GLASNIK MATEMATIČKI
Tom 6 (26) — No. 2 — 1971.

ON AN EXAMPLE OF NORMED KÖTHE SPACES

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In this paper we consider a generalization of and discuss an interesting example of a function space defined in [3] (cf. also [4]). The example is as follows: let ϱ be the function norm

$$\varrho(u) = \sum_{n=1}^{\infty} 2^{-n} u(n) + \limsup_{n\to\infty} u(n), \quad u \geq 0, \tag{1}$$

defined on R^N, where N denotes the set of positive integers, and R is the extended real continuum. Then, the Köthe function space $L_\varrho(N) = \{u : \varrho(|u|) < +\infty\}$ is not a complete function space; even more, the completion of $L_\varrho(N)$ cannot be realized as a set of classes of functions on N, i. e. $L_\varrho(N)$ has no function completion on N.

We say that a real vector space F is a regular function space of type $(X, \mathfrak{A})$, where X is a given set and $\mathfrak{A} \subseteq \mathfrak{P}(X)$ an ideal, provided there exists a vector subspace $\mathfrak{F} \subseteq X^R$ and a linear surjection $\varphi : \mathfrak{F} \to F$ such that

$$u \in \varphi^{-1}(0) \quad \text{iff} \quad u \equiv 0 \pmod{\mathfrak{A}}; \tag{2}$$

here for $u, v \in R^X$ we write $u \equiv v \pmod{\mathfrak{A}}$ if there is a set $A \in \mathfrak{A}$ such that $u(x) = v(x)$ for all $x \in X \setminus A$. For the example quoted, it is obvious that $L_\varrho(N)$ has no regular function completion.

In the remainder of this paper X denotes a locally compact space and μ a given Radon measure on X. Let $C(X)$ denote the set of all real continuous functions on X, $CB(X) \subseteq C(X)$ the set of bounded functions and $C_0(X) \subseteq CB(X)$ the set of all functions with compact support.

We say that a function norm ϱ, defined on μ-measurable functions (cf. [4]), is bounded provided $\varrho(u) < +\infty$ for every $u \in CB(X)$. Obviously, the norm (1) is a bounded function norm if we take the discrete topology and also a discrete measure μ on N. The boundedness of ϱ implies the inclusion $CB(X) \subseteq L_\varrho(X)$, where $L_\varrho(X)$ is the normed Köthe space (cf. [4]), and we denote by $K_\varrho(X)$ the Hausdorff completion of $CB(X)$ in the topology generated by ϱ.

Ovaj rad je financirao Savezni fond za naučni rad i Republički fond za naučni rad SRH.

THEOREM 1. *The space $K_\varrho(X)$ has a function completion on the Stone-Čech compactification bX of X, i. e. there exists an ideal $\mathfrak{A} \subseteq \mathfrak{P}(bX)$ such that $K_\varrho(X)$ is a regular function space of type $(X, \mathfrak{A})$.*

Proof. Let

$$C : CB(X) \to C(bX)$$

be the extension operator from X to bX and let $S_X = C^{-1}$ (cf. [2]). For every $u \in C(bX)$ we set

$$\beta(u) = \varrho(S_X u).$$

β is a Radon seminorm in the terminology of [1] (More generally, for an arbitrary locally compact space X, by a Radon seminorm on X we mean a monotone seminorm on $C_0(X)$). We denote by $E_\beta(bX)$ the Hausdorff completion of the space $C(bX)$ in the topology given by β. Then, by the main theorem of [1], $E_\beta(bX)$ is a regular function space of type $(bX, \mathfrak{A})$ for a σ-ideal $\mathfrak{A}$. Since C is an isometric isomorphism of the dense subspace $CB(X) \subseteq K_\varrho(X)$ to the dense subspace $C(bX) \subseteq E_\beta(bX)$, the theorem is proved.

Starting from this theorem we can give a simple characterization of bounded function norms:

THEOREM 2. *For a bounded function seminorm ϱ there exist two Radon seminorms ϱ_0 and ϱ_Δ on X and $\Delta = bX \setminus X$ respectively such that*

$$\tfrac{1}{2}[\varrho_0(u) + \varrho_\Delta(Bu)] \leq \varrho(u) \leq \varrho_0(u) + \varrho_\Delta(Bu), \quad u \in CB(X), \tag{3}$$

where Bu is the restriction of Cu to Δ.

Proof. For $u \in C(bX)$ we can write

$$u = u\xi_X + u\xi_\Delta,$$

where ξ_A denotes the characteristic function of the set A, and since β is a seminorm, we obtain

$$\beta(u) \leq \beta(u\xi_X) + \beta(u\xi_\Delta).$$

Furthermore, since β is a monotone seminorm, it follows from the identity

$$|u| = |u\xi_X| + |u\xi_\Delta|$$

that

$$\beta(u\xi_X) \leq \beta(u) \quad \text{and} \quad \beta(u\xi_\Delta) \leq \beta(u)$$

and we obtain finally

$$\tfrac{1}{2}[\beta(u\xi_X) + \beta(u\xi_\Delta)] \leq \beta(u) \leq \beta(u\xi_X) + \beta(u\xi_\Delta) \tag{4}$$

for $u \in C(bX)$. It remains to see that $\beta(u\xi_X)$ and $\beta(u\xi_\Delta)$ can be interpreted as Radon seminorms on X and Δ respectively. For this purpose we define a Radon seminorm ϱ_0 on X by setting

$$\varrho_0(u) = \varrho(u), \qquad u \in C_0(X);$$

we also set

$$\varrho_0(u) = \sup\{\varrho_0(v) : v \in C_0(X), 0 \leq v \leq |u|\}, \qquad u \in CB(X).$$

Because of the inequality $\varrho_0(u) \leq \varrho(u)$, the real number $\varrho(u)$ is finite for every $u \in CB(X)$ and in fact ϱ_0 is a seminorm on $CB(X)$ (cf. [1]; the proof of this assertion is identical to the proof of the analogous assertion for the Radon measure). Moreover we can prove that

$$\varrho_0(S_X u) = \beta(u\xi_X), \qquad u \in C(bX). \tag{5}$$

Indeed, for $S_X u$ we have (because of the lower semi-continuity)

$$S_X u = \sup\{v : 0 \leq v \leq |S_X u|\}, \qquad v \in C_0(X) \tag{6}$$

and because of $Cv(x) = 0$, $x \in \Delta$, we also have

$$u\xi_X = \sup\{Cv : 0 \leq Cv \leq |u\xi_X|\}, \qquad v \in C_0(X). \tag{7}$$

By the argument as given above we obtain finally

$$\beta(u\xi_X) = \sup \beta(Cv) = \sup \varrho_0(v) = \varrho_0(S_X u),$$

where the supremum is taken over the same set as in (6) or (7). Thus (5) is proved.

For $u \in C(bX)$ we have $S_\Delta u \in C(\Delta)$, where S_Δ is the restriction operator on Δ. Since bX is a normal topological space, for every $v \in C(\Delta)$ there exists a function $u \in C(bX)$ such that $v = S_\Delta u$. If we set

$$\varrho_\Delta(v) = \beta(S_\Delta u), \qquad v \in C(\Delta), \tag{8}$$

where $S_\Delta u = v$, then ϱ_Δ is a Radon seminorm on Δ.

Finally, if we write $w = Cu$ for an arbitrary $u \in CB(X)$ and if we set $Bu = S_\Delta w$, then, by (5) and (8), we obtain (3).

In such a way the nonexistence of a functional completion on X for a bounded function seminorm ϱ is caused by the term $\varrho_\Delta(Bu)$ in (3). For example, the norm given by (1) can be written in the form

$$\varrho(u) = \sum_{n=1}^{\infty} 2^{-n} u(n) + \max_{x \in \Delta} u(x), \tag{9}$$

where $\Delta = bN \setminus N$ and this shows that the unusual property of the normed Köthe space $L_\varrho(N)$ with respect to completion is in fact understandable.

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(Received 21. XII 1971.)

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O JEDNOM PRIMJERU PROSTORA FUNKCIJA

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Sadržaj

U članku je generaliziran i razjašnjen zanimljiv primjer »neobičnog« prostora nizova realnih brojeva na kojem je norma zadana pomoću formule (1). Taj se prostor razlikuje od prostora koji se obično koriste u analizi po tome što se njegovo popunjenje nemože realizirati kao prostor klasa nizova. Uz korištenje osnovnog teorema iz [1] dokazano je međutim, u generalnom slučaju omeđene polunorme zadane na lokalno kompaktnom prostoru X, da se tada ustvari popunjenje općenito realizira na Stone-Čechovoj kompaktifikaciji bX prostora X (Teorem 1). U Teoremu 2 je taj rezultat nadopunjen time što je za svaku ograničenu polunormu ϱ dokazano postojanje dvaju Radonovih polunormi ϱ_0 i ϱ_Δ zadanih na X odnosno na $\Delta = bX \setminus X$ i takvih da je ϱ ekvivalentno sa $\varrho_0 + \varrho_\Delta$.