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ON THE DIOPHANTINE SYSTEM INVOLVING PAIRS OF TRIANGLES WITH THE SAME AREA AND THE SAME PERIMETER

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Abstract. Many authors studied the problem that rational triangle pairs (triangle-parallelogram pairs) with the same area and the same perimeter. In this paper, we give a unified description of this problem by using the affine transformation in a rectangular coordinate system. According to the fact that two triangles with the same area and the same perimeter determine an affine transformation, this problem can be reduced to solving a specific Diophantine system. Moreover, we will give some rational parameter solutions to this Diophantine system.

1. Introduction

A rational (Heron) triangle is a triangle with rational (integral) sides and rational (integral) area. Many authors studied two or more triangles with the same area and the same perimeter. Prielipp [23] proved that there are no two such distinct right triangles, Bradley [1] (also see [26]) showed that there are no three such isosceles triangles, Hirakawa and Matsumura [12] obtained a unique pair of right triangle-isosceles triangle, Choudhry [5] (also see [2, 15, 21]) investigated two or more distinct Heron (rational) triangles, Juyal and Moody [14] studied Heron triangle and right triangle pairs. More results can be found in [6, 20, 25].

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For triangle and quadrilateral pairs, Guy [11] proved that there is no non-degenerate such integral right triangle-rectangle pair. Several authors studied other cases, such as Zhang [27] (integral right triangle-parallelogram and Heron triangle-parallelogram pairs), Lalín and Ma [16] (θ-triangle-ω-parallelogram pairs). For more results, we can refer to [3, 4, 8, 28, 29, 30, 31].

Suppose the areas of two polygons are A_1, A_2, and the perimeters are P_1, P_2, respectively. We say that the areas and perimeters of these two polygons are in fixed proportions (α, β) respectively, if they satisfy the following relationship

$$A_2 = \alpha A_1, \quad P_2 = \beta P_1,$$

where α and β are positive rational numbers. When $\alpha = \beta = 1$, it corresponds to two polygons with the same area and the same perimeter.

Li and Zhang [17, 18, 19] obtained infinitely many pairs of rational triangles (pairs of cyclic quadrilaterals, pairs of triangles and some special quadrilaterals) with areas and perimeters in fixed proportions (α, β) respectively.

In this paper, we use affine transformations to uniformly describe the problem of two rational triangles having the same area and the same perimeter. In this process, we can not only get all the existing results but also obtain more rational solutions.

The structure of this paper is as follows. In Section 2, we introduce several equivalence relations and discuss some of their simple properties. In Section 3, we focus on the weak metric equivalence relation and obtain the set $VL_2(\mathbb{Q})$ to describe the affine transformation between two weakly metric equivalent rational triangles. In Section 4, the set $VL_2(\mathbb{Q})$ is used to uniformly describe the existing conclusions that two rational triangles are weakly metric equivalent, and to get more such pairs. In Section 5, the affine transformation is also used for describing the problem that triangle and parallelogram pairs having the same area and the same perimeter. In Section 6, we consider the rational triangle pairs or rational triangle-parallelogram pairs with areas and perimeters in fixed proportions (α, β), respectively, where α and β are positive rational numbers.

2. Some equivalence relations

The purpose of this paper is to study the problem of two rational triangles with the same area and the same perimeter by using affine transformations.

On an affine plane, the affine transformation is a one-to-one, onto function which preserve parallelism. All affine transformations form a transformation group G_A. In this paper, the affine transformation is confined to a rectangular coordinate system. Let f be an affine transformation and P be a point with coordinates (x, y). Suppose that the point P becomes $f(P)$ with coordinates

(x', y'). We have the following transformation formula:

$$\begin{cases} x' = a_{11}x + a_{12}y + b_1, \\ y' = a_{21}x + a_{22}y + b_2. \end{cases} \tag{2.1}$$

Let T be a triangle and $f(T)$ be the image of T. Let A and A' be the areas of T and $f(T)$ respectively, then we have $A' = |\det(M_f)|A$, where M_f is the transformation matrix of f, i.e.,

$$M_f = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

We first introduce several equivalent relations, which enable us to relate this problem to number theory, algebra, geometry, and topology.

Affine equivalence relation: For any two rational triangles T_1 and T_2, if there exists an affine transformation f such that $f(T_1) = T_2$, then we say that T_1 and T_2 are affine equivalent and denote it as $T_1 \sim_a T_2$.

Metric equivalence relation: For any two rational triangles T_1 and T_2, if there exists an isometric transformation f such that $f(T_1) = T_2$, then T_1 and T_2 are metric equivalent, denoted as $T_1 \sim_m T_2$, or simply denoted as $T_1 = T_2$.

Area-preserving equivalence relation: Two rational triangles T_1 and T_2 are area-preserving equivalent if the areas of T_1 and T_2 are the same, denoted as $T_1 \sim_A T_2$.

Perimeter-preserving equivalence relation: Two rational triangles T_1 and T_2 are perimeter-preserving equivalent if the perimeters of T_1 and T_2 are the same, denoted as $T_1 \sim_P T_2$.

Weakly metric equivalence relation: Two rational triangles T_1 and T_2 are weakly metric equivalent if they have the same area and the same perimeter, denoted as $T_1 \sim_w T_2$, or simply denoted as $T_1 \sim T_2$.

Similarity equivalence relation: Two rational triangles T_1 and T_2 are similarity equivalent if they are similar, denoted as $T_1 \sim_s T_2$.

Let us point out some simple properties of these equivalence relations.

(1) An affine transformation is uniquely determined by the images of three ordered non-collinear vertices; therefore, any two non-degenerate triangles are affine equivalent.

(2) If two triangles T_1 and T_2 are metric equivalent, then they are congruent, and vice versa. The transformation between T_1 and T_2 is an isometric transformation. It should be noted that isometric transformations only include translation, rotation and reflection. All isometric transformations form a group, denoted as G_M, which is a subgroup of the affine transformation group G_A.

(3) In order to make the areas of triangles T_1 and T_2 the same, we only need to make the determinant of the matrix corresponding to the affine transformation between T_1 and T_2 equal to ± 1. Obviously, all such transformations also form a group, denoted as G_a. Moreover, $G_M < G_a < G_A$ holds.

In order to discuss other equivalence relations, we first consider the correspondence between sets, group actions, and equivalence relations from a general perspective.

We know that a group action can naturally induce an equivalence relation. Conversely, given an equivalence relation on a set X, we can also find a group G such that the group action of G on X just induces this given equivalence relation.

Let X be a non-empty set, and S_X be the transformation group on set X. $\sim$ is any equivalence relation on X. $X/\sim$ is the corresponding quotient set. For the convenience of description, we introduce a mapping

$$\varphi : \ X/\sim \to S,$$

where S is a certain sufficiently large set, and φ is injective. Let τ be the natural mapping from X to $X/\sim$. A mapping φ_0 can be induced by the mapping φ, that is, $\varphi_0 = \varphi \circ \tau$. The mapping φ_0 has a simple property: $\forall x, y \in X, \varphi_0(x) = \varphi_0(y) \Leftrightarrow x \sim y$.

THEOREM 2.1. *The set*

$$G = \{g \in S_X \mid \forall x \in X, \varphi_0(gx) = \varphi_0(x)\}$$

is a group, and the equivalence relation induced by the action of group G on X is exactly the given equivalence relation.

PROOF. The condition for constructing the set G is equivalent to that for any $x \in D$ (where D is any equivalence class of X), and for any $g \in S_X$, if $gx \in D$, then $g \in G$. Now we prove that the set G forms a group. First, for any $x \in D$, the identity transformation e is in G because $ex = x \in D$. Let $g_1, g_2 \in G$. For any $x \in X$, we have $\varphi_0(g_2 g_1 x) = \varphi_0(g_1 x) = \varphi_0(x)$, so $g_2 g_1 \in G$. Also, suppose $g \in G$. Since $gx \in D$, there exists $y \in D$ such that $gx = y$. Then $g^{-1}y = g^{-1}gx = x \in D$, so $g^{-1} \in G$. Therefore, the set G forms a group. ∎

3. Weak metric equivalence relation and the set $VL_2(\mathbb{Q})$

The focus of our next research is weak metric equivalence relation. For a rational triangle T, we wish to find an affine transformation f such that $f(T)$ is weakly equivalent to T, i.e., $f(T) \sim T$. Let V_1, V_2 and V_3 be the three vertices of the rational triangle T. Since an isometric transformation maps a rational triangle T to a congruent rational triangle, we can adjust the position and orientation of T in the plane via translation, rotation and reflection. Hence we impose the following constraints on T and $f(T)$.

In a rectangular coordinate system, we put the rational triangle T in the first quadrant. The vertex V_1 is at the origin of the coordinate system, the vertex V_2 is on the positive x-axis and the vertex V_3 is above the x-axis (see Figure 1). Let $f(V_1), f(V_2)$ and $f(V_3)$ be the three vertices of the rational triangle $f(T)$. We can make $f(V_1)$ coincide with V_1, that is still at the origin. $f(V_2)$ is on the positive x-axis and $f(V_3)$ is above the x-axis.

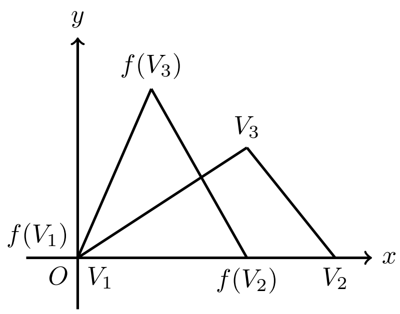

FIGURE 1. Rational triangles T and $f(T)$.

In other words, the rational triangles to be discussed are confined to the following set

$$\mathfrak{T} = \{T \mid V_1 = (0,0), V_2 = (r,0), V_3 = (s,t), r \in \mathbb{R}^*, s \in \mathbb{R}_{\geq 0}, t \in \mathbb{R}^*\},$$

where $\mathbb{R}^* = \{x \in \mathbb{R} \mid x > 0\}$ and $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$. We first prove

THEOREM 3.1. *If $V_1 = (0,0), V_2 = (r,0)$ and $V_3 = (s,t)$ are the three vertices of the rational triangle T, then r, s, t are rational numbers.*

PROOF. Since T is a rational triangle, the side length $|V_1V_2|$ is a positive rational number, therefore, $r = |V_1V_2|$ is a positive rational number. Since the area $A = \frac{1}{2}rt$ of the rational triangle T is a positive rational number, therefore $t = \frac{2A}{r}$ is also a positive rational number. Let us denote $w_1 = |V_1V_3|$ and $w_2 = |V_2V_3|$. Then we have

$$s^2 + t^2 = w_1^2,$$
$$(s-r)^2 + t^2 = w_2^2.$$

Subtracting the two equations yields

$$s = \frac{r^2 + w_1^2 - w_2^2}{2r}.$$

Thus, s is a rational number. □

According to Theorem 3.1, we can describe the set $\mathfrak{T}$ more precisely as

$$\mathfrak{T} = \{T \mid V_1 = (0,0), V_2 = (r,0), V_3 = (s,t), r \in \mathbb{Q}^*, s \in \mathbb{Q}_{\geq 0}, t \in \mathbb{Q}^*\},$$

where $\mathbb{Q}^* = \{x \in \mathbb{Q} \mid x > 0\}$ and $\mathbb{Q}_{\geq 0} = \{x \in \mathbb{Q} \mid x \geq 0\}$. Therefore, the set $\mathfrak{T}$ is a countable set. We first consider the partition of $\mathfrak{T}$ generated by the equivalence relation.

(1) Since any two rational triangles are affine equivalent, therefore $\mathfrak{T}$ has only one affine equivalence class, that is, $\mathfrak{T} = [T]_a$ for all $T \in \mathfrak{T}$, and any rational triangle can be a representative element.

(2) In any metric equivalence class in set $\mathfrak{T}$, there are only finitely many elements. Specifically, the cardinality of $[T]_m$ is at most 12, that is, $|[T]_m| \leq 12$. In fact, a triangle has only three vertices. It is required that one of them is at the origin (there are three selection methods), another one is on the x-axis (there are two selection methods), and the last one is in the first quadrant (there are two choices because of symmetry). Therefore, for any $T \in \mathfrak{T}$, we have $|[T]_m| \leq 12$.

(3) The following lemma shows that for any positive rational number n, there exists an area-preserving equivalence class in which the area of a rational triangle T is n, and $|[T]_A| = \infty$.

Lemma 3.2. [10] *Fix a positive integer n. There are infinitely many rational triangles with area n.*

(4) For perimeter, we can ask the following question:

Question 3.3. Given $P_0 \in \mathbb{Q}^*$, does there exist a rational triangle T such that the perimeter of the rational triangle T is exactly P_0?

Question 3.3 is easily solved: for any $P_0 \in \mathbb{Q}^*$, take the parameter $k = \dfrac{P_0}{2m(m+1)}$ in the right triangle $(k(m^2-1), 2km, k(m^2+1))$, where $m > 1$.

(5) For weak metric equivalence, we can also ask the following question:

Question 3.4. Given $A_0, P_0 \in \mathbb{Q}^*$, does there exist a rational triangle T such that the area and perimeter of the rational triangle T are exactly A_0 and P_0, respectively.

It is not difficult to find that to make $f(T) \in \mathfrak{T}$, we only need $b_1 = b_2 = a_{21} = 0$ in the transformation formula (2.1). In addition, to make the areas of T and $f(T)$ be the same, as long as the determinant $\det(M_f)$ of M_f is equal to ± 1. The sign of the discriminant $\det(M_f)$ determines the orientation of the rational triangle $f(T)$. Up to an isometry, we may always assume that $\det(M_f) = 1$. Therefore, M_f can be assumed to be in the following form

$$M_f = \begin{bmatrix} \frac{1}{a} & b \\ 0 & a \end{bmatrix},$$

where $a \in \mathbb{Q}^*, b \in \mathbb{Q}$. Thus, the transformation formula (2.1) becomes

$$\begin{cases} x' = \dfrac{1}{a}x + by, \\ y' = ay. \end{cases}$$

Let

$$UL_2(\mathbb{Q}) = \left\{ \begin{bmatrix} \frac{1}{a} & b \\ 0 & a \end{bmatrix} \middle| \ a \in \mathbb{Q}^*, b \in \mathbb{Q} \right\}.$$

It is easy to check that the set $UL_2(\mathbb{Q})$ forms a group under the multiplication of matrices. Let

$$U = \{f \mid M_f \in UL_2(\mathbb{Q})\},$$

This is a subgroup of G_a. Obviously, the natural action of U on $\mathfrak{T}$ is given by the map

$$U \times \mathfrak{T} \to \mathfrak{T}, \quad (f, T) \mapsto f(T).$$

Suppose the area of $T \in \mathfrak{T}$ is A_T, then the orbit of T in $\mathfrak{T}$ is the set

$$O_T = \{T' \in \mathfrak{T} \mid \text{the area of } T' \text{ is } A_T\}.$$

But the group $UL_2(\mathbb{Q})$ is not the set that we need, because we cannot guarantee that T and $f(T)$ have the same perimeter.

For any $T \in \mathfrak{T}$, take the set

$$U_T = \{f \mid f(T) \sim T, f \in U\}, \text{ and } U_T L_2(\mathbb{Q}) = \{M_f \mid f \in U_T\}.$$

Obviously, for any $T \in \mathfrak{T}$, there is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in U_T L_2(\mathbb{Q})$. Moreover, $I \in U_{T_1} L_2(\mathbb{Q}) \cap U_{T_2} L_2(\mathbb{Q})$ for any $T_1, T_2 \in \mathfrak{T}$. Let

$$V = \bigcup_{T \in \mathfrak{T}} U_T.$$

Correspondingly, let

$$VL_2(\mathbb{Q}) = \{M_f \mid f \in V\},$$

which is the set we will discuss next.

From the set V, we have

Theorem 3.5. *If $T_1 \sim T_2$, then there is a $f \in U_{T_1} \subset V$ such that $f(T_1) = T_2$.*

Proof. This is obvious because any two rational triangles are affine equivalent. More specifically, suppose the vertices of the rational triangle T_1 are $(0,0), (r,0), (s,t)$, and those of T_2 are $(0,0), (R,0), (S,T)$. Then we can take $a = \frac{r}{R} = \frac{T}{t}$ and $b = \frac{S - \frac{s}{a}}{t}$, and the corresponding affine transformation f satisfies $f(T_1) = T_2$. ∎

The set $VL_2(\mathbb{Q})$ is not empty, because $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in VL_2(\mathbb{Q})$. Obviously, for any rational triangle T, $f_I(T) = T$.

For any $f \in V$, there is a rational triangle T such that $f(T) \sim T$. Obviously, $f^{-1}(f(T)) = T$ is weakly metric equivalent to $f(T)$. Therefore, for any $f \in U_T$, $f^{-1} \in U_{f(T)} \subset V$.

However, the set $VL_2(\mathbb{Q})$ may not form a group, because it is not clear whether the composition of weak metric affine transformation is closed. Taking any $f, g \in V$ such that $f(T_1) \sim T_1$ and $g(T_2) \sim T_2$. The question is whether we can find a rational triangle T such that $g(f(T)) \sim T$ or

$f(g(T)) \sim T$. If there is a positive answer to this question, the set $VL_2(\mathbb{Q})$ can form a group.

Taking any $f \in V$, there is a rational triangle T such that $f(T) \sim T$. Assuming that rational triangle T' is similar to T, then $f(T') \sim T'$.

For the set $VL_2(\mathbb{Q})$ or V, we have the following two basic questions:

(1) Given a rational triangle T, is there a $f \in V, f \neq I$, such that $f(T) \sim T$? In other words, in the set U_T, besides the element I, are there any other elements?

(2) Given any $f \in V, f \neq I$, such that $f(T) \sim T$, is there a rational triangle $T' \in \mathfrak{T}$ that is not similar to T such that $f(T') \sim T'$? That is, for two different and non-similar T_1 and T_2, in the intersection of the sets U_{T_1} and U_{T_2}, besides the element I, are there any other elements?

To solve question (1), we give the following two lemmas:

Lemma 3.6. [5] *Given any scalene rational triangle T_1, whose sides do not, in any order, satisfy the Diophantine equation*

$$\begin{aligned} &x^4 - x^3y - 2x^3z + 3x^2yz - xy^3 + 3xy^2z \\ &- 6xyz^2 + 3xz^3 + y^4 - 2y^3z + 3yz^3 - 2z^4 = 0, \end{aligned} \tag{3.1}$$

there exists a second rational triangle T_2 such that $T_1 \sim T_2$.

The result of Lemma 3.6 [5] explicitly constructs the three sides of T_2 from the three sides of triangle T_1, and proves that T_1 and T_2 are not congruent. However, the explicit expressions are too complex to be presented here.

Lemma 3.7. [5] *Given any scalene rational triangle T_1, there exist infinitely many rational triangles that are weakly metric equivalent to T_1.*

Lemma 3.6 requires that the three sides of the scalene triangle T_1 do not, in any order, satisfy Eq. (3.1). In fact, this condition can be removed. We first present the following lemma.

Lemma 3.8. ([7],[22, Theorem 5.3 (b)]) *Let C be a curve of genus $g \geq 2$ over $\mathbb{Q}$, and J be its Jacobian variety. Let p be a prime number. Suppose that the rank of $J(\mathbb{Q})$ is smaller than g, and $p > 2g$. Then, we have*

$$\#C(\mathbb{Q}) \leq \#C(\mathbb{F}_p) + (2g-2),$$

where $\#C(\mathbb{Q})$ denotes the number of rational points of C over $\mathbb{Q}$, and $\#C(\mathbb{F}_p)$ denotes the number of rational points of C over $\mathbb{F}_p$.

Remark 3.9. In the same paper, Choudhry [5] gives all the rational solutions of Eq. (3.1) as follows:

$$\begin{aligned} x &= \rho(m+n)(2m^3 + m^2n + 2mn^2 - n^3), \\ y &= \rho(m-n)(2m^3 - m^2n + 2mn^2 + n^3), \\ z &= 2\rho(m^2 + 3n^2)m^2, \end{aligned} \tag{3.2}$$

where m, n and ρ are all rational numbers. In fact, with at most one exception, the triangle given by (3.2) is not a rational triangle. To illustrate this point, we only need to show that the area of such a triangle is not a rational number. Since (3.2) is homogeneous, we can let $\rho = n = 1$. According to Heron's formula, the square of the area of such a triangle is

$$(m^2+1)(3m^4+6m^2-1)(m-1)^4(m+1)^4m^2,\ m>1.$$

Therefore, we need $(m^2+1)(3m^4+6m^2-1)$ to be a perfect square, say w^2. This is a hyperelliptic curve

$$\mathcal{C}_0 : w^2 = (m^2+1)(3m^4+6m^2-1).$$

This is a curve of genus 2. By the package of Magma, the Mordell-Weil rank of the Jacobian variety of $\mathcal{C}_0$ over $\mathbb{Q}$ is at most 1. Here are the inputs and outputs:

```
> _<m>:=PolynomialRing(Rationals());
> C:=HyperellipticCurve((m^2 + 1)*(3*m^4 + 6*m^2 - 1));
> ptsC:=RationalPoints(C:Bound:=1000000);
> ptsC;
{@ (-1 : -4 : 1), (-1 : 4 : 1), (1 : -4 : 1), (1 : 4 : 1) @}
> J:=Jacobian(C);
> RankBound(J);
1
```

We check that $\mathcal{C}_0$ has good reduction at 7 and $\#\mathcal{C}_0(\mathbb{F}_7) = 4$. Therefore, by applying Lemma 3.8, we obtain that $\#\mathcal{C}_0(\mathbb{Q}) \leq 6$. Excluding the four points $(\pm1, \pm4)$, the curve $\mathcal{C}_0$ contains at most two more rational points (which may not exist at all). If they exist, one satisfies $m > 0$ and the other satisfies $m < 0$. Obviously, $(\pm1, \pm4)$ are degenerate points. Therefore, with at most one exception, Lemma 3.6 holds for any scalene rational triangle.

Lemma 3.6 gives a positive answer to question (1), and Lemma 3.7 shows that there can be an infinite number of such $f \in U_T$. By Lemma 3.7, we give the concept of weak similarity of affine transformation.

DEFINITION 3.10. *Let $f, g \in V$, if there is a rational triangle T and $h \in V$ such that $f(T) = h(g(T))$, then f and g are weakly similar, we write it as $f \sim g$.*

THEOREM 3.11. *For any $T \in \mathfrak{T}$, if $f \in U_T$ and $g \in U_T$, then f and g are weakly similar. The converse also holds.*

PROOF. Since $f(T) \sim T$ and $g(T) \sim T$, then $f(T) \sim g(T)$. By Theorem 3.5, there exists $h \in U_{g(T)} \subset V$ such that $f(T) = h(g(T))$. The converse also holds. □

For question (2), it is easy to see that $f(T) \sim T$ for any rational triangle T when $f = f_I$. For a general $f \in V$, there should be $f(T) \sim T$ only for some specific rational triangles T. Therefore, we have the following conjecture:

Conjecture 3.12. *If $f \in V, f \neq f_I$, there are only a finite number of dissimilar rational triangles $T \in \mathfrak{T}$ such that $f(T) \sim T$. In other words, for two different and non-similar T_1 and T_2, the set $U_{T_1} \cap U_{T_2}$ is a finite set.*

It is not clear whether the set V is closed under the composition of weakly metric affine transformation. However, in some special cases, this is possible. By lemma 3.7, we can get a sequence of rational triangles $\{T_n\}, n \geq 1$, in which any two rational triangles are weakly metric equivalent. By Theorem 3.5, we can use $\{T_n\}$ to construct a sequence $\{f_n\}$ such that $f_i(T_i) = T_{i+1}, i \geq 1$. We have the following theorem:

Theorem 3.13. *If $f_1, \cdots, f_k \in \{f_n\}$, $k \geq 1$, then $f_k \cdots f_1 \in V$. Generally, $f_k \cdots f_i \in V$, $k \geq i \geq 1$.*

We have a certain grasp of the properties of the set $VL_2(\mathbb{Q})$, but the conditions satisfied by a and b are not specific. Next, we show that the values of a and b are determined by a Diophantine system.

For a rational triangle $T \in \mathfrak{T}$, the coordinates of the three vertices of T are

$$V_1 = (0,0),\ V_2 = (r,0),\ V_3 = (s,t),$$

where $r \in \mathbb{Q}^*, s \in \mathbb{Q}_{\geq 0}, t \in \mathbb{Q}^*$. Thus, the sides of the rational triangle T are

$$|V_1V_2| = r,\ |V_1V_3| = \sqrt{s^2+t^2},\ |V_2V_3| = \sqrt{(s-r)^2+t^2}.$$

Since the sides of rational triangle T are rationals, we thus have

$$s^2+t^2 = \square,\ (s-r)^2+t^2 = \square,\ r \in \mathbb{Q}^*, s \in \mathbb{Q}_{\geq 0}, t \in \mathbb{Q}^*.$$

By the transformation f, the coordinates of the three vertices of $f(T)$ are

$$f(V_1) = (0,0),\ f(V_2) = \left(\frac{r}{a}, 0\right),\ f(V_3) = \left(\frac{s}{a}+bt, at\right).$$

Then, the sides of the rational triangle $f(T)$ are

$$|f(V_1)f(V_2)| = \frac{r}{a},\ |f(V_1)f(V_3)| = \sqrt{\left(\frac{s}{a}+bt\right)^2+(at)^2},$$
$$|f(V_2)f(V_3)| = \sqrt{\left(\frac{s-r}{a}+bt\right)^2+(at)^2}.$$

To ensure that the sides of $f(T)$ are rational, we have

$$\left(\frac{s}{a}+bt\right)^2+(at)^2 = \square,\ \left(\frac{s-r}{a}+bt\right)^2+(at)^2 = \square.$$

Since T and $f(T)$ are weakly metric equivalent, they have the same perimeter, i.e.,

$$r+\sqrt{s^2+t^2}+\sqrt{(s-r)^2+t^2}=\frac{r}{a}+\sqrt{\left(\frac{s}{a}+bt\right)^2+(at)^2}+\sqrt{\left(\frac{s-r}{a}+bt\right)^2+(at)^2}.$$

Therefore, for any rational triangle $T \in \mathfrak{T}$, its corresponding weakly metric affine transformation f is completely characterized by the following Diophantine system

$$\begin{cases} s^2+t^2=w_1^2, \\ (s-r)^2+t^2=w_2^2, \\ \left(\frac{s}{a}+bt\right)^2+(at)^2=w_3^2, \\ \left(\frac{s-r}{a}+bt\right)^2+(at)^2=w_4^2, \\ r+w_1+w_2=\frac{r}{a}+w_3+w_4, \end{cases} \tag{3.3}$$

where $b \in \mathbb{Q}$, $s \in \mathbb{Q}_{\geq 0}$, $a, r, t, w_1, w_2, w_3, w_4 \in \mathbb{Q}^*$.

THEOREM 3.14. *For any pair of rational triangles T_1 and T_2 that are weakly metrically equivalent, there is at least one rational solution of Eq. (3.3) corresponding to them.*

PROOF. It is obvious from Theorem 3.5. □

Assume that r, s and t satisfy the first two equations, then $(a, b) = (1, 0)$ is a solution of Eq. (3.3). The corresponding affine transformation is f_I. Moreover, $(a, b) = \left(1, \frac{r-2s}{t}\right)$ is also a solution of Eq. (3.3). This corresponds to reflecting the rational triangle T about the line $x = \frac{r}{2}$.

Suppose (a, b, r, s, t) is any solution of Eq. (3.3) and k is any positive rational number, then (a, b, kr, ks, kt) is also a solution of Eq. (3.3), which means that Eq. (3.3) is homogeneous with respect to r, s and t.

THEOREM 3.15. *Assume that r, s, t satisfy the first two equations, and that the corresponding rational triangle T_1 is scalene. Then there are infinitely many pairs (a, b) that satisfy Eq. (3.3).*

PROOF. It can be obtained from Lemmas 3.6 and 3.7. □

Let S be the solution set of Eq. (3.3). In particular, let $S_{(r,s,t)}$ be the solution set with respect to (a, b). Thus, the set $VL_2(\mathbb{Q})$ can be expressed as

$$VL_2(\mathbb{Q}) = \{M \mid M \in UL_2(\mathbb{Q}), (a, b) \in S_{(r,s,t)}\},$$

which gives an accurate description of the set $VL_2(\mathbb{Q})$.

Now we discuss whether the composition of weakly metric affine transformations in the set V is closed according to the solution set S. The question can be described as: Suppose $(a_1, b_1, r_1, s_1, t_1) \in S$ and $(a_2, b_2, r_2, s_2, t_2) \in S$, if $(a, b) = \left(a_1a_2, \frac{b_2}{a_1} + a_2b_1\right)$ or $(a, b) = \left(a_1a_2, \frac{b_1}{a_2} + a_1b_2\right)$, does Eq. (3.3) have a rational solution? If there is a positive answer to this question, the set $VL_2(\mathbb{Q})$ can obviously form a group. Thus, we have the following question.

Question 3.16. Does the set $VL_2(\mathbb{Q})$ form a group?

4. Concrete weakly metric affine transformation

In this section, we will discuss two problems. One is using the set $VL_2(\mathbb{Q})$ to uniformly describe the existing results that two rational triangles with the same area and the same perimeter. The other is applying the set $VL_2(\mathbb{Q})$ to find more weakly metric equivalent rational triangles.

4.1. *Existing results.* We first give a unified description of the existing results, focusing on the following four cases.

Case 4.1.1. T and $f(T)$ are right triangles. This result was originally obtained by Prielipp [23]; we provide a new proof here.

Theorem 4.1. [23] *There are no two distinct right triangles that are weakly metric equivalent.*

Proof. After an isometric transformation, we can always align the two legs of any right triangle exactly with the x-axis and y-axis, which gives $s = 0$. Therefore, without loss of generality, we may assume that $r = 2km, s = 0, t = k(m^2 - 1), k > 0, m > 1$. Then the coordinates of the three vertices of T are

$$V_1 = (0, 0),\ V_2 = (2km, 0),\ V_3 = (0, k(m^2 - 1)).$$

The coordinates of the three vertices of $f(T)$ are

$$f(V_1) = (0, 0),\ f(V_2) = \left(\frac{2km}{a}, 0\right),\ f(V_3) = \left(bk(m^2 - 1), ak(m^2 - 1)\right).$$

Since $f(T)$ is a right triangle, we can take $b = 0$, then $w_3 = ak(m^2 - 1)$. Eq. (3.3) now becomes

$$\begin{cases} \dfrac{4k^2m^2}{a^2} + a^2k^2(m^2 - 1)^2 = w_4^2, \\ 2km(m + 1) = \dfrac{2km}{a} + ak(m^2 - 1) + w_4. \end{cases} \tag{4.1}$$

Eliminating w_4 in Eq. (4.1), we have

$$(a - 1)((m^2 - 1)a - 2m) = 0.$$

Then $a = 1$ or $a = \frac{2m}{m^2-1}$. Obviously, when $a = 1$ or $a = \frac{2m}{m^2-1}$, $f(T)$ and T are congruent. Therefore, there are no two distinct right triangles are weakly metric equivalent, which is obtained by Prielipp [23]. ∎

Case 4.1.2. T and $f(T)$ are isosceles triangles. This result was originally obtained by Bradley [1] and Yiu [26]; we provide a new proof here.

THEOREM 4.2. [1, 26] *There are no three isosceles triangles that are weakly metric equivalent at the same time.*

PROOF. After an isometric transformation, we can always align the base of any isosceles triangle exactly with the x-axis, in which case $s = \frac{r}{2}$.

Case 4.1.2.1. Take $r = 4km, s = 2km, t = k(m^2 - 1), k > 0, m > 1$, then the coordinates of the three vertices of T are

$$V_1 = (0, 0),\ V_2 = (4km, 0),\ V_3 = (2km, k(m^2 - 1)).$$

The coordinates of the three vertices of $f(T)$ are

$$f(V_1) = (0, 0),\ f(V_2) = \left(\frac{4km}{a}, 0\right),\ f(V_3) = \left(\frac{2km}{a} + bk(m^2 - 1), ak(m^2 - 1)\right).$$

Since $f(T)$ is an isosceles triangle, we can take $b = 0$, then $w_3 = w_4$. Eq. (3.3) now becomes

$$\begin{cases} \dfrac{4k^2m^2}{a^2} + a^2k^2(m^2 - 1)^2 = w_4^2, \\ 2k(m + 1)^2 = \dfrac{4km}{a} + 2w_4. \end{cases} \tag{4.2}$$

Eliminating w_4 in Eq. (4.2), we have

$$(a - 1)((m - 1)^2a^2 + (m - 1)^2a - 4m) = 0.$$

Then

$$a = 1 \text{ or } a = \frac{1 - m \pm \sqrt{m^2 + 14m + 1}}{2(m - 1)}.$$

When $a = 1$, $f = f_I$, this is a trivial case. Since $a \in \mathbb{Q}^*$, then a has at most one nontrivial positive rational solution, which leads to the fact that there are no three isosceles triangles are weakly metric equivalent at the same time.

Case 4.1.2.2. Take $r = 2k(m^2 - 1), s = k(m^2 - 1), t = 2km, k > 0, m > 1$, then the coordinates of the three vertices of T are

$$V_1 = (0, 0),\ V_2 = (2k(m^2 - 1), 0),\ V_3 = (k(m^2 - 1), 2km).$$

The coordinates of the three vertices of $f(T)$ are

$$f(V_1) = (0, 0),\ f(V_2) = \left(\frac{2k(m^2 - 1)}{a}, 0\right),\ f(V_3) = \left(\frac{k(m^2 - 1)}{a} + 2bkm, 2akm\right).$$

Since $f(T)$ is an isosceles triangle, we can take $b = 0$, then $w_3 = w_4$. Eq. (3.3) now becomes

$$\begin{cases} \dfrac{k^2(m^2 - 1)^2}{a^2} + 4a^2k^2m^2 = w_4^2, \\ 4km^2 = \dfrac{2k(m^2 - 1)}{a} + 2w_4. \end{cases} \tag{4.3}$$

Eliminating w_4 in Eq. (4.3), we have

$$(a-1)(a^2+a-(m^2-1))=0.$$

Then

$$a=1 \text{ or } a=\frac{-1\pm\sqrt{4m^2-3}}{2}.$$

Hence, a has at most one nontrivial positive rational solution, then there are no three isosceles triangles are weakly metric equivalent at the same time, this case was studied by Bradley [1] and Yiu [26]. ∎

Case 4.1.3. T is a right triangle and $f(T)$ is an isosceles triangle. This result was obtained by Hirakawa and Matsumura [12, 13], and we present a new proof here.

THEOREM 4.3. [12, 13] *There exists only one pair of right triangle and isosceles triangle that are weakly metric equivalent.*

PROOF. After isometric transformation, we can always align the two legs of right triangle T with the x-axis and y-axis, and place the base of isosceles triangle $f(T)$ on the x-axis.

Take $r=2km, s=0, t=k(m^2-1), k>0, m>1$. Then the coordinates of the three vertices of T are

$$V_1=(0,0),\ V_2=(2km,0),\ V_3=(0,k(m^2-1)).$$

The coordinates of the three vertices of $f(T)$ are

$$f(V_1)=(0,0),\ f(V_2)=\left(\frac{2km}{a},0\right),\ f(V_3)=\left(bk(m^2-1),ak(m^2-1)\right).$$

Since $f(T)$ is an isosceles triangle, we can take $bk(m^2-1)=\frac{km}{a}$, then $w_3=w_4$. Eq. (3.3) now becomes

$$\begin{cases} \dfrac{k^2m^2}{a^2}+a^2k^2(m^2-1)^2=w_4^2, \\ 2km(m+1)=\dfrac{2km}{a}+2w_4. \end{cases} \tag{4.4}$$

Eliminating w_4 in Eq. (4.4), we have

$$(m+1)(m-1)^2a^3-m^2(m+1)a+2m^2=0. \tag{4.5}$$

By the map

$$U=\frac{a(m-1)}{m},\quad V=-\frac{2(m^2-2m-1)}{m(m+1)},$$

we can transform Eq. (4.5) into the hyperelliptic curve

$$\mathcal{C}_1: V^2=U^6-2U^4+12U^3+U^2-12U+4.$$

This is a curve of genus 2. By the package of Magma, the Mordell-Weil rank of the Jacobian variety of $\mathcal{C}_1$ over $\mathbb{Q}$ is at most 1. Here are the inputs and outputs:

```
> _<U>:=PolynomialRing(Rationals());
> C:=HyperellipticCurve(U^6 - 2*U^4 + 12*U^3 + U^2  - 12*U + 4);
> ptsC:=RationalPoints(C:Bound:=1000);
> ptsC;
{@ (1 : -1 : 0), (1 : 1 : 0), (-1 : -2 : 1), (-1 : 2 : 1),
(0 : -2 : 1), (0 : 2 : 1), (1 : -2 : 1), (1 : 2 : 1),
(5 : -217 : 6), (5 : 217 : 6) @}
> J:=Jacobian(C);
> RankBound(J);
1
```

Finally, we check that $\mathcal{C}_1$ has good reduction at 5 and $\#\mathcal{C}_1(\mathbb{F}_5) = 8$. Therefore, by applying Lemma 3.8, we obtain that $\#\mathcal{C}_1(\mathbb{Q}) \leq 10$. Hence, the only finite rational points in $\mathcal{C}_1$ are

$$(0, \pm 2),\ (\pm 1, \pm 2),\ \left(\frac{5}{6}, \pm\frac{217}{216}\right).$$

Obviously, $(0, \pm 2)$ and $(\pm 1, \pm 2)$ are degenerate points. From the rational points $\left(\frac{5}{6}, \pm\frac{217}{216}\right)$, we obtain

$$(a, m) = \left(\frac{8}{3}, \frac{16}{11}\right) \text{ or } (a, m) = \left(\frac{45}{44}, \frac{27}{5}\right).$$

When $k = 121$ or $k = \frac{25}{2}$ respectively, it corresponds to the unique pair of right triangle and isosceles triangle obtained by Hirakawa and Matsumura [12, 13], i.e., $(135, 352, 377), (132, 366, 366)$. ∎

Case 4.1.4. T and $f(T)$ are rational triangles. We present a result originally obtained by Choudhry [5] and give a new proof herein.

THEOREM 4.4. [5] *There exist infinitely many pairs of rational triangles that are weakly metric equivalent.*

PROOF. In this case, we only give the affine transformation corresponding to a parameter solution obtained in [5]. The sides of T and $f(T)$ are given respectively by

$$\begin{aligned}
a_1 &= (R^2 + 1)(S^2 + S + 1),\\
b_1 &= S(S + 1)(R^2 + S^2 + S + 1),\\
c_1 &= R^2 + (S^2 + S + 1)^2,
\end{aligned}$$

and

$$a_2 = (R^2 + S^2)(S^2 + S + 1),$$
$$b_2 = (S + 1)(R^2 + S^2 + S + 1),$$
$$c_2 = R^2S^2 + (S^2 + S + 1)^2,$$

where R, S are arbitrary positive rational numbers. Take

$$r = R^2 + (S^2 + S + 1)^2,$$
$$s = \frac{S(S+1)(R^2 + S^2 + S + 1)(S^2 + R + S + 1)(S^2 - R + S + 1)}{S^4 + 2S^3 + R^2 + 3S^2 + 2S + 1},$$
$$t = \frac{(2S^2 + 2S + 2)(R^2 + S^2 + S + 1)RS(S+1)}{S^4 + 2S^3 + R^2 + 3S^2 + 2S + 1}.$$

Then

$$a = \frac{S^4 + 2S^3 + R^2 + 3S^2 + 2S + 1}{R^2S^2 + S^4 + 2S^3 + 3S^2 + 2S + 1},$$
$$b = \frac{(S-1)b(R,S)}{2SR(S^4 + 2S^3 + R^2 + 3S^2 + 2S + 1)(R^2S^2 + S^4 + 2S^3 + 3S^2 + 2S + 1)},$$

where

$$b(R,S) = R^6S^2 - (S^2 + S + 1)(S^4 + S^3 - S^2 + S + 1)R^4 - (2S^2 + 3S + 2)(S^2 + S + 1)^3R^2 - (S^2 + S + 1)^5.$$

□

4.2. *New parametric solutions of Eq. (3.3).* We can get more solutions by directly solving Eq. (3.3). We are concerned with the following two cases.

Case 4.2.1. $s = 0$, T is a right triangle.

THEOREM 4.5. *There exist infinitely many pairs of right triangles and rational triangles that are weakly metric equivalent.*

PROOF. Take $r = 2km, s = 0, t = k(m^2 - 1), k > 0, m > 1$, then

$$w_1 = k(m^2 - 1),\ w_2 = k(m^2 + 1).$$

From the third equation of Eq. (3.3), we have

$$k^2(m^2 - 1)^2(a^2 + b^2) = w_3^2.$$

Then

$$a = 2cp,\ b = c(p^2 - 1),\ w_3 = ck(m^2 - 1)(p^2 + 1),\ c > 0,\ p > 1.$$

Eq. (3.3) now becomes

$$\begin{cases} \left(ck(p^2-1)(m^2-1)-\dfrac{km}{cp}\right)^2+4c^2p^2k^2(m^2-1)^2=w_4^2, \\ 2km(m+1)=\dfrac{km}{cp}+ck(m^2-1)(p^2+1)+w_4. \end{cases} \tag{4.6}$$

Eliminating w_4 in Eq. (4.6), we have

$$p(p^2+1)(m^2-1)c^2-p(m^2+(p+1)m-p)c+m=0, \tag{4.7}$$

then

$$c=\frac{p((m-1)p+m(m+1))\pm\sqrt{\Delta(c)}}{2p(p^2+1)(m^2-1)},$$

where

$$\Delta(c)=p^2m^4-2p(p^2-p+2)m^3+p^2(p^2+1)m^2-2p(p^3-p^2-2)m+p^4.$$

To get the rational solutions of Eq. (4.7) with respect to c, it needs $\Delta(c)$ be a perfect square, say w^2. Using a method described by Fermat (see [9, p. 639]), it's easy to get the following solutions

$$m=\frac{p^2+p+1}{p^2},\ m=\frac{p^2-p+1}{p(p-2)},\ m=\frac{p^3-p^2-1}{p^2(p+1)}$$

such that $\Delta(c)$ is a perfect square. Take $m=\frac{p^2+p+1}{p^2}$, by Eq. (4.7), we get

$$c=\frac{p}{p^2+1},\ c=\frac{p^2+p+1}{2p^3+3p^2+2p+1}.$$

Take $c=\frac{p}{p^2+1}$, then

$$\begin{aligned} &s=0,\ r=\frac{2k(p^2+p+1)}{p^2},\ t=\frac{k(p+1)(2p^2+p+1)}{p^4}, \\ &a=\frac{2p^2}{p^2+1},\ b=\frac{p(p^2-1)}{p^2+1}, \\ &w_1=\frac{k(p+1)(2p^2+p+1)}{p^4},\ w_2=\frac{k(p^2+1)(2p^2+2p+1)}{p^4}, \\ &w_3=\frac{k(p+1)(2p^2+p+1)}{p^3},\ w_4=\frac{(p^4+2p^3+4p^2+2p+1)k}{p^4}, \end{aligned}$$

this is a parametric solution of Eq. (3.3). When $k>0$ and $p>1$, the parameters r,t,a,w_1,w_2,w_3,w_4 are all positive, which implies that both the rational triangles T and $f(T)$ are non-degenerate. Verified that when $k>0$ and $p>1$, the rational triangles T and $f(T)$ are distinct.

Let us consider the rational points on the quartic curve

$$\mathcal{C}: w^2=\Delta(c).$$

By the map φ:

$$X = \frac{3p(p(p^2+1)m^2 - 6(p^3-p^2-2)m + 6p^3 + 6pw)}{m^2},$$
$$Y = -\frac{Y(m,w)}{m^3},$$

where

$$Y(m,w) = 54p(p^2((p^2-p+2)m^3 - p(p^2+1)m^2 + 3(p^3-p^2-2)m - 2p^3) + ((p^3-p^2-2)m - 2p^3)w).$$

We can transform $\mathcal{C}$ into the elliptic curve

$$\mathcal{E}_p:\ Y^2 = X^3 - 27p^2(p^6 - 12p^5 + 38p^4 - 36p^3 + 49p^2 - 24p + 48)X + 54p^4(p^3 - 6p^2 + p - 12)(p^5 - 12p^4 + 38p^3 - 36p^2 + p - 24).$$

The discriminant of $\mathcal{E}_p$ is

$$\Delta(p) = 544195584p^6(p^4 - 10p^3 + 17p^2 + 8p + 16)(p-1)^2(p^2+1)^2.$$

Hence, $\Delta(p) \neq 0$, so $\mathcal{E}_p$ is non-singular.

It is easy to check that the elliptic curve $\mathcal{E}_p$ contains two rational points

$$Q = (3p(p^3 - 6p^2 + p - 12), 0),\ P = (3p^2(p^2 - 6p + 1), 108p^2(p+1)).$$

By the inverse map φ^{-1}, we have

$$m = \frac{6p(Yp - 3(p^3-p^2-2)X + 9p^2(p^5 - 7p^4 + 7p^3 - 15p^2 - 2))}{(X - 3p^2(p^2-6p+1))(X - 3p^2(p^2+6p+1))}.$$

We can get more parametric solutions by the group law, such as

$$[2]P = \Bigg(\frac{3(p^6 - 4p^5 + 38p^4 - 16p^3 + 25p^2 - 12p + 12)}{(p+1)^2}, \frac{108(2p^2+1)(2p^5 - 4p^4 + 5p^3 - 2p^2 + 3p - 2)}{(p+1)^3}\Bigg).$$

From the rational point $[2]P$, we obtain

$$m = \frac{p(p+1)}{p^2-p+1},$$

then

$$c = \frac{p+1}{2p^2+1},\ c = \frac{p^2-p+1}{2p^3-p^2+2p-1}.$$

Take $c = \frac{p+1}{2p^2+1}$, then

$$\begin{aligned}
&s = 0,\ r = \frac{2kp(p+1)}{p^2-p+1},\ t = \frac{k(2p-1)(2p^2+1)}{(p^2-p+1)^2},\\
&a = \frac{2(p+1)p}{2p^2+1},\ b = \frac{(p+1)^2(p-1)}{2p^2+1},\\
&w_1 = \frac{k(2p-1)(2p^2+1)}{(p^2-p+1)^2},\ w_2 = \frac{k(2p^4+4p^2-2p+1)}{(p^2-p+1)^2},\\
&w_3 = \frac{(p^2+1)(2p-1)k(p+1)}{(p^2-p+1)^2},\ w_4 = \frac{kp(5p^2-2p+2)}{(p^2-p+1)^2}.
\end{aligned}$$

When $k > 0$ and $p > 1$, the parameters $r, t, a, w_1, w_2, w_3, w_4$ are all positive, which implies that both the rational triangles T and $f(T)$ are non-degenerate. We check that the rational triangles T and $f(T)$ are distinct provided $p \neq 2$. We have verified that these parametric solutions are not presented in existing literature. ∎

Case 4.2.2. $s \neq 0$, T is a rational triangle.

THEOREM 4.6. *There exist infinitely many pairs of rational triangles that are weakly metric equivalent.*

PROOF. Take $s = 2km, t = k(m^2-1), k > 0, m > 1$, then

$$w_1 = k(m^2+1).$$

From the second equation of Eq. (3.3), we have

$$r = \frac{2k((m^2+1)c-2m)}{c^2-1},\ w_2 = \frac{k((m^2+1)c^2-4mc+m^2+1)}{c^2-1},$$

where c is a rational number.

From the third equation of Eq. (3.3), we have

$$\left(\frac{2km}{a} + bk(m^2-1)\right)^2 + a^2k^2(m^2-1)^2 = w_3^2.$$

Let

$$\frac{2km}{a} + bk(m^2-1) = d(p^2-1),\ ak(m^2-1) = 2dp,$$

then

$$a = \frac{2dp}{k(m^2-1)},\ b = \frac{p(p^2-1)d^2 - k^2m(m^2-1)}{dkp(m^2-1)},\ w_3 = d(p^2+1),$$

where $p>1$. Eq. (3.3) now becomes

$$\begin{cases} \left(\dfrac{d^2p(p^2-1)(c^2-1)-(m^2-1)((m^2+1)c-2m)k^2}{(c^2-1)dp}\right)^2+4d^2p^2=w_4^2, \\ \dfrac{2k((m^2+1)c-2m)}{c-1}=\dfrac{(m^2-1)((m^2+1)c-2m)k^2}{(c^2-1)dp}+d(p^2+1)+w_4. \end{cases} \tag{4.8}$$

Eliminating w_4 in Eq. (4.8), we have

$$\begin{aligned} &dp((p^2+1)d-k(m^2+1))c^2 \\ &-k(m-1)(p((m+1)p+m-1)d-k(m+1)(m^2+1))c \\ &-p(p^2+1)d^2+kp((m^2-1)p+2m)d-2k^2m(m^2-1)=0, \end{aligned} \tag{4.9}$$

then

$$c=\frac{(m^4-1)k^2-p((m^2-1)p+(m-1)^2)k\pm\sqrt{\Delta(c)}}{2p(k(m^2+1)-(p^2+1))},$$

where

$$\begin{aligned} \Delta(c)=&(m^4-1)^2k^4-2dp(m^4-1)((m^2-1)p+(m+1)^2)k^3 \\ &+d^2p((m^2-1)^2p^3+2(m^2-1)(3m^2+2m+3)p^2 \\ &+(m+1)^4p+8m(m^2-1))k^2 \\ &-4d^3p^2(p^2+1)((m^2-1)p+(m+1)^2)k+4d^4p^2(p^2+1)^2. \end{aligned}$$

To get the rational solutions of Eq. (4.9) with respect to c, it needs $\Delta(c)$ be a perfect square, say w^2. Since the discriminant $\Delta(c)$ is homogeneous with respect to k and d, we can let $d=1$. Using a method described by Fermat (see [9, p. 639]), it's easy to get the following solution

$$k=\frac{2(m^2+1)p^2+(m+1)^2}{(m^2+1)((m^2-1)p+(m+1)^2)}$$

such that $\Delta(c)$ is a perfect square. By Eq. (4.9), we get

$$c=\frac{(m+1)(m^2+1)p^2+(m-1)(m^2+1)p+2m(m+1)}{(m^2+1)((m+1)p^2-(m-1)p+m+1)}.$$

By some calculations, we obtain

$$\begin{aligned}
r &= \frac{2((m+1)p^2-(m-1)p+m+1)(m^2+1)p}{(m+1)(2(m^2+1)p-m^2+1)},\\
s &= \frac{2(2(m^2+1)p^2+(m+1)^2)m}{(m^2+1)((m^2-1)p+(m+1)^2)},\\
t &= \frac{(m-1)(2(m^2+1)p^2+(m+1)^2)}{((m-1)p+m+1)(m^2+1)},\\
a &= \frac{2((m-1)p+m+1)(m^2+1)p}{(m-1)(2(m^2+1)p^2+(m+1)^2)},\\
b &= \frac{b(m,p)}{(m^4-1)(2(m^2+1)p^2+(m+1)^2)((m^2-1)p+(m+1)^2)p},
\end{aligned}$$

where

$$\begin{aligned}
b(m,p) = {} & (m^4-1)^2p^5+2(m^2-1)(m^2+1)^3p^4+4(m+1)^2m(m^2+1)^2p^3\\
& -2(m+1)^4(m^4-1)p^2-(m+1)^4(m^2+1)^2p-(m+1)^5m(m-1).
\end{aligned}$$

And

$$\begin{aligned}
w_1 &= \frac{2(m^2+1)p^2+(m+1)^2}{(m+1)((m-1)p+m+1)},\\
w_2 &= \frac{(m-1)(2(m^2+1)p^4+4(m^2+m+1)p^2-2(m^2-1)p+(m+1)^2)}{(2(m^2+1)p-(m^2-1))((m-1)p+m+1)},\\
w_3 &= p^2+1,\\
w_4 &= \frac{((5m^4+6m^2+5)p^2-2(m+1)(m-1)^3p+2(m^2+1)(m+1)^2)p}{(m+1)(2(m^2+1)p-(m^2-1))((m-1)p+m+1)},
\end{aligned}$$

this is a parametric solution of Eq. (3.3). When $m>1$ and $p>1$, the parameters r,t,a,w_1,w_2,w_3,w_4 are all positive, which implies that both the rational triangles T and $f(T)$ are non-degenerate. After verification, when $p\neq\frac{m+1}{m-1}$ and $p\neq\frac{2(m-1)}{m+1}$, the rational triangles T and $f(T)$ are distinct.

Let us consider the rational points on the quartic curve

$$\mathcal{C}: w^2=\Delta(c).$$

By the map φ:

$$\begin{aligned}
X = {} & \frac{3pX(k,w)}{k^2},\\
Y = {} & -\frac{108p^2(p^2+1)Y(k,w)}{k^3},
\end{aligned}$$

where

$$\begin{aligned}
X(k,w) = {} & ((m^2-1)^2p^3 + 2(m^2-1)(3m^2+2m+3)p^2 + (m+1)^4p \\
& + 8m(m^2-1))k^2 - 12p(p^2+1)((m^2-1)p + (m+1)^2)k \\
& + 24p(p^2+1)^2 - 12(p^2+1)w, \\
Y(k,w) = {} & (m^4-1)((m^2-1)p + (m+1)^2)k^3 - ((m^2-1)^2p^3 \\
& + 2(m^2-1)(3m^2+2m+3)p^2 + (m+1)^4p + 8m(m^2-1))k^2 \\
& + 6p(p^2+1)((m^2-1)p + (m+1)^2)k - 8p(p^2+1)^2 \\
& - (((m^2-1)p + (m+1)^2)k - 4(p^2+1))w.
\end{aligned}$$

We can transform $\mathcal{C}$ into the elliptic curve

$$\mathcal{E}_{(m,p)}:\ Y^2 = X^3 - 27p^2A_4(m,p)X + 54p^3A_6(m,p),$$

where

$$\begin{aligned}
A_4(m,p) = {} & (m^2-1)^4p^6 - 4(3m^2-2m+3)(m^2-1)^3p^5 \\
& + 2(19m^4 - 20m^3 + 50m^2 - 20m + 19)(m^2-1)^2p^4 \\
& - 4(m+1)^3(m-1)(3m^2-2m+3)^2p^3 \\
& + (m+1)^2(49m^6 - 90m^5 + 223m^4 - 300m^3 + 223m^2 - 90m + 49)p^2 \\
& - 8(m+1)^5(m-1)(3m^2-2m+3)p + 16(m^2+3)(3m^2+1)(m^2-1)^2, \\
A_6(m,p) = {} & ((m^2-1)^2p^3 - 2(m^2-1)(3m^2-2m+3)p^2 + (m+1)^4p \\
& - 4(m^2-1)(3m^2-2m+3))((m^2-1)^4p^6 \\
& - 4(3m^2-2m+3)(m^2-1)^3p^5 \\
& + 2(19m^4 - 20m^3 + 50m^2 - 20m + 19)(m^2-1)^2p^4 \\
& - 4(m+1)^3(m-1)(3m^2-2m+3)^2p^3 \\
& + (m+1)^2(m^6 + 102m^5 - 113m^4 + 84m^3 - 113m^2 + 102m + 1)p^2 \\
& - 8(m+1)^5(m-1)(3m^2-2m+3)p + 32m(3m^2+2m+3)(m^2-1)^2).
\end{aligned}$$

By some calculations, we can check that when $m \neq \frac{p+1}{p-1}$, the discriminant $\Delta(m,p)$ of $\mathcal{E}_{(m,p)}$ is not 0, then $\mathcal{E}_{(m,p)}$ is non-singular.

It is easy to check that the elliptic curve $\mathcal{E}_{(m,p)}$ contains two rational points

$$\begin{aligned}
Q &= (X_1(m,p), 0), \\
P &= (X_2(m,p), 216p^2(p^2+1)(m^4-1)((m^2-1)p + (m+1)^2)),
\end{aligned}$$

where

$$\begin{aligned}X_1(m,p) &= 3p((m^2-1)^2p^3 - 2(m^2-1)(3m^2-2m+3)p^2\\ &\quad + (m+1)^4p - 4(m^2-1)(3m^2-2m+3)),\\ X_2(m,p) &= 3p((m^2-1)^2p^3 + 2(m^2-1)(9m^2+2m+9)p^2\\ &\quad + (m+1)^4p + 4(m^2-1)(3m^2+2m+3)).\end{aligned}$$

By the inverse map φ^{-1}, we have

$$k = -\frac{12p(p^2+1)k(X,Y)}{(X-X_1(m,p))(X-X_2(m,p))},$$

where

$$\begin{aligned}k(X,Y) = Y &+ 3p((m^2-1)p + (m+1)^2)(X - 3p((m^2-1)^2p^3\\ &- 2(m^2-1)(3m^2-2m+3)p^2 + (m+1)^4p\\ &- 4(m^2-1)(3m^2-2m+3))).\end{aligned}$$

Similar to Case 4.2.1, we can get more parametric solutions by the group law. From the rational point $[2]P$, we get

$$k = \frac{2(p^2+1)p((m-1)p+m+1)}{(m-1)(2(m^2+1)p^2+(m+1)^2)},$$

then

$$c = \frac{2(m^2-1)p^2+4pm+m^2-1}{2(m^2+1)p-m^2+1}.$$

By some calculations, we obtain

$$\begin{aligned}r &= \frac{(p^2+1)(2(m^2+1)p-m^2+1)}{(m^2-1)p^2-(m-1)^2p+m^2-1},\\ s &= \frac{4(p^2+1)p((m-1)p+m+1)m}{(m-1)(2(m^2+1)p^2+(m+1)^2)},\\ t &= \frac{2(p^2+1)p((m^2-1)p+(m+1)^2)}{2(m^2+1)p^2+(m+1)^2},\\ a &= \frac{2(m^2+1)p^2+(m+1)^2}{(p^2+1)((m^2-1)p+(m+1)^2)},\\ b &= -\frac{b(m,p)}{2(m^2-1)p((m^2-1)p+(m+1)^2)(p^2+1)(2(m^2+1)p^2+(m+1)^2)},\end{aligned}$$

where

$$\begin{aligned}b(m,p) &= 4m(m^2-1)^2p^7 - 4(m^2-1)(m^4-2m^3-2m^2-2m+1)p^6\\ &\quad + 4(m+1)^2m(3m^2-2m+3)p^5 + 8m(m^2-1)(m^2+4m+1)p^4\\ &\quad + 4(m+1)^2m(3m^2+2m+3)p^3 + 3(m-1)(m+1)^5p^2\\ &\quad + 4(m+1)^4mp + (m-1)(m+1)^5.\end{aligned}$$

And

$$\begin{aligned}w_1 &= \frac{2p(p^2+1)((m-1)p+m+1)(m^2+1)}{(m-1)(2(m^2+1)p^2+(m+1)^2)},\\ w_2 &= \frac{(p^2+1)(m+1)(2(m^2+1)p^4+4(m^2+m+1)p^2-2(m^2-1)p+(m+1)^2)}{((m+1)p^2-(m-1)p+m+1)(2(m^2+1)p^2+(m+1)^2)},\\ w_3 &= p^2+1,\\ w_4 &= \frac{(p^2+1)((5m^4+6m^2+5)p^2-2(m+1)(m-1)^3p+2(m^2+1)(m+1)^2)p}{(m-1)((m+1)p^2-(m-1)p+m+1)(2(m^2+1)p^2+(m+1)^2)},\end{aligned}$$

this is a parametric solution of Eq. (3.3). When $m > 1$ and $p > 1$, the parameters $r, t, a, w_1, w_2, w_3, w_4$ are all positive, which implies that both the rational triangles T and $f(T)$ are non-degenerate. After verification, when $p \neq \frac{m+1}{m-1}$ and $p \neq \frac{2(m-1)}{m+1}$, the rational triangles T and $f(T)$ are distinct. We have verified that these parametric solutions are not presented in existing literature. □

5. TRIANGLE AND PARALLELOGRAM PAIRS

In this section, we consider the problem that triangle and parallelogram pairs have the same area and the same perimeter.

5.1. *The corresponding Diophantine system.* A triangle cannot be transformed into a parallelogram by affine transformation, but the parallelogram is composed of two congruent triangles (see Figure 2), which allows us to study this problem by the affine transformation.

Let V_1, V_2 and V_3 be the three vertices of rational triangle T, and $T \in \mathfrak{T}$. Let $f(V_1), f(V_2), f(V_3)$ and $f(V_4)$ be the four vertices of parallelogram L. Let $f(T)$ be a triangle with $f(V_1), f(V_2)$ and $f(V_3)$ as vertices. Let A and A' be the areas of rational triangle T and $f(T)$, respectively. Considering the affine transformation f from T to $f(T)$ such that $A = 2A'$, then the area of parallelogram L is equal to the area of rational triangle T. The transformation matrix M_f can be taken as

$$M_f = \begin{bmatrix} \frac{1}{2a} & b \\ 0 & a \end{bmatrix},$$

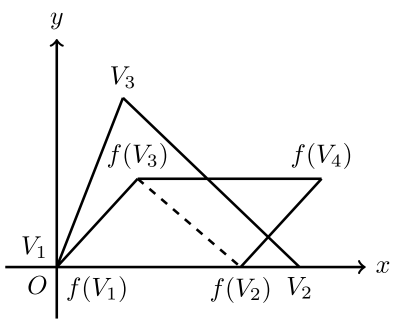

FIGURE 2. Rational triangle T and parallelogram L.

where $a \in \mathbb{Q}^*, b \in \mathbb{Q}$. In order to ensure that rational triangle T and parallelogram L have the same perimeter, we have the following Diophantine system.

$$\begin{cases} s^2 + t^2 = w_1^2, \\ (s-r)^2 + t^2 = w_2^2, \\ \left(\dfrac{s}{2a} + bt\right)^2 + (at)^2 = w_3^2, \\ r + w_1 + w_2 = 2\left(\dfrac{r}{2a} + w_3\right), \end{cases} \tag{5.1}$$

where $b \in \mathbb{Q}$, $s \in \mathbb{Q}_{\geq 0}$, $a, r, t, w_1, w_2, w_3 \in \mathbb{Q}^*$.

THEOREM 5.1. *For any pair of rational triangle and parallelogram with the same area and the same perimeter, there is at least one rational solution of Eq. (5.1) corresponding to them.*

PROOF. This is obvious. □

5.2. *Existing results.* We use Eq. (5.1) to discuss the existing results, focusing on the following two cases.

Case 5.2.1. T is a right triangle and L is a rectangle. This result was obtained by Guy [11], and we give a new proof here.

THEOREM 5.2. [11] *There is no right triangle and rectangle pair with the same area and the same perimeter.*

PROOF. Take $r = 2km, s = 0, t = k(m^2 - 1), k > 0, m > 1$, then

$$w_1 = k(m^2 - 1),\ w_2 = k(m^2 + 1).$$

Since L is a rectangle, we can take $b = 0$. Eq. (5.1) now becomes

$$2km(m+1) = 2\left(\frac{km}{a} + ak(m^2 - 1)\right). \tag{5.2}$$

Solving Eq. (5.2), we get

$$a = \frac{m^2 + m \pm \sqrt{\Delta(a)}}{2(m^2-1)},$$

where

$$\Delta(a) = m^4 - 2m^3 + m^2 + 4m.$$

To get the rational solutions of Eq. (5.2), we need to make the discriminant $\Delta(a)$ be a perfect square, say w^2. By the map

$$\begin{aligned} X =& \frac{3(m^2+10m-12w+13)}{(m-1)^2}, \\ Y =& -\frac{108(m^3-2m^2+7m+2-(m+3)w)}{(m-1)^3}, \end{aligned}$$

We get an elliptic curve

$$\mathcal{E}: \ Y^2 = \ X^3 - 675X + 13662.$$

The discriminant of $\mathcal{E}$ is -3809369088, so $\mathcal{E}$ is nonsingular.

The inverse map is

$$\begin{aligned} m =& \frac{X^2+30X-99-12Y}{(X+33)(X-39)}, \\ w =& -\frac{2(X^3-27X^2+2727X-31293-6(X+33)Y)}{(X+33)(X-39)^2}. \end{aligned}$$

By means of SageMath software, the rank of $\mathcal{E}$ is 0, and $\mathcal{E}$ only has degenerate solutions

$$(X,Y) = (-33,0), (3,\pm 108), (39,\pm 216).$$

Here are the inputs and outputs:

```
sage: E = EllipticCurve([0, 0, 0, -675, 13662])
sage: E.rank()
0
sage: E.torsion_points()
[(-33 : 0 : 1), (0 : 1 : 0), (3 : -108 : 1),
 (3 : 108 : 1), (39 : -216 : 1), (39 : 216 : 1)]
```

Therefore, there is no right triangle and rectangle pair with the same area and the same perimeter, this case was obtained by Guy [11]. ∎

Case 5.2.2. T is an isosceles triangle and L is a rhombus. This result was obtained by Zhang and Peng [28], and we provide a new proof here.

THEOREM 5.3. [28] *There is no isosceles triangle and rhombus pair with the same area and the same perimeter, with at most four exceptions.*

PROOF. Case 5.2.2.1. Take $r = 4km, s = 2km, t = k(m^2 - 1), k > 0, m > 1$, then

$$w_1 = w_2 = k(m^2 + 1).$$

Since L is a rhombus, then Eq. (5.1) becomes

$$\begin{cases} \left(\dfrac{2km}{a}\right)^2 = \left(\dfrac{km}{a} + bk(m^2 - 1)\right)^2 + a^2k^2(m^2 - 1)^2, \\ 2k(m + 1)^2 = \dfrac{8km}{a}. \end{cases} \tag{5.3}$$

Eliminating a in Eq. (5.3), we have

$$\begin{aligned} &16(m - 1)^2(m + 1)^4b^2 + 8(m - 1)(m + 1)^5b \\ &- 3m^6 - 18m^5 + 211m^4 - 572m^3 + 211m^2 - 18m - 3 = 0. \end{aligned} \tag{5.4}$$

The discriminant of Eq. (5.4) with respect to b is

$$\Delta(b) = (m^3 + 11m^2 - 5m + 1)(m^3 - 5m^2 + 11m + 1).$$

Obviously, we need to make $\Delta(b)$ be a perfect square, say w^2. We consider the curve

$$\mathcal{C}_1 : w^2 = (m^3 + 11m^2 - 5m + 1)(m^3 - 5m^2 + 11m + 1).$$

This is a curve of genus 2. By the package of Magma, the Mordell-Weil rank of the Jacobian variety of $\mathcal{C}_1$ over $\mathbb{Q}$ is at most 1. Here are the inputs and outputs:

```
> _<m>:=PolynomialRing(Rationals());
> C:=HyperellipticCurve((m^3+11*m^2-5*m+1)*(m^3-5*m^2+11*m+1));
> ptsC:=RationalPoints(C:Bound:=1000000);
> ptsC;
{@ (1 : -1 : 0), (1 : 1 : 0), (0 : -1 : 1),
(0 : 1 : 1), (1 : -8 : 1), (1 : 8 : 1) @}
> J:=Jacobian(C);
> RankBound(J);
1
```

We check that $\mathcal{C}_1$ has good reduction at 5 and $\#\mathcal{C}_1(\mathbb{F}_5) = 8$. Therefore, by applying Lemma 3.8, we obtain that $\#\mathcal{C}_1(\mathbb{Q}) \leq 10$. Excluding the four points $(0, \pm 1)$ and $(1, \pm 8)$, the curve $\mathcal{C}_1$ contains at most four more rational points (which may not exist at all). If they exist, two satisfy $m > 0$ and the other two satisfy $m < 0$. Obviously, $(0, \pm 1), (1, \pm 8)$ are degenerate points. Hence, this case admits at most two exceptions.

Case 5.2.2.2. Take $r = 2k(m^2 - 1), s = k(m^2 - 1), t = 2km, k > 0, m > 1$, then

$$w_1 = w_2 = k(m^2 + 1).$$

Since L is a rhombus, then Eq. (5.1) becomes

$$\begin{cases} \left(\dfrac{k(m^2-1)}{a}\right)^2 = \left(\dfrac{k(m^2-1)}{2a} + 2bkm\right)^2 + (2akm)^2, \\ 4km^2 = \dfrac{4k(m^2-1)}{a}. \end{cases} \tag{5.5}$$

Eliminating a in Eq. (5.5), we have

$$16b^2m^4 + 8bm^5 - 3m^6 + 16m^4 - 32m^2 + 16 = 0. \tag{5.6}$$

The discriminant of Eq. (5.6) with respect to b is

$$\Delta(b) = (m^3 - 2m^2 + 2)(m^3 + 2m^2 - 2).$$

We consider the curve

$$\mathcal{C}_2 : w^2 = (m^3 - 2m^2 + 2)(m^3 + 2m^2 - 2).$$

This is a curve of genus 2. By the package of Magma, the Mordell-Weil rank of the Jacobian variety of $\mathcal{C}_2$ over $\mathbb{Q}$ is at most 1. Here are the inputs and outputs:

```
> _<m>:=PolynomialRing(Rationals());
> C:=HyperellipticCurve((m^3 - 2*m^2 + 2)*(m^3 + 2*m^2 - 2));
> ptsC:=RationalPoints(C:Bound:=1000000);
> ptsC;
{@ (1 : -1 : 0), (1 : 1 : 0), (-1 : -1 : 1),
(-1 : 1 : 1), (1 : -1 : 1), (1 : 1 : 1) @}
> J:=Jacobian(C);
> RankBound(J);
1
```

We check that $\mathcal{C}_2$ has good reduction at 5 and $\#\mathcal{C}_2(\mathbb{F}_5) = 8$. Therefore, by applying Lemma 3.8, we obtain that $\#\mathcal{C}_2(\mathbb{Q}) \leq 10$. Excluding the four points $(\pm 1, \pm 1)$, the curve $\mathcal{C}_2$ contains at most four more rational points (which may not exist at all). If they exist, two satisfy $m > 0$ and the other two satisfy $m < 0$. Obviously, $(\pm 1, \pm 1)$ are degenerate points. Hence, this case admits at most two exceptions. ∎

5.3. *New parametric solutions of Eq. (5.1).* We can get more solutions by directly solving Eq. (5.1). We are concerned with the following two cases.

Case 5.3.1. $s = 0$, T is a right triangle.

Theorem 5.4. *There exist infinitely many right triangles and parallelograms with the same area and the same perimeter.*

Proof. Take $r = 2km, s = 0, t = k(m^2 - 1), k > 0, m > 1$, then

$$w_1 = k(m^2 - 1),\ w_2 = k(m^2 + 1).$$

From the third equation of Eq. (5.1), we have

$$k^2(m^2-1)^2(a^2+b^2)=w_3^2,$$

then

$$a=2cp,\ b=c(p^2-1),\ w_3=ck(m^2-1)(p^2+1),\ c>0,\ p>1.$$

Eq. (5.1) now becomes

$$2p(p^2+1)(m^2-1)c^2-2m(m+1)pc+m=0, \tag{5.7}$$

then

$$c=\frac{m(m+1)p\pm\sqrt{\Delta(c)}}{2p(p^2+1)(m^2-1)},$$

where

$$\Delta(c)=p^2m^4-2p(p^2-p+1)m^3+p^2m^2+2p(p^2+1)m.$$

To get the rational solutions of Eq. (5.7) with respect to c, it needs $\Delta(c)$ be a perfect square, say w^2. Using a method described by Fermat (see [9, p. 639]), it's easy to get the following solutions

$$m=-\frac{(p-1)^4}{4p(p^2-3p+1)}$$

such that $\Delta(c)$ is a perfect square. By Eq. (5.7), we get

$$c=-\frac{(p-1)^2}{(p^2+1)(p^2-4p+1)}.$$

By some calculations, we obtain

$$s=0,\ r=-\frac{k(p-1)^4}{2p(p^2-3p+1)},\ t=\frac{k(p^2+2p-1)(p^2-2p-1)(p^2-4p+1)^2}{16p^2(p^2-3p+1)^2},$$

$$a=-\frac{2(p-1)^2p}{(p^2+1)(p^2-4p+1)},\ b=-\frac{(p-1)^3(p+1)}{(p^2+1)(p^2-4p+1)},$$

and

$$w_1=\frac{k(p^2+2p-1)(p^2-2p-1)(p^2-4p+1)^2}{16p^2(p^2-3p+1)^2},$$

$$w_2=\frac{k(p^8-8p^7+44p^6-152p^5+246p^4-152p^3+44p^2-8p+1)}{16p^2(p^2-3p+1)^2},$$

$$w_3=-\frac{k(p^2-4p+1)(p^2+2p-1)(p^2-2p-1)(p-1)^2}{16p^2(p^2-3p+1)^2},$$

this is a parametric solution of Eq. (5.1). When $k>0, 1+\sqrt{2}<p<\frac{3+\sqrt{5}}{2}$, the parameters r,t,a,w_1,w_2,w_3 are all positive, which implies that both T and $f(T)$ are non-degenerate.

Let us consider the rational points on the quartic curve

$$\mathcal{C}: w^2=\Delta(c).$$

By the map φ:

$$\begin{aligned}X =& -\frac{3p(6(m^2-1)p^2 - p(13m^2+10m+1) + 6(m^2-1) + 12w)}{(m-1)^2},\\ Y =& \frac{108pY(m,w)}{(m-1)^3},\end{aligned}$$

where

$$\begin{aligned}Y(m,w) = & \ p((m-1)(m^2+4m+1)p^2 - m(m+1)(3m+1)p \\ & + (m-1)(m^2+4m+1)) - ((m-1)p^2 - (3m+1)p + m - 1)w.\end{aligned}$$

We can transform $\mathcal{C}$ into the elliptic curve

$$\begin{aligned}\mathcal{E}_p :\ Y^2 = & \ X^3 - 27p^2(12p^4 - 12p^3 + 25p^2 - 12p + 12)X \\ & + 54p^4(12p^2 - p + 12)(6p^2 - p + 6).\end{aligned}$$

The discriminant of $\mathcal{E}_p$ is

$$\Delta(p) = 34012224p^6(4p^4 - 12p^3 + 9p^2 - 12p + 4)(p^2+1)^4.$$

Hence, $\Delta(p) \neq 0$, so $\mathcal{E}_p$ is non-singular.

It is easy to check that the elliptic curve $\mathcal{E}_p$ contains two rational points

$$Q = (-3p(6p^2 - p + 6), 0),\ P = (-3p(6p^2 - 25p + 6), -216p^2(p^2 - 3p + 1)).$$

By the inverse map φ^{-1}, we have

$$m = \frac{(X + 3p(6p^2 - p + 6))(X - 3p(6p^2 - 11p + 6)) - 12pY}{(X + 3p(6p^2 - p + 6))(X + 3p(6p^2 - 25p + 6))}.$$

We can get more parametric solutions by the group law. From the rational point $[2]P$, we obtain

$$m = \frac{(p^2 - 4p + 1)^2}{(p^2 - 2p - 1)(p^2 + 2p - 1)},$$

then

$$c = -\frac{p^2 - 4p + 1}{4(p-1)^2 p}.$$

By some calculations, we have

$$\begin{aligned}& s = 0,\ r = \frac{2k(p^2 - 4p + 1)^2}{(p^2 + 2p - 1)(p^2 - 2p - 1)},\ t = -\frac{16k(p-1)^4 p(p^2 - 3p + 1)}{(p^2 + 2p - 1)^2(p^2 - 2p - 1)^2},\\ & a = -\frac{p^2 - 4p + 1}{2(p-1)^2},\ b = -\frac{(p+1)(p^2 - 4p + 1)}{4p(p-1)},\end{aligned}$$

and

$$
\begin{aligned}
w_1 &= -\frac{16k(p-1)^4p(p^2-3p+1)}{(p^2+2p-1)^2(p^2-2p-1)^2},\\
w_2 &= \frac{2k(p^8-8p^7+44p^6-152p^5+246p^4-152p^3+44p^2-8p+1)}{(p^2+2p-1)^2(p^2-2p-1)^2},\\
w_3 &= \frac{4(p^2+1)(p^2-3p+1)(p-1)^2k(p^2-4p+1)}{(p^2+2p-1)^2(p^2-2p-1)^2},
\end{aligned}
$$

this is a parametric solution of Eq. (5.1). When $k>0, 1+\sqrt{2}<p<\frac{3+\sqrt{5}}{2}$, the parameters r,t,a,w_1,w_2,w_3 are all positive, which implies that both T and $f(T)$ are non-degenerate. We have verified that these parametric solutions are not presented in existing literature. □

<u>Case 5.3.2.</u> $s\neq 0$, T is a rational triangle.

THEOREM 5.5. *There exist infinitely many rational triangles and parallelograms with the same area and the same perimeter.*

PROOF. Take $s=2km, t=k(m^2-1), k>0, m>1$, then

$$w_1 = k(m^2+1).$$

From the second equation of Eq. (5.1), we have

$$r=\frac{2k((m^2+1)c-2m)}{c^2-1},\ w_2=\frac{k((m^2+1)c^2-4mc+m^2+1)}{c^2-1},$$

where c is a rational number.

From the third equation of Eq. (5.1), we have

$$\left(\frac{km}{a}+bk(m^2-1)\right)^2+a^2k^2(m^2-1)^2=w_3^2.$$

Let

$$\frac{km}{a}+bk(m^2-1)=d(p^2-1),\ ak(m^2-1)=2dp,$$

then

$$a=\frac{2dp}{k(m^2-1)},\ b=\frac{2p(p^2-1)d^2-k^2m(m^2-1)}{2dkp(m^2-1)},\ w_3=d(p^2+1),$$

where $p>1$. Eq. (5.1) now becomes

$$
\begin{aligned}
&2dp((p^2+1)d-k(m^2+1))c^2\\
&-k(m-1)(2p(m-1)d-k(m+1)(m^2+1))c \qquad (5.8)\\
&-2p(p^2+1)d^2+4dkmp-2k^2m(m^2-1)=0,
\end{aligned}
$$

then

$$c=\frac{k(2(m-1)^2pd-(m^4-1)k)\pm\sqrt{\Delta(c)}}{4dp((p^2+1)d-k(m^2+1))},$$

where

$$\begin{aligned}\Delta(c) = {} & (m^4-1)^2k^4 - 4dp(m^4-1)(m+1)^2k^3 \\ & + 4d^2p(4m(m^2-1)p^2 + (m+1)^4p + 4m(m^2-1))k^2 \\ & - 16d^3p^2(p^2+1)(m+1)^2k + 16d^4p^2(p^2+1)^2.\end{aligned}$$

To get the rational solutions of Eq. (5.8) with respect to c, it needs $\Delta(c)$ be a perfect square, say w^2. Since the discriminant $\Delta(c)$ is homogeneous with respect to k and d, we can let $d = 1$. Using a method described by Fermat (see [9, p. 639]), it's easy to get the following solution

$$k = \frac{p^2+1}{m^2+1}$$

such that $\Delta(c)$ is a perfect square. By Eq. (5.8), we have

$$c = \frac{2(m(m+1)p^2 + (m-1)(m^2+1)p + m(m+1))}{(m^2+1)((m+1)p^2 - 2(m-1)p + m + 1)}.$$

By some calculations, we obtain

$$\begin{aligned}& r = -\frac{4((m+1)p^2 - 2(m-1)p + m + 1)p(m^2+1)}{(m+1)((m^2-1)p^2 - 4(m^2+1)p + m^2 - 1)}, \\ & s = \frac{2(p^2+1)m}{m^2+1},\ t = \frac{(p^2+1)(m^2-1)}{m^2+1}, \\ & a = \frac{2p(m^2+1)}{(p^2+1)(m^2-1)}, \\ & b = -\frac{m(m^2-1)p^4 - 2(m^2+1)^2p^3 + 2m(m^2-1)p^2 + 2(m^2+1)^2p + m(m^2-1)}{2(m^2-1)p(p^2+1)(m^2+1)}, \\ & w_1 = w_3 = p^2 + 1, \\ & w_2 = -\frac{(m-1)\cdot w_2(m,p)}{(m+1)((m^2-1)p^2 - 4(m^2+1)p + m^2 - 1)},\end{aligned}$$

where

$$w_2(m,p) = (m+1)^2p^4 - 4(m^2-1)p^3 + (10m^2+4m+10)p^2 - 4(m^2-1)p + (m+1)^2.$$

When $m > 1, 1 < p < \frac{2(m^2+1)+\sqrt{3m^4+10m^2+3}}{m^2-1}$, the parameters r, t, a, w_1, w_2, w_3 are all positive, which implies that both T and $f(T)$ are non-degenerate.

Let us consider the rational points on the quartic curve

$$\mathcal{C} : w^2 = \Delta(c).$$

By the map φ:

$$\begin{aligned} X =& \frac{12pX(k,w)}{k^2}, \\ Y =& \frac{432p^2(p^2+1)Y(k,w)}{k^3}, \end{aligned}$$

where

$$\begin{aligned} X(k,w) =& \ (4m(m^2-1)p^2+(m+1)^4p+4m(m^2-1))k^2 \\ & -12p(p^2+1)(m+1)^2k+24p(p^2+1)^2-6(p^2+1)w, \\ Y(k,w) =& \ (m^4-1)(m+1)^2k^3-2(4m(m^2-1)p^2+(m+1)^4p+4m(m^2-1))k^2 \\ & +12p(p^2+1)(m+1)^2k-16p(p^2+1)^2-((m+1)^2k-4(p^2+1))w. \end{aligned}$$

We can transform $\mathcal{C}$ into the elliptic curve

$$\mathcal{E}_{(m,p)}: \ Y^2 = \ X^3-432p^2A_4(m,p)X-3456p^3A_6(m,p),$$

where

$$\begin{aligned} A_4(m,p) =& \ 4(m^2+3)(3m^2+1)(m^2-1)^2p^4-4(m-1)(3m^2-2m+3)(m+1)^5p^3 \\ & +(25m^6-42m^5+119m^4-140m^3+119m^2-42m+25)(m+1)^2p^2 \\ & -4(m-1)(3m^2-2m+3)(m+1)^5p+4(m^2+3)(3m^2+1)(m^2-1)^2, \\ A_6(m,p) =& \ (2(m^2-1)(3m^2-2m+3)p^2-(m+1)^4p \\ & +2(m^2-1)(3m^2-2m+3))(8m(3m^2+2m+3)(m^2-1)^2p^4 \\ & -4(m-1)(3m^2-2m+3)(m+1)^5p^3 \\ & +(m^6+54m^5-49m^4+52m^3-49m^2+54m+1)(m+1)^2p^2 \\ & -4(m-1)(3m^2-2m+3)(m+1)^5p+8m(3m^2+2m+3)(m^2-1)^2). \end{aligned}$$

By some calculations, we can check that the discriminant $\Delta(m,p)$ of $\mathcal{E}_{(m,p)}$ is not 0, then $\mathcal{E}_{(m,p)}$ is non-singular.

It is easy to check that the elliptic curve $\mathcal{E}_{(m,p)}$ contains two rational points

$$\begin{aligned} Q &= (X_1(m,p),0), \\ P &= (X_2(m,p),864p^2(p^2+1)(m^4-1)(m+1)^2), \end{aligned}$$

where

$$\begin{aligned} X_1(m,p) =& -12p(2(m^2-1)(3m^2-2m+3)p^2 \\ & -(m+1)^4p+2(m^2-1)(3m^2-2m+3)), \\ X_2(m,p) =& \ 12p(2(m^2-1)(3m^2+2m+3)p^2 \\ & +(m+1)^4p+2(m^2-1)(3m^2+2m+3)). \end{aligned}$$

By the inverse map φ^{-1}, we have

$$k=-\frac{24p(p^2+1)k(X,Y)}{(X-X_1(m,p))(X-X_2(m,p))},$$

where

$$\begin{aligned}k(X,Y)=Y&+6p(m+1)^2(X+12p(2(m^2-1)(3m^2-2m+3)p^2\\&-(m+1)^4p+2(m^2-1)(3m^2-2m+3))).\end{aligned}$$

By the group law, we get

$$\begin{aligned}[2]P=(&12(3(m^2-1)^2p^4-2(m^2-1)(3m^2-2m+3)p^3\\&+(m+1)^2(7m^2-10m+7)p^2-2(m^2-1)(3m^2-2m+3)p+3(m^2-1)^2),\\&-216(p^2+1)^2(m^2-1)^2((m^2-1)p^2-(3m^2-2m+3)p+m^2-1)).\end{aligned}$$

From the rational point $[2]P$, we have

$$k=\frac{4p}{m^2-1},$$

then

$$c=-\frac{(m^2-1)p^2+8mp+m^2-1}{(m^2-1)p^2-4(m^2+1)p+m^2-1}.$$

By some calculations, we obtain

$$\begin{aligned}&r=-\frac{(p^2+1)((m^2-1)p^2-(4(m^2+1))p+m^2-1)}{(m^2-1)p^2-2(m-1)^2p+m^2-1},\ s=\frac{8mp}{m^2-1},\ t=4p,\\&a=\frac{1}{2},\ b=\frac{(m^2-1)p^2-8mp-m^2+1}{4p(m^2-1)},\\&w_1=\frac{4p(m^2+1)}{m^2-1},\ w_3=p^2+1,\\&w_2=\frac{(m+1)^2p^4-4(m^2-1)p^3+(10m^2+4m+10)p^2-4(m^2-1)p+(m+1)^2}{(m+1)((m+1)p^2-2(m-1)p+m+1)},\end{aligned}$$

this is a parametric solution of Eq. (5.1). When

$$1<p<2+\sqrt{3},\ m>1 \text{ or } p>2+\sqrt{3},\ 1<m<\frac{\sqrt{(p^2-4p+1)(p^2+4p+1)}}{p^2-4p+1},$$

the parameters r,t,a,w_1,w_2,w_3 are all positive, which implies that both T and $f(T)$ are non-degenerate. We have verified that these parametric solutions are not presented in existing literature. ∎

6. AREAS AND PERIMETERS ARE PROPORTIONAL

In this section, we consider the problem that rational triangle pairs or rational triangle-parallelogram pairs with areas and perimeters in fixed proportions (α, β) respectively, where α and β are positive rational numbers. The existing results can be found in [18, 17, 19]. We only give the Diophantine systems corresponding to Eq. (3.3) and Eq. (5.1), respectively.

Case 6.1. Rational triangle pairs.

The transformation matrix M_f can be taken as

$$M_f = \begin{bmatrix} \frac{\alpha}{a} & b \\ 0 & a \end{bmatrix},$$

where $a \in \mathbb{Q}^*, b \in \mathbb{Q}$. The corresponding Diophantine system is

$$\begin{cases} s^2 + t^2 = w_1^2, \\ (s-r)^2 + t^2 = w_2^2, \\ \left(\dfrac{\alpha s}{a} + bt\right)^2 + (at)^2 = w_3^2, \\ \left(\dfrac{\alpha(s-r)}{a} + bt\right)^2 + (at)^2 = w_4^2, \\ \dfrac{\alpha r}{a} + w_3 + w_4 = \beta\left(r + w_1 + w_2\right), \end{cases} \tag{6.1}$$

where $b \in \mathbb{Q}$, $s \in \mathbb{Q}_{\geq 0}$, $\alpha, \beta, a, r, t, w_1, w_2, w_3, w_4 \in \mathbb{Q}^*$. A parametric solution of Eq. (6.1) is as follows.

$$\begin{aligned} & s = 0,\ r = \frac{2k(\alpha p^3 - \beta^2)}{p^2(\alpha p - \beta^2)},\ t = \frac{k\beta^2(p^2-1)(2\alpha p^3 - (p^2+1)\beta^2)}{p^4(\alpha p - \beta^2)^2}, \\ & a = \frac{2p^2\alpha}{\beta(p^2+1)},\ b = \frac{p\alpha(p^2-1)}{\beta(p^2+1)}, \\ & w_1 = \frac{k\beta^2(p^2-1)(2\alpha p^3 - (p^2+1)\beta^2)}{p^4(\alpha p - \beta^2)^2}, \\ & w_2 = \frac{k((p^4+1)\beta^4 - 2\alpha p^3(p^2+1)\beta^2 + 2\alpha^2 p^6)}{p^4(\alpha p - \beta^2)^2}, \\ & w_3 = \frac{k\alpha\beta(p^2-1)(2\alpha p^3 - (p^2+1)\beta^2)}{p^3(\alpha p - \beta^2)^2}, \\ & w_4 = \frac{(\beta^4(p^2+1) - 4\alpha\beta^2 p^3 + \alpha^2 p^4(p^2+1))\beta k}{(\alpha p - \beta^2)^2 p^4}, \end{aligned}$$

where $p > \max\left\{1, \frac{\beta^2}{\alpha}\right\}$.

Case 6.2. Rational triangle and parallelogram pairs.

The transformation matrix M_f can be taken as

$$M_f = \begin{bmatrix} \frac{\alpha}{2a} & b \\ 0 & a \end{bmatrix},$$

where $a \in \mathbb{Q}^*, b \in \mathbb{Q}$. The corresponding Diophantine system is

$$(6.2) \qquad \begin{cases} s^2 + t^2 = w_1^2, \\ (s-r)^2 + t^2 = w_2^2, \\ \left(\dfrac{\alpha s}{2a} + bt\right)^2 + (at)^2 = w_3^2, \\ 2\left(\dfrac{\alpha r}{2a} + w_3\right) = \beta(r + w_1 + w_2), \end{cases}$$

where $b \in \mathbb{Q}$, $s \in \mathbb{Q}_{\geq 0}$, $\alpha, \beta, a, r, t, w_1, w_2, w_3 \in \mathbb{Q}^*$. A parametric solution of Eq. (6.2) is as follows.

$$s = 0,\ r = -\frac{k((p^2+1)\alpha - 2p\beta^2)^2}{2((p^2+1)\alpha - 3p\beta^2)\beta^2 p},\ t = \frac{k((p^2+1)^2\alpha^2 - 8p^2\beta^4)((p^2+1)\alpha - 4p\beta^2)^2}{16((p^2+1)\alpha - 3p\beta^2)^2\beta^4 p^2},$$

$$a = -\frac{2((p^2+1)\alpha - 2p\beta^2)\beta p}{(p^2+1)((p^2+1)\alpha - 4p\beta^2)},\ b = -\frac{((p^2+1)\alpha - 2p\beta^2)\beta(p^2-1)}{(p^2+1)((p^2+1)\alpha - 4p\beta^2)},$$

$$w_1 = \frac{k((p^2+1)^2\alpha^2 - 8p^2\beta^4)((p^2+1)\alpha - 4p\beta^2)^2}{16((p^2+1)\alpha - 3p\beta^2)^2\beta^4 p^2},$$

$$w_2 = \frac{k((p^2+1)^4\alpha^4 - 8p\beta^2(p^2+1)^3\alpha^3 + 40p^2\beta^4(p^2+1)^2\alpha^2 - 128p^3\beta^6(p^2+1)\alpha + 160p^4\beta^8)}{16((p^2+1)\alpha - 3p\beta^2)^2\beta^4 p^2},$$

$$w_3 = -\frac{((p^2+1)\alpha - 2p\beta^2)k((p^2+1)\alpha - 4p\beta^2)((p^2+1)^2\alpha^2 - 8p^2\beta^4)}{16\beta^3((p^2+1)\alpha - 3p\beta^2)^2 p^2},$$

where $p > 1$, $\frac{2\sqrt{2}p\beta^2}{p^2+1} < \alpha < \frac{3p\beta^2}{p^2+1}$.

The solutions of Eq. (6.1) and Eq. (6.2) can be discussed similarly to Eq. (3.3) and Eq. (5.1), respectively.

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