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A LOWER BOUND ON THE RELATIVE L^1-NORM OF POINCARÉ SERIES

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Abstract. In this paper, we adapt methods from Muić's theory of integral non-vanishing criteria for Poincaré series to establish a lower bound on the relative L^1-norm of a Poincaré series constructed from a vector-valued L^1-function φ on a unimodular locally compact Hausdorff group G. In the case when $G = \mathrm{Sp}_{2n}(\mathbb{R})$, we transfer this bound to the context of Poincaré series on the Siegel upper half-space $\mathcal{H}_n$ in a classical way, deducing results on the classical Poincaré series and on Siegel cusp forms given by Poincaré series of polynomial type.

1. Introduction

Let G be a unimodular, second-countable, locally compact Hausdorff group, and let Λ and Γ be discrete subgroups of G such that $\Lambda \subseteq \Gamma$. Let V be a finite-dimensional complex Banach space. Given $\varphi \in L^1(\Lambda\backslash G, V) \setminus \{0\}$, the Poincaré series

$$\left(P_{\Lambda\backslash\Gamma}\varphi\right)(g) = \sum_{\gamma\in\Lambda\backslash\Gamma} \varphi(\gamma g)$$

converges absolutely almost everywhere on G and defines an element of $L^1(\Gamma\backslash G, V)$. For appropriate choices of G and Γ, such series are an important tool for constructing automorphic forms on $\Gamma\backslash G$ (see, e.g., [1, §5]). Consequently, they are of great importance in number theory and in harmonic analysis on $\Gamma\backslash G$.

Studying certain basic properties of a given Poincaré series $P_{\Lambda\backslash\Gamma}\varphi$, such as deciding whether it vanishes identically, is often highly non-trivial. One

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approach to establishing non-vanishing is given by the theory of integral non-vanishing criteria for Poincaré series, developed by G. Muić in [12] in the scalar-valued setting and subsequently extended to the vector-valued setting in [20]. In this paper, we adapt the techniques from this theory to study the L^p-norms of Poincaré series, which are of considerable interest in contemporary research. Indeed, for various families of cuspidal automorphic and modular forms, the behaviour of L^p-norms has been widely investigated by numerous researchers and a variety of methods, especially in recent years [2,6,10,15].

Our main result is the following theorem, giving a lower bound on the relative L^1-norm of the Poincaré series $P_{\Lambda\backslash\Gamma}\varphi$,

$$\frac{\left\|P_{\Lambda\backslash\Gamma}\varphi\right\|_{L^1(\Gamma\backslash G,V)}}{\|\varphi\|_{L^1(\Lambda\backslash G,V)}},$$

which is known to take values in $[0,1]$ (see (2.2)).

Theorem 1.1. *Let G be a unimodular, second-countable, locally compact Hausdorff group, and let dg be a Haar measure on G. Let Λ and Γ be discrete subgroups of G such that $\Lambda\subseteq\Gamma$. Let C be a Borel measurable subset of G such that*

$$CC^{-1}\cap\Gamma\subseteq\Lambda.$$

Let V be a finite-dimensional complex Banach space, and let $\varphi\in L^1(\Lambda\backslash G,V)\setminus\{0\}$. Then, denoting by $(\Lambda C)^c$ the complement of the set ΛC in G, we have

$$\frac{\left\|P_{\Lambda\backslash\Gamma}\varphi\right\|_{L^1(\Gamma\backslash G,V)}}{\|\varphi\|_{L^1(\Lambda\backslash G,V)}}\geq 1-2\cdot\frac{\int_{\Lambda\backslash(\Lambda C)^c}\|\varphi(g)\|_V\,dg}{\|\varphi\|_{L^1(\Lambda\backslash G,V)}}.$$

As a corollary of Theorem 1.1, for every $p\in\mathbb{R}_{\geq 1}\cup\{\infty\}$, we obtain the following lower bound on the L^p-norm of the Poincaré series $P_{\Lambda\backslash\Gamma}\varphi$.

Corollary 1.2. *Let G, dg, Λ, Γ, C, and V be as in Theorem 1.1. Suppose that $\operatorname{vol}(\Gamma\backslash G)<\infty$. Let $p,q\in\mathbb{R}_{\geq 1}\cup\{\infty\}$ be such that $\frac{1}{p}+\frac{1}{q}=1$. Let $\varphi\in L^1(\Lambda\backslash G,V)$ be such that $P_{\Lambda\backslash\Gamma}\varphi\in L^p(\Gamma\backslash G,V)$. Then, we have*

$$\begin{aligned}&\left\|P_{\Lambda\backslash\Gamma}\varphi\right\|_{L^p(\Gamma\backslash G,V)}\\&\qquad\geq\operatorname{vol}(\Gamma\backslash G)^{-\frac{1}{q}}\left(\int_{\Lambda\backslash\Lambda C}\|\varphi(g)\|_V\,dg-\int_{\Lambda\backslash(\Lambda C)^c}\|\varphi(g)\|_V\,dg\right).\end{aligned}$$

In Sections 3 and 4, we turn our attention to the case when $G=\mathrm{Sp}_{2n}(\mathbb{R})$ and, by applying the inverse of the classical lift of Siegel cusp forms to cuspidal automorphic forms on $\mathrm{Sp}_{2n}(\mathbb{R})$ (see (3.4); cf. [17, §4]), prove analogous estimates for the standardly defined Poincaré series on the Siegel upper half-space $\mathcal{H}_n$.

In Sections 5 and 7, we apply the results of Section 3 to the following two families of (Siegel) cusp forms:

(1) A family of Siegel cusp forms on $\mathcal{H}_n$ given by Poincaré series of polynomial type that was studied in [21]. These Siegel cusp forms correspond, via the classical lift, to Poincaré series of certain K-finite matrix coefficients of integrable antiholomorphic discrete series representations of $\mathrm{Sp}_{2n}(\mathbb{R})$. The representation-theoretic terminology used here is standard; see, e.g., [8] and [16].
(2) The family of classical Poincaré series, which are certain holomorphic cusp forms on the upper half-plane $\mathcal{H}_1$ defined by Poincaré series of exponential type. The classical Poincaré series are famous for encoding the Fourier coefficients of holomorphic cusp forms via the Riesz representation theorem (see (7.3)).

We provide illustrative numerical evaluations of the obtained lower bounds, computed using Wolfram Mathematica 14.3 [18]; the code can be found in [23].

Notation. Throughout the paper, given a set S, we denote by id_S the identity map on S. Given a Banach space E, we write $\|\cdot\|_E$ for its norm. Given a topological space S, we denote by $C_c(S)$ the space of compactly supported, continuous functions $S \to \mathbb{C}$. Let $i \in \mathbb{C}$ be the imaginary unit. Given a matrix $z = x + iy \in M_n(\mathbb{C})$ with $x, y \in M_n(\mathbb{R})$, we write $\Re(z) = x$ and $\Im(z) = y$ and denote by $z^\top$ the transpose of z.

2. Estimates for Poincaré series on locally compact Hausdorff groups

Let G be a unimodular, second-countable, locally compact Hausdorff group, and let dg be a Haar measure on G. Given a discrete subgroup Γ of G, we equip the quotient space $\Gamma\backslash G$ with the unique right-G-invariant Radon measure such that

$$\int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} \varphi(\gamma g)\, dg = \int_G \varphi(g)\, dg, \quad \varphi \in C_c(G)$$

(see, e.g., [3, §2.6]). If Λ is a subgroup of Γ, then we have

$$\int_{\Gamma\backslash G} \sum_{\gamma\in\Lambda\backslash\Gamma} \varphi(\gamma g)\, dg = \int_{\Lambda\backslash G} \varphi(g)\, dg, \quad \varphi \in C_c(\Lambda\backslash G).$$

Let V be a finite-dimensional complex Banach space. Given $p \in \mathbb{R}_{\geq 1}$, let $L^p(\Gamma\backslash G, V)$ (resp., $L^\infty(\Gamma\backslash G, V)$) denote the Banach space of equivalence classes (under equality almost everywhere) of measurable functions $\varphi : \Gamma\backslash G \to V$ such that the function $\|\varphi(\,\cdot\,)\|_V : \Gamma\backslash G \to \mathbb{R}$ is p-integrable (resp., essentially bounded). The Banach space norm on $L^p(\Gamma\backslash G, V)$ is given by

$$\|\varphi\|_{L^p(\Gamma\backslash G,V)} = \left(\int_{\Gamma\backslash G} \|\varphi(g)\|_V^p \, dg\right)^{\frac{1}{p}}, \quad \varphi \in L^p(\Gamma\backslash G, V),$$

whereas the Banach space norm on $L^\infty(\Gamma\backslash G, V)$ is given by

$$\|\varphi\|_{L^\infty(\Gamma\backslash G,V)} = \operatorname*{ess\,sup}_{g\in\Gamma\backslash G} \|\varphi(g)\|_V\,, \quad \varphi \in L^\infty(\Gamma\backslash G, V).$$

The Banach spaces $L^p(G, V)$ and $L^\infty(G, V)$ are defined analogously.

Let Λ be a subgroup of Γ. For every $\varphi \in L^1(\Lambda\backslash G, V)$, the Poincaré series

$$\left(P_{\Lambda\backslash\Gamma}\varphi\right)(g) = \sum_{\gamma\in\Lambda\backslash\Gamma} \varphi(\gamma g) \tag{2.1}$$

converges absolutely almost everywhere on G and defines an element of $L^1(\Gamma\backslash G, V)$. Moreover, we have the well-known estimate

(2.2)

$$\begin{aligned}\left\|P_{\Lambda\backslash\Gamma}\varphi\right\|_{L^1(\Gamma\backslash G,V)} &= \int_{\Gamma\backslash G} \left\|\sum_{\gamma\in\Lambda\backslash\Gamma}\varphi(\gamma g)\right\|_V \,dg \le \int_{\Gamma\backslash G} \sum_{\gamma\in\Lambda\backslash\Gamma} \|\varphi(\gamma g)\|_V \,dg\\ &= \int_{\Lambda\backslash G} \|\varphi(g)\|_V \,dg = \|\varphi\|_{L^1(\Lambda\backslash G,V)}\,,\end{aligned}$$

where equality holds if $\varphi(\Lambda\backslash G) \subseteq \mathbb{R}_{\ge 0}v$ for some $v \in V$.

We recall that, given complex normed spaces $E \neq 0$ and F, the operator norm $\|A\|_{\mathrm{op}}$ of a linear operator $A : E \to F$ is given by

$$\|A\|_{\mathrm{op}} = \sup_{v\in E\setminus\{0\}} \frac{\|Av\|_F}{\|v\|_E}.$$

Our preceding observation implies that if $V \neq 0$, then the operator norm $\left\|P_{\Lambda\backslash\Gamma}\right\|_{\mathrm{op}}$ of the Poincaré series operator $P_{\Lambda\backslash\Gamma} : L^1(\Lambda\backslash G, V) \to L^1(\Gamma\backslash G, V)$, given by

$$\left\|P_{\Lambda\backslash\Gamma}\right\|_{\mathrm{op}} = \sup_{\varphi\in L^1(\Lambda\backslash G,V)\setminus\{0\}} \frac{\left\|P_{\Lambda\backslash\Gamma}\varphi\right\|_{L^1(\Gamma\backslash G,V)}}{\|\varphi\|_{L^1(\Lambda\backslash G,V)}},$$

equals 1.

The main result of this paper, Theorem 1.1, provides a lower bound on the relative L^1-norm

$$\frac{\left\|P_{\Lambda\backslash\Gamma}\varphi\right\|_{L^1(\Gamma\backslash G,V)}}{\|\varphi\|_{L^1(\Lambda\backslash G,V)}}$$

of the Poincaré series $P_{\Lambda\backslash\Gamma}\varphi$, where $\varphi \in L^1(\Lambda\backslash G, V) \setminus \{0\}$. For the reader's convenience, we restate it here as Theorem 2.1.

Theorem 2.1. *Let G be a unimodular, second-countable, locally compact Hausdorff group, and let dg be a Haar measure on G. Let Λ and Γ be discrete subgroups of G such that $\Lambda \subseteq \Gamma$. Let C be a Borel measurable subset of G such that*

$$CC^{-1} \cap \Gamma \subseteq \Lambda. \tag{2.3}$$

Let V be a finite-dimensional complex Banach space, and let $\varphi \in L^1(\Lambda\backslash G, V)\setminus\{0\}$. Then, denoting by $(\Lambda C)^c$ the complement of the set ΛC in G, we have

$$\frac{\left\|P_{\Lambda\backslash\Gamma}\varphi\right\|_{L^1(\Gamma\backslash G,V)}}{\|\varphi\|_{L^1(\Lambda\backslash G,V)}} \geq 1 - 2\cdot\frac{\int_{\Lambda\backslash(\Lambda C)^c}\|\varphi(g)\|_V\,dg}{\|\varphi\|_{L^1(\Lambda\backslash G,V)}}. \tag{2.4}$$

PROOF. We follow the idea of proof of the version [19, Theorem 1] of Muić's integral non-vanishing criterion [12, Theorem 4.1] for Poincaré series.

Given a set $S \subseteq \Lambda\backslash G$, let $\mathbf{1}_S : \Lambda\backslash G \to \mathbb{C}$ be the indicator function of S. As explained in detail in [19, proof of Theorem 1], the assumption (2.3) implies that for every $g \in G$, we have

$$\#\left\{\gamma \in \Lambda\backslash\Gamma : \mathbf{1}_{\Lambda\backslash\Lambda C}(\gamma g) \neq 0\right\} \leq 1. \tag{2.5}$$

Hence, we have

$$\begin{aligned}
\left\|P_{\Lambda\backslash\Gamma}(\mathbf{1}_{\Lambda\backslash\Lambda C}\cdot\varphi)\right\|_{L^1(\Gamma\backslash G,V)} &= \int_{\Gamma\backslash G}\left\|\sum_{\gamma\in\Lambda\backslash\Gamma}\mathbf{1}_{\Lambda\backslash\Lambda C}(\gamma g)\,\varphi(\gamma g)\right\|_V dg\\
&\overset{(2.5)}{=} \int_{\Gamma\backslash G}\sum_{\gamma\in\Lambda\backslash\Gamma}\mathbf{1}_{\Lambda\backslash\Lambda C}(\gamma g)\,\|\varphi(\gamma g)\|_V\,dg\\
&= \int_{\Lambda\backslash G}\mathbf{1}_{\Lambda\backslash\Lambda C}(g)\,\|\varphi(g)\|_V\,dg\\
&= \int_{\Lambda\backslash\Lambda C}\|\varphi(g)\|_V\,dg.
\end{aligned} \tag{2.6}$$

On the other hand, we have

$$\begin{aligned}
\left\|P_{\Lambda\backslash\Gamma}(\mathbf{1}_{\Lambda\backslash(\Lambda C)^c}\cdot\varphi)\right\|_{L^1(\Gamma\backslash G,V)} &\overset{(2.2)}{\leq} \left\|\mathbf{1}_{\Lambda\backslash(\Lambda C)^c}\cdot\varphi\right\|_{L^1(\Lambda\backslash G,V)}\\
&= \int_{\Lambda\backslash(\Lambda C)^c}\|\varphi(g)\|_V\,dg.
\end{aligned} \tag{2.7}$$

Thus, we obtain

$$\begin{aligned}
\left\|P_{\Lambda\backslash\Gamma}\varphi\right\|_{L^1(\Gamma\backslash G,V)} &= \left\|P_{\Lambda\backslash\Gamma}(\mathbf{1}_{\Lambda\backslash\Lambda C}\cdot\varphi) + P_{\Lambda\backslash\Gamma}(\mathbf{1}_{\Lambda\backslash(\Lambda C)^c}\cdot\varphi)\right\|_{L^1(\Gamma\backslash G,V)}\\
&\geq \left\|P_{\Lambda\backslash\Gamma}(\mathbf{1}_{\Lambda\backslash\Lambda C}\cdot\varphi)\right\|_{L^1(\Gamma\backslash G,V)} - \left\|P_{\Lambda\backslash\Gamma}(\mathbf{1}_{\Lambda\backslash(\Lambda C)^c}\cdot\varphi)\right\|_{L^1(\Gamma\backslash G,V)}\\
&\underset{(2.7)}{\overset{(2.6)}{\geq}} \int_{\Lambda\backslash\Lambda C}\|\varphi(g)\|_V\,dg - \int_{\Lambda\backslash(\Lambda C)^c}\|\varphi(g)\|_V\,dg\\
&= \|\varphi\|_{L^1(\Lambda\backslash G,V)} - 2\int_{\Lambda\backslash(\Lambda C)^c}\|\varphi(g)\|_V\,dg.
\end{aligned} \tag{2.8}$$

Dividing this estimate by $\|\varphi\|_{L^1(\Lambda\backslash G,V)}$, we obtain (2.4). ∎

Remark 2.2. (i) Since the left-hand side of estimate (2.4) is obviously nonnegative, the estimate is only of interest when its right-hand side is strictly positive, i.e., by the last equality in (2.8), when we have

$$\int_{\Lambda\backslash\Lambda C}\|\varphi(g)\|_V\,dg>\int_{\Lambda\backslash(\Lambda C)^c}\|\varphi(g)\|_V\,dg. \tag{2.9}$$

(ii) By all but the final equality in (2.8), the inequality (2.9) presents, when combined with the remaining assumptions of Theorem 2.1, a sufficient condition for $\left\|P_{\Lambda\backslash\Gamma}\varphi\right\|_{L^1(\Gamma\backslash G,V)}$ to be strictly positive, i.e., for $P_{\Lambda\backslash\Gamma}\varphi$ not to vanish identically. This is exactly the content of (the vector-valued, Λ-relative version of) Muić's integral non-vanishing criterion [12, Theorem 4.1].

In the theory of automorphic forms, many prominent examples of cuspidal automorphic forms are given by Poincaré series of the form (2.1) that belong to

$$L^1(\Gamma\backslash G,V)\cap L^\infty(\Gamma\backslash G,V)=\bigcap_{p\in\mathbb{R}_{\geq1}\cup\{\infty\}}L^p(\Gamma\backslash G,V).$$

Examples of such scalar-valued cuspidal automorphic forms are the Poincaré series on $\mathrm{SL}_2(\mathbb{R})$ studied in [13, Lemma 3-1]. Their vector-valued analogues were studied in [21], arising as classical lifts of the vector-valued Siegel Poincaré series of polynomial type considered in [21, Theorem 1.1]. Corollary 1.2, which we restate below, for the reader's convenience, as Corollary 2.3, provides a lower bound on the L^p-norm of such Poincaré series.

Corollary 2.3. *Let G be a unimodular, second-countable, locally compact Hausdorff group, and let dg be a Haar measure on G. Let Λ and Γ be discrete subgroups of G such that $\Lambda\subseteq\Gamma$ and $\mathrm{vol}(\Gamma\backslash G)<\infty$. Let C be a Borel measurable subset of G such that*

$$CC^{-1}\cap\Gamma\subseteq\Lambda.$$

Let $p,q\in\mathbb{R}_{\geq1}\cup\{\infty\}$ be such that $\frac{1}{p}+\frac{1}{q}=1$, and let V be a finite-dimensional complex Banach space. Let $\varphi\in L^1(\Lambda\backslash G,V)$ be such that $P_{\Lambda\backslash\Gamma}\varphi\in L^p(\Gamma\backslash G,V)$. Then, we have

$$\begin{aligned}&\left\|P_{\Lambda\backslash\Gamma}\varphi\right\|_{L^p(\Gamma\backslash G,V)}\\ &\quad\geq\mathrm{vol}(\Gamma\backslash G)^{-\frac{1}{q}}\left(\int_{\Lambda\backslash\Lambda C}\|\varphi(g)\|_V\,dg-\int_{\Lambda\backslash(\Lambda C)^c}\|\varphi(g)\|_V\,dg\right).\end{aligned}\tag{2.10}$$

Proof. Following the proof of Theorem 2.1 up to the penultimate line of (2.8), we have

$$\left\|P_{\Lambda\backslash\Gamma}\varphi\right\|_{L^1(\Gamma\backslash G,V)}\geq\int_{\Lambda\backslash\Lambda C}\|\varphi(g)\|_V\,dg-\int_{\Lambda\backslash(\Lambda C)^c}\|\varphi(g)\|_V\,dg. \tag{2.11}$$

The inequality (2.10) follows by applying to the left-hand side of (2.11) the estimate

$$\|\phi\|_{L^1(\Gamma\backslash G,V)} \leq \mathrm{vol}(\Gamma\backslash G)^{\frac{1}{q}} \ \|\phi\|_{L^p(\Gamma\backslash G,V)}\,, \quad \phi\in L^p(\Gamma\backslash G,V),$$

which holds by Hölder's inequality. ∎

3. Poincaré series on the Siegel upper half-space

Following the notation of [21] and [22, §3], let $n\in\mathbb{Z}_{>0}$ and $\omega=(\omega_1,\ldots,\omega_n)\in\mathbb{Z}^n$ with $\omega_1\geq\ldots\geq\omega_n\geq 0$. Let (ρ,V) be an irreducible polynomial representation of $\mathrm{GL}_n(\mathbb{C})$ of highest weight ω. Thus, (ρ,V) is the up to equivalence unique irreducible polynomial representation of $\mathrm{GL}_n(\mathbb{C})$ such that there exists $v_0\in V\setminus\{0\}$ such that

$$\rho(g)v_0=\left(\prod_{r=1}^n g_{r,r}^{\omega_r}\right)v_0$$

for all upper-triangular matrices $g\in\mathrm{GL}_n(\mathbb{C})$. We equip V with a Hermitian inner product $\langle\,\cdot\,,\,\cdot\,\rangle_V$ with respect to which the restriction $\rho\big|_{\mathrm{U}(n)}$ is unitary and denote by $\|\cdot\|_V$ the corresponding Hilbert space norm on V.

Let $J_n=\begin{pmatrix} & I_n\\ -I_n & \end{pmatrix}\in\mathrm{GL}_{2n}(\mathbb{C})$, where $I_n\in M_n(\mathbb{C})$ is the identity matrix. The group

$$\mathrm{Sp}_{2n}(\mathbb{R})=\left\{g\in\mathrm{GL}_{2n}(\mathbb{R}):g^\top J_n g=J_n\right\}$$

acts on the left on the Siegel upper half-space

$$\mathcal{H}_n=\left\{z\in M_n(\mathbb{C}):z^\top=z \text{ and } \Im(z)>0\right\}$$

by

$$g.z=(Az+B)(Cz+D)^{-1} \tag{3.1}$$

for all $g=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\mathrm{Sp}_{2n}(\mathbb{R})$ and $z\in\mathcal{H}_n$, and it acts on the right on the space $V^{\mathcal{H}_n}$ of functions $\mathcal{H}_n\to V$ by

$$\left(f\big|_\rho g\right)(z)=\rho(Cz+D)^{-1}f(g.z) \tag{3.2}$$

for all $f\in V^{\mathcal{H}_n}$, $g=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\mathrm{Sp}_{2n}(\mathbb{R})$, and $z\in\mathcal{H}_n$.

The stabilizer of iI_n under the action (3.1) is the maximal compact subgroup

$$K=\mathrm{Sp}_{2n}(\mathbb{R})\cap\mathrm{U}(2n)=\left\{k_{A+iB}\coloneqq\begin{pmatrix}A&B\\-B&A\end{pmatrix}\in M_{2n}(\mathbb{R}):A+iB\in\mathrm{U}(n)\right\}$$

of $\mathrm{Sp}_{2n}(\mathbb{R})$. Every $g \in \mathrm{Sp}_{2n}(\mathbb{R})$ admits a unique factorization in $\mathrm{Sp}_{2n}(\mathbb{R})$ of the form

$$g = \begin{pmatrix} I_n & x \\ 0 & I_n \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} k, \tag{3.3}$$

where $k \in K$ and $x + iy = g.(iI_n)$. The factorization (3.3) is a special case of the Iwasawa decomposition for connected semisimple Lie groups (see, e.g., [9, Ch. VI, §4]).

Let us denote the first two factors on the right-hand side of (3.3), from left to right, by n_x and a_y. Every function $f : \mathcal{H}_n \to V$ lifts to the function $\Phi_\rho f : \mathrm{Sp}_{2n}(\mathbb{R}) \to V$,

$$(\Phi_\rho f)(g) = \left(f\big|_\rho g\right)(iI_n) = \rho\left(u^\top y^{\frac{1}{2}}\right) f(x+iy), \quad g = n_x a_y k_u \in \mathrm{Sp}_{2n}(\mathbb{R}). \tag{3.4}$$

Lemma 3.1. *Let $f : \mathcal{H}_n \to V$ and $S \subseteq \mathcal{H}_n$. Let*

$$C = \{n_x a_y k : x + iy \in S,\ k \in K\}. \tag{3.5}$$

Let $\mathbf{1}_S : \mathcal{H}_n \to \mathbb{C}$ and $\mathbf{1}_C : \mathrm{Sp}_{2n}(\mathbb{R}) \to \mathbb{C}$ be the indicator functions of the sets S and C, respectively. Then, we have

$$\Phi_\rho(\mathbf{1}_S f) = \mathbf{1}_C \, \Phi_\rho f. \tag{3.6}$$

Proof. For every $g = n_x a_y k_u \in \mathrm{Sp}_{2n}(\mathbb{R})$, we have

$$(\Phi_\rho(\mathbf{1}_S f))(g) \overset{(3.4)}{=} \mathbf{1}_S(x+iy)(\Phi_\rho f)(g) \overset{(3.5)}{=} \mathbf{1}_C(g)(\Phi_\rho f)(g).$$

□

We fix the following Haar measure on $\mathrm{Sp}_{2n}(\mathbb{R})$:

$$\int_{\mathrm{Sp}_{2n}(\mathbb{R})} \varphi(g)\, dg = \int_{\mathcal{H}_n} \int_{\mathrm{U}(n)} \varphi(n_x a_y k_u)\, du\, d\mathsf{v}(x+iy), \quad \varphi \in C_c(\mathrm{Sp}_{2n}(\mathbb{R})),$$

where du is the Haar measure on $\mathrm{U}(n)$ such that $\int_{\mathrm{U}(n)} du = 1$, and v is the $\mathrm{Sp}_{2n}(\mathbb{R})$-invariant Radon measure on $\mathcal{H}_n$ given by

$$d\mathsf{v}(x+iy) = \det y^{-n-1} \prod_{1 \le r \le s \le n} dx_{r,s}\, dy_{r,s}.$$

Let Γ be a subgroup of $\mathrm{Sp}_{2n}(\mathbb{Z})$. Denoting

$$\varepsilon_\Gamma = |\Gamma \cap \{\pm I_{2n}\}|,$$

we have

$$\int_{\Gamma\backslash\mathrm{Sp}_{2n}(\mathbb{R})} \varphi(g)\, dg = \varepsilon_\Gamma^{-1} \int_{\Gamma\backslash\mathcal{H}_n} \int_{\mathrm{U}(n)} \varphi(n_x a_y k_u)\, du\, d\mathsf{v}(x+iy) \tag{3.7}$$

for all $\varphi \in C_c(\Gamma\backslash\mathrm{Sp}_{2n}(\mathbb{R}))$.

Given $p \in \mathbb{R}_{\geq 1} \cup \{\infty\}$, let $L^p_{\rho,\Gamma}(\mathcal{H}_n, V)$ denote the Banach space of measurable functions $f : \mathcal{H}_n \to V$, modulo equality almost everywhere, that have the following two properties:

(1) $f\big|_\rho \gamma = f$ for all $\gamma \in \Gamma$.

(2) $\|f\|_{L^p_{\rho,\Gamma}(\mathcal{H}_n,V)} < \infty$, where

(3.8)
$$\|f\|_{L^p_{\rho,\Gamma}(\mathcal{H}_n,V)} = \begin{cases} \left(\varepsilon_\Gamma^{-1} \int_{\Gamma\backslash\mathcal{H}_n} \left\|\rho\left(\Im(z)^{\frac{1}{2}}\right) f(z)\right\|_V^p \, d\mathsf{v}(z)\right)^{\frac{1}{p}}, & \text{if } p \in \mathbb{R}_{\geq 1} \\ \operatorname*{ess\,sup}_{z\in\mathcal{H}_n} \left\|\rho\left(\Im(z)^{\frac{1}{2}}\right) f(z)\right\|_V, & \text{if } p = \infty. \end{cases}$$

We write $L^p_\rho(\mathcal{H}_n, V) = L^p_{\rho,\{1_{\mathrm{Sp}_{2n}(\mathbb{Z})}\}}(\mathcal{H}_n, V)$. By (3.7) and (3.4), we have the following lemma.

LEMMA 3.2. *Let Γ be a subgroup of $\mathrm{Sp}_{2n}(\mathbb{Z})$. Let $p \in \mathbb{R}_{\geq 1} \cup \{\infty\}$. Then, the operator Φ_ρ restricts to an isometry $L^p_{\rho,\Gamma}(\mathcal{H}_n, V) \to L^p(\Gamma\backslash\mathrm{Sp}_{2n}(\mathbb{R}), V)$.*

Let Λ be a subgroup of Γ. One of our main results, Theorem 4.1 below, is a version of Theorem 1.1 for Poincaré series

$$P_{\Lambda\backslash\Gamma,\rho} f = \sum_{\gamma\in\Lambda\backslash\Gamma} f\big|_\rho \gamma \tag{3.9}$$

with $f \in L^1_{\rho,\Lambda}(\mathcal{H}_n, V)$. Such Poincaré series are an important tool for constructing Siegel modular forms; see Proposition 5.1 and (7.2) below.

The following Λ-relative version of [21, Lemma 6.6] is elementary.

LEMMA 3.3. *Let Λ and Γ be subgroups of $\mathrm{Sp}_{2n}(\mathbb{Z})$ such that $\Lambda \subseteq \Gamma$. Let $f : \mathcal{H}_n \to V$ be such that*

$$f\big|_\rho \lambda = f, \quad \lambda \in \Lambda.$$

Then, the Poincaré series $P_{\Lambda\backslash\Gamma,\rho} f$ given by (3.9) converges absolutely almost everywhere on $\mathcal{H}_n$ if and only if the series $P_{\Lambda\backslash\Gamma}\Phi_\rho f$ converges absolutely almost everywhere on $\mathrm{Sp}_{2n}(\mathbb{R})$. Moreover, assuming convergence, we have

$$\Phi_\rho P_{\Lambda\backslash\Gamma,\rho} f = P_{\Lambda\backslash\Gamma}\Phi_\rho f \quad \textit{a.e. on } \mathrm{Sp}_{2n}(\mathbb{R}). \tag{3.10}$$

LEMMA 3.4. *Let Λ and Γ be subgroups of $\mathrm{Sp}_{2n}(\mathbb{Z})$ such that $\Lambda \subseteq \Gamma$. Let $f \in L^1_{\rho,\Lambda}(\mathcal{H}_n, V)$. Then, we have the following:*

(i) *The Poincaré series $P_{\Lambda\backslash\Gamma,\rho} f$ converges absolutely almost everywhere on $\mathcal{H}_n$ and defines an element of $L^1_{\rho,\Gamma}(\mathcal{H}_n, V)$.*

(ii) *The Poincaré series $P_{\Lambda\backslash\Gamma}\Phi_\rho f$ converges absolutely almost everywhere on $\mathrm{Sp}_{2n}(\mathbb{R})$ and defines an element of $L^1(\Gamma\backslash\mathrm{Sp}_{2n}(\mathbb{R}), V)$.*

(iii) $\left\|P_{\Lambda\backslash\Gamma,\rho} f\right\|_{L^1_{\rho,\Gamma}(\mathcal{H}_n,V)} = \left\|P_{\Lambda\backslash\Gamma}\Phi_\rho f\right\|_{L^1(\Gamma\backslash\mathrm{Sp}_{2n}(\mathbb{R}),V)}$.

(iv) $\left\|P_{\Lambda\backslash\Gamma,\rho} f\right\|_{L^1_{\rho,\Gamma}(\mathcal{H}_n,V)} \leq \|f\|_{L^1_{\rho,\Lambda}(\mathcal{H}_n,V)}$.

PROOF. By Lemma 3.2, we have that $\Phi_\rho f \in L^1(\Lambda\backslash \mathrm{Sp}_{2n}(\mathbb{R}), V)$, hence (ii) follows from the elementary facts on Poincaré series (2.1) recalled in Section 2. The claim (i) follows from (ii) by Lemma 3.3. To prove (iii), it suffices to note that

$$\begin{aligned}\left\|P_{\Lambda\backslash\Gamma,\rho} f\right\|_{L^1_{\rho,\Gamma}(\mathcal{H}_n,V)} &\overset{\text{Lem. } 3.2}{=} \left\|\Phi_\rho P_{\Lambda\backslash\Gamma,\rho} f\right\|_{L^1(\Gamma\backslash \mathrm{Sp}_{2n}(\mathbb{R}),V)} \\ &\overset{(3.10)}{=} \left\|P_{\Lambda\backslash\Gamma} \Phi_\rho f\right\|_{L^1(\Gamma\backslash \mathrm{Sp}_{2n}(\mathbb{R}),V)}.\end{aligned}$$

Finally, the inequality (iv) holds since we have

$$\begin{aligned}\left\|P_{\Lambda\backslash\Gamma,\rho} f\right\|_{L^1_{\rho,\Gamma}(\mathcal{H}_n,V)} &\overset{\text{(iii)}}{=} \left\|P_{\Lambda\backslash\Gamma} \Phi_\rho f\right\|_{L^1(\Gamma\backslash \mathrm{Sp}_{2n}(\mathbb{R}),V)} \\ &\overset{(2.2)}{\leq} \left\|\Phi_\rho f\right\|_{L^1(\Lambda\backslash \mathrm{Sp}_{2n}(\mathbb{R}),V)} \overset{\text{Lem. } 3.2}{=} \|f\|_{L^1_{\rho,\Lambda}(\mathcal{H}_n,V)}.\end{aligned}$$

□

Since $-I_n$ is central in $\mathrm{GL}_n(\mathbb{C})$, Schur's lemma implies that $\rho(-I_n)$ is a scalar operator (see, e.g., [7, Corollary 4.30]). As $\rho(-I_n)^2 = \rho(I_n) = \mathrm{id}_V$, it follows that $\rho(-I_n) \in \{\mathrm{id}_V, -\mathrm{id}_V\}$.

LEMMA 3.5. *Let Λ and Γ be subgroups of $\mathrm{Sp}_{2n}(\mathbb{Z})$ such that $\Lambda \subseteq \Gamma$. Suppose that $-I_{2n} \in \Gamma$ and $\rho(-I_n) = -\mathrm{id}_V$. Then, for every $f \in L^1_{\rho,\Lambda}(\mathcal{H}_n, V)$, we have that $P_{\Lambda\backslash\Gamma,\rho} f = 0$.*

PROOF. Let $f \in L^1_{\rho,\Lambda}(\mathcal{H}_n, V)$. Then, $P_{\Lambda\backslash\Gamma,\rho} f \in L^1_{\rho,\Gamma}(\mathcal{H}_n, V)$ by Lemma 3.4(i). But we have $L^1_{\rho,\Gamma}(\mathcal{H}_n, V) = 0$ since every $\phi \in L^1_{\rho,\Gamma}(\mathcal{H}_n, V)$ satisfies

$$\phi = \phi\big|_\rho (-I_{2n}) \overset{(3.2)}{=} \rho(-I_n)\phi = -\phi,$$

where the first and last equality hold, respectively, since $-I_{2n} \in \Gamma$ and $\rho(-I_n) = -\mathrm{id}_V$. It follows that $P_{\Lambda\backslash\Gamma,\rho} f = 0$. □

4. ESTIMATES FOR POINCARÉ SERIES ON THE SIEGEL UPPER HALF-SPACE

In this section, we retain the notation and assumptions of Section 3 and prove analogues of Theorem 1.1 and Corollary 1.2 for Poincaré series on the Siegel upper half-space, namely Theorem 4.1 and Corollary 4.2 below.

THEOREM 4.1. *Let Λ and Γ be subgroups of $\mathrm{Sp}_{2n}(\mathbb{Z})$ such that $\Lambda \subseteq \Gamma$. Suppose that $-I_{2n} \notin \Gamma$ or $\rho(-I_n) = \mathrm{id}_V$. Let $f \in L^1_{\rho,\Lambda}(\mathcal{H}_n, V) \setminus \{0\}$. Then, we have the following:*

(i) *Let C be a Borel measurable subset of $\mathrm{Sp}_{2n}(\mathbb{R})$ such that*

$$CC^{-1} \cap \Gamma \subseteq \Lambda. \tag{4.1}$$

Then, denoting by $(\Lambda C)^c$ the complement of the set ΛC in $\mathrm{Sp}_{2n}(\mathbb{R})$, *we have*

$$\frac{\left\|P_{\Lambda\backslash\Gamma,\rho}f\right\|_{L^1_{\rho,\Gamma}(\mathcal{H}_n,V)}}{\|f\|_{L^1_{\rho,\Lambda}(\mathcal{H}_n,V)}} \geq 1-2\cdot\frac{\int_{\Lambda\backslash(\Lambda C)^c}\|(\Phi_\rho f)(g)\|_V\,dg}{\|\Phi_\rho f\|_{L^1(\Lambda\backslash\mathrm{Sp}_{2n}(\mathbb{R}),V)}}. \tag{4.2}$$

(ii) *Let S be a Borel measurable subset of $\mathcal{H}_n$ such that for all $z_1, z_2 \in S$, the following implication holds:*

$$\Gamma.z_1 = \Gamma.z_2 \;\Rightarrow\; \Lambda.z_1 = \Lambda.z_2. \tag{4.3}$$

Then, denoting by $(\Lambda.S)^c$ the complement of the set $\Lambda.S$ in $\mathcal{H}_n$, we have

$$\frac{\left\|P_{\Lambda\backslash\Gamma,\rho}f\right\|_{L^1_{\rho,\Gamma}(\mathcal{H}_n,V)}}{\|f\|_{L^1_{\rho,\Lambda}(\mathcal{H}_n,V)}} \geq 1-2\cdot\frac{\int_{\Lambda\backslash(\Lambda.S)^c}\left\|\rho\left(\Im(z)^{\frac{1}{2}}\right)f(z)\right\|_V\,d\mathsf{v}(z)}{\varepsilon_\Lambda\,\|f\|_{L^1_{\rho,\Lambda}(\mathcal{H}_n,V)}}. \tag{4.4}$$

Proof. (i) By Lemma 3.2, we have that $\Phi_\rho f \in L^1(\Lambda\backslash\mathrm{Sp}_{2n}(\mathbb{R}),V)\setminus\{0\}$. Thus, Theorem 2.1 implies that

$$\frac{\left\|P_{\Lambda\backslash\Gamma}\Phi_\rho f\right\|_{L^1(\Gamma\backslash\mathrm{Sp}_{2n}(\mathbb{R}),V)}}{\|\Phi_\rho f\|_{L^1(\Lambda\backslash\mathrm{Sp}_{2n}(\mathbb{R}),V)}} \geq 1-2\cdot\frac{\int_{\Lambda\backslash(\Lambda C)^c}\|(\Phi_\rho f)(g)\|_V\,dg}{\|\Phi_\rho f\|_{L^1(\Lambda\backslash\mathrm{Sp}_{2n}(\mathbb{R}),V)}}. \tag{4.5}$$

Since the left-hand sides in (4.2) and (4.5) coincide by Lemma 3.4(iii) and Lemma 3.2, this proves (i).

(ii) Suppose that $\varepsilon_\Lambda = \varepsilon_\Gamma$. Writing $\Gamma_z = \{\gamma\in\Gamma : \gamma.z = z\}$ for $z\in\mathcal{H}_n$, let

$$S_0 = \{z\in S : \Gamma_z \subseteq \{\pm I_{2n}\}\}.$$

Let us prove that the subset

$$C = \{n_x a_y k : x+iy\in S_0,\ k\in K\}$$

of $\mathrm{Sp}_{2n}(\mathbb{R})$ satisfies the condition (4.1). In other words, assuming that $g, g'\in C$ are such that $\gamma := gg'^{-1}\in\Gamma$, let us show that $\gamma\in\Lambda$. Let $z = g.(iI_n)$ and $z' = g'.(iI_n)$. By acting on iI_n by $g = \gamma g'$, we obtain

$$z = \gamma.z', \tag{4.6}$$

hence by (4.3) we have that

$$z = \lambda.z' \tag{4.7}$$

for some $\lambda\in\Lambda$. The equalities (4.6) and (4.7) imply that

$$\lambda^{-1}\gamma.z' = z'.$$

Since $z'\in S_0$, it follows that $\lambda^{-1}\gamma\in\{\pm I_{2n}\}$, i.e., we have

$$\gamma\in\{\pm\lambda\}\cap\Gamma\subseteq\Lambda,$$

where the last inclusion holds since $\varepsilon_\Lambda = \varepsilon_\Gamma$.

Thus, the set C satisfies (4.1), hence by (i) we have

$$\frac{\left\|P_{\Lambda\backslash\Gamma,\rho}f\right\|_{L^1_{\rho,\Gamma}(\mathcal{H}_n,V)}}{\|f\|_{L^1_{\rho,\Lambda}(\mathcal{H}_n,V)}}\geq 1-2\cdot\frac{\int_{\Lambda\backslash(\Lambda C)^c}\|(\Phi_\rho f)(g)\|_V\,dg}{\|\Phi_\rho f\|_{L^1(\Lambda\backslash\mathrm{Sp}_{2n}(\mathbb{R}),V)}}. \tag{4.8}$$

Let $\mathbf{1}_{(\Lambda C)^c}:\mathrm{Sp}_{2n}(\mathbb{R})\to\mathbb{C}$ and $\mathbf{1}_{(\Lambda.S_0)^c}:\mathcal{H}_n\to\mathbb{C}$ be the indicator functions of the sets $(\Lambda C)^c$ and $(\Lambda.S_0)^c$, respectively. We have

(4.9)

$$\begin{aligned}\int_{\Lambda\backslash(\Lambda C)^c}\|(\Phi_\rho f)(g)\|_V\,dg &= \left\|\mathbf{1}_{(\Lambda C)^c}\,\Phi_\rho f\right\|_{L^1(\Lambda\backslash\mathrm{Sp}_{2n}(\mathbb{R}),V)}\\ &\overset{(3.6)}{=} \left\|\Phi_\rho\left(\mathbf{1}_{(\Lambda.S_0)^c}\,f\right)\right\|_{L^1(\Lambda\backslash\mathrm{Sp}_{2n}(\mathbb{R}),V)}\\ &\overset{\text{Lem. 3.2}}{=} \left\|\mathbf{1}_{(\Lambda.S_0)^c}\,f\right\|_{L^1_{\rho,\Lambda}(\mathcal{H}_n,V)}\\ &\overset{(3.8)}{=} \varepsilon_\Lambda^{-1}\int_{\Lambda\backslash(\Lambda.S_0)^c}\left\|\rho\left(\Im(z)^{\frac{1}{2}}\right)f(z)\right\|_V\,d\mathsf{v}(z)\\ &= \varepsilon_\Lambda^{-1}\int_{\Lambda\backslash(\Lambda.S)^c}\left\|\rho\left(\Im(z)^{\frac{1}{2}}\right)f(z)\right\|_V\,d\mathsf{v}(z),\end{aligned}$$

where the last equality holds since $(\Lambda.S)^c$ and $(\Lambda.S_0)^c$ differ by a v-null set. Namely, the sets S and S_0 differ by a subset of the set

$$\{z\in\mathcal{H}_n:\Gamma_z\setminus\{\pm I_{2n}\}\neq\emptyset\},$$

which is a v-null set by [5, Lemma 3]. By the countability of Λ, it follows that the sets $\Lambda.S$ and $\Lambda.S_0$ also differ by a v-null set. Consequently, their complements in $\mathcal{H}_n$ differ by a v-null set as well.

By Lemma 3.2, we have

$$\|\Phi_\rho f\|_{L^1(\Lambda\backslash\mathrm{Sp}_{2n}(\mathbb{R}),V)}=\|f\|_{L^1_{\rho,\Lambda}(\mathcal{H}_n,V)}. \tag{4.10}$$

By substituting (4.9) and (4.10) into the right-hand side of (4.8), we obtain (4.4). This finishes the proof of (ii) in the case when $\varepsilon_\Lambda=\varepsilon_\Gamma$.

Finally, suppose that $\varepsilon_\Lambda\neq\varepsilon_\Gamma$. Then, $-I_{2n}\in\Gamma\setminus\Lambda$. Thus, by the assumptions in the statement of the theorem, we have $\rho(-I_n)=\mathrm{id}_V$ and hence, by (3.2), $f|_\rho(-I_{2n})=f$. Consequently, f belongs to $L^1_{\rho,\pm\Lambda}(\mathcal{H}_n,V)$ and satisfies

$$\|f\|_{L^1_{\rho,\Lambda}(\mathcal{H}_n,V)}=2\,\|f\|_{L^1_{\rho,\pm\Lambda}(\mathcal{H}_n,V)}, \tag{4.11}$$

where $\pm\Lambda$ denotes the subgroup of Γ generated by $-I_{2n}$ and Λ. Moreover, we have

$$P_{\Lambda\backslash\Gamma,\rho}f=2P_{(\pm\Lambda)\backslash\Gamma,\rho}f. \tag{4.12}$$

Since $\varepsilon_{\pm\Lambda} = \varepsilon_\Gamma$, we can apply (ii) to the Poincaré series $P_{(\pm\Lambda)\backslash\Gamma,\rho}f$, obtaining the estimate

$$\frac{\left\|P_{(\pm\Lambda)\backslash\Gamma,\rho}f\right\|_{L^1_{\rho,\Gamma}(\mathcal{H}_n,V)}}{\|f\|_{L^1_{\rho,\pm\Lambda}(\mathcal{H}_n,V)}} \geq 1 - 2\cdot\frac{\int_{\Lambda\backslash(\Lambda.S)^c}\left\|\rho\left(\Im(z)^{\frac{1}{2}}\right)f(z)\right\|_V\, d\mathsf{v}(z)}{2\,\|f\|_{L^1_{\rho,\pm\Lambda}(\mathcal{H}_n,V)}},$$

which is equivalent to (4.4) by (4.11) and (4.12). This finishes the proof of (ii). $\square$

Corollary 4.2. *Let Γ be a subgroup of finite index in $\mathrm{Sp}_{2n}(\mathbb{Z})$, and let Λ be a subgroup of Γ. Suppose that $-I_{2n}\notin\Gamma$ or $\rho(-I_n) = \mathrm{id}_V$. Let $p,q\in\mathbb{R}_{\geq 1}\cup\{\infty\}$ such that $\frac{1}{p}+\frac{1}{q} = 1$. Let $f\in L^1_{\rho,\Lambda}(\mathcal{H}_n,V)$ such that $P_{\Lambda\backslash\Gamma,\rho}f\in L^p_{\rho,\Gamma}(\mathcal{H}_n,V)$. Then, we have the following:*

(i) *Let C be a Borel measurable subset of $\mathrm{Sp}_{2n}(\mathbb{R})$ such that*

$$CC^{-1}\cap\Gamma\subseteq\Lambda.$$

Then, denoting by $(\Lambda C)^c$ the complement of the set ΛC in $\mathrm{Sp}_{2n}(\mathbb{R})$, we have

$$\begin{aligned}\left\|P_{\Lambda\backslash\Gamma,\rho}f\right\|_{L^p_{\rho,\Gamma}(\mathcal{H}_n,V)} &\\ \geq \frac{1}{\mathrm{vol}(\Gamma\backslash\mathrm{Sp}_{2n}(\mathbb{R}))^{\frac{1}{q}}}&\left(\int_{\Lambda\backslash\Lambda C}\left\|\left(\Phi_\rho f\right)(g)\right\|_V\, dg\right.\\ &\left.-\int_{\Lambda\backslash(\Lambda C)^c}\left\|\left(\Phi_\rho f\right)(g)\right\|_V\, dg\right).\end{aligned}\tag{4.13}$$

(ii) *Let S be a Borel measurable subset of $\mathcal{H}_n$ such that for all $z_1,z_2\in S$, the following implication holds:*

$$\Gamma.z_1 = \Gamma.z_2 \;\Rightarrow\; \Lambda.z_1 = \Lambda.z_2.$$

Then, denoting by $(\Lambda.S)^c$ the complement of the set $\Lambda.S$ in $\mathcal{H}_n$, we have

$$\begin{aligned}\left\|P_{\Lambda\backslash\Gamma,\rho}f\right\|_{L^p_{\rho,\Gamma}(\mathcal{H}_n,V)} &\\ \geq \frac{1}{\varepsilon_\Lambda\,\mathrm{vol}(\Gamma\backslash\mathrm{Sp}_{2n}(\mathbb{R}))^{\frac{1}{q}}}&\left(\int_{\Lambda\backslash\Lambda.S}\left\|\rho\left(\Im(z)^{\frac{1}{2}}\right)f(z)\right\|_V\, d\mathsf{v}(z)\right.\\ &\left.-\int_{\Lambda\backslash(\Lambda.S)^c}\left\|\rho\left(\Im(z)^{\frac{1}{2}}\right)f(z)\right\|_V\, d\mathsf{v}(z)\right).\end{aligned}\tag{4.14}$$

Proof. (i) Applying Theorem 4.1(i) and multiplying (4.2) by $\|\Phi_\rho f\|_{L^1(\Lambda\backslash \mathrm{Sp}_{2n}(\mathbb{R}),V)}$, by (4.10) we obtain

$$\begin{aligned}
&\left\|P_{\Lambda\backslash\Gamma,\rho} f\right\|_{L^1_{\rho,\Gamma}(\mathcal{H}_n,V)}\\
&\geq \int_{\Lambda\backslash \mathrm{Sp}_{2n}(\mathbb{R})} \|(\Phi_\rho f)(g)\|_V \, dg - 2\int_{\Lambda\backslash(\Lambda C)^c} \|(\Phi_\rho f)(g)\|_V \, dg \\
&= \int_{\Lambda\backslash\Lambda C} \|(\Phi_\rho f)(g)\|_V \, dg - \int_{\Lambda\backslash(\Lambda C)^c} \|(\Phi_\rho f)(g)\|_V \, dg.
\end{aligned} \tag{4.15}$$

The inequality (4.13) follows by applying to the left-hand side of (4.15) the estimate

(4.16)

$$\begin{aligned}
\|\phi\|_{L^1_{\rho,\Gamma}(\mathcal{H}_n,V)} &\leq \left(\varepsilon_\Gamma^{-1} \operatorname{vol}(\Gamma\backslash\mathcal{H}_n)\right)^{\frac{1}{q}} \|\phi\|_{L^p_{\rho,\Gamma}(\mathcal{H}_n,V)}\\
&\overset{(3.7)}{=} \operatorname{vol}(\Gamma\backslash \mathrm{Sp}_{2n}(\mathbb{R}))^{\frac{1}{q}} \|\phi\|_{L^p_{\rho,\Gamma}(\mathcal{H}_n,V)}, \quad \phi \in L^p_{\rho,\Gamma}(\mathcal{H}_n,V),
\end{aligned}$$

which holds by Hölder's inequality.

(ii) We apply Theorem 4.1(ii) and multiply (4.4) by $\|f\|_{L^1_{\rho,\Lambda}(\mathcal{H}_n,V)}$, obtaining

$$\begin{aligned}
\left\|P_{\Lambda\backslash\Gamma,\rho} f\right\|_{L^1_{\rho,\Gamma}(\mathcal{H}_n,V)} &\geq \varepsilon_\Lambda^{-1}\Bigg(\int_{\Lambda\backslash\mathcal{H}_n} \left\|\rho\left(\Im(z)^{\frac{1}{2}}\right) f(z)\right\|_V d\mathsf{v}(z)\\
&\qquad - 2\int_{\Lambda\backslash(\Lambda.S)^c} \left\|\rho\left(\Im(z)^{\frac{1}{2}}\right) f(z)\right\|_V d\mathsf{v}(z)\Bigg)\\
&= \varepsilon_\Lambda^{-1}\Bigg(\int_{\Lambda\backslash\Lambda.S} \left\|\rho\left(\Im(z)^{\frac{1}{2}}\right) f(z)\right\|_V d\mathsf{v}(z)\\
&\qquad - \int_{\Lambda\backslash(\Lambda.S)^c} \left\|\rho\left(\Im(z)^{\frac{1}{2}}\right) f(z)\right\|_V d\mathsf{v}(z)\Bigg).
\end{aligned} \tag{4.17}$$

The inequality (4.14) follows by applying (4.16) to the left-hand side of (4.17). □

5. Application to Siegel Poincaré series of polynomial type

Let $n \in \mathbb{Z}_{>0}$ and $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{Z}^n$ with $\omega_1 \geq \ldots \geq \omega_n > 2n$. Let ρ be an irreducible polynomial representation of $\mathrm{GL}_n(\mathbb{C})$ of highest weight ω, realized on a complex finite-dimensional Hilbert space V, such that the restriction $\rho\big|_{\mathrm{U}(n)}$ is unitary. The principal congruence subgroup of $\mathrm{Sp}_{2n}(\mathbb{Z})$ of level $N \in \mathbb{Z}_{>0}$ is the group

$$\Gamma^n(N) = \{\gamma \in \mathrm{Sp}_{2n}(\mathbb{Z}) : \gamma \equiv I_{2n} \pmod{N}\}.$$

The space $S_\rho(\Gamma^n(N))$ of Siegel cusp forms of weight ρ for $\Gamma^n(N)$ is the finite-dimensional complex vector space of holomorphic functions $f : \mathcal{H}_n \to V$ that have the following two properties:

(1) $f\big|_\rho \gamma = f$ for all $\gamma \in \Gamma^n(N)$.

(2) $\sup_{z\in\mathcal{H}_n} \left\|\rho\left(\Im(z)^{\frac{1}{2}}\right) f(z)\right\|_V < \infty$.

Let $\mathbb{C}[X_{r,s} : 1 \le r, s \le n]$ denote the ring of polynomials with complex coefficients in the n^2 variables $X_{r,s}$ with $r, s \in \{1, \dots, n\}$. As in [21, (16)], given $\mu \in \mathbb{C}[X_{r,s} : 1 \le r, s \le n]$ and $v \in V$, let $f_{\mu,v} \in L^1_\rho(\mathcal{H}_n, V)$ be given by

$$f_{\mu,v}(z) = \mu\left((z - iI_n)(z + iI_n)^{-1}\right) \rho\left(\frac{1}{2i}(z + iI_n)\right)^{-1} v, \quad z \in \mathcal{H}_n.$$

The following proposition is a special case of [21, Theorem 1.1].

Proposition 5.1. *Let $N \in \mathbb{Z}_{>0}$. For every $\mu \in \mathbb{C}[X_{r,s} : 1 \le r, s \le n]$ and $v \in V$, the Poincaré series*

$$\begin{aligned}\left(P_{\Gamma^n(N),\rho} f_{\mu,v}\right)(z) &= \sum_{\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\in\Gamma^n(N)} \mu\bigg(\Big(Az + B - i(Cz + D)\Big)\Big(Az + B + i(Cz + D)\Big)^{-1}\bigg) \\ &\qquad \cdot \rho\left(\frac{1}{2i}\Big(Az + B + i(Cz + D)\Big)\right)^{-1} v\end{aligned}$$

converges absolutely and uniformly on compact subsets of $\mathcal{H}_n$ and defines an element of $S_\rho(\Gamma^n(N))$. Moreover, we have

$$S_\rho(\Gamma^n(N)) = \operatorname{span}_{\mathbb{C}}\left\{P_{\Gamma^n(N),\rho} f_{\mu,v} : \mu \in \mathbb{C}[X_{r,s} : 1 \le r, s \le n],\, v \in V\right\}.$$

Given $x \in \mathbb{R}^n$, let $h_x = \operatorname{diag}\left(e^{x_1}, \dots, e^{x_n}, e^{-x_1}, \dots, e^{-x_n}\right) \in \mathrm{Sp}_{2n}(\mathbb{R})$ and $d_x = \operatorname{diag}(x_1, \dots, x_n) \in M_n(\mathbb{R})$. By Cartan's KAK decomposition [8, Theorem 5.20], every $g \in \mathrm{Sp}_{2n}(\mathbb{R})$ has a factorization

$$g = k h_x k' \tag{5.1}$$

with $k, k' \in K$ and $x \in \mathbb{R}^n$. By [21, Lemma 6.5], given $\mu \in \mathbb{C}[X_{r,s} : 1 \le r, s \le n]$ and $v \in V$, the function $\Phi_\rho f_{\mu,v} : \mathrm{Sp}_{2n}(\mathbb{R}) \to V$ is given by

$$\left(\Phi_\rho f_{\mu,v}\right)(k_u h_x k_{u'}) = \mu\left(u \tanh(d_x) u^\top\right) \rho\left((u')^\top \cosh(d_x)^{-1} u^\top\right) v$$

for all $u, u' \in \mathrm{U}(n)$ and $x \in \mathbb{R}^n$.

For every $t \in \mathbb{R}_{>0}$, let

$$A_t^+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : t > x_1 > \dots > x_n > 0\}.$$

Given $\mu \in \mathbb{C}[X_{r,s} : 1 \le r, s \le n]$ and $v \in V$, let $\varphi_{\rho,\mu,v} : A_1^+ \times \mathrm{U}(n) \to \mathbb{R}_{\ge 0}$,

$$\varphi_{\rho,\mu,v}(x, u) = \left|\mu\left(u d_x^{\frac{1}{2}} u^\top\right)\right| \left\|\rho\left((I_n - d_x)^{\frac{1}{2}} u^\top\right) v\right\|_V \frac{\prod_{1\le r<s\le n}(x_r - x_s)}{\prod_{r=1}^n (1 - x_r)^{n+1}}.$$

Given $N \in \mathbb{Z}_{>0}$, let

$$M_n(N) = \left(\sqrt{1 + \frac{4n}{N^2}} + \sqrt{\frac{4n}{N^2}} \right)^{-2}.$$

THEOREM 5.2. *Let $\mu \in \mathbb{C}[X_{r,s} : 1 \le r, s \le n] \setminus \{0\}$, $v \in V \setminus \{0\}$, and $N \in \mathbb{Z}_{\ge 3}$. Then, we have*

$$\frac{\left\| P_{\Gamma^n(N),\rho} f_{\mu,v} \right\|_{L^1_{\rho,\Gamma^n(N)}(\mathcal{H}_n,V)}}{\left\| f_{\mu,v} \right\|_{L^1_\rho(\mathcal{H}_n,V)}} \ge 2 \cdot \frac{\int_{A^+_{M_n(N)}} \int_{\mathrm{U}(n)} \varphi_{\rho,\mu,v}(x,u)\, du\, dx}{\int_{A^+_1} \int_{\mathrm{U}(n)} \varphi_{\rho,\mu,v}(x,u)\, du\, dx} - 1. \tag{5.2}$$

PROOF. We apply Theorem 4.1(i) with $\Gamma = \Gamma^n(N)$, $\Lambda = \left\{ 1_{\mathrm{Sp}_{2n}(\mathbb{Z})} \right\}$, $f = f_{\mu,v}$, and

$$C = K \left\{ h_x : x \in \left[0, \operatorname{artanh} \sqrt{M_n(N)} \right[^n \right\} K.$$

By [21, proof of Theorem 8.2], we have

$$CC^{-1} \cap \Gamma^n(N) = \left\{ 1_{\mathrm{Sp}_{2n}(\mathbb{R})} \right\}.$$

Thus, Theorem 4.1(i) implies that

$$\begin{aligned} \frac{\left\| P_{\Gamma^n(N),\rho} f_{\mu,v} \right\|_{L^1_{\rho,\Gamma^n(N)}(\mathcal{H}_n,V)}}{\left\| f_{\mu,v} \right\|_{L^1_\rho(\mathcal{H}_n,V)}} &\ge 1 - 2 \cdot \frac{\int_{\mathrm{Sp}_{2n}(\mathbb{R}) \setminus C} \left\| \left(\Phi_\rho f_{\mu,v} \right)(g) \right\|_V \, dg}{\left\| \Phi_\rho f_{\mu,v} \right\|_{L^1(\mathrm{Sp}_{2n}(\mathbb{R}),V)}} \\ &= 2 \cdot \frac{\int_C \left\| \left(\Phi_\rho f_{\mu,v} \right)(g) \right\|_V \, dg}{\left\| \Phi_\rho f_{\mu,v} \right\|_{L^1(\mathrm{Sp}_{2n}(\mathbb{R}),V)}} - 1. \end{aligned} \tag{5.3}$$

By [21, (52) and (53)], there exists $c \in \mathbb{R}_{>0}$ such that

$$\int_C \left\| \left(\Phi_\rho f_{\mu,v} \right)(g) \right\|_V \, dg = c \int_{A^+_{M_n(N)}} \int_{\mathrm{U}(n)} \varphi_{\rho,\mu,v}(x,u)\, du\, dx \tag{5.4}$$

and

$$\left\| \Phi_\rho f_{\mu,v} \right\|_{L^1(\mathrm{Sp}_{2n}(\mathbb{R}),V)} = c \int_{A^+_1} \int_{\mathrm{U}(n)} \varphi_{\rho,\mu,v}(x,u)\, du\, dx. \tag{5.5}$$

Substituting (5.4) and (5.5) into (5.3), we obtain (5.2). □

6. THE CASE OF DETERMINANTAL PARAMETERS

Let us retain the notation and assumptions of Section 5. In the case when $\rho = \det^m : \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}^\times$ for some $m \in \mathbb{Z}_{>2n}$, $\mu = \det^l$ for some $l \in \mathbb{Z}_{\ge 0}$, and $v = 1$, the function $f_{\mu,v}$ is given by

$$f_{\det^l,1}(z) = (2i)^m \frac{\det(z - iI_n)^l}{\det(z + iI_n)^{l+m}}, \quad z \in \mathcal{H}_n,$$

the Siegel cusp form $P_{\Gamma^n(N),\rho} f_{\mu,v} \in S_\rho(\Gamma^n(N))$ is given by

$$P_{\Gamma^n(N),\det^m} f_{\det^l,1}(z) = (2i)^m \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^n(N)} \frac{\det\big(Az+B-i(Cz+D)\big)^l}{\det\big(Az+B+i(Cz+D)\big)^{l+m}}$$

for all $z \in \mathcal{H}_n$, and Theorem 5.2 simplifies into the part (i) of the following corollary.

Corollary 6.1. *Let $n \in \mathbb{Z}_{>0}$, $m \in \mathbb{Z}_{>2n}$, $l \in \mathbb{Z}_{\geq 0}$, and $p \in [0,1[$. Let $\varphi_{n,m,l} : A_1^+ \to \mathbb{R}_{\geq 0}$,*

$$\varphi_{n,m,l}(x) = \left(\prod_{r=1}^n x_r^{\frac{l}{2}} (1-x_r)^{\frac{m}{2}-n-1} \right) \prod_{1 \leq r < s \leq n} (x_r - x_s).$$

Then, we have the following:

(i) *For every $N \in \mathbb{Z}_{\geq 3}$, we have*

$$\frac{\left\| P_{\Gamma^n(N),\det^m} f_{\det^l,1} \right\|_{L^1_{\det^m,\Gamma^n(N)}(\mathcal{H}_n,\mathbb{C})}}{\left\| f_{\det^l,1} \right\|_{L^1_{\det^m}(\mathcal{H}_n,\mathbb{C})}} \geq 2 \cdot \frac{\int_{A^+_{M_n(N)}} \varphi_{n,m,l}(x)\, dx}{\int_{A_1^+} \varphi_{n,m,l}(x)\, dx} - 1. \tag{6.1}$$

(ii) *There exists $N^{(p)}_{n,m,l}$, defined as the smallest integer $N \geq 3$ such that*

$$\frac{\int_{A^+_{M_n(N)}} \varphi_{n,m,l}(x)\, dx}{\int_{A_1^+} \varphi_{n,m,l}(x)\, dx} > \frac{p+1}{2}. \tag{6.2}$$

(iii) *For every integer $N \geq N^{(p)}_{n,m,l}$, we have*

$$\frac{\left\| P_{\Gamma^n(N),\det^m} f_{\det^l,1} \right\|_{L^1_{\det^m,\Gamma^n(N)}(\mathcal{H}_n,\mathbb{C})}}{\left\| f_{\det^l,1} \right\|_{L^1_\rho(\mathcal{H}_n,\mathbb{C})}} > p.$$

Proof of (ii) and (iii). Since the set A_1^+ equals the increasing union $\bigcup_{N=3}^\infty A^+_{M_n(N)}$, by the monotone convergence theorem the left-hand side of (6.2) is a non-decreasing function of $N \in \mathbb{Z}_{\geq 3}$ and tends to 1 as N tends to infinity. Thus, (ii) holds, and every integer $N \geq N^{(p)}_{n,m,l}$ satisfies the inequality (6.2), which is equivalent to the right-hand side of (6.1) being strictly greater than p. Thus, (6.1) implies (iii). □

Remark 6.2. In the case when $p = 0$, Corollary 6.1(iii) implies that the Siegel cusp form $P_{\Gamma^n(N),\det^m} f_{\det^l,1}$ does not vanish identically for all integers $N \geq N^{(0)}_{n,m,l}$. This non-vanishing result has been proved in [21, Theorem 8.2] by applying a variant [21, Proposition 8.1] of Muić's integral non-vanishing criterion [12, Theorem 4.1] for Poincaré series.

The values of integers $N^{(p)}_{n,m,l}$ for $n \in \{1, 2\}$, $p \in \{0, 0.5, 0.9\}$, $m \in \{2n + 1\ , 2n + 2, \ldots, 2n + 10\}$, and $l \in \{0, 1, 2, 3\}$ are given in Tables 1 and 2. We computed these values using Wolfram Mathematica 14.3 [18]; the code can be found in [23].

Table 1. Values of $N^{(p)}_{1,m,l}$ for $m \in \{3, 4, \ldots, 12\}$, $l \in \{0, 1, 2, 3\}$, and $p \in \{0, 0.5, 0.9\}$

m	$p = 0$				$p = 0.5$				$p = 0.9$			
	l				l				l			
	0	1	2	3	0	1	2	3	0	1	2	3
3	14	23	32	40	62	101	140	179	1598	2591	3596	4606
4	6	9	12	15	14	21	28	35	78	117	156	195
5	4	6	8	10	8	12	15	18	28	40	51	63
6	4	5	6	7	6	8	11	13	16	22	28	34
7	3	4	5	6	5	7	8	10	12	15	19	23
8	3	4	5	5	4	6	7	8	9	12	14	17
9	3	3	4	5	4	5	6	7	8	10	12	14
10	3	3	4	4	4	5	5	6	7	8	10	12
11	3	3	4	4	3	4	5	6	6	7	9	10
12	3	3	3	4	3	4	5	5	5	7	8	9

Table 2. Values of $N^{(p)}_{2,m,l}$ for $m \in \{5, 6, \ldots, 14\}$, $l \in \{0, 1, 2, 3\}$, and $p \in \{0, 0.5, 0.9\}$

m	$p = 0$				$p = 0.5$				$p = 0.9$			
	l				l				l			
	0	1	2	3	0	1	2	3	0	1	2	3
5	77	107	137	167	349	484	620	756	9037	12543	16066	19598
6	25	33	41	49	59	79	99	118	331	442	552	662
7	15	20	24	28	29	38	46	54	99	128	157	185
8	11	14	17	20	19	24	29	34	51	65	78	91
9	9	11	14	16	15	18	22	25	33	41	49	57
10	8	10	11	13	12	15	17	20	25	30	36	41
11	7	8	10	11	10	12	14	17	20	24	28	32
12	6	8	9	10	9	11	13	14	16	20	23	26
13	6	7	8	9	8	10	11	13	14	17	19	22
14	5	6	7	8	7	9	10	11	12	15	17	19

7. Application to the classical Poincaré series

Let $m \in \mathbb{Z}_{\geq 3}$ and $N \in \mathbb{Z}_{>0}$. In this section, we study a family of Siegel cusp forms in $S_m(\Gamma(N)) := S_{\rho_m}(\Gamma(N))$, where ρ_m is the character of $\mathrm{GL}_1(\mathbb{C})$ given by

$$\rho_m(z) = z^m, \quad z \in \mathrm{GL}_1(\mathbb{C}) \equiv \mathbb{C}^\times,$$

and $\Gamma(N)$ is the principal congruence subgroup $\Gamma^1(N)$ of $\mathrm{Sp}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$. As the notation suggests, the space $S_m(\Gamma(N))$ coincides with the standardly defined space of holomorphic cusp forms of weight m for $\Gamma(N)$ on the upper half-plane $\mathcal{H} = \mathcal{H}_1$. We endow $S_m(\Gamma(N))$ with the Petersson inner product

$$\langle f_1, f_2 \rangle_{S_m(\Gamma(N))} = \varepsilon_{\Gamma(N)}^{-1} \int_{\Gamma(N)\backslash\mathcal{H}} f_1(z)\, \overline{f_2(z)}\, \Im(z)^m \, d\mathsf{v}(z)$$

for all $f_1, f_2 \in S_m(\Gamma(N))$. The elements of $S_m(\Gamma(N))$ lift via Φ_{ρ_m} to cuspidal automorphic forms on $\mathrm{SL}_2(\mathbb{R}) = \mathrm{Sp}_2(\mathbb{R})$ (see, e.g., [4, Proposition 2.1]).

We recall that by [11, (1.1.7)], we have

$$\Im(g.z) = \frac{\Im(z)}{|cz+d|^2}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \ z \in \mathcal{H}. \tag{7.1}$$

Let

$$\Gamma(N)_\infty = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N) : c = 0 \right\}.$$

For every $l \in \mathbb{Z}_{>0}$, the function $f_{\frac{l}{N}} : \mathcal{H} \to \mathbb{C}$,

$$f_{\frac{l}{N}}(z) = e^{2\pi i \frac{l}{N} z},$$

belongs to $L^1_{\rho_m, \Gamma(N)_\infty}(\mathcal{H}, \mathbb{C})$. The following proposition is well known (see, e.g., [11, Corollary 2.6.11]).

Proposition 7.1. *Let $m \in \mathbb{Z}_{\geq 3}$ and $N \in \mathbb{Z}_{>0}$. Then, for every $l \in \mathbb{Z}_{>0}$, the Poincaré series*

$$\left(P_{\Gamma(N)_\infty\backslash\Gamma(N),\rho_m} f_{\frac{l}{N}}\right)(z) = \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(N)_\infty\backslash\Gamma(N)} (cz+d)^{-m} e^{2\pi i \frac{l}{N} \cdot \frac{az+b}{cz+d}}, \quad z \in \mathcal{H}, \tag{7.2}$$

converges absolutely and uniformly on compact subsets of $\mathcal{H}$. Moreover, we have

$$S_m(\Gamma(N)) = \mathrm{span}_{\mathbb{C}} \left\{ P_{\Gamma(N)_\infty\backslash\Gamma(N),\rho_m} f_{\frac{l}{N}} : l \in \mathbb{Z}_{>0} \right\}.$$

Every cusp form $f \in S_m(\Gamma(N))$ has an absolutely convergent Fourier expansion

$$f(z) = \sum_{l=1}^{\infty} a_l(f)\, e^{2\pi i \frac{l}{N} z}, \quad z \in \mathcal{H},$$

for some $a_l(f) \in \mathbb{C}$. The cusp forms $P_{\Gamma(N)_\infty \backslash \Gamma(N), \rho_m} f_{\frac{l}{N}}$, known as the classical Poincaré series, are famous for their following property [11, Theorem 2.6.10]: for all $f \in S_m(\Gamma(N))$ and $l \in \mathbb{Z}_{>0}$, we have

$$a_l(f) = \frac{\varepsilon_{\Gamma(N)}(4\pi l)^{m-1}}{N^m(m-2)!} \left\langle f, P_{\Gamma(N)_\infty \backslash \Gamma(N), \rho_m} f_{\frac{l}{N}} \right\rangle_{S_m(\Gamma(N))}. \tag{7.3}$$

We note that if $N \in \{1,2\}$, then $-I_2 \in \Gamma(N)$, hence by [11, (2.1.11)] we have that $S_m(\Gamma(N)) = 0$ if m is odd.

Theorem 7.2. *Let $m \in \mathbb{Z}_{\geq 3}$ and $N \in \mathbb{Z}_{>0}$ such that m is even if $N \in \{1,2\}$. Then, for every $l \in \mathbb{Z}_{>0}$, we have*

$$\frac{\left\| P_{\Gamma(N)_\infty \backslash \Gamma(N), \rho_m} f_{\frac{l}{N}} \right\|_{L^1_{\rho_m, \Gamma(N)}(\mathcal{H}, \mathbb{C})}}{\left\| f_{\frac{l}{N}} \right\|_{L^1_{\rho_m, \Gamma(N)_\infty}(\mathcal{H}, \mathbb{C})}} \geq 1 - 2 \cdot \frac{\gamma\left(\frac{m}{2} - 1, \frac{2\pi l}{N^2}\right)}{\Gamma\left(\frac{m}{2} - 1\right)}, \tag{7.4}$$

where the symbol Γ on the right-hand side denotes the gamma function, given by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt, \quad \Re(s) > 0,$$

while γ denotes the lower incomplete gamma function, given by

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t}\, dt, \quad \Re(s) > 0,\ x \in \mathbb{R}_{>0}.$$

Proof. We apply Theorem 4.1(ii) with $\Gamma = \Gamma(N)$, $\Lambda = \Gamma(N)_\infty$, $\rho = \rho_m$, $f = f_{\frac{l}{N}}$, and

$$S = \left\{ z \in \mathcal{H} : \Im(z) > \frac{1}{N} \right\}. \tag{7.5}$$

Let us prove that S satisifes (4.3), i.e., that for all $z_1, z_2 \in S$, the following implication holds:

$$\Gamma(N).z_1 = \Gamma(N).z_2 \quad \Rightarrow \quad \Gamma(N)_\infty.z_1 = \Gamma(N)_\infty.z_2.$$

Let $z_1, z_2 \in S$ such that $\Gamma(N).z_1 = \Gamma(N).z_2$. Then, $z_2 = \gamma.z_1$ for some $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$. If $c \neq 0$, then $|c| \geq N$, hence we have

$$\begin{aligned} \frac{1}{N} &\overset{(7.5)}{<} \Im(z_2) = \Im(\gamma.z_1) \overset{(7.1)}{=} \frac{\Im(z_1)}{|cz_1 + d|^2} = \frac{\Im(z_1)}{(c\Re(z_1) + d)^2 + (c\Im(z_1))^2} \\ &\leq \frac{\Im(z_1)}{(c\Im(z_1))^2} = \frac{1}{c^2 \Im(z_1)} < \frac{1}{N^2 \cdot \frac{1}{N}} = \frac{1}{N}, \end{aligned}$$

which is a contradiction. Thus, we have $c = 0$, which means that $\gamma \in \Gamma(N)_\infty$ and hence $\Gamma(N)_\infty.z_1 = \Gamma(N)_\infty.z_2$.

By Theorem 4.1(ii), it follows that, denoting $S^c = \mathcal{H} \setminus S$, we have

$$\frac{\left\| P_{\Gamma(N)_\infty \backslash \Gamma(N), \rho_m} f_{\frac{l}{N}} \right\|_{L^1_{\rho_m, \Gamma(N)}(\mathcal{H}, \mathbb{C})}}{\left\| f_{\frac{l}{N}} \right\|_{L^1_{\rho_m, \Gamma(N)_\infty}(\mathcal{H}, \mathbb{C})}} \geq 1 - 2 \cdot \frac{\int_{\Gamma(N)_\infty \backslash S^c} \Im(z)^{\frac{m}{2}} \left| e^{2\pi i \frac{l}{N} z} \right| d\mathsf{v}(z)}{\varepsilon_{\Gamma(N)_\infty} \left\| f_{\frac{l}{N}} \right\|_{L^1_{\rho_m, \Gamma(N)_\infty}(\mathcal{H}, \mathbb{C})}}. \tag{7.6}$$

We have

$$\begin{aligned} \int_{\Gamma(N)_\infty \backslash S^c} \Im(z)^{\frac{m}{2}} \left| e^{2\pi i \frac{l}{N} z} \right| d\mathsf{v}(z) &= \int_0^{\frac{1}{N}} \int_0^N y^{\frac{m}{2}} e^{-\frac{2\pi l}{N} y} \frac{dx\, dy}{y^2} \\ &= \frac{N^{\frac{m}{2}}}{(2\pi l)^{\frac{m}{2}-1}} \int_0^{\frac{2\pi l}{N^2}} t^{\frac{m}{2}-2} e^{-t}\, dt = \frac{N^{\frac{m}{2}}}{(2\pi l)^{\frac{m}{2}-1}} \gamma\left(\frac{m}{2}-1, \frac{2\pi l}{N^2}\right), \end{aligned} \tag{7.7}$$

where the next-to-last equality follows from the change of variables $t = \frac{2\pi l}{N} y$. Similarly, we have

$$\begin{aligned} \varepsilon_{\Gamma(N)_\infty} \left\| f_{\frac{l}{N}} \right\|_{L^1_{\rho_m, \Gamma(N)_\infty}(\mathcal{H}, \mathbb{C})} &\overset{(3.8)}{=} \int_{\Gamma(N)_\infty \backslash \mathcal{H}} \Im(z)^{\frac{m}{2}} \left| e^{2\pi i \frac{l}{N} z} \right| d\mathsf{v}(z) \\ &= \int_0^\infty \int_0^N y^{\frac{m}{2}} e^{-\frac{2\pi l}{N} y} \frac{dx\, dy}{y^2} = \frac{N^{\frac{m}{2}}}{(2\pi l)^{\frac{m}{2}-1}} \Gamma\left(\frac{m}{2}-1\right). \end{aligned} \tag{7.8}$$

By substituting (7.7) and (7.8) into (7.6), we obtain (7.4). ∎

Corollary 7.3. *Let $m \in \mathbb{Z}_{\geq 3}$, $l \in \mathbb{Z}_{>0}$, and $p \in [0, 1[$. Then, we have the following:*

(i) *There exists $N^{(p)}_{m,l}$, defined as the smallest integer $N \geq 2 - (-1)^m$ such that*

$$\gamma\left(\frac{m}{2}-1, \frac{2\pi l}{N^2}\right) < \frac{1-p}{2} \Gamma\left(\frac{m}{2}-1\right). \tag{7.9}$$

(ii) *For every integer $N \geq N^{(p)}_{m,l}$, we have*

$$\frac{\left\| P_{\Gamma(N)_\infty \backslash \Gamma(N), \rho_m} f_{\frac{l}{N}} \right\|_{L^1_{\rho_m, \Gamma(N)}(\mathcal{H}, \mathbb{C})}}{\left\| f_{\frac{l}{N}} \right\|_{L^1_{\rho_m, \Gamma(N)_\infty}(\mathcal{H}, \mathbb{C})}} > p. \tag{7.10}$$

Proof. Since $\gamma\left(\frac{m}{2}-1, \frac{2\pi l}{N^2}\right)$ is a non-increasing function of $N \in \mathbb{Z}_{>0}$ and tends to 0 as N tends to infinity, (i) holds, and every integer $N \geq N^{(p)}_{m,l}$ satisfies the inequality (7.9), which is equivalent to the right-hand side of (7.4) being strictly greater than p. Thus, (7.4) implies (ii). ∎

Table 3. Values of $N_{m,l}^{(p)}$ for $m \in \{3,4,\ldots,12\}$, $l \in \{1,2,3,4,5\}$, and $p \in \{0,0.5,0.9\}$

m	$p=0$					$p=0.5$					$p=0.9$				
	l					l					l				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
3	6	8	10	11	12	12	16	20	23	25	57	80	98	114	127
4	4	5	6	7	7	5	7	9	10	11	12	16	20	23	25
5	3	4	4	5	6	4	5	6	7	8	6	9	11	12	14
6	2	3	4	4	5	3	4	5	6	6	5	6	8	9	10
7	3	3	3	4	4	3	4	4	5	5	4	5	6	7	8
8	2	3	3	4	4	2	3	4	4	5	3	4	5	6	7
9	3	3	3	3	4	3	3	3	4	4	3	4	5	5	6
10	2	2	3	3	3	2	3	3	4	4	3	4	4	5	5
11	3	3	3	3	3	3	3	3	3	4	3	3	4	4	5
12	2	2	3	3	3	2	2	3	3	4	2	3	4	4	4

Remark 7.4. In the case when $p=0$, the inequality (7.10) implies that the cusp form $P_{\Gamma(N)_\infty\backslash\Gamma(N),\rho_m} f_{\frac{l}{N}}$ does not vanish identically for all integers $N \geq N_{m,l}^{(0)}$. This non-vanishing result was established in [14] as an application of Muić's integral non-vanishing criterion for Poincaré series.

The values of integers $N_{m,l}^{(p)}$ for $p \in \{0,0.5,0.9\}$, $m \in \{3,4,\ldots,12\}$, and $l \in \{1,2,3,4,5\}$ are given in Table 3. We computed these values using Wolfram Mathematica 14.3 [18]; the code can be found in [23].

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