vb) se razmatra metoda za računanje faktora korekcije onih

vezanja, koja u nerelativističkoj aproksimaciji sadrže derivacije leptonskih funkcija. Kao primjer smo pokazali račun faktora korekcije za prvi dio matričkog elementa (80). Rezultat je dan u (85). U Vc) je izvršena specijalizacija općih formula (86) za slučaj $(0 \longrightarrow 0+)$ prelaza (87). Definirajući parametre (88) i (89), korekcioni faktor poprima oblik (90) sa (91)—(96). Neutrinska valna funkcija je računata pomoću razvoja (35), uzevši u obzir samo prva dva člana, gdje drugi član daje tzv. korekciju zbog konačne valne dužine. Ta je korekcija dobra sve do 10 MeV. Formule (91)—(96) vrijede, kako za slučaj čistog, tako i za slučaj modificiranog Coulombovog polja, samo što veličine (73) treba zamijeniti sa veličinama korigiranima za (74). Jedino što je zanemareno je korekcija koja potječe od varijacije leptonskih funkcija po području jezgre. Ta je korekcija razmatrana u V d). Ispravan postupak, koji uzima u obzir i varijaciju leptonskih funkcija preko jezgre je ilustriran primjerom (97) sa razvojem leptonskih funkcija (98). Taj se razvoj dobije pomoću formula (42) i (43), a za razmatrani slučaj $(0 \longrightarrow 0+)$ prijelaza dan je u dodatku. Definirajući omjere matričnih elemenata kao nove parametre (99) može se korekcija pisati u obliku (100), (101), (101 a). Formalno se svodi na zamjenu veličina (73), korigiranih za (74), novim veličinama (103), prema uputi (102). Korekcija se svodi na numeričko računanje veličina (103a). U V e) su sumirani rezultati numeričkih računa za prijelaz $Ho^{166} \rightarrow Er^{166}$. Čisto tenzorsko vezanje, korigirano i za (74) nije bilo u stanju protumačiti ravan Fermijev dijagram, kao što se vidi iz *Sl.* 1. Na *Sl.* 2 dane su korekcije (74) za $r_0 = \alpha/2\, A^{1/3}$ i $r'_0 = 0{\cdot}8\ r_0$ uz parabolički oblik gustoće naboja sa $\varrho(r_0)/\varrho(0) = 1 + \varepsilon = 1{\cdot}05$. Destruktivna interferencija tenzorskog i pseudoskalarnog vezanja, $\Gamma \approx 10$, daje patološke oblike spektra i nije u mogućnosti da objasni eksperimentalne nalaze. Zato smo prešli na detaljnije ispitivanje druge mogućnosti, t. j. slučaja malog tenzorskog matričnog elementa. Tu korekcija, koju smo uveli u V d), igra dominantnu ulogu. Za mješavinu tenzora i pseudoskalara imaćemo nove parametre (104a) i (104 b) od kojih su mjerodavni samo Γ_3 i Γ'_3. *Sl.* 3. i *Sl.* 4. prikazuje veličine u (103a), a *Sl.* 5. ovisnost maksimalne varijacije korekcionog faktora preko čitavog područja energije u ovisnosti o parametrima Γ, Γ_3, Γ'_3. Kako se sa slike vidi, moguće je uvijek dobiti maleni ΔC, recimo $\Delta C < 6\%$, ako se uzme $\Gamma_3 \geq 6$. Ostala dva parametra nisu tako kritična i nemoguće ih je za sada jednoznačno odrediti. Prema tome alternativa malog tenzorskog matričnog elementa objašnjava oba eksperimentalna nalaza za prijelaz $Ho^{166} \longrightarrow Er^{166}$. Bitnu ulogu pri tome igrala je korekcija uveden u V d) kojom se egzaktno uzima u obzir varijacija leptonskih funkcija na području jezgre.

U *Dodatku* su složene neke formule, koje su služile za izračunavanje korekcionih faktora i korekcija.

(Primljeno 20. X. 1956.)