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CLIFFORD ALGEBRAS AND LITTLEWOOD-RICHARDSON COEFFICIENTS

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ABSTRACT. We show how to use Clifford algebra techniques to describe the de Rham cohomology ring of equal rank compact symmetric spaces G/K . In particular, for $G/K = U(n)/U(k) \times U(n-k)$, we obtain a new way of multiplying Schur polynomials, i.e., computing the Littlewood-Richardson coefficients. The corresponding multiplication on the Clifford algebra side is, in a convenient basis given by projections of the spin module, simply the componentwise multiplication of vectors in $\mathbb{C}^{\binom{n}{k}}$ also known as the Hadamard product. The quantum moment map induces a filtered algebra structure on $\mathbb{C}^{\binom{n}{k}}$. The associated graded product of this filtered algebra recovers the Littlewood-Richardson coefficients.

1. INTRODUCTION

In this paper we consider the classical compact symmetric space $G/K = U(n)/U(k) \times U(n-k)$. It is well known that this space is diffeomorphic

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to the complex Grassmannian $\text{Gr}(k, \mathbb{C}^n)$ of k -dimensional subspaces of $\mathbb{C}^n$. The de Rham cohomology $H(G/K)$ (with complex coefficients) is very well understood through Schubert calculus. One of the well known results about $H(G/K)$ is that it is isomorphic to $(\bigwedge \mathfrak{p}^*)^{\mathfrak{k}}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the decomposition of the complexified Lie algebra of G into eigenspaces of the (differentiated) involution defining K . This result is due to É. Cartan and de Rham, see [6, §4.1] for an exposition of a proof. By classical results of Borel, H. Cartan and others,

$$(1.1) \quad \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}} / I_+ \cong (\bigwedge \mathfrak{p}^*)^{\mathfrak{k}}.$$

Here $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{g}$, $W_{\mathfrak{k}}$ is the Weyl group of $(\mathfrak{k}, \mathfrak{t})$, and I_+ is the ideal in $\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}$ generated by the polynomials without constant term invariant under the Weyl group $W_{\mathfrak{g}}$ of $(\mathfrak{g}, \mathfrak{t})$.

Furthermore, there is a basis for $(\bigwedge \mathfrak{p}^*)^{\mathfrak{k}}$ given by the images under the isomorphism (1.1) of Schur polynomials $s_{\lambda} \in \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}$ associated to partitions λ with Young diagrams inside the $k \times (n - k)$ box. The Schur polynomial s_{λ} is defined as follows. Let

$$a_{\lambda} = \det \begin{bmatrix} x_1^{\lambda_1+k-1} & x_2^{\lambda_1+k-1} & \dots & x_k^{\lambda_1+k-1} \\ x_1^{\lambda_2+k-2} & x_2^{\lambda_2+k-2} & \dots & x_k^{\lambda_2+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_k} & x_2^{\lambda_k} & \dots & x_k^{\lambda_k} \end{bmatrix},$$

with $\{x_i\}$ an orthonormal basis of $\mathfrak{t}$, then define $s_{\lambda} = a_{\lambda}/a_0$.

For more details see e.g. [18, Chapter 1]. The multiplication of Schur polynomials is described by the Littlewood-Richardson rule [16]:

$$s_{\lambda} s_{\mu} = \sum_{\nu} c_{\lambda\mu}^{\nu} s_{\nu},$$

where $c_{\lambda\mu}^{\nu}$ are the Littlewood-Richardson (LR) coefficients. The LR coefficients also appear in other places. For example, let V_{λ} , V_{μ} and V_{ν} be finite-dimensional representations of $\text{GL}_m(\mathbb{C})$ with highest weights given by partitions λ, μ, ν . Then one has

$$\dim \text{Hom}_{\text{GL}_m(\mathbb{C})}(V_{\nu}, V_{\lambda} \otimes V_{\mu}) = c_{\lambda\mu}^{\nu}.$$

The LR coefficients also appear in the representation theory of symmetric groups, combinatorics, theory of symmetric functions, and topology of Grassmannians, in particular Schubert calculus, which is our main point of interest. For details see the classical books [7, 21] and historical reviews in [8, 15].

The multiplication of the basis of Schur polynomials in $H(\text{Gr}(k, \mathbb{C}^n))$ is again given by the Littlewood-Richardson rule, but we replace $c_{\lambda\mu}^{\nu}$ by zero if ν does not fit in the $k \times (n - k)$ box; for example, see [14, 4, 1] and [20, Theorem 3.1].

In the following, we denote the subspace of $\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{t}}}$ spanned by the above Schur polynomials by $\mathcal{P}_{k \times (n-k)}$. Let $\mathbf{P}_{k \times (n-k)}$ denote the set of partitions inside the $k \times (n-k)$ box.

In [6] and [5] we study a filtered deformation of $(\bigwedge \mathfrak{p}^*)^{\mathfrak{k}} \cong (\bigwedge \mathfrak{p})^{\mathfrak{k}}$, given as the $\mathfrak{k}$ -invariants $\text{Cl}(\mathfrak{p})^{\mathfrak{k}}$ in the Clifford algebra $\text{Cl}(\mathfrak{p})$ with respect to the trace form B . Recall that $\text{Cl}(\mathfrak{p})$ is defined as the quotient of the tensor algebra $T(\mathfrak{p})$ by the ideal generated by

$$X \otimes Y + Y \otimes X - 2B(X, Y), \quad X, Y \in \mathfrak{p}.$$

The grading by degree of $T(\mathfrak{p})$ induces a filtration of $\text{Cl}(\mathfrak{p})$, and the associated graded algebra of the filtered algebra $\text{Cl}(\mathfrak{p})$ is $\bigwedge \mathfrak{p}$. Passing to $\mathfrak{k}$ -invariants, we conclude that $\text{Cl}(\mathfrak{p})^{\mathfrak{k}}$ is a filtered algebra with the associated graded algebra equal to $(\bigwedge \mathfrak{p})^{\mathfrak{k}}$.

The main advantage of $\text{Cl}(\mathfrak{p})^{\mathfrak{k}}$ over $(\bigwedge \mathfrak{p})^{\mathfrak{k}}$ is that the algebra structure is much simpler. Namely, $\text{Cl}(\mathfrak{p})$ is the endomorphism algebra of the spin module S , which becomes a $\mathfrak{k}$ -module through $\alpha : U(\mathfrak{k}) \rightarrow \text{Cl}(\mathfrak{p})$ defined on $X \in \mathfrak{k}$ by

$$(1.2) \quad \alpha(X) = \frac{1}{4} \sum_i [X, b_i] d_i,$$

where b_i is any basis of $\mathfrak{p}$ and d_i is the dual basis with respect to B . We note the algebra morphism α is a quantum moment map as defined in [17], see [13] for details. One knows that the $\mathfrak{k}$ -module S is multiplicity free, with $|W_{\mathfrak{g}}|/|W_{\mathfrak{k}}| = \binom{n}{k}$ components. In fact, these components have highest weights $\sigma\rho - \rho_{\mathfrak{k}}$, where ρ and $\rho_{\mathfrak{k}}$ are the half sums of compatible positive root systems for $(\mathfrak{g}, \mathfrak{k})$ respectively $(\mathfrak{k}, \mathfrak{t})$, and σ runs over the set

$$(1.3) \quad W^1 = \{\sigma \in W_{\mathfrak{g}} \mid \sigma\rho \text{ is } \mathfrak{k}\text{-dominant}\}.$$

It now follows from Schur's lemma that $\text{Cl}(\mathfrak{p})^{\mathfrak{k}} = \text{End}_{\mathfrak{k}} S$ is spanned by the orthogonal projections to these components. In this way the algebra $\text{Cl}(\mathfrak{p})^{\mathfrak{k}}$ becomes isomorphic to $\mathbb{C}^{\binom{n}{k}}$ with componentwise multiplication, also called the Hadamard product. The quantum moment map α endows $\mathbb{C}^{\binom{n}{k}}$ with a filtration: $F^m \mathbb{C}^{\binom{n}{k}} = \alpha(U^m(\mathfrak{k})^{\mathfrak{k}})$, $m \in \mathbb{Z}_+$.

THEOREM 1.4. *The algebra homomorphism α of (1.2) restricted to $U(\mathfrak{k})^{\mathfrak{k}} \cong \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{t}}}$ induces a filtered algebra isomorphism*

$$\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{t}}}/I_{\rho} \cong \text{Cl}(\mathfrak{p})^{\mathfrak{k}},$$

where I_{ρ} is the ideal in $\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{t}}}$ generated by the $W_{\mathfrak{g}}$ -invariant polynomials vanishing at ρ .

The Cartan–Borel theorem (1.1) can be deduced from Theorem 1.4 by passing to associated graded algebras. In particular, the isomorphism (1.1) is induced by $\text{gr } \alpha$.

Through the above identifications, (images of) Schur polynomials define a basis for $\text{Cl}(\mathfrak{p})^\mathfrak{k}$, and since the multiplication in $\text{Cl}(\mathfrak{p})^\mathfrak{k}$ is equal to the multiplication in $(\bigwedge \mathfrak{p}^*)^\mathfrak{k} \cong (\bigwedge \mathfrak{p})^\mathfrak{k}$ modulo lower order terms, we can recover the Littlewood-Richardson coefficients from the multiplication in $\text{Cl}(\mathfrak{p})^\mathfrak{k}$, which is simply the Hadamard product in the basis of projections. Following Kostant [12, §1.2], we view $\bigwedge \mathfrak{p}$ and $\text{Cl}(\mathfrak{p})$ as the same space with two multiplications, $\wedge$ respectively $\bullet$. In particular, this space is graded, and we denote by $a_{[d]}$ the d th graded component of a in this space. (We point out that while $\wedge$ is compatible with this grading, $\bullet$ is not.) The Clifford product $\bullet$ restricted to the algebra of projections is the Hadamard product, which we denote by $\bullet_H$. In coordinates related to the basis given by the projections

$$(a_1, \dots, a_N) \bullet_H (b_1, \dots, b_N) = (a_1 b_1, \dots, a_N b_N), \quad N = \binom{n}{k}.$$

The main result of this paper is

THEOREM 1.5. *Given Schur polynomials s_λ and s_μ of degree $|\lambda|$ respectively $|\mu|$, we can calculate the wedge product of their images in $H(\text{Gr}(k, \mathbb{C}^n)) \cong \mathbb{C}[\mathfrak{t}^*]^{W^\mathfrak{k}}/I_+$ as*

$$\text{gr } \alpha(s_\lambda) \wedge \text{gr } \alpha(s_\mu) = (\alpha(s_\lambda) \bullet_H \alpha(s_\mu))_{[2|\lambda|+2|\mu|]}.$$

This describes a novel way to calculate the Littlewood-Richardson coefficients using the Hadamard product (componentwise multiplication) $\bullet_H$, a linear isomorphism from the space $\mathcal{P}_{k \times (n-k)}$ of Schur polynomials to $\text{Cl}(\mathfrak{p})^\mathfrak{k}$ induced by α , and the inverse of this isomorphism. Let M be the matrix sending the basis $s_\lambda \in \mathcal{P}_{k \times (n-k)}$ to $\alpha(s_\lambda) \in \text{Span}\{\text{pr}_\sigma\}$, and $P_{[d]}$ be the projection to degree d polynomials.

COROLLARY 1.6. *The multiplication of Schur polynomials s_λ and s_μ in the cohomology ring $H(\text{Gr}(k, \mathbb{C}^n)) \cong (\bigwedge \mathfrak{p})^\mathfrak{k}$ is*

$$s_\lambda \cdot s_\mu = P_{[|\lambda|+|\mu|]} M^{-1}(M(s_\lambda) \bullet_H M(s_\mu)).$$

The above is illustrated by Algorithm 1 and examples in Section 3. We leave the evaluation of the complexity of this algorithm to future work. We do note, however, that the calculation of LR coefficients has been shown to be $\#P$ -complete, see [19, 2]. Furthermore, the linear isomorphism corresponding to M is easily described as evaluation of Schur polynomials at points $\sigma\rho$ for $\sigma \in W^1$, see (2.1).

To obtain the full list of LR coefficients for a given λ and μ , one can consider the case of $n = 2(|\lambda| + |\mu|)$, $k = |\lambda| + |\mu|$. But in order to compute $c_{\lambda\mu}^\nu$ for given λ , μ , and ν it is enough to consider, for example, the minimal $k \times (n-k)$ box such that $\lambda, \mu, \nu \in \mathbf{P}_{k \times (n-k)}$.

Algorithm 1 An algorithm to calculate $s_\lambda \cdot s_\mu$

Require: $\lambda, \mu \in \mathbf{P}_{k \times (n-k)}$
Ensure: the list of $c_{\lambda\mu}^\nu$ for $\nu \in \mathbf{P}_{k \times (n-k)}$

Poly $\leftarrow \{s_\nu : \nu \in \mathbf{P}_{k \times (n-k)}\}$, $N = \binom{n}{k}$, Pts $\leftarrow \{\}$.
 $E_\lambda \leftarrow$ the coordinate vector of s_λ in the basis Poly
 $E_\mu \leftarrow$ the coordinate vector of s_μ in the basis Poly
 $\rho \leftarrow \{\frac{n-1}{2}, \frac{n-1}{2} - 1, \dots, \frac{n-1}{2} - (n-1)\}$ $\triangleright$ Calculate ρ
for σ a k shuffle of n **do**
 Pts $\leftarrow$ Pts $\cup \sigma(\rho)$ $\triangleright$ Calculate the points $\{\sigma\rho : \sigma \in W^1\}$
end for
for $i = 1, \dots, N$ **do**
 for $j = 1, \dots, N$ **do**
 $M[i, j] \leftarrow \text{Poly}[i](\text{Pts}[j])$ $\triangleright M$ is the basis change $\{s_\lambda\}$ to $\{\text{pr}_\sigma\}$
 end for
end for
 $X_\lambda \leftarrow M \cdot E_\lambda$, $X_\mu \leftarrow M \cdot E_\mu$ $\triangleright$ Coordinates of s_λ, s_μ in the basis of $\{\text{pr}_\sigma\}$
for $i = 1, \dots, N$ **do**
 $W[i] \leftarrow X_\lambda[i] \times X_\mu[i]$ $\triangleright$ Hadamard product
end for
 $Z \leftarrow M^{-1} \cdot W$ $\triangleright$ Back to the basis of $\{s_\lambda\}$
for $i = 1, \dots, N$ **do**
 if $\deg \text{Poly}[i] \neq \deg s_\lambda + \deg s_\mu$ **then**
 $Z[i] \leftarrow 0$ $\triangleright$ Removing lower order terms
 end if
end for
return Z $\triangleright$ The list of $c_{\lambda\mu}^\nu$

We believe the techniques in this paper will be applicable to computing the small quantum cohomology of Grassmannians. We intend to pursue this direction in the near future.

In Section 2 we discuss the $\mathfrak{k}$ -decomposition of the spin module S and prove Theorems 1.4 and 1.5. In Section 3 we compute a couple of examples, calculated using [22].

Our algorithm is similar to the algorithm proposed by Kostant and Kumar [11] applied to Schubert polynomials and later refined by Billey [3]. However, neither of these uses Clifford algebras which provide a natural way to define the relevant filtrations.

2. DESCRIPTION OF $\text{Cl}(\mathfrak{p})^\mathfrak{k}$ FOR EQUAL RANK SYMMETRIC PAIRS

In this section G and K are as in the introduction, but (G, K) could equally well be any equal rank compact symmetric pair with G and K connected (for disconnected K , see [6, Section 3]). Since G and K are connected

it is enough to work with Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$. In this more general situation, the form B is replaced by any nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, and $\mathfrak{p}$ is still the -1 eigenspace of the involution defining $\mathfrak{k}$ (or equivalently, the orthogonal of $\mathfrak{k}$ with respect to B).

Let us first review some facts about the spin module S for $\text{Cl}(\mathfrak{p})$. Since $\dim \mathfrak{p}$ is even, the Clifford algebra $\text{Cl}(\mathfrak{p})$ has only one simple module S , and moreover $\text{Cl}(\mathfrak{p}) \cong \text{End } S$. To construct S , write

$$\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-,$$

where $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are maximal isotropic subspaces of $\mathfrak{p}$ in duality under B . Then one can take $S = \bigwedge \mathfrak{p}^+$, with elements of $\mathfrak{p}^+$ acting by wedging, and elements of $\mathfrak{p}^-$ by contracting. As mentioned in the introduction, S becomes a $\mathfrak{k}$ -module through the map α of (1.2). Furthermore, the $\mathfrak{k}$ -module S is multiplicity free and decomposes as [9, §2.3.6]

$$S = \bigoplus_{\sigma \in W^1} E_{\sigma\rho - \rho_{\mathfrak{k}}},$$

where ρ , $\rho_{\mathfrak{k}}$ and W^1 are as in the introduction, see (1.3), and $E_{\sigma\rho - \rho_{\mathfrak{k}}}$ denotes the irreducible finite-dimensional $\mathfrak{k}$ -module with highest weight $\sigma\rho - \rho_{\mathfrak{k}}$.

Now $\text{Cl}(\mathfrak{p}) \cong \text{End } S$ implies

$$\text{Cl}(\mathfrak{p})^{\mathfrak{k}} \cong \text{End}_{\mathfrak{k}} S,$$

and by Schur's Lemma this is the algebra of projections $\text{pr}_{\sigma} : S \rightarrow E_{\sigma\rho - \rho_{\mathfrak{k}}}$, which we denote by $\text{Pr}(S)$. This is a very simple commutative algebra, isomorphic to $\mathbb{C}^{|W^1|}$ with coordinatewise multiplication (Hadamard product), by identifying pr_{σ} with $(0, \dots, 0, 1, 0, \dots, 0)$, with 1 in the place corresponding to σ .

Since the map α defines the action of $\mathfrak{k}$ on the spin module S , the restriction of α to the center $U(\mathfrak{k})^{\mathfrak{k}}$ of $U(\mathfrak{k})$ has a very simple description in terms of the Harish-Chandra isomorphism $U(\mathfrak{k})^{\mathfrak{k}} \cong \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}$. Namely, under the Harish-Chandra isomorphism, the infinitesimal character of $E_{\sigma\rho - \rho_{\mathfrak{k}}}$ corresponds to evaluation at $\sigma\rho$. Therefore, the restriction of α to $U(\mathfrak{k})^{\mathfrak{k}}$ corresponds to the algebra morphism $\text{ev}_{\rho} : \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}} \rightarrow \text{Pr}(S) \cong \text{Cl}(\mathfrak{p})^{\mathfrak{k}}$, given by

$$(2.1) \quad \text{ev}_{\rho}(p) = \sum_{\sigma \in W^1} p(\sigma\rho) \text{pr}_{\sigma}.$$

LEMMA 2.2. *The map ev_{ρ} is surjective.*

PROOF. We need to prove that for any choice of scalars a_{σ} , $\sigma \in W^1$, there is $P \in \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}$ such that $P(\sigma\rho) = a_{\sigma}$ for all $\sigma \in W^1$. The polynomial P is similar to the Lagrange interpolation polynomial, but we have to ensure it is $W_{\mathfrak{k}}$ -invariant.

It is enough to find the polynomials $P_{\sigma} \in \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}$, $\sigma \in W^1$, such that $P_{\sigma}(\tau\rho) = \delta_{\sigma\tau}$ for any $\tau \in W^1$.

We define

$$Q_\sigma(\gamma) = \prod_{\tau \in W^1 \setminus \sigma} \frac{\|\gamma - \tau\rho\|^2}{\|\sigma\rho - \tau\rho\|^2}, \quad \gamma \in \mathfrak{t}^*.$$

which is well defined since $\sigma\rho$ and $\tau\rho$ for $\sigma \neq \tau$ are in different Weyl chambers and thus are not equal. Furthermore, the function $\gamma \mapsto \|\gamma - a\|^2$ for any $a \in \mathfrak{t}^*$ is a polynomial function so Q_σ is a product of polynomials, thus a polynomial function. Then $Q_\sigma(\tau\rho) = \delta_{\sigma\tau}$ as required, but the Q_σ are not necessarily $W_{\mathfrak{t}}$ -invariant. So we set

$$P_\sigma = \frac{1}{|W_{\mathfrak{t}}|} \sum_{w \in W_{\mathfrak{t}}} w Q_\sigma$$

and obtain the polynomials we were looking for. $\square$

The kernel of ev_ρ clearly contains the ideal I_ρ generated by $\{p \in \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{g}}} : p(\rho) = 0\}$. To see that in fact $\ker \text{ev}_\rho = I_\rho$, we prove

LEMMA 2.3. *The ideal I_ρ is of codimension $|W^1|$ in $\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{t}}}$.*

PROOF. Let J be the ideal of $\mathbb{C}[\mathfrak{t}^*]$ generated by $\{p \in \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{g}}} : p(0) = 0\}$, then $\mathbb{C}[\mathfrak{t}^*]/J$ is the coinvariant algebra associated to $W_{\mathfrak{g}}$ and is isomorphic as a $W_{\mathfrak{g}}$ -module to $\mathbb{C}[W_{\mathfrak{g}}]$, see [10, Section 3.6], hence has dimension $|W_{\mathfrak{g}}|$. Therefore J has codimension $|W_{\mathfrak{g}}|$. The associated graded ideal of I_ρ is I_+ . Applying $W_{\mathfrak{t}}$ invariance to the exact sequence of $W_{\mathfrak{g}}$ -modules

$$0 \rightarrow J \rightarrow \mathbb{C}[\mathfrak{t}^*] \rightarrow \mathbb{C}[\mathfrak{t}^*]/J \cong \mathbb{C}[W_{\mathfrak{g}}] \rightarrow 0$$

we obtain the exact sequence

$$0 \rightarrow J^{W_{\mathfrak{t}}} \cong I_+ \rightarrow \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{t}}} \rightarrow (\mathbb{C}[\mathfrak{t}^*]/J)^{W_{\mathfrak{t}}} \cong \mathbb{C}[W_{\mathfrak{g}}]^{W_{\mathfrak{t}}} \rightarrow 0.$$

We can conclude that $I_+ = J^{W_{\mathfrak{t}}}$ is of codimension $\dim \mathbb{C}[W_{\mathfrak{g}}]^{W_{\mathfrak{t}}} = |W^1|$ and hence, since the functor gr preserves codimension, so is I_ρ . $\square$

PROOF OF THEOREM 1.4. We have seen that the map $\alpha : U(\mathfrak{t})^{\mathfrak{t}} \rightarrow \text{Cl}(\mathfrak{p})^{\mathfrak{t}}$ becomes ev_ρ under identifications $U(\mathfrak{t})^{\mathfrak{t}} = \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{t}}}$ and $\text{Cl}(\mathfrak{p})^{\mathfrak{t}} = \text{Pr}(S)$. By Lemma 2.2, ev_ρ is onto, and by Lemma 2.3, its kernel is exactly I_ρ . $\square$

The above discussion can be summarized by the following commutative diagram, with bottom row exact:

$$\begin{array}{ccccccc} U(\mathfrak{t})^{\mathfrak{t}} & \xrightarrow{\alpha} & \text{Cl}(\mathfrak{p})^{\mathfrak{t}} & & & & \\ \downarrow \text{hc} & & \downarrow \cong & & & & \\ 0 & \longrightarrow & I_\rho & \longrightarrow & \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{t}}} & \xrightarrow{\text{ev}_\rho} & \text{Pr}(S) \longrightarrow 0 \end{array}$$

REMARK 2.4. The above proof of Theorem 1.4 works equally well for an unequal rank symmetric pair $(\mathfrak{g}, \mathfrak{k})$. In this case, the statement is that α (or ev_ρ) maps $U(\mathfrak{k})^\mathfrak{k} \cong \mathbb{C}[\mathfrak{t}^*]^{W_\mathfrak{k}}$ onto $\text{Pr}(S)$, which is however no longer all of $\text{Cl}(\mathfrak{p})^\mathfrak{k}$, and that the kernel of α is generated by the polynomials invariant under the Weyl group of the pair $(\mathfrak{g}, \mathfrak{t})$. Here $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$, which is no longer a Cartan subalgebra of $\mathfrak{g}$.

In the more general case when $\mathfrak{k}$ is a reductive quadratic subalgebra of $\mathfrak{g}$ which is not necessarily symmetric, the above proof still shows that α maps $U(\mathfrak{k})^\mathfrak{k}$ onto $\text{Pr}(S)$, but it is more difficult to describe its kernel.

PROOF OF THEOREM 1.5. Recall that we view $\text{Cl}(\mathfrak{p})$ and $\bigwedge \mathfrak{p}$ as the same space with two multiplications, $\bullet$ and $\wedge$. Note that for elements a and b of this space such that a (resp. b) is in the k th (resp. l th) filtered piece but not the $(k-1)$ st (resp. $(l-1)$ st) filtered piece, we have

$$a_{[k]} \wedge b_{[l]} = (a \bullet b)_{[k+l]}.$$

Also, $\text{gr } \alpha(s_\lambda) = \alpha(s_\lambda)_{[2|\lambda|]}$, hence

$$\text{gr } \alpha(s_\lambda) \wedge \text{gr } \alpha(s_\mu) = \alpha(s_\lambda)_{[2|\lambda|]} \wedge \alpha(s_\mu)_{[2|\mu|]} = (\alpha(s_\lambda) \bullet \alpha(s_\mu))_{[2|\lambda|+2|\mu|]}.$$

□

3. EXAMPLES

We highlight the application of Theorem 1.5 to the calculation of Littlewood-Richardson coefficients below with a couple of examples.

EXAMPLE 3.1. Let $\mathfrak{g} = \mathfrak{gl}_4$ and $\mathfrak{k} = \mathfrak{gl}_2 \oplus \mathfrak{gl}_2$, which corresponds to $\text{Gr}(2, \mathbb{C}^4)$. The set $\sigma\rho$ for $\sigma \in W^1$ is

$$\begin{array}{lll} (\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}), & (\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}), & (\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}), \\ (\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}), & (\frac{1}{2}, -\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}), & (-\frac{1}{2}, -\frac{3}{2}, \frac{3}{2}, \frac{1}{2}). \end{array}$$

Let $\{\text{pr}_i : i = 1, \dots, 6\} = \{\text{pr}_\sigma : \sigma \in W^1\}$ be the basis of orthogonal projections in $\text{Pr}(S)$. By (2.1),

$$\alpha(s_\lambda) = \text{ev}_\rho(s_\lambda) = \sum_{\sigma \in W^1} s_\lambda(\sigma\rho) \text{pr}_\sigma, \quad s_\lambda \in \mathcal{P}_{k \times (n-k)}.$$

For simplicity of notation, for the remainder of these examples we will replace $\alpha(s_\lambda)$ by s_λ . Expressing s_λ as a vector in $\mathbb{C}^6 \cong \text{Pr}(S)$

$$\begin{array}{ll} s_{(0,0)} = (1, 1, 1, 1, 1, 1), & s_{(1,0)} = (2, 1, 0, 0, -1, -2), \\ s_{(2,0)} = (\frac{13}{4}, \frac{7}{4}, \frac{9}{4}, \frac{1}{4}, \frac{7}{4}, \frac{13}{4}), & s_{(1,1)} = (\frac{3}{4} - \frac{3}{4}, -\frac{9}{4}, -\frac{1}{4}, -\frac{3}{4}, \frac{3}{4}), \\ s_{(2,1)} = (\frac{3}{2}, -\frac{3}{4}, 0, 0, \frac{3}{4}, -\frac{3}{2}), & s_{(2,2)} = (\frac{9}{16}, \frac{9}{16}, \frac{81}{16}, \frac{1}{16}, \frac{9}{16}, \frac{9}{16}). \end{array}$$

The change of basis from s_λ to pr_i and its inverse are given by the following matrices, which can be computed using Algorithm 1,

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 & -1 & -2 \\ \frac{13}{4} & \frac{7}{4} & \frac{9}{4} & \frac{1}{4} & \frac{7}{4} & \frac{13}{4} \\ \frac{3}{2} & \frac{-3}{4} & 0 & 0 & \frac{3}{4} & \frac{-3}{2} \\ \frac{9}{16} & \frac{9}{16} & \frac{81}{16} & \frac{1}{16} & \frac{9}{16} & \frac{9}{16} \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} \frac{3}{64} & \frac{1}{8} & \frac{1}{16} & \frac{13}{48} & \frac{1}{6} & \frac{1}{12} \\ \frac{-3}{16} & \frac{1}{4} & \frac{1}{4} & \frac{-7}{12} & \frac{-1}{3} & \frac{-1}{3} \\ \frac{1}{16} & 0 & \frac{-1}{16} & \frac{1}{16} & 0 & \frac{1}{4} \\ \frac{-3}{16} & \frac{-1}{4} & \frac{1}{4} & \frac{-7}{12} & \frac{1}{3} & \frac{-1}{3} \\ \frac{3}{64} & \frac{-1}{8} & \frac{1}{16} & \frac{13}{48} & \frac{-1}{6} & \frac{1}{12} \end{bmatrix}.$$

The multiplication $\bullet$ in $\text{Cl}(\mathfrak{p})^\mathfrak{k}$ is given by expressing s_λ and s_μ as vectors in $\mathbb{C}^6 = \text{Span}_{\mathbb{C}}(\text{pr}_i : i = 1, \dots, 6)$ using M , then multiplying with the Hadamard product. The resulting vector is then expressed as a combination of s_λ by applying M^{-1} . Explicitly $s_\lambda \bullet s_\mu = M^{-1}(M(s_\lambda) \bullet_H M(s_\mu))$, where $\bullet_H$ is the Hadamard (component-wise) product. This gives the new multiplication below: (The terms vanishing in the associated graded algebra $(\wedge \mathfrak{p})^\mathfrak{k}$, equivalently those vanishing under the projection $P_{[|\lambda|+|\mu|]}$, are underlined.)

$$\begin{aligned} s_{(1,0)} \bullet s_{(1,0)} &= s_{(1,1)} + s_{(2,0)}, & s_{(1,0)} \bullet s_{(2,0)} &= \frac{5}{2} s_{(1,0)} + s_{(2,1)}, \\ s_{(1,0)} \bullet s_{(1,1)} &= s_{(2,1)}, & s_{(1,0)} \bullet s_{(2,1)} &= \frac{9}{16} s_{(0,0)} + \frac{5}{2} s_{(1,1)} + s_{(2,2)}, \\ s_{(1,0)} \bullet s_{(2,2)} &= \frac{9}{16} s_{(1,0)}, & s_{(2,0)} \bullet s_{(2,0)} &= \frac{5}{2} s_{(1,1)} + \frac{5}{2} s_{(2,0)} + s_{(2,2)}, \\ s_{(1,1)} \bullet s_{(2,0)} &= \frac{9}{16} s_{(0,0)} + \frac{5}{2} s_{(1,1)}, & s_{(2,0)} \bullet s_{(2,1)} &= \frac{9}{16} s_{(1,0)} + \frac{5}{2} s_{(2,1)}, \\ s_{(2,0)} \bullet s_{(2,2)} &= \frac{9}{16} s_{(1,1)} + \frac{5}{2} s_{(2,2)}, & s_{(1,1)} \bullet s_{(1,1)} &= s_{(2,2)}, \\ s_{(1,1)} \bullet s_{(2,1)} &= \frac{9}{16} s_{(1,0)}, & s_{(1,1)} \bullet s_{(2,2)} &= \frac{9}{16} s_{(2,0)} - \frac{5}{2} s_{(2,2)}, \\ s_{(2,1)} \bullet s_{(2,1)} &= \frac{9}{16} s_{(1,1)} + \frac{9}{16} s_{(2,0)}, & s_{(2,1)} \bullet s_{(2,2)} &= \frac{9}{16} s_{(2,1)}, \\ s_{(2,2)} \bullet s_{(2,2)} &= \frac{81}{256} s_{(0,0)} + \frac{45}{32} s_{(1,1)} \\ &\quad - \frac{45}{32} s_{(2,0)} + \frac{25}{4} s_{(2,2)}. \end{aligned}$$

EXAMPLE 3.2. In this example $\mathfrak{g} = \mathfrak{gl}_5$ and $\mathfrak{k} = \mathfrak{gl}_3 \oplus \mathfrak{gl}_2$. Then $|W^1| = 10$. The change of bases matrices are:

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 & 1 & 0 & -1 & -1 & -2 & -3 \\ 7 & 4 & 3 & 4 & 1 & 1 & 3 & 1 & 4 & 7 \\ 2 & 0 & -2 & -4 & 0 & -1 & -2 & 0 & 0 & 2 \\ 15 & 8 & 5 & 0 & 1 & 0 & -5 & -1 & -8 & -15 \\ 6 & 0 & -2 & 0 & 0 & 0 & 2 & 0 & 0 & -6 \\ 14 & 0 & -6 & -16 & 0 & -1 & -6 & 0 & 0 & 14 \\ 4 & 0 & 4 & 16 & 0 & 1 & 4 & 0 & 0 & 4 \\ 12 & 0 & 4 & 0 & 0 & 0 & -4 & 0 & 0 & -12 \\ 8 & 0 & -8 & -64 & 0 & -1 & -8 & 0 & 0 & 8 \end{bmatrix},$$

$$M^{-1} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{36} & 0 & \frac{1}{24} & \frac{1}{72} & \frac{7}{144} & \frac{1}{48} & \frac{1}{144} \\ \frac{-1}{6} & \frac{-1}{12} & \frac{1}{6} & \frac{-5}{24} & \frac{1}{12} & 0 & \frac{1}{24} & \frac{-5}{24} & \frac{-1}{12} & \frac{-1}{24} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{-1}{8} & \frac{-1}{8} & \frac{3}{16} & \frac{1}{16} & \frac{1}{16} \\ 0 & 0 & 0 & \frac{-1}{36} & 0 & 0 & \frac{1}{36} & \frac{-1}{36} & 0 & \frac{-1}{36} \\ \frac{2}{3} & \frac{2}{3} & \frac{-1}{6} & \frac{5}{6} & \frac{-1}{6} & 0 & \frac{-1}{6} & \frac{5}{24} & \frac{1}{24} & \frac{1}{24} \\ 0 & 0 & 0 & \frac{-16}{9} & 0 & 0 & \frac{4}{9} & \frac{-4}{9} & 0 & \frac{-1}{9} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{8} & \frac{-1}{8} & \frac{3}{16} & \frac{-1}{16} & \frac{1}{16} \\ \frac{2}{3} & \frac{-2}{3} & \frac{-1}{6} & \frac{4}{6} & \frac{1}{6} & 0 & \frac{-1}{6} & \frac{16}{24} & \frac{-1}{24} & \frac{1}{24} \\ \frac{-1}{6} & \frac{1}{12} & \frac{1}{6} & \frac{-5}{24} & \frac{-1}{12} & 0 & \frac{1}{24} & \frac{-5}{24} & \frac{1}{12} & \frac{-1}{24} \\ 0 & 0 & 0 & \frac{1}{36} & 0 & \frac{-1}{24} & \frac{1}{72} & \frac{7}{144} & \frac{-1}{48} & \frac{1}{144} \end{bmatrix}.$$

Again the new multiplication is calculated by expressing s_λ and s_μ as vectors in $\mathbb{C}^{10}$, performing the Hadamard product then applying the change of basis from $\{\text{pr}_i\}$ to $\{s_\lambda\}$. (Again the terms vanishing in the associated graded algebra $(\bigwedge \mathfrak{p})^\natural$ are underlined.)

$$\begin{aligned} s_{(1,0)} \bullet s_{(1,0)} &= s_{(1,1)} + s_{(2,0)}, & s_{(1,0)} \bullet s_{(1,1)} &= s_{(2,1)}, \\ s_{(1,0)} \bullet s_{(2,0)} &= s_{(2,1)} + s_{(3,0)}, & s_{(1,0)} \bullet s_{(2,1)} &= s_{(2,2)} + s_{(3,1)}, \\ s_{(1,0)} \bullet s_{(3,0)} &= s_{(3,1)} - \underline{4s_{(0,0)} + 5s_{(2,0)}}, & s_{(1,0)} \bullet s_{(2,2)} &= s_{(3,2)}, \\ s_{(1,0)} \bullet s_{(3,1)} &= s_{(3,2)} + \underline{5s_{(2,1)}}, & s_{(1,0)} \bullet s_{(3,3)} &= \underline{4s_{(2,1)}}, \\ s_{(1,0)} \bullet s_{(3,2)} &= s_{(3,3)} + \underline{4s_{(1,1)} + 5s_{(2,2)}}, & s_{(1,1)} \bullet s_{(2,0)} &= s_{(3,1)}, \\ s_{(2,0)} \bullet s_{(2,0)} &= s_{(2,2)} + s_{(3,1)} - \underline{4s_{(0,0)} + 5s_{(2,0)}}, & s_{(2,0)} \bullet s_{(2,1)} &= s_{(3,2)} + \underline{5s_{(2,1)}}, \\ s_{(2,0)} \bullet s_{(3,0)} &= s_{(3,2)} - \underline{4s_{(1,0)} + 5s_{(2,1)} + 5s_{(3,0)}}, & s_{(2,0)} \bullet s_{(2,2)} &= \underline{4s_{(1,1)} + 5s_{(2,2)}}, \\ s_{(2,0)} \bullet s_{(3,1)} &= s_{(3,3)} + \underline{5s_{(2,2)} + 5s_{(3,1)}}, & s_{(2,0)} \bullet s_{(3,3)} &= \underline{4s_{(2,2)} + 5s_{(3,3)}}, \\ s_{(2,0)} \bullet s_{(3,2)} &= \underline{4s_{(2,1)} + 5s_{(3,2)}}, \\ s_{(1,1)} \bullet s_{(1,1)} &= s_{(2,2)}, & s_{(1,1)} \bullet s_{(2,1)} &= s_{(3,2)}, \\ s_{(1,1)} \bullet s_{(3,0)} &= \underline{5s_{(2,1)}}, & s_{(1,1)} \bullet s_{(2,2)} &= s_{(3,3)}, \\ s_{(1,1)} \bullet s_{(3,1)} &= \underline{4s_{(1,1)} + 5s_{(2,2)}}, & s_{(1,1)} \bullet s_{(3,3)} &= \underline{4s_{(3,1)} - 5s_{(3,3)}}, \\ s_{(1,1)} \bullet s_{(3,2)} &= \underline{4s_{(2,1)}}, & s_{(2,1)} \bullet s_{(3,0)} &= \underline{5s_{(2,2)} + 5s_{(3,1)}}, \\ s_{(3,0)} \bullet s_{(3,0)} &= s_{(3,3)} - \underline{20s_{(0,0)} + 21s_{(2,0)} + 5s_{(2,2)} + 5s_{(3,1)}}, \\ s_{(3,0)} \bullet s_{(3,1)} &= \underline{25s_{(2,1)} + 5s_{(3,2)}}, & s_{(2,2)} \bullet s_{(3,0)} &= \underline{5s_{(3,2)}}, \\ s_{(3,0)} \bullet s_{(3,2)} &= \underline{20s_{(1,1)} + 25s_{(2,2)} + 5s_{(3,3)}}, & s_{(3,0)} \bullet s_{(3,3)} &= \underline{20s_{(2,1)}}, \\ s_{(2,1)} \bullet s_{(2,1)} &= s_{(3,3)} + \underline{4s_{(1,1)} + 5s_{(2,2)}}, & s_{(2,1)} \bullet s_{(3,1)} &= \underline{4s_{(2,1)} + 5s_{(3,2)}}, \\ s_{(2,1)} \bullet s_{(2,2)} &= \underline{4s_{(2,2)}}, & s_{(2,1)} \bullet s_{(3,2)} &= \underline{4s_{(2,2)} + 4s_{(3,1)}}, \\ s_{(2,1)} \bullet s_{(3,3)} &= \underline{4s_{(3,2)}}, & s_{(3,1)} \bullet s_{(3,1)} &= \underline{20s_{(1,1)} + 25s_{(2,2)} + 4s_{(3,1)}}, \\ s_{(2,2)} \bullet s_{(3,1)} &= \underline{4s_{(2,2)} + 5s_{(3,3)}}, \\ s_{(3,1)} \bullet s_{(3,2)} &= \underline{20s_{(2,1)} + 4s_{(3,2)}}, \end{aligned}$$

$$\begin{aligned}
s_{(3,1)} \bullet s_{(3,3)} &= \underline{20s_{(3,1)}} - \underline{21s_{(3,3)}}, \\
s_{(2,2)} \bullet s_{(2,2)} &= \underline{4s_{(3,1)}} - \underline{5s_{(3,3)}}, \\
s_{(2,2)} \bullet s_{(3,2)} &= \underline{4s_{(3,2)}}, \\
s_{(2,2)} \bullet s_{(3,3)} &= \underline{16s_{(1,1)}} + \underline{20s_{(2,2)}} - \underline{20s_{(3,1)}} + \underline{25s_{(3,3)}}, \\
s_{(3,2)} \bullet s_{(3,2)} &= \underline{16s_{(1,1)}} + \underline{20s_{(2,2)}} + \underline{4s_{(3,3)}}, \\
s_{(3,2)} \bullet s_{(3,3)} &= \underline{16s_{(2,1)}}, \\
s_{(3,3)} \bullet s_{(3,3)} &= \underline{-80s_{(1,1)}} - \underline{84s_{(2,2)}} + \underline{100s_{(3,1)}} - \underline{105s_{(3,3)}}.
\end{aligned}$$

In both examples, as well as in general, deleting the underlined terms gives the Littlewood-Richardson coefficients.

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