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Mohit Mittal

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ON A VARIANT OF PILLAI'S PROBLEM INVOLVING CONVERGENT DENOMINATORS OF QUADRATIC IRRATIONALS

MOHIT MITTAL
Birla Institute of Technology and Science, Pilani, India

ABSTRACT. Let $(q_{\alpha,n})_{n\geq 0}$ be the sequence of convergent denominators to the simple continued fraction expansion of α. In this paper, using Baker's theory of linear forms in logarithms, we show that there are only finitely many integers c such that the equation $q_{\alpha,n} - q_{\beta,m} = c$ has at least two distinct solutions (n, m), with n, m *large*. Here, α, β are quadratic irrationals with $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\beta)$. In specific instances, we further employ the Baker-Davenport reduction technique to solve the equation $q_{\alpha,n} - q_{\beta,m} = c$ completely and explicitly list all solutions.

1. INTRODUCTION

In 1931, Pillai [28] proved that for fixed positive integers a and b, with $\min(a, b) \geq 2$ and ${}^{\log a}/_{\log b}$ irrational, the number of solutions (x, y) to the Diophantine inequalities $0 < a^x - b^y \leq c$ is asymptotically equal to

$$\frac{(\log c)^2}{2(\log a)(\log b)}$$

as $c \to \infty$. This work drew Herschfeld's [19, 20] attention, who proved, in 1935, that the Diophantine equation

$$2^x - 3^y = c,$$

has at most one solution when $|c|$ is sufficiently large and fixed. Following this, in 1936, Pillai [29, 30] considered the more general Diophantine equation

$$a^x - b^y = c, \tag{1.1}$$

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where a, b, c are fixed nonzero integers with $a > b \geq 1$. He showed that for coprime a and b, sufficiently large values of c allow at most one positive integer solution (x, y). Further, he put forth two conjectures. The first conjecture states that the only integers c that admit at least two representations of the form $2^x - 3^y$ come from the set $\{-1, 5, 13\}$, with explicit representations

$$\begin{aligned} 2^3 - 3^2 &= -1 = 2 - 3, \\ 2^5 - 3^3 &= 5 = 2^3 - 3, \\ 2^8 - 3^5 &= 13 = 2^4 - 3. \end{aligned}$$

This conjecture was confirmed by Stroeker and Tijdeman [33] in 1982. (See also [4].)

Secondly, he conjectured that, for fixed nonzero c, equation (1.1) has at most finitely many solutions in integers a, b, x, and y, all greater than one. The case $c = 1$ is Catalan's conjecture, one of the famous classical problems in number theory. In 1976, Tijdeman [34] proved that equation (1.1) with $c = 1$ has only finitely many solutions. Catalan's conjecture was proved in 2002 by Mihăilescu [24]. However, Pillai's conjecture remains unproven for $c \geq 2$.

The first work in the direction of replacing powers of positive integers by terms of a linear recurrence sequence was done in 2015 by Ddamulira, Luca & Rakotomalala [14]. They considered the Diophantine equation

$$F_n - 2^m = c,$$

and explicitly determined all integers c that have at least two distinct representations.

In 2016, Chim, Pink & Ziegler [7] considered the Diophantine equation

$$F_n - T_m = c,$$

where c is a fixed integer and $(F_n)_{n\geq 0}$, $(T_m)_{m\geq 0}$ are the sequences of Fibonacci numbers and Tribonacci numbers, respectively. This type of equation can be viewed as a variation of Pillai's equation. They showed that the only integers c having at least two distinct representations of the form $F_n - T_m$ come from the set

$$\mathcal{C} = \{0, 1, -1, -2, -3, 4, -5, 6, 8, -10, 11, -11, -22, -23, -41, -60, -271\},$$

and also gave all representations of each $c \in \mathcal{C}$.

In the same year, Chim, Pink & Ziegler [8] considered the Diophantine equation

$$U_n - V_m = c,$$

where c is a fixed integer and $(U_n)_{n\geq 0}$, $(V_m)_{m\geq 0}$ are linear recurrence sequences satisfying the *dominant root condition*. By finding an effective upper bound for $|c|$, they showed that there are finitely many choices of c which have at least two distinct representations of the form $U_n - V_m$. We refer to

[5, 9, 10, 11, 12, 13, 16, 17, 18] and the references therein to get more insight into the theme of this paper.

In this paper, for a fixed integer c, we study the equation

$$q_{\alpha,n} - q_{\beta,m} = c,$$

which is a variation of Pillai's equation. Here, $(q_{\alpha,n})_{n\geq 0}$ and $(q_{\beta,m})_{m\geq 0}$ are the sequences of convergent denominators arising from the continued fraction expansions of quadratic irrationals α and β, respectively. This problem has not been addressed previously, as the linear recurrence sequences do not satisfy the dominant root condition (see Remark 1). We refer to [25] and [31] for some more results that involve sequences of convergent denominators.

For specific choices of quadratic irrationals α and β, the sequences $(q_{\alpha,n})_{n\geq 0}$ and $(q_{\beta,m})_{m\geq 0}$ become Lehmer sequences [3] of the type

$$L_n(R,-1) = \begin{cases} \frac{\xi^n-\eta^n}{\xi-\eta} & \text{if } n \text{ is odd,} \\ \frac{\xi^n-\eta^n}{\xi^2-\eta^2} & \text{if } n \text{ is even.} \end{cases}$$

Here, $R \in \mathbb{Z}_+$ and ξ, η are the roots of the quadratic equation $x^2 - \sqrt{R}x - 1 = 0$. In these cases, we explicitly solve certain Diophantine equations and exhibit all of their solutions. We begin with the following theorem:

Theorem 1.1. *Let α, β be real quadratic irrational numbers with simple continued fraction expansions*

$$\alpha = [a_{0,0}; \ldots, a_{0,r_0-1}, \overline{b_{0,0}, \ldots, b_{0,s_0-1}}], r_0 \geq 0, s_0 \geq 1,$$
$$\beta = [a_{1,0}; \ldots, a_{1,r_1-1}, \overline{b_{1,0}, \ldots, b_{1,s_1-1}}], r_1 \geq 0, s_1 \geq 1,$$

and $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\beta)$. Let $(q_{\alpha,i})_{i\geq 0}, (q_{\beta,i})_{i\geq 0}$ be the sequences of denominators of the convergents to the continued fraction expansions of α, β, respectively. Then there exist effectively computable constants N^*, M^* depending only on α, β, respectively, and an effectively computable finite set $\mathcal{C}$ with the following property: an integer c has at least two distinct representations of the form $q_{\alpha,N} - q_{\beta,M}$ with $N \geq N^*$ and $M \geq M^*$ if and only if $c \in \mathcal{C}$.

Let $a, b \in \mathbb{N}$ such that $2 \leq a < b$. For $\alpha = [0; \overline{1,a}]$ and $\beta = [0; \overline{1,b}]$, we explicitly compute an upper bound for the size of the solutions. Moreover, for $2 \leq a < b \leq 5$, we effectively compute the set $\mathcal{C}$. We note that for $\alpha = [0; \overline{1,a}]$, the sequence of convergent denominators is a Lehmer sequence with $R = a$. We prove

Theorem 1.2. *Let $a, b \in \mathbb{N}$ with $2 \leq a < b$. Let α, β be real quadratic irrational numbers with simple continued fraction expansions*

$$\alpha = [0; \overline{1,a}], \ \beta = [0; \overline{1,b}],$$

and $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\beta)$. Let $(q_{\alpha,i})_{i\geq 0}, (q_{\beta,i})_{i\geq 0}$ be the sequences of denominators of the convergents to the continued fraction expansions of α, β, respectively.

If c is an integer that has at least two distinct representations of the form $q_{\alpha,N} - q_{\beta,M}$, i.e. $c = q_{\alpha,N_1} - q_{\beta,M_1} = q_{\alpha,N_2} - q_{\beta,M_2}$ with $N_1 > N_2 \geq 2$ and $M_1 > M_2 \geq 2$, then

$$max(N_1, M_1) \leq 1.1793 \times 10^{46} \log^6 \theta_{\beta,1} \times \log^3(9.06 \times 10^{14} \log^2 \theta_{\beta,1}),$$

where $\theta_{\beta,1} = {}^{(b+2+\sqrt{b^2+4b})}/_2$.

In particular, for $a = 2$ and $b = 3$ we extend Theorem 1.2 to find the effectively computable set $\mathcal{C}$. Therefore, we have the following theorem:

THEOREM 1.3. *Let $\alpha = [0; \overline{1,2}]$ and $\beta = [0; \overline{1,3}]$. The only integers c that have at least two distinct representations of the form $q_{\alpha,N} - q_{\beta,M}$ with $N, M \geq 1$, belong to the set*

$$\mathcal{C} = \{-4, -1, 0, 10, 37\}.$$

Furthermore, for each $c \in \mathcal{C}$, two distinct representations of the form $c = q_{\alpha,N} - q_{\beta,M}$ is shown below:

$$\begin{aligned} q_{\alpha,5} - q_{\beta,4} = 15 - 19 = \boxed{-4} &= 1 - 5 = q_{\alpha,1} - q_{\beta,3}, \\ q_{\alpha,3} - q_{\beta,3} = 4 - 5 = \boxed{-1} &= 3 - 4 = q_{\alpha,2} - q_{\beta,2}, \\ q_{\alpha,3} - q_{\beta,2} = 4 - 4 = \boxed{0} &= 1 - 1 = q_{\alpha,1} - q_{\beta,1}, \\ q_{\alpha,5} - q_{\beta,3} = 15 - 5 = \boxed{10} &= 11 - 1 = q_{\alpha,4} - q_{\beta,1}, \\ q_{\alpha,7} - q_{\beta,4} = 56 - 19 = \boxed{37} &= 41 - 4 = q_{\alpha,6} - q_{\beta,2}. \end{aligned}$$

For other choices of $\alpha = [0; \overline{1,a}]$ and $\beta = [0; \overline{1,b}]$, where $(a, b) \in \{(2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$, the set $\mathcal{C}$ along with two distinct representations of each $c \in \mathcal{C}$ is shown in Section 7.

Note that it is important to choose $N \geq 1$ and $M \geq 1$. Otherwise, there would be infinitely many c's having two distinct representations. As an example, for $\alpha = [0; \overline{1,a}]$ there are infinitely many integers $c_M = 1 - q_{\beta,M}$, $M \geq 0$, that have two distinct representations of the form

$$q_{\alpha,0} - q_{\beta,M} = q_{\alpha,1} - q_{\beta,M} = 1 - q_{\beta,M}.$$

Remark 1.4. We note that for a real quadratic irrational number α, the sequence of convergent denominators of its continued fraction expansion satisfies a linear recurrence relation, but the associated characteristic polynomial may not have a dominating root. For instance, if $\alpha = \sqrt{27}$, then its continued fraction expansion is $[5; \overline{5, 10}]$ and $q_{i+4} = 52q_{i+2} - q_i$, $i \geq 0$. The characteristic polynomial is $x^4 - 52x^2 + 1$ and its roots are $\pm\sqrt{26 + 15\sqrt{3}}, \pm\sqrt{26 - 15\sqrt{3}}$. To circumscribe this issue, we use an argument of Pethő [26] (see also Lenstra and Shallit [21]) to split the sequence into finitely many subsequences, each satisfying the same binary recurrence relation (with different initial terms, see Section 2).

In Section 2, we mention some preliminary results. In Sections 3, 4, and 5, we prove Theorems 1.1, 1.2, and 1.3, respectively.

2. Preliminaries

Let α be a real quadratic irrational number. Let its simple continued fraction expansion be $\alpha = [a_0; \ldots, a_{r-1}, \overline{b_0, \ldots, b_{s-1}}]$, $r \geq 0, s \geq 1$. Let $(q_{\alpha,i})_{i\geq 0}$ be the sequence of denominators of the convergents to α. Then by Pethő [26, Lemmata 1 & 2] (see also [21, p. 352]), $(q_{\alpha,i})_{i\geq 0}$ satisfies the linear recurrence

$$q_{\alpha,i+2s} - t_\alpha q_{\alpha,i+s} + (-1)^s q_{\alpha,i} = 0, \; i \geq r, \tag{2.2}$$

where t_α is defined as

$$t_\alpha = \operatorname{trace}\left(\prod_{0\leq j<s} \begin{pmatrix} b_j & 1 \\ 1 & 0 \end{pmatrix}\right).$$

The next lemma gives a Binet-type formula for the convergent denominators.

Lemma 2.1. *Let α be a real quadratic irrational number with simple continued fraction expansion $[a_0; \ldots, a_{r-1}, \overline{b_0, \ldots, b_{s-1}}]$, $r \geq 0, s \geq 1$. For $j = 0, \ldots, s-1$, let $q_{\alpha,i}^{(j)} = q_{\alpha,si+j+r}$, $i \geq 0$. We have*

$$q_{\alpha,i}^{(j)} = c_{\alpha,1}^{(j)} \theta_{\alpha,1}^i - c_{\alpha,2}^{(j)} \theta_{\alpha,2}^i, \; i \geq 0, \tag{2.3}$$

where

$$\theta_{\alpha,1} = \frac{t_\alpha + \sqrt{t_\alpha^2 - 4(-1)^s}}{2}, \; \theta_{\alpha,2} = \frac{t_\alpha - \sqrt{t_\alpha^2 - 4(-1)^s}}{2},$$
$$c_{\alpha,1}^{(j)} = \frac{q_{\alpha,1}^{(j)} - \theta_{\alpha,2} q_{\alpha,0}^{(j)}}{\theta_{\alpha,1} - \theta_{\alpha,2}} > 0, \; c_{\alpha,2}^{(j)} = \frac{q_{\alpha,1}^{(j)} - \theta_{\alpha,1} q_{\alpha,0}^{(j)}}{\theta_{\alpha,1} - \theta_{\alpha,2}}.$$

Proof. From (2.2) and the definition of $q_{\alpha,i}^{(j)}$, we have

$$q_{\alpha,i+2}^{(j)} - t_\alpha q_{\alpha,i+1}^{(j)} + (-1)^s q_{\alpha,i}^{(j)} = 0, \; i \geq 0.$$

For a fixed j, the subsequence $\left(q_{\alpha,i}^{(j)}\right)_{i\geq 0}$ satisfies the above equation. The corresponding characteristic equation is $x^2 - t_\alpha x + (-1)^s = 0$, whose roots are $\theta_{\alpha,1}$ and $\theta_{\alpha,2}$. By [32, Theorem C.1(a)], the Binet-type formula can be written as in (2.3), where $c_{\alpha,1}^{(j)}$ and $c_{\alpha,2}^{(j)}$ are to be determined. Setting $i = 0$ and $i = 1$ in (2.3) gives

$$q_{\alpha,0}^{(j)} = c_{\alpha,1}^{(j)} - c_{\alpha,2}^{(j)},$$
$$q_{\alpha,1}^{(j)} = c_{\alpha,1}^{(j)} \theta_{\alpha,1} - c_{\alpha,2}^{(j)} \theta_{\alpha,2}.$$

Solving these yields the stated expressions. □

Using the above lemma, we note that $1 < \theta_{\alpha,1} < t_\alpha + 1$ and $-1 < \theta_{\alpha,2} < 1$.

We recall that two complex numbers u_1 and u_2 are said to be *multiplicatively independent* if for integers k_1 and k_2, $u_1^{k_1} u_2^{k_2} = 1$ implies $k_1 = k_2 = 0$. The following lemma provides a sufficient condition for the multiplicative independence of $\theta_{\alpha,1}$ and $\theta_{\beta,1}$.

LEMMA 2.2. *Let α, β be real quadratic irrational numbers with simple continued fraction expansions*

$$\alpha = [a_{0,0}; \dots, a_{0,r_0-1}, \overline{b_{0,0}, \dots, b_{0,s_0-1}}], r_0 \geq 0, s_0 \geq 1,$$
$$\beta = [a_{1,0}; \dots, a_{1,r_1-1}, \overline{b_{1,0}, \dots, b_{1,s_1-1}}], r_1 \geq 0, s_1 \geq 1.$$

If $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\beta)$, then $\theta_{\alpha,1}$ and $\theta_{\beta,1}$ are multiplicatively independent.

PROOF. Let $(p_{\alpha,n})_{n\geq 0}$ and $(q_{\alpha,n})_{n\geq 0}$ be the sequences of convergent numerators and denominators, respectively, to the continued fraction expansion of α. Referring to [21], $(p_{\alpha,i})_{i\geq 0}$ satisfies the linear recurrence

$$p_{\alpha,i+2s_0} - t_\alpha p_{\alpha,i+s_0} + (-1)^{s_0} p_{\alpha,i} = 0, \ i \geq r_0.$$

In a manner similar to Lemma 2.1, we have

$$p_{\alpha,i}^{(j)} = d_{\alpha,1}^{(j)} \theta_{\alpha,1}^i - d_{\alpha,2}^{(j)} \theta_{\alpha,2}^i, \ i \geq 0,$$

where $d_{\alpha,1}^{(j)} = \frac{p_{\alpha,1}^{(j)} - \theta_{\alpha,2} p_{\alpha,0}^{(j)}}{\theta_{\alpha,1} - \theta_{\alpha,2}}$ and $d_{\alpha,2}^{(j)} = \frac{p_{\alpha,1}^{(j)} - \theta_{\alpha,1} p_{\alpha,0}^{(j)}}{\theta_{\alpha,1} - \theta_{\alpha,2}}$. Since $\alpha = \lim_{n\to\infty} \frac{p_{\alpha,n}}{q_{\alpha,n}}$, for any fixed j, we have

$$\alpha = \lim_{i\to\infty} \frac{p_{\alpha,i}^{(j)}}{q_{\alpha,i}^{(j)}} = \lim_{i\to\infty} \frac{d_{\alpha,1}^{(j)} \theta_{\alpha,1}^i - d_{\alpha,2}^{(j)} \theta_{\alpha,2}^i}{c_{\alpha,1}^{(j)} \theta_{\alpha,1}^i - c_{\alpha,2}^{(j)} \theta_{\alpha,2}^i} = \frac{d_{\alpha,1}^{(j)}}{c_{\alpha,1}^{(j)}} = \frac{p_{\alpha,1}^{(j)} - \theta_{\alpha,2} p_{\alpha,0}^{(j)}}{q_{\alpha,1}^{(j)} - \theta_{\alpha,2} q_{\alpha,0}^{(j)}}.$$

Hence, $\alpha \in \mathbb{Q}(\theta_{\alpha,2})$ and therefore $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\theta_{\alpha,2})$. As $[\mathbb{Q}(\theta_{\alpha,2}) : \mathbb{Q}] = 2$, we have either $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 1$ or $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$. If $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 1$, then α is a rational number, which is not possible. So, $\mathbb{Q}(\alpha) = \mathbb{Q}(\theta_{\alpha,2}) = \mathbb{Q}(\sqrt{t_\alpha^2 - 4(-1)^{s_0}}) = \mathbb{Q}(\theta_{\alpha,1})$.

Now suppose $\theta_{\alpha,1}$ and $\theta_{\beta,1}$ are multiplicatively dependent. Then there exist integers u, v such that $\theta_{\alpha,1}^u = \theta_{\beta,1}^v$. As $\mathbb{Q}(\theta_{\alpha,1}) \cap \mathbb{Q}(\theta_{\beta,1}) = \mathbb{Q}$, we get $\theta_{\alpha,1}^u \in \mathbb{Q}$. This gives $\theta_{\alpha,1}^u = \theta_{\alpha,2}^u = \pm\theta_{\alpha,1}^{-u}$, which yields $\theta_{\alpha,1}^{2u} = \pm 1$. Since $\mathbb{Q}(\theta_{\alpha,1})$ is a real quadratic field, we get $\theta_{\alpha,1} = \pm 1$, which is a contradiction. □

LEMMA 2.3. *Let α be a real quadratic irrational number with simple continued fraction expansion $[a_0; \dots, a_{r-1}, \overline{b_0, \dots, b_{s-1}}]$, $r \geq 0, s \geq 1$. Then there exists a positive integer N_0^* depending only on α such that for $N_1 > N_2 \geq N_0^*$, we have*

$$C_{\alpha,3} \theta_{\alpha,1}^{n_1} \leq |q_{\alpha,N_1} - q_{\alpha,N_2}| \leq C_{\alpha,4} \theta_{\alpha,1}^{n_1}.$$

Proof. Recall that $q_{\alpha,i}^{(j)} = q_{\alpha,si+j+r}$, from this we obtain $q_{\alpha,N} = q_{\alpha,n}^{(j)}$. Using equation (2.3), we have

$$q_{\alpha,n}^{(j)} = c_{\alpha,1}^{(j)}\theta_{\alpha,1}^{n} - c_{\alpha,2}^{(j)}\theta_{\alpha,2}^{n}.$$

As $n \to \infty$

$$q_{\alpha,n}^{(j)} \sim c_{\alpha,1}^{(j)}\theta_{\alpha,1}^{n}.$$

Hence,

$$q_{\alpha,N} = q_{\alpha,n}^{(j)} \sim c_{\alpha,1}^{(j)}\theta_{\alpha,1}^{n}, \quad \text{as } N \to \infty.$$

Therefore, there are positive constants $C_{\alpha,5}$, $C_{\alpha,6}$ such that ${C_{\alpha,6}}/{C_{\alpha,5}} < \theta_{\alpha,1}$ and

$$C_{\alpha,5}\theta_{\alpha,1}^{n} \leq q_{\alpha,N} \leq C_{\alpha,6}\theta_{\alpha,1}^{n}, \quad N \geq N_0,$$

where N_0 is a sufficiently large integer.
For $N_1 > N_2 \geq N_0$,

$$\begin{aligned} |q_{\alpha,N_1} - q_{\alpha,N_2}| \leq |q_{\alpha,N_1}| + |q_{\alpha,N_2}| &\leq C_{\alpha,6}(\theta_{\alpha,1}^{n_1} + \theta_{\alpha,1}^{n_2}) \\ &= C_{\alpha,6}\left(1 + \frac{1}{\theta_{\alpha,1}^{n_1-n_2}}\right)\theta_{\alpha,1}^{n_1} \\ &\leq C_{\alpha,4}\theta_{\alpha,1}^{n_1}, \end{aligned} \tag{2.4}$$

where $C_{\alpha,4} = 2C_{\alpha,6}$.
For $N_1 > N_2 \geq N_0$. If $n_1 \neq n_2$, then

$$\begin{aligned} |q_{\alpha,N_1} - q_{\alpha,N_2}| \geq |q_{\alpha,N_1}| - |q_{\alpha,N_2}| &\geq C_{\alpha,5}\theta_{\alpha,1}^{n_1} - C_{\alpha,6}\theta_{\alpha,1}^{n_2} \\ &= C_{\alpha,5}\left(1 - \frac{C_{\alpha,6}}{C_{\alpha,5}\theta_{\alpha,1}^{n_1-n_2}}\right)\theta_{\alpha,1}^{n_1} \\ &\geq C_{\alpha,5}\left(1 - \frac{C_{\alpha,6}}{C_{\alpha,5}\theta_{\alpha,1}}\right)\theta_{\alpha,1}^{n_1}. \end{aligned} \tag{2.5}$$

If $n_1 = n_2$, then using equation (2.3), we have

$$\begin{aligned} |q_{\alpha,N_1} - q_{\alpha,N_2}| &= \left|(c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)})\theta_{\alpha,1}^{n_1} - (c_{\alpha,2}^{(j_1)} - c_{\alpha,2}^{(j_2)})\theta_{\alpha,2}^{n_1}\right| \\ &= \left|(c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}) - (c_{\alpha,2}^{(j_1)} - c_{\alpha,2}^{(j_2)})\left(\frac{\theta_{\alpha,2}}{\theta_{\alpha,1}}\right)^{n_1}\right|\theta_{\alpha,1}^{n_1}. \end{aligned}$$

Since $N_i = sn_i + j_i + r$ for $i = 1, 2$, with $n_1 = n_2$ and $N_1 > N_2$, we obtain $j_1 > j_2$. For, suppose $c_{\alpha,1}^{(j_1)} = c_{\alpha,1}^{(j_2)}$. Using Lemma 2.1, we deduce

$$q_{\alpha,1}^{(j_1)} - q_{\alpha,1}^{(j_2)} = \theta_{\alpha,2}(q_{\alpha,0}^{(j_1)} - q_{\alpha,0}^{(j_2)}).$$

This gives $\theta_{\alpha,2}$ is a rational number, which is a contradiction. Therefore, $c_{\alpha,1}^{(j_1)} \neq c_{\alpha,1}^{(j_2)}$. Hence, $c_{\alpha,2}^{(j_1)} \neq c_{\alpha,2}^{(j_2)}$. Since $\lim_{n_1\to\infty}\left(\frac{\theta_{\alpha,2}}{\theta_{\alpha,1}}\right)^{n_1} = 0$, there exists

$n^* \in \mathbb{N}$ such that

$$\left|\left(\frac{\theta_{\alpha,2}}{\theta_{\alpha,1}}\right)^{n_1}\right| \le \frac{|c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}|}{2|c_{\alpha,2}^{(j_1)} - c_{\alpha,2}^{(j_2)}|}, \quad n_1 \ge n^*.$$

Let $N_0^* = \max(N_0, sn^* + j_2 + r)$. Therefore

$$|q_{\alpha,N_1} - q_{\alpha,N_2}| \ge \frac{\left|c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\right|}{2}\theta_{\alpha,1}^{n_1}, \tag{2.6}$$

for $N_1 > N_2 \ge N_0^*$.
Combining equations (2.5) and (2.6), we have

$$|q_{\alpha,N_1} - q_{\alpha,N_2}| \ge C_{\alpha,3}\theta_{\alpha,1}^{n_1}, \tag{2.7}$$

where $C_{\alpha,3} = \min\left(C_{\alpha,5}\left(1 - \frac{C_{\alpha,6}}{C_{\alpha,5}\theta_{\alpha,1}}\right), \min_{0\le j_2<j_1\le s-1} \frac{\left|c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\right|}{2}\right)$.
Combining inequalities (2.4) and (2.5), we get the desired result. □

The following lemma establishes the irrationality of the algebraic expression that arises in the proof of Theorem 1.2.

Lemma 2.4. *Let α be a real quadratic irrational number with simple continued fraction expansion $[0; \overline{1, a}]$. Let $c_{\alpha,1}^{(j)}, \theta_{\alpha,1}$, and $\theta_{\alpha,2}$ be as defined in Lemma 2.1. If $n_1 > n_2$, then*

$$c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} \notin \mathbb{Q}.$$

Proof. By virtue of Lemma 2.1, $j_1, j_2 \in \{0, 1\}$ with

$$c_{\alpha,1}^{(0)} = \frac{a + \sqrt{d_a}}{2\sqrt{d_a}},\ c_{\alpha,1}^{(1)} = \frac{\theta_{\alpha,1}}{\sqrt{d_a}},$$

where $d_a = a^2+4a$ and $\theta_{\alpha,1} = \dfrac{a + 2 + \sqrt{d_a}}{2}$. It is worth noting that d_a is not a perfect square. For, suppose $d_a = w^2$ for some integer w. This gives $(a+2)^2 = w^2 + 2^2$. If $\gcd(w, 2) = 1$, then $(w, 2, a + 2)$ is a primitive Pythagorean triple, which is not possible. If $\gcd(w, 2) = 2$, then $(w/2)^2 + 1 = (a+2/2)^2$, which is not possible. Thus the claim follows.

Now, depending upon the parity of n_1 and n_2, there are four cases. It suffices to consider two, as the remaining cases follow analogously.
Case(i): n_1 and n_2 are even (that is, $j_1 = j_2 = 0$).
Suppose to the contrary that $c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} = \frac{a+\sqrt{d_a}}{2\sqrt{d_a}}(\theta_{\alpha,1}^{n_1} - \theta_{\alpha,1}^{n_2}) \in \mathbb{Q}$.
Consider the automorphism of $\mathbb{Q}(\sqrt{d_a})$ that takes $\sqrt{d_a}$ to $-\sqrt{d_a}$. This gives,

$$\frac{a + \sqrt{d_a}}{2\sqrt{d_a}}(\theta_{\alpha,1}^{n_1} - \theta_{\alpha,1}^{n_2}) = \frac{a - \sqrt{d_a}}{-2\sqrt{d_a}}(\theta_{\alpha,2}^{n_1} - \theta_{\alpha,2}^{n_2}).$$

Since $\theta_{\alpha,1} \cdot \theta_{\alpha,2} = 1$, we obtain

$$\theta_{\alpha,1}^{n_1+n_2} = \frac{a-\sqrt{d_a}}{a+\sqrt{d_a}}.$$

As the right-hand side is negative and the left-hand side is positive, this is a contradiction. The same argument applies when n_1 and n_2 are odd.
Case(ii): n_1 is odd and n_2 is even (that is, $j_1 = 1, j_2 = 0$).
Suppose to the contrary that $c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} = \frac{\theta_{\alpha,1}}{\sqrt{d_a}}\theta_{\alpha,1}^{n_1} - \frac{a+\sqrt{d_a}}{2\sqrt{d_a}}\theta_{\alpha,1}^{n_2} \in \mathbb{Q}$. Consider the automorphism of $\mathbb{Q}(\sqrt{d_a})$ that takes $\sqrt{d_a}$ to $-\sqrt{d_a}$. This gives,

$$\frac{\theta_{\alpha,1}}{\sqrt{d_a}}\theta_{\alpha,1}^{n_1} - \frac{a+\sqrt{d_a}}{2\sqrt{d_a}}\theta_{\alpha,1}^{n_2} = \frac{\theta_{\alpha,2}}{-\sqrt{d_a}}\theta_{\alpha,2}^{n_1} - \frac{a-\sqrt{d_a}}{-2\sqrt{d_a}}\theta_{\alpha,2}^{n_2}.$$

Since $\theta_{\alpha,1} \cdot \theta_{\alpha,2} = 1$, we obtain

$$\theta_{\alpha,1}^{n_1+1} + \theta_{\alpha,1}^{-n_1-1} = \theta_{\alpha,1}^{n_2+1} + \theta_{\alpha,1}^{-n_2-1} - (\theta_{\alpha,1}^{n_2} + \theta_{\alpha,1}^{-n_2}).$$

Let us define the sequence $S = (S_i)_{i\geq 0}$, as $S_i = \theta_{\alpha,1}^{i} + \theta_{\alpha,1}^{-i}$. Then $S_{n_1+1} = S_{n_2+1} - S_{n_2} < S_{n_2+1}$. Since the sequence S is strictly increasing and $n_1 > n_2$, this is a contradiction. A similar argument follows when n_1 is even and n_2 is odd. ∎

Next, we recall the definition and some basic properties of the absolute logarithmic height function $h(\cdot)$. If δ is an algebraic number with minimal polynomial $f(X) = d_0(X - \delta^{(1)}) \cdots (X - \delta^{(d)}) \in \mathbb{Z}[X]$,

$$h(\delta) = \frac{1}{d}\left(\log d_0 + \sum_{i=1}^{d} \max(0, \log|\delta^{(i)}|)\right)$$

denotes its absolute logarithmic height. If δ_1, δ_2 are algebraic numbers, then

$$\left.\begin{aligned} h(\delta_1 \pm \delta_2) &\leq h(\delta_1) + h(\delta_2) + \log 2, \\ h(\delta_1\delta_2^{\pm 1}) &\leq h(\delta_1) + h(\delta_2), \\ h(\delta_1^m) &= |m| h(\delta_1) \ (m \in \mathbb{Z}) \end{aligned}\right\} \tag{2.8}$$

(see [35, Property 3.3]).

The following lemma due to Chim, Pink & Ziegler [8] gives a lower bound for the absolute logarithmic height of the product of powers of multiplicatively independent algebraic numbers.

Lemma 2.5. *Let K be a number field and suppose that $\delta_1, \delta_2 \in K$ are multiplicatively independent. There exists an effectively computable constant $D_0 > 0$ such that for $n, m \in \mathbb{Z}$, we have*

$$h\left(\frac{\delta_1^n}{\delta_2^m}\right) \geq D_0 \max(|n|, |m|).$$

We will use the following lower bound for linear forms in logarithms due to Matveev [22, 23]. (See also [2, 6].)

Lemma 2.6. *Let T be a positive integer. Let $\delta_1, \ldots, \delta_T$ be non-zero real algebraic numbers and $\log\delta_1, \ldots, \log\delta_T$ be some determinations of their complex logarithms. Let D be the degree of the number field generated by $\delta_1, \ldots, \delta_T$ over $\mathbb{Q}$. For $j = 1, \ldots, T$, let A'_j be a real number satisfying*

$$A'_j \geq \max\left(Dh(\delta_j), |\log\delta_j|, 0.16\right).$$

Let $k_1, \ldots, k_T$ be rational integers. Set

$$B = \max(|k_1|, \ldots, |k_T|) \text{ and } \Lambda = \delta_1^{k_1}\cdots\delta_T^{k_T} - 1.$$

If $\Lambda \neq 0$, then

$$\log|\Lambda| > -1.4 \times 30^{T+3}T^{4.5}D^2\log(eD)A'_1\cdots A'_T\log(eB).$$

We shall need the following lemma due to Pethő & de Weger [27].

Lemma 2.7. *Let $a \geq 0$, $c \geq 1$, $g > (e^2/c)^c$, and let $x \in \mathbb{R}$ be the largest solution of $x = a + g(\log x)^c$. Then,*

$$x < 2^c(a^{1/c} + g^{1/c}\log(c^c g))^c.$$

The next result is a slight variation of a result due to Dujella and Pethő [15], which itself is a generalization of the result due to Baker and Davenport [1]. For $x \in \mathbb{R}$, let $\|x\| = \min\left(x - \lfloor x\rfloor, \lceil x\rceil - x\right)$ be the distance from x to the nearest integer.

Lemma 2.8. *Let M be a positive integer. Let q be a convergent denominator to the continued fraction of the irrational γ such that $q > 6M$, and let A_1, A_2, κ be real numbers with $A_1 > 0$ and $A_2 > 1$. Let $\varepsilon := \|\kappa q\| - M\|\gamma q\|$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < m\gamma - n + \kappa < A_1 A_2^{-k}$$

in positive integers m, n and k with $m \leq M$ and $k \geq \dfrac{\log(A_1 q/\varepsilon)}{\log A_2}$.

3. Proof of Theorem 1.1

In what follows, $C_1, C_2, \ldots$ and the constants implicit in the symbol $\ll$ depend only upon $s_0, s_1, a_{0,0}, \ldots, a_{0,r_0-1}, b_{0,0}, \ldots, b_{0,s_0-1}, a_{1,0}, \ldots, a_{1,r_1-1}, b_{1,0}, \ldots, b_{1,s_1-1}$.

Let c be an integer. Assume that $(N_1, M_1) \neq (N_2, M_2)$ are pairs of indices with $N_2 \leq N_1$ such that

$$c = q_{\alpha,N_1} - q_{\beta,M_1} = q_{\alpha,N_2} - q_{\beta,M_2}, \tag{3.9}$$

i.e.

$$q_{\alpha,N_1} - q_{\alpha,N_2} = q_{\beta,M_1} - q_{\beta,M_2}. \tag{3.10}$$

Since the sequences $(q_{\alpha,i})_{i\geq 0}$, $(q_{\beta,i})_{i\geq 0}$ are either strictly increasing or would eventually be strictly increasing, there exists $N_1^*, M_1^* \in \mathbb{N}$ such that for $i_1 >$

$i_2 \geq N_1^*$ and $j_1 > j_2 \geq M_1^*$, $q_{\alpha,i_1} > q_{\alpha,i_2}$ and $q_{\beta,j_1} > q_{\beta,j_2}$, respectively. We will choose $N^* \geq N_1^*$ and $M^* \geq M_1^*$. Hence, $N_1 \geq N_2 \geq N_1^*$ and $M_1 \geq M_2 \geq M_1^*$. Further $N_1 \neq N_2$, since otherwise we either have $M_1 = M_2$ or $M_1 \neq M_2$ with both $M_1, M_2 \leq M_1^*$. Without loss of generality, we assume $N_1 > N_2$, which implies $q_{\alpha,N_1} - q_{\alpha,N_2} > 0$. Using equation (3.10), $q_{\beta,M_1} - q_{\beta,M_2} > 0$ and therefore $M_1 > M_2$.
Write $N_i = s_0 n_i + j_i + r_0$, $0 \leq j_i \leq s_0 - 1$ and $M_i = s_1 m_i + p_i + r_1$, $0 \leq p_i \leq s_1 - 1$ for $i = 1, 2$. For technical reasons, we assume that $N_1 \geq 2s_0 + r_0$, so that $n_1 \geq 2$. In fact, for $N_1 < 2s_0 + r_0$, there will only be finitely many possibilities for c.

Applying Lemma 2.3 for β, there exists M_0^* such that for $M_1 > M_2 \geq M^* = \max(M_0^*, M_1^*)$, we have

$$C_{\beta,3}\theta_{\beta,1}^{m_1} \leq |q_{\beta,M_1} - q_{\beta,M_2}| \leq C_{\beta,4}\theta_{\beta,1}^{m_1}.$$

Let $N^* = \max(N_0^*, N_1^*)$. For $N_1 > N_2 \geq N^*$, we combine the above equation with (2.4), (2.7), and (3.10), to get

$$C_{\alpha,3}\theta_{\alpha,1}^{n_1} \leq |q_{\alpha,N_1} - q_{\alpha,N_2}| = |q_{\beta,M_1} - q_{\beta,M_2}| \leq C_{\beta,4}\theta_{\beta,1}^{m_1} \tag{3.11}$$

and

$$C_{\beta,3}\theta_{\beta,1}^{m_1} \leq |q_{\beta,M_1} - q_{\beta,M_2}| = |q_{\alpha,N_1} - q_{\alpha,N_2}| \leq C_{\alpha,4}\theta_{\alpha,1}^{n_1}.$$

Using the above equation, we get

$$m_1 < C_1 n_1, \quad \text{where} \quad C_1 = \frac{\log\theta_{\alpha,1} + 2\log^+(C_{\alpha,4}/C_{\beta,3})}{\log\theta_{\beta,1}} \tag{3.12}$$

since $n_1 \geq 2$ and $\log^+(x) = \max(\log x, 0)$.
Now, using equations (2.3) and (3.10), we get

$$\begin{aligned} c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\alpha,2}^{(j_1)}\theta_{\alpha,2}^{n_1} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} + c_{\alpha,2}^{(j_2)}\theta_{\alpha,2}^{n_2} = c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1} - c_{\beta,2}^{(p_1)}\theta_{\beta,2}^{m_1} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2} \\ + c_{\beta,2}^{(p_2)}\theta_{\beta,2}^{m_2}. \end{aligned} \tag{3.13}$$

There are 4 possible cases:

(i) $n_1 = n_2$ and $m_1 = m_2$;
(ii) $n_1 = n_2$ and $m_1 > m_2$;
(iii) $n_1 > n_2$ and $m_1 = m_2$;
(iv) $n_1 > n_2$ and $m_1 > m_2$.

In each of these cases, we show that $m_1 \ll \log^3 n_1$, where $\ll$ indicates that the constant involved depends only on α and β.
Case (i): $n_1 = n_2$ and $m_1 = m_2$.
Using equation (3.13), we have

$$(c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)})\theta_{\alpha,1}^{n_1} - (c_{\alpha,2}^{(j_1)} - c_{\alpha,2}^{(j_2)})\theta_{\alpha,2}^{n_1} = (c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)})\theta_{\beta,1}^{m_1} - (c_{\beta,2}^{(p_1)} - c_{\beta,2}^{(p_2)})\theta_{\beta,2}^{m_1},$$

i.e.
$(c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)})\theta_{\alpha,1}^{n_1} - (c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)})\theta_{\beta,1}^{m_1} = (c_{\alpha,2}^{(j_1)} - c_{\alpha,2}^{(j_2)})\theta_{\alpha,2}^{n_1} - (c_{\beta,2}^{(p_1)} - c_{\beta,2}^{(p_2)})\theta_{\beta,2}^{m_1}$.
Since $|\theta_{\alpha,2}| < 1$, $|\theta_{\beta,2}| < 1$, we get

$$\left|(c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)})\theta_{\alpha,1}^{n_1} - (c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)})\theta_{\beta,1}^{m_1}\right| \le C_2.$$

If $c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)} = 0$, then, using Lemma 2.1, we obtain

$$q_{\beta,1}^{(p_1)} - q_{\beta,1}^{(p_2)} = \theta_{\beta,2}(q_{\beta,0}^{(p_1)} - q_{\beta,0}^{(p_2)}).$$

We observe that the left-hand side is an integer, while the right-hand side is an irrational. Therefore, $p_1 = p_2$. We recall that $M_i = s_1 m_i + p_i + r_1$, this gives $M_1 = M_2$, which contradicts our assumption that $M_1 > M_2$. Hence, $c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)} \neq 0$. Dividing by $\left|(c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)})\theta_{\beta,1}^{m_1}\right|$, we get

$$\begin{aligned}\left|\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}}\theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1} - 1\right| &\le \frac{C_2}{\left|c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\right|\theta_{\beta,1}^{m_1}} \\ &= \frac{C_3}{\theta_{\beta,1}^{m_1}}, \end{aligned} \tag{3.14}$$

where $C_3 = \dfrac{C_2}{\left|c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\right|}$.
Let

$$\Gamma_1 = \frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}}\theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1}.$$

If $\Gamma_1 = 1$, then $\dfrac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}} = \dfrac{\theta_{\beta,1}^{m_1}}{\theta_{\alpha,1}^{n_1}}$. Taking heights and applying Lemma 2.5, we get

$$\max(n_1, m_1) \ll 1. \tag{3.15}$$

If $\Gamma_1 \neq 1$, then we apply Lemma 2.6 with $T = 3$, $D = 4$,

$$\begin{aligned}&\delta_1 = \theta_{\alpha,1},\ \delta_2 = \theta_{\beta,1},\ \delta_3 = \frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}}, \\ &k_1 = n_1,\ k_2 = -m_1,\ k_3 = 1, \\ &A_1' \ll 1,\ A_2' \ll 1,\ A_3' \ll 1.\end{aligned}$$

Using (3.12), we can take $B = \max(C_1, 1)n_1$. Then

$$\log|\Gamma_1 - 1| > -C_4 \log(e \max(C_1, 1)n_1).$$

Comparing with (3.14), we obtain

$$m_1 < \frac{C_4(1+\log\max(C_1,1)+\log n_1)+\log C_3}{\log\theta_{\beta,1}}.$$

Therefore,

$$m_1 < C_5 \log n_1,$$

where $C_5 = \dfrac{2C_4\left(2+\log^+ C_1\right)+2\log^+ C_3}{\log\theta_{\beta,1}}$ since $n_1 \geq 2$.
From the above inequality and (3.15), we obtain

$$m_1 \ll \log n_1 \ll \log^3 n_1.$$

Case (ii): $n_1 = n_2$ and $m_1 > m_2$.
Using equation (3.13), we have
(3.16)

$$(c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)})\theta_{\alpha,1}^{n_1} - (c_{\alpha,2}^{(j_1)} - c_{\alpha,2}^{(j_2)})\theta_{\alpha,2}^{n_1} = c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1} - c_{\beta,2}^{(p_1)}\theta_{\beta,2}^{m_1} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2} + c_{\beta,2}^{(p_2)}\theta_{\beta,2}^{m_2},$$

i.e.

$$(c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)})\theta_{\alpha,1}^{n_1} - c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1} = (c_{\alpha,2}^{(j_1)} - c_{\alpha,2}^{(j_2)})\theta_{\alpha,2}^{n_1} - c_{\beta,2}^{(p_1)}\theta_{\beta,2}^{m_1} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2} + c_{\beta,2}^{(p_2)}\theta_{\beta,2}^{m_2}.$$

Since $|\theta_{\alpha,2}| < 1$, $|\theta_{\beta,2}| < 1$, we get

$$\left|(c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)})\theta_{\alpha,1}^{n_1} - c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}\right| \leq \left|c_{\alpha,2}^{(j_1)} - c_{\alpha,2}^{(j_2)}\right| + \left|c_{\beta,2}^{(p_1)}\right| + c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2} + \left|c_{\beta,2}^{(p_2)}\right|.$$

Dividing by $c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}$, we get

$$\begin{aligned}
\left|\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)}}\theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1} - 1\right| &\leq \frac{\left|c_{\alpha,2}^{(j_1)} - c_{\alpha,2}^{(j_2)}\right| + \left|c_{\beta,2}^{(p_1)}\right| + \left|c_{\beta,2}^{(p_2)}\right|}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}} + \frac{c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}} \\
&\leq \frac{C_6}{\theta_{\beta,1}^{m_1}} + \frac{C_7}{\theta_{\beta,1}^{m_1-m_2}} \\
&\leq \frac{C_8}{\theta_{\beta,1}^{m_1-m_2}}. \qquad (3.17)
\end{aligned}$$

Let

$$\Gamma_2 = \frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)}}\theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1}.$$

If $\Gamma_2 = 1$, then $\dfrac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)}} = \dfrac{\theta_{\beta,1}^{m_1}}{\theta_{\alpha,1}^{n_1}}$. Taking heights and applying Lemma 2.5, we get

$$\max(n_1, m_1) \ll 1. \qquad (3.18)$$

If $\Gamma_2 \neq 1$, then we apply Lemma 2.6 with $T = 3$, $D = 4$,

$$\begin{aligned}
&\delta_1 = \theta_{\alpha,1},\ \delta_2 = \theta_{\beta,1},\ \delta_3 = \frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)}},\\
&k_1 = n_1,\ k_2 = -m_1,\ k_3 = 1,\\
&A_1' \ll 1,\ A_2' \ll 1,\ A_3' \ll 1.
\end{aligned}$$

Using (3.12), we can take $B = \max(C_1, 1)n_1$. Then

$$\log|\Gamma_2 - 1| > -C_9 \log(e \max(C_1, 1)n_1).$$

Comparing with (3.17), we obtain

$$m_1 - m_2 < \frac{C_9(1 + \log\max(C_1, 1) + \log n_1) + \log C_8}{\log\theta_{\beta,1}}.$$

Therefore,

$$m_1 - m_2 < C_{10} \log n_1,$$

where $C_{10} = \dfrac{2C_9\left(2 + \log^+ C_1\right) + 2\log^+ C_8}{\log\theta_{\beta,1}}$ since $n_1 \geq 2$.

Again using equation (3.16), we have

$$(c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)})\theta_{\alpha,1}^{n_1} - c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1} + c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2} = (c_{\alpha,2}^{(j_1)} - c_{\alpha,2}^{(j_2)})\theta_{\alpha,2}^{n_1} - c_{\beta,2}^{(p_1)}\theta_{\beta,2}^{m_1} + c_{\beta,2}^{(p_2)}\theta_{\beta,2}^{m_2}.$$

Since $|\theta_{\alpha,2}| < 1$, $|\theta_{\beta,2}| < 1$, we get

$$\left|(c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)})\theta_{\alpha,1}^{n_1} - c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1} + c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}\right| \leq \left|(c_{\alpha,2}^{(j_1)} - c_{\alpha,2}^{(j_2)})\right| + \left|c_{\beta,2}^{(p_1)}\right| + \left|c_{\beta,2}^{(p_2)}\right|.$$

If $\left|c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}\right| = 0$, then

$$\left|(c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)})\theta_{\alpha,1}^{n_1}\right| \leq \left|(c_{\alpha,2}^{(j_1)} - c_{\alpha,2}^{(j_2)})\right| + \left|c_{\beta,2}^{(p_1)}\right| + \left|c_{\beta,2}^{(p_2)}\right|.$$

Since $c_{\alpha,1}^{(j_1)} \neq c_{\alpha,1}^{(j_2)}$, we obtain that $n_1 \ll 1$. Hence, by (3.12), $m_1 \ll 1$. Assuming $\left|c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}\right| \neq 0$, we divide by it to obtain

$$\left|\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}}\theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1} - 1\right| \leq \frac{\left|(c_{\alpha,2}^{(j_1)} - c_{\alpha,2}^{(j_2)})\right| + \left|c_{\beta,2}^{(p_1)}\right| + \left|c_{\beta,2}^{(p_2)}\right|}{\left|c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{-(m_1-m_2)}\right|} \cdot \frac{1}{\theta_{\beta,1}^{m_1}}.$$

We now find an upper bound for the expression $1/\left|c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{-(m_1-m_2)}\right|$. The sequence $(1/\theta_{\beta,1}^k)_{k\geq 1}$ decreases to 0. Therefore, there exists a $k_0 \in \mathbb{N}$ depending only on β such that

$$\left|\frac{1}{\theta_{\beta,1}^k}\right| \leq \frac{c_{\beta,1}^{(p_1)}}{2c_{\beta,1}^{(p_2)}},\ \text{for all } k \geq k_0.$$

Therefore,

$$\left|c_{\beta,1}^{(p_1)} - \frac{c_{\beta,1}^{(p_2)}}{\theta_{\beta,1}^{m_1-m_2}}\right| \geq \min\left(\left|c_{\beta,1}^{(p_1)} - \frac{c_{\beta,1}^{(p_2)}}{\theta_{\beta,1}}\right|, \ldots, \left|c_{\beta,1}^{(p_1)} - \frac{c_{\beta,1}^{(p_2)}}{\theta_{\beta,1}^{k_0-1}}\right|, \frac{c_{\beta,1}^{(p_1)}}{2}\right).$$

Finally, we get

$$\left|\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}}\theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1} - 1\right| \leq \frac{C_{11}}{\theta_{\beta,1}^{m_1}}. \tag{3.19}$$

Let

$$\Gamma_3 = \frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}}\theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1}.$$

If $\Gamma_3 = 1$, then $\dfrac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}} = \dfrac{\theta_{\beta,1}^{m_1}}{\theta_{\alpha,1}^{n_1}}$. Applying Lemma 2.5, there exists an effectively computable constant D_0, such that

$$D_0 \max(n_1, m_1) \leq h\left(\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}}\right).$$

Applying (2.8), we get

$$\begin{aligned}
h\left(\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}}\right) &\leq h\left(c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\right) + h\left(c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}\right) \\
&\leq h(c_{\alpha,1}^{(j_1)}) + h(c_{\alpha,1}^{(j_2)}) + h(c_{\beta,1}^{(p_1)}) + h(c_{\beta,1}^{(p_2)}) \\
&\quad + (m_1 - m_2)\frac{\log\theta_{\beta,1}}{2} + 2\log 2.
\end{aligned}$$

As noted above,

$$m_1 - m_2 \leq C_{10}\log n_1.$$

Hence,

$$\max(n_1, m_1) \ll \log n_1. \tag{3.20}$$

If $\Gamma_3 \neq 1$, then we apply Lemma 2.6 with $T = 3$, $D = 4$,

$$\begin{aligned}
&\delta_1 = \theta_{\alpha,1},\ \delta_2 = \theta_{\beta,1},\ \delta_3 = \frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}}, \\
&k_1 = n_1,\ k_2 = -m_1,\ k_3 = 1, \\
&A_1' \ll 1,\ A_2' \ll 1,\ A_3' \ll \log n_1.
\end{aligned}$$

Using (3.12), we can take $B = \max(C_1, 1)n_1$. Then

$$\log|\Gamma_3 - 1| > -C_{12}\log(e\max(C_1, 1)n_1)\log n_1.$$

Comparing with (3.19), we obtain

$$m_1 < \frac{C_{12}(1+\log\max(C_1,1)+\log n_1)\log n_1 + \log C_{11}}{\log\theta_{\beta,1}}.$$

Therefore,

$$m_1 < C_{13}\log^2 n_1,$$

where $C_{13} = \dfrac{2C_{12}\left(2+\log^+ C_1\right)+3\log^+ C_{11}}{\log\theta_{\beta,1}}$ since $n_1 \geq 2$.
From the above inequality, (3.18), and (3.20), we obtain

$$m_1 \ll \log^2 n_1 \ll \log^3 n_1.$$

Case (iii): It is similar to case (ii). Following a similar procedure, we get

$$n_1 \ll \log^2 n_1.$$

Hence,

$$m_1 \ll \log^2 n_1 \ll \log^3 n_1.$$

Case (iv): $n_1 > n_2$ and $m_1 > m_2$.
Using equation (3.13), we get

$$\begin{aligned} c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1} =& c_{\alpha,2}^{(j_1)}\theta_{\alpha,2}^{n_1} + c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} - c_{\alpha,2}^{(j_2)}\theta_{\alpha,2}^{n_2} - c_{\beta,2}^{(p_1)}\theta_{\beta,2}^{m_1} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2} \\ &+ c_{\beta,2}^{(p_2)}\theta_{\beta,2}^{m_2}. \end{aligned}$$

Since $|\theta_{\alpha,2}| < 1$, $|\theta_{\beta,2}| < 1$, we get

$$\left|c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}\right| \leq C_{14} + c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} + c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}.$$

Dividing by $c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}$ and using (3.11), we get

$$\begin{aligned} \left|\frac{c_{\alpha,1}^{(j_1)}}{c_{\beta,1}^{(p_1)}}\theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1} - 1\right| &\leq \frac{C_{14}}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}} + \frac{c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)}}\cdot\frac{\theta_{\alpha,1}^{n_2}}{\theta_{\beta,1}^{m_1}} + \frac{c_{\beta,1}^{(p_2)}}{c_{\beta,1}^{(p_1)}}\cdot\frac{\theta_{\beta,1}^{m_2}}{\theta_{\beta,1}^{m_1}} \\ &\leq \frac{C_{14}}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}} + \frac{c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)}}\cdot\frac{\theta_{\alpha,1}^{n_1}}{\theta_{\beta,1}^{m_1}}\cdot\frac{\theta_{\alpha,1}^{n_2}}{\theta_{\alpha,1}^{n_1}} + \frac{c_{\beta,1}^{(p_2)}}{c_{\beta,1}^{(p_1)}}\cdot\frac{1}{\theta_{\beta,1}^{m_1-m_2}} \\ &\leq \frac{C_{14}}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}} + \frac{c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)}}\cdot\frac{C_{\beta,4}}{C_{\alpha,3}}\cdot\frac{1}{\theta_{\alpha,1}^{n_1-n_2}} + \frac{c_{\beta,1}^{(p_2)}}{c_{\beta,1}^{(p_1)}}\cdot\frac{1}{\theta_{\beta,1}^{m_1-m_2}} \\ &\leq C_{15}\max\left\{\frac{1}{\theta_{\alpha,1}^{n_1-n_2}}, \frac{1}{\theta_{\beta,1}^{m_1-m_2}}\right\} \\ &\leq \frac{C_{15}}{\min(\theta_{\alpha,1},\theta_{\beta,1})^{\min(n_1-n_2,m_1-m_2)}}. \end{aligned} \tag{3.21}$$

Let

$$\Gamma_4 = \frac{c_{\alpha,1}^{(j_1)} \theta_{\alpha,1}^{n_1}}{c_{\beta,1}^{(p_1)} \theta_{\beta,1}^{m_1}}.$$

If $\Gamma_4 = 1$, then $\dfrac{c_{\alpha,1}^{(j_1)}}{c_{\beta,1}^{(p_1)}} = \dfrac{\theta_{\beta,1}^{m_1}}{\theta_{\alpha,1}^{n_1}}$. Taking heights and applying Lemma 2.5, we get

$$\max(n_1, m_1) \ll 1. \tag{3.22}$$

If $\Gamma_4 \neq 1$, then we apply Lemma 2.6 with $T = 3$, $D = 4$,

$$\begin{aligned}
&\delta_1 = \theta_{\alpha,1},\ \delta_2 = \theta_{\beta,1},\ \delta_3 = \frac{c_{\alpha,1}^{(j_1)}}{c_{\beta,1}^{(p_1)}},\\
&k_1 = n_1,\ k_2 = -m_1,\ k_3 = 1,\\
&A_1' \ll 1,\ A_2' \ll 1,\ A_3' \ll 1.
\end{aligned}$$

Using (3.12), we can take $B = \max(C_1, 1)n_1$. Then

$$\log|\Gamma_4 - 1| > -C_{16} \log(e \max(C_1, 1)n_1).$$

Comparing with (3.21), we obtain

$$\min(n_1 - n_2, m_1 - m_2) < \frac{C_{16}(1 + \log\max(C_1, 1) + \log n_1) + \log C_{15}}{\log\min(\theta_{\alpha,1}, \theta_{\beta,1})}.$$

Therefore,

$$\min(n_1 - n_2, m_1 - m_2) < C_{17} \log n_1,$$

where $C_{17} = \dfrac{2C_{16}\left(2 + \log^+ C_1\right) + 2\log^+ C_{15}}{\log\min(\theta_{\alpha,1}, \theta_{\beta,1})}$ since $n_1 \geq 2$.

If $\min(n_1 - n_2, m_1 - m_2) = n_1 - n_2$, then, using equation (3.13), we have

$$\begin{aligned}
c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} - c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1} =& c_{\alpha,2}^{(j_1)}\theta_{\alpha,2}^{n_1} - c_{\alpha,2}^{(j_2)}\theta_{\alpha,2}^{n_2} - c_{\beta,2}^{(p_1)}\theta_{\beta,2}^{m_1} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}\\
&+ c_{\beta,2}^{(p_2)}\theta_{\beta,2}^{m_2}.
\end{aligned}$$

Since $|\theta_{\alpha,2}| < 1$, $|\theta_{\beta,2}| < 1$, we get

$$\left|c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} - c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}\right| \leq C_{18} + c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}.$$

Dividing by $c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}$, we get

$$\left|\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)}}\theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1} - 1\right| \leq \frac{C_{19}}{\theta_{\beta,1}^{m_1-m_2}}. \tag{3.23}$$

Let

$$\Gamma_5 = \frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)}}\theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1}.$$

If $\Gamma_5 = 1$, then $\dfrac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)}} = \dfrac{\theta_{\beta,1}^{m_1}}{\theta_{\alpha,1}^{n_1}}$. Applying Lemma 2.5, there exists an effectively computable constant D_0, such that

$$D_0 \max(n_1, m_1) \leq h\left(\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)}}\right).$$

Applying (2.8), we get

$$\begin{aligned} h\left(\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)}}\right) &\leq h\left(c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}\right) + h(c_{\beta,1}^{(p_1)}) \\ &\leq h(c_{\alpha,1}^{(j_1)}) + h(c_{\alpha,1}^{(j_2)}) + (n_1 - n_2)\frac{\log\theta_{\alpha,1}}{2} + h(c_{\beta,1}^{(p_1)}) \\ &\quad + \log 2. \end{aligned}$$

As noted above,

$$n_1 - n_2 \leq C_{17} \log n_1.$$

Hence,

$$\max(n_1, m_1) \ll \log n_1. \tag{3.24}$$

If $\Gamma_5 \neq 1$, then we apply Lemma 2.6 with $T = 3$, $D = 4$,

$$\begin{aligned} &\delta_1 = \theta_{\alpha,1},\ \delta_2 = \theta_{\beta,1},\ \delta_3 = \frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)}}, \\ &k_1 = n_1,\ k_2 = -m_1,\ k_3 = 1, \\ &A_1' \ll 1,\ A_2' \ll 1,\ A_3' \ll \log n_1. \end{aligned}$$

Using (3.12), we can take $B = \max(C_1, 1)n_1$. Then

$$\log|\Gamma_5 - 1| > -C_{20} \log(e \max(C_1, 1)n_1) \log n_1.$$

Comparing with (3.23), we obtain

$$m_1 - m_2 < \frac{C_{20}(1 + \log\max(C_1, 1) + \log n_1)\log n_1 + \log C_{19}}{\log\theta_{\beta,1}}.$$

Therefore,

$$m_1 - m_2 < C_{21} \log^2 n_1,$$

where $C_{21} = \dfrac{2C_{20}\left(2 + \log^+ C_1\right) + 3\log^+ C_{19}}{\log\theta_{\beta,1}}$ since $n_1 \geq 2$.

If $\min(n_1 - n_2, m_1 - m_2) = m_1 - m_2$, then, we get

$$n_1 - n_2 < C_{22} \log^2 n_1.$$

Using both cases,

$$\max(n_1 - n_2, m_1 - m_2) \ll \log^2 n_1.$$

By again considering equation (3.13), we get

$$c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} - c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1} + c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2} = c_{\alpha,2}^{(j_1)}\theta_{\alpha,2}^{n_1} - c_{\alpha,2}^{(j_2)}\theta_{\alpha,2}^{n_2} - c_{\beta,2}^{(p_1)}\theta_{\beta,2}^{m_1} + c_{\beta,2}^{(p_2)}\theta_{\beta,2}^{m_2}.$$

Since $|\theta_{\alpha,2}| < 1$, $|\theta_{\beta,2}| < 1$, we get

$$\left|c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} - c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1} + c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}\right| \le C_{23}.$$

We now proceed as in Case (ii). Either $m_1 \ll 1$ or we have

$$\begin{aligned}\left|\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}}\theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1} - 1\right| &\le \frac{C_{23}}{\left|c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}\right|} \cdot \frac{1}{\theta_{\beta,1}^{m_1}} \\ &\le \frac{C_{24}}{\theta_{\beta,1}^{m_1}}. \qquad (3.25)\end{aligned}$$

Let

$$\Gamma_6 = \frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}}\theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1}.$$

If $\Gamma_6 = 1$, then $\dfrac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}} = \dfrac{\theta_{\beta,1}^{m_1}}{\theta_{\alpha,1}^{n_1}}$. Applying Lemma 2.5, there exists an effectively computable constant D_0, such that

$$D_0 \max(n_1, m_1) \le h\left(\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}}\right).$$

Applying (2.8), we get

$$\begin{aligned}h\left(\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}}\right) &\le h\left(c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}\right) + h\left(c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}\right) \\ &\le h(c_{\alpha,1}^{(j_1)}) + h(c_{\alpha,1}^{(j_2)}) + h(c_{\beta,1}^{(p_1)}) + h(c_{\beta,1}^{(p_2)}) + 2\log 2 \\ &\quad + (m_1 - m_2)\frac{\log\theta_{\beta,1}}{2} + (n_1 - n_2)\frac{\log\theta_{\alpha,1}}{2}.\end{aligned}$$

As noted above,

$$\max(n_1 - n_2, m_1 - m_2) \ll \log^2 n_1.$$

Hence,

$$\max(n_1, m_1) \ll \log^2 n_1. \qquad (3.26)$$

If $\Gamma_6 \neq 1$, then we apply Lemma 2.6 with $T = 3$, $D = 4$,

$$\delta_1 = \theta_{\alpha,1},\ \delta_2 = \theta_{\beta,1},\ \delta_3 = \frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}},$$
$$k_1 = n_1,\ k_2 = -m_1,\ k_3 = 1,$$
$$A_1' \ll 1,\ A_2' \ll 1,\ A_3' \ll \log^2 n_1.$$

Using (3.12), we can take $B = \max(C_1, 1)n_1$. Then

$$\log|\Gamma_6 - 1| > -C_{26}\log(e\max(C_1, 1)n_1)\log^2 n_1.$$

Comparing with (3.25), we obtain

$$m_1 < \frac{C_{26}(1 + \log\max(C_1, 1) + \log n_1)\log^2 n_1 + \log C_{24}}{\log\theta_{\beta,1}}.$$

Therefore,

$$m_1 < C_{27}\log^3 n_1,$$

where $C_{27} = \dfrac{2C_{26}\left(2 + \log^+ C_1\right) + 4\log^+ C_{24}}{\log\theta_{\beta,1}}$ since $n_1 \geq 2$.

From the above inequality, (3.22), (3.24), and (3.26), we obtain

$$m_1 \ll \log^3 n_1.$$

In all four cases, we have proved that

$$m_1 \ll \log^3 n_1.$$

Now, using equation (3.11), we get

$$n_1 < C_{28}\log^3 n_1.$$

Therefore, $n_1 < C_{29}\log^3 n_1$. We now use Lemma 2.7 with $a = 0, c = 3, g = C_{29}^3 = C_{28}$. (If necessary, we can replace C_{29} by e^2.) We thus obtain

$$n_1 \leq 216C_{29}^3\log^3(3C_{29}).$$

Since n_1 is bounded, m_1 will also be bounded. Hence, N_1 and M_1 are bounded. Therefore, the cardinality of the set $\mathcal{C}$ is finite. Hence, Theorem 1.1 is proved.

4. Proof of Theorem 1.2

Let c be an integer which has two distinct representations as shown in equation (3.9). We note that for our choices of α and β, we have $s_0 = s_1 = 2$ and $r_0 = r_1 = 0$, thus, $N_i = 2n_i + j_i$, $0 \leq j_i \leq 1$ and $M_i = 2m_i + p_i$, $0 \leq p_i \leq 1$, $i = 1, 2$. For technical reasons, we assume that $n_1 \geq 6$. Since $\alpha = [0; \overline{1, a}]$, we have $q_{\alpha,0} = 1 = q_{\alpha,1}$ and

$$q_{\alpha,N} = \begin{cases} aq_{\alpha,N-1} + q_{\alpha,N-2} & \text{if } N \text{ is even,} \\ q_{\alpha,N-1} + q_{\alpha,N-2} & \text{if } N \text{ is odd.} \end{cases}$$

Let $d_a = a^2 + 4a$ and $\theta_{\alpha,1} = \dfrac{a+2+\sqrt{d_a}}{2}$. Then

$$q_{\alpha,2\ell} = \frac{a+\sqrt{d_a}}{2\sqrt{d_a}}\theta_{\alpha,1}^{\ell} - \frac{a-\sqrt{d_a}}{2\sqrt{d_a}}\theta_{\alpha,1}^{-\ell},$$

$$q_{\alpha,2\ell+1} = \frac{\theta_{\alpha,1}}{\sqrt{d_a}}\theta_{\alpha,1}^{\ell} - \frac{\theta_{\alpha,1}^{-1}}{\sqrt{d_a}}\theta_{\alpha,1}^{-\ell}.$$

We have the following inequalities:

$$\frac{1}{2}\theta_{\alpha,1}^{\ell} < q_{\alpha,2\ell} < \frac{11}{10}\cdot\frac{a+\sqrt{d_a}}{2\sqrt{d_a}}\theta_{\alpha,1}^{\ell},$$

$$\frac{7}{8}\cdot\frac{\theta_{\alpha,1}}{\sqrt{d_a}}\theta_{\alpha,1}^{\ell} < q_{\alpha,2\ell+1} < \frac{9}{8}\cdot\frac{\theta_{\alpha,1}}{\sqrt{d_a}}\theta_{\alpha,1}^{\ell}.$$

From the above inequalities, we have

$$\frac{1}{2}\theta_{\alpha,1}^{\lfloor N/2\rfloor} < q_{\alpha,N} < \frac{9}{8}\cdot\frac{\theta_{\alpha,1}}{\sqrt{d_a}}\theta_{\alpha,1}^{\lfloor N/2\rfloor},\ N \geq 0.$$

We note that $\dfrac{\frac{9\theta_{\alpha,1}}{8\sqrt{d_a}}}{1/2} < \theta_{\alpha,1}$. Referring to equation (2.7), for $N_1 > N_2 \geq 2$, we have

$$\frac{1}{2\sqrt{d_a}}\theta_{\alpha,1}^{\lfloor N_1/2\rfloor} \leq |q_{\alpha,N_1} - q_{\alpha,N_2}| = |q_{\beta,M_1} - q_{\beta,M_2}| < q_{\beta,M_1} < \frac{9}{8}\cdot\frac{\theta_{\beta,1}}{\sqrt{d_b}}\theta_{\beta,1}^{\lfloor M_1/2\rfloor}.$$

As $\lfloor \frac{N_1}{2}\rfloor = n_1$ and $\lfloor\frac{M_1}{2}\rfloor = m_1$, we have

$$\frac{1}{2\sqrt{d_a}}\theta_{\alpha,1}^{n_1} < \frac{9}{8}\cdot\frac{\theta_{\beta,1}}{\sqrt{d_b}}\theta_{\beta,1}^{m_1}, \tag{4.27}$$

therefore

$$n_1 < \frac{\log(\frac{9}{8}\cdot\frac{\theta_{\beta,1}}{\sqrt{d_b}}) + \log(2\sqrt{d_a})}{\log\theta_{\alpha,1}} + m_1\frac{\log\theta_{\beta,1}}{\log\theta_{\alpha,1}}.$$

As $y = \log\left(\dfrac{9}{8}\cdot\dfrac{x+2+\sqrt{x^2+4x}}{2\sqrt{x^2+4x}}\right)$ and $y = \dfrac{0.17+\log(2\sqrt{x^2+4x})}{\log(\frac{x+2+\sqrt{x^2+4x}}{2})}$ are decreasing functions of x for $x \geq 2$, we get $\log\left(\dfrac{9}{8}\cdot\frac{\theta_{\beta,1}}{\sqrt{d_b}}\right) < 0.17$ for $b \geq 3$, and $\frac{0.17+\log(2\sqrt{d_a})}{\log\theta_{\alpha,1}} < 1.60$ for $a \geq 2$. Therefore,

$$n_1 < 1.60 + m_1\frac{\log\theta_{\beta,1}}{\log\theta_{\alpha,1}}. \tag{4.28}$$

Similarly for $M_1 > M_2 \geq 2$, we have

$$m_1 < \frac{\log(\frac{9}{8}\cdot\frac{\theta_{\alpha,1}}{\sqrt{d_a}}) + \log(2\sqrt{d_b})}{\log\theta_{\beta,1}} + n_1\frac{\log\theta_{\alpha,1}}{\log\theta_{\beta,1}}.$$

Since $\dfrac{\log(\frac{9}{8}\cdot\frac{\theta_{\alpha,1}}{\sqrt{d_a}})+\log(2\sqrt{d_b})}{\log\theta_{\beta,1}}<1.55$ for $2\le a<b$, we have

$$m_1<1.55+n_1\frac{\log\theta_{\alpha,1}}{\log\theta_{\beta,1}}. \tag{4.29}$$

As in the proof of Theorem 1.1, the bound obtained in Case (iv) is the largest among all cases; therefore, we discuss only Case (iv).

Case (iv): $n_1>n_2$ and $m_1>m_2$.

Using equation (3.13), we get

$$\left|c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1}-c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}\right|\le\left|c_{\alpha,2}^{(j_1)}\right|+\left|c_{\alpha,2}^{(j_2)}\right|+\left|c_{\beta,2}^{(p_1)}\right|+\left|c_{\beta,2}^{(p_2)}\right|+c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2}+c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}.$$

Since $\dfrac{\sqrt{d_a}-a}{2\sqrt{d_a}}>\dfrac{\theta_{\alpha,1}^{-1}}{\sqrt{d_a}}$, $y=\dfrac{\sqrt{x^2+4x}-x}{2\sqrt{x^2+4x}}$ is a decreasing function of x for $x\ge2$, and $2\le a<b$, we get

$$\left|c_{\alpha,2}^{(j_1)}\right|+\left|c_{\alpha,2}^{(j_2)}\right|+\left|c_{\beta,2}^{(p_1)}\right|+\left|c_{\beta,2}^{(p_2)}\right|\le2\left(\frac{\sqrt{d_a}-a}{2\sqrt{d_a}}+\frac{\sqrt{d_b}-b}{2\sqrt{d_b}}\right)\le0.77. \tag{4.30}$$

Therefore,

$$\left|c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1}-c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}\right|\le0.77+c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2}+c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}.$$

Dividing by $c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}$, we get

$$\left|\frac{c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1}}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}}-1\right|\le\frac{0.77}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}}+\frac{c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2}}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}}+\frac{c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}}.$$

First, we compute an upper bound for the term $\dfrac{0.77}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}}+\dfrac{c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}}$ by using that the functions $y=\dfrac{\sqrt{x^2+4x}}{x+\sqrt{x^2+4x}}$ and $y=\dfrac{x+2+\sqrt{x^2+4x}}{x+\sqrt{x^2+4x}}$ are decreasing functions of x for $x\ge3$. So,

$$\begin{aligned}\frac{0.77}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}}+\frac{c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}}&\le\frac{0.77}{\frac{b+\sqrt{d_b}}{2\sqrt{d_b}}}\cdot\frac{1}{\theta_{\beta,1}^{m_1}}+\frac{\frac{\theta_{\beta,1}}{\sqrt{d_b}}}{\frac{b+\sqrt{d_b}}{2\sqrt{d_b}}}\cdot\frac{1}{\theta_{\beta,1}^{m_1-m_2}}\\&=\frac{1.54\sqrt{d_b}}{b+\sqrt{d_b}}\cdot\frac{1}{\theta_{\beta,1}^{m_1}}+\frac{b+2+\sqrt{d_b}}{b+\sqrt{d_b}}\cdot\frac{1}{\theta_{\beta,1}^{m_1-m_2}}\\&\le\frac{0.94}{\theta_{\beta,1}^{m_1}}+\frac{1.27}{\theta_{\beta,1}^{m_1-m_2}}\\&\le\frac{2.21}{\theta_{\beta,1}^{m_1-m_2}}\quad(\text{because}\ \ m_1>m_2).\end{aligned} \tag{4.31}$$

Next, we compute an upper bound for $c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2}/c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}$ by using that the function $y = \dfrac{x+2+\sqrt{x^2+4x}}{2(x+\sqrt{x^2+4x})}$ is a decreasing function of x for $x \geq 3$. Using equation (4.27), we deduce

$$\begin{aligned}
\frac{c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2}}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}} = \frac{c_{\alpha,1}^{(j_2)}}{c_{\beta,1}^{(p_1)}} \cdot \frac{\theta_{\alpha,1}^{n_1}}{\theta_{\beta,1}^{m_1}} \cdot \frac{\theta_{\alpha,1}^{n_2}}{\theta_{\alpha,1}^{n_1}} &\leq \frac{\theta_{\alpha,1}}{\sqrt{d_a}} \cdot \frac{2\sqrt{d_b}}{b+\sqrt{d_b}} \cdot \frac{\theta_{\alpha,1}^{n_1}}{\theta_{\beta,1}^{m_1}} \cdot \frac{1}{\theta_{\alpha,1}^{n_1-n_2}} \\
&\leq \frac{\theta_{\alpha,1}}{\sqrt{d_a}} \cdot \frac{2\sqrt{d_b}}{b+\sqrt{d_b}} \cdot \frac{9\theta_{\beta,1}\sqrt{d_a}}{4\sqrt{d_b}} \cdot \frac{1}{\theta_{\alpha,1}^{n_1-n_2}} \\
&= \frac{9\theta_{\alpha,1}\theta_{\beta,1}}{2(b+\sqrt{d_b})} \cdot \frac{1}{\theta_{\alpha,1}^{n_1-n_2}} \leq \frac{2.85\ \theta_{\alpha,1}}{\theta_{\alpha,1}^{n_1-n_2}}. \qquad (4.32)
\end{aligned}$$

Finally,

$$\begin{aligned}
\left| \frac{c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1}}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}} - 1 \right| &\leq \frac{2.21}{\theta_{\beta,1}^{m_1-m_2}} + \frac{2.85\theta_{\alpha,1}}{\theta_{\alpha,1}^{n_1-n_2}} \\
&\leq 5.06\theta_{\alpha,1} \left(\frac{1}{\theta_{\beta,1}^{m_1-m_2}} + \frac{1}{\theta_{\alpha,1}^{n_1-n_2}} \right) \\
&\leq 10.12\theta_{\alpha,1} \max \left(\frac{1}{\theta_{\alpha,1}^{n_1-n_2}}, \frac{1}{\theta_{\beta,1}^{m_1-m_2}} \right). \qquad (4.33)
\end{aligned}$$

Let $\Gamma_7 = \dfrac{c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1}}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}}$. If $\Gamma_7 = 1$, then $c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} = c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}$ which is not possible because $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\beta)$. As $\Gamma_7 \neq 1$, we apply Lemma 2.6 with $T = 3$, $D = 4$,

$$\begin{aligned}
&\delta_1 = \theta_{\alpha,1},\ \delta_2 = \theta_{\beta,1},\ \delta_3 = \frac{c_{\alpha,1}^{(j_1)}}{c_{\beta,1}^{(p_1)}}, \\
&k_1 = n_1,\ k_2 = -m_1,\ k_3 = 1, \\
&A_1' = 2\log\theta_{\alpha,1},\ A_2' = 2\log\theta_{\beta,1}.
\end{aligned}$$

Using (2.8), $h(\delta_3) \leq h(c_{\alpha,1}^{(j_1)}) + h(c_{\beta,1}^{(p_1)})$.
Now,

$$h\left(\frac{a+\sqrt{d_a}}{2\sqrt{d_a}}\right) \leq h(a+\sqrt{d_a}) + h(2\sqrt{d_a}) = \frac{\log 4a}{2} + \frac{\log 4d_a}{2},$$

and

$$h\left(\frac{\theta_{\alpha,1}}{\sqrt{d_a}}\right) \leq h(\theta_{\alpha,1}) + h(\sqrt{d_a}) = \frac{\log\theta_{\alpha,1}}{2} + \frac{\log d_a}{2}.$$

Since $\log 4a+\log 4d_a > \log\theta_{\alpha,1}+\log d_a$, we have $h\left(c_{\alpha,1}^{(j_1)}\right) \le \dfrac{\log 4a}{2} + \dfrac{\log 4d_a}{2}$. Similarly, $h\left(c_{\beta,1}^{(p_1)}\right) \le \dfrac{\log 4b}{2} + \dfrac{\log 4d_b}{2}$. Since $2 \le a < b$, $h\left(c_{\alpha,1}^{(j_1)}/c_{\beta,1}^{(p_1)}\right) \le \log 4b + \log 4d_b < 5\log\theta_{\beta,1}$. Therefore, we may take $A_3' = 20\log\theta_{\beta,1}$. By using equation (4.29), we can take $B = 2n_1$. Then

$$\log|\Gamma_7 - 1| > -4.3736\times 10^{14}(1+\log(2n_1))(\log\theta_{\alpha,1})\log^2\theta_{\beta,1}.$$

Comparing with (4.33), we obtain

$$\min(n_1-n_2, m_1-m_2) < 8.75\times 10^{14}\log^2\theta_{\beta,1}\log n_1.$$

If $\min(n_1-n_2, m_1-m_2) = n_1-n_2$, then using equation (3.13), we have

$$\begin{aligned} c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} - c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1} &= c_{\alpha,2}^{(j_1)}\theta_{\alpha,2}^{n_1} - c_{\alpha,2}^{(j_2)}\theta_{\alpha,2}^{n_2} - c_{\beta,2}^{(p_1)}\theta_{\beta,2}^{m_1} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2} \\ &\quad + c_{\beta,2}^{(p_2)}\theta_{\beta,2}^{m_2}. \end{aligned}$$

Using $|\theta_{\alpha,2}| < 1$, $|\theta_{\beta,2}| < 1$, and (4.30), we get

$$\begin{aligned} \left|c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} - c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}\right| &\le \left|c_{\alpha,2}^{(j_1)}\right| + \left|c_{\alpha,2}^{(j_2)}\right| + \left|c_{\beta,2}^{(p_1)}\right| + \left|c_{\beta,2}^{(p_2)}\right| + c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2} \\ &\le 0.77 + c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}. \end{aligned}$$

Dividing by $c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}$ and using inequality (4.31), we get

$$\begin{aligned} \left|\frac{c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2}}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}} - 1\right| &\le \frac{0.77}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}} + \frac{c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}}{c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}} \\ &\le \frac{2.21}{\theta_{\beta,1}^{m_1-m_2}}. \end{aligned} \tag{4.34}$$

Let

$$\Gamma_8 = \frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)}}\theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1}.$$

Suppose $\Gamma_8 = 1$. By Lemma 2.4, we have $c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} \notin \mathbb{Q}$. Since $\mathbb{Q}(\alpha) \ne \mathbb{Q}(\beta)$, the equality $c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} = c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1}$ is impossible. Hence, $\Gamma_8 \ne 1$. We apply Lemma 2.6 with $T = 3$, $D = 4$,

$$\begin{aligned} &\delta_1 = \theta_{\alpha,1},\ \delta_2 = \theta_{\beta,1},\ \delta_3 = \frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)}}, \\ &k_1 = n_1,\ k_2 = -m_1,\ k_3 = 1, \\ &A_1' = 2\log\theta_{\alpha,1},\ A_2' = 2\log\theta_{\beta,1}. \end{aligned}$$

Applying (2.8), we get

$$\begin{aligned} h\left(\delta_3\right) &\leq h\left(c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}\right) + h\left(c_{\beta,1}^{(p_1)}\right) \\ &\leq h(c_{\alpha,1}^{(j_1)}) + h(c_{\alpha,1}^{(j_2)}) + h(c_{\beta,1}^{(p_1)}) + (n_1-n_2)\frac{\log\theta_{\alpha,1}}{2} + \log 2. \end{aligned}$$

As

$$h(c_{\alpha,1}^{(j_1)}) + h(c_{\alpha,1}^{(j_2)}) + h(c_{\beta,1}^{(p_1)}) \leq \frac{3}{2}(\log 4b + \log 4d_b) \leq \frac{15}{2}\log\theta_{\beta,1},$$

$$\text{and } n_1 - n_2 < 8.75\times 10^{14}\log^2\theta_{\beta,1}\log n_1,$$

we get

$$\begin{aligned} h\left(\delta_3\right) &\leq 8.75\times 10^{14}\log^2\theta_{\beta,1}\log n_1\frac{\log\theta_{\alpha,1}}{2} + \frac{15}{2}\log\theta_{\beta,1} + \log 2 \\ &< 8.75\times 10^{14}\log^3\theta_{\beta,1}\log n_1. \end{aligned}$$

Therefore, we may take $A_3' = 3.5\times 10^{15}\log^3\theta_{\beta,1}\log n_1$.
Using equation (4.29), we can take $B = 2n_1$. Then

$$\log|\Gamma_8 - 1| > -7.3567\times 10^{28}(1+\log 2n_1)\log\theta_{\alpha,1}\log^4\theta_{\beta,1}\log n_1.$$

Comparing with (4.34), we obtain

$$m_1 - m_2 < 1.54\times 10^{29}\log^4\theta_{\beta,1}\log^2 n_1.$$

In a similar manner, we can show that if the $\min(n_1-n_2, m_1-m_2) = m_1-m_2$, then $n_1 - n_2 < 1.54\times 10^{29}\log^4\theta_{\beta,1}\log^2 n_1$.
Hence,

$$\max(n_1-n_2, m_1-m_2) < 1.54\times 10^{29}\log^4\theta_{\beta,1}\log^2 n_1.$$

By again considering equation (3.13), we get

$$\begin{aligned} c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} - c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1} + c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2} = c_{\alpha,2}^{(j_1)}\theta_{\alpha,2}^{n_1} - c_{\alpha,2}^{(j_2)}\theta_{\alpha,2}^{n_2} - c_{\beta,2}^{(p_1)}\theta_{\beta,2}^{m_1} \\ + c_{\beta,2}^{(p_2)}\theta_{\beta,2}^{m_2}. \end{aligned}$$

Using $|\theta_{\alpha,2}| < 1$, $|\theta_{\beta,2}| < 1$, and (4.30), we get

$$\begin{aligned} \left|c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} - c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1} + c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}\right| &\leq \left|c_{\alpha,2}^{(j_1)}\right| + \left|c_{\alpha,2}^{(j_2)}\right| + \left|c_{\beta,2}^{(p_1)}\right| + \left|c_{\beta,2}^{(p_2)}\right| \\ &\leq 0.77. \end{aligned} \tag{4.35}$$

Since $\frac{1}{\theta_{\beta,1}^{k}} \leq \frac{c_{\beta,1}^{(p_1)}}{2c_{\beta,1}^{(p_2)}}$ holds for all $k \geq 1$, in particular for $k = m_1 - m_2$, we deduce $c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1} \geq c_{\beta,1}^{(p_1)}/2 > 0$. Dividing by $\left|c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}\right| \neq 0$, we get

$$\left|\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}}\theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1} - 1\right| \leq \frac{0.77}{\left|c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}\right|}\cdot\frac{1}{\theta_{\beta,1}^{m_1}}.$$

As $y = \frac{x+\sqrt{x^2+4x}}{2\sqrt{x^2+4x}}$ is a decreasing function of x for $x \geq 3$, we get

$$\left| \frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}} \theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1} - 1 \right| \leq \frac{0.77}{\frac{c_{\beta,1}^{(p_1)}}{2}} \cdot \frac{1}{\theta_{\beta,1}^{m_1}} \leq \frac{1.54}{\frac{b+\sqrt{d_b}}{2\sqrt{d_b}}} \cdot \frac{1}{\theta_{\beta,1}^{m_1}}$$

$$\leq \frac{1.87}{\theta_{\beta,1}^{m_1}}. \tag{4.36}$$

Let

$$\Gamma_9 = \frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}} \theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1}.$$

Suppose $\Gamma_9 = 1$. By Lemma 2.4, we have $c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} \notin \mathbb{Q}$. Since $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\beta)$, the equality $c_{\alpha,1}^{(j_1)}\theta_{\alpha,1}^{n_1} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2} = c_{\beta,1}^{(p_1)}\theta_{\beta,1}^{m_1} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2}$ is impossible. As $\Gamma_9 \neq 1$, we apply Lemma 2.6 with $T = 3$, $D = 4$,

$$\delta_1 = \theta_{\alpha,1},\ \delta_2 = \theta_{\beta,1},\ \delta_3 = \frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}},$$
$$k_1 = n_1,\ k_2 = -m_1,\ k_3 = 1,$$
$$A_1' = 2\log\theta_{\alpha,1},\ A_2' = 2\log\theta_{\beta,1}.$$

Applying (2.8), we get

$$h(\delta_3) \leq h(c_{\alpha,1}^{(j_1)}) + h(c_{\alpha,1}^{(j_2)}) + h(c_{\beta,1}^{(p_1)}) + h(c_{\beta,1}^{(p_2)}) + 2\log 2 + (m_1 - m_2)\frac{\log\theta_{\beta,1}}{2} + (n_1 - n_2)\frac{\log\theta_{\alpha,1}}{2}.$$

As noted above,

$$\max(n_1 - n_2, m_1 - m_2) \leq 1.54 \times 10^{29} \log^4\theta_{\beta,1}\log^2 n_1,$$
and
$$h(c_{\alpha,1}^{(j_1)}) + h(c_{\alpha,1}^{(j_2)}) + h(c_{\beta,1}^{(p_1)}) + h(c_{\beta,1}^{(p_2)}) \leq 10\log\theta_{\beta,1}.$$

This gives

$$h(\delta_3) \leq 1.55 \times 10^{29}\log^5\theta_{\beta,1}\log^2 n_1$$

and therefore, we may take $A_3' = 6.2 \times 10^{29}\log^5\theta_{\beta,1}\log^2 n_1$.
Using equation (4.29), we can take $B = 2n_1$. Then

$$\log|\Gamma_9 - 1| > -1.3558 \times 10^{43}(1 + \log 2n_1)\log\theta_{\alpha,1}\log^6\theta_{\beta,1}\log^2 n_1.$$

Comparing with (4.36), we obtain

$$m_1 < 2.72 \times 10^{43}\log\theta_{\alpha,1}\log^5\theta_{\beta,1}\log^3 n_1.$$

Now, using equation (4.28), we get

$$n_1 < 2.73 \times 10^{43}\log^6\theta_{\beta,1}\log^3 n_1.$$

On combining all four cases we have $n_1 < (3.02 \times 10^{14} \log^2 \theta_{\beta,1} \log n_1)^3$. We now use Lemma 2.7 with $a = 0, c = 3, g = 2.73 \times 10^{43} \log^6 \theta_{\beta,1}$. We thus obtain

$$\begin{aligned} n_1 &\leq 216 \times 2.73 \times 10^{43} \log^6 \theta_{\beta,1} \log^3(3 \times 3.02 \times 10^{14} \log^2 \theta_{\beta,1}) \\ &\leq 5.897 \times 10^{45} \log^6 \theta_{\beta,1} \log^3(9.06 \times 10^{14} \log^2 \theta_{\beta,1}). \end{aligned}$$

Hence,

$$\max(N_1, M_1) \leq 1.1793 \times 10^{46} \log^6 \theta_{\beta,1} \times \log^3(9.06 \times 10^{14} \log^2 \theta_{\beta,1}).$$

5. Proof of Theorem 1.3

Using Theorem 1.2 for $\alpha = [0; \overline{1, 2}]$ and $\beta = [0; \overline{1, 3}]$, we get

$$\max(N_1, M_1) \leq 7.7 \times 10^{51}.$$

Now, using equation (4.29), we have

$$m_1 < n_1 \leq 3.9 \times 10^{51}.$$

We applied a brute force computer search for solutions in the range $1 \leq N_2 < N_1 \leq 400$ and $1 \leq M_2 < M_1 \leq 340$. These solutions are listed in Theorem 1.3. Therefore, from now onwards $401 \leq N_1 \leq 7.71 \times 10^{51}$. In particular $n_1 \geq 200$.

Now we consider the inequality (4.33) and recall that $\Gamma_7 = \dfrac{c_{\alpha,1}^{(j_1)} \theta_{\alpha,1}^{n_1}}{c_{\beta,1}^{(p_1)} \theta_{\beta,1}^{m_1}}$. Taking $\Lambda := \log \Gamma_7$, we get

$$\Lambda = n_1 \log \theta_{\alpha,1} - m_1 \log \theta_{\beta,1} + \log\left(\frac{c_{\alpha,1}^{(j_1)}}{c_{\beta,1}^{(p_1)}}\right).$$

Assuming that $\left|e^\Lambda - 1\right| < \frac{1}{4}$, we get $|\Lambda| < \frac{1}{2}$. Since the inequality $|x| < 2|e^x - 1|$ is true for all nonzero $x \in (-\frac{1}{2}, \frac{1}{2})$, we get

$$|\Lambda| < 20.24\theta_{\alpha,1} \times \max\left(\frac{1}{\theta_{\alpha,1}^{n_1-n_2}}, \frac{1}{\theta_{\beta,1}^{m_1-m_2}}\right).$$

If $\Lambda > 0$, then we have the following inequality

$$0 < n_1 \frac{\log \theta_{\alpha,1}}{\log \theta_{\beta,1}} - m_1 + \frac{\log\left(c_{\alpha,1}^{(j_1)}/c_{\beta,1}^{(p_1)}\right)}{\log \theta_{\beta,1}} < \frac{20.24\theta_{\alpha,1}}{\log \theta_{\beta,1}} \times \max\left(\theta_{\alpha,1}^{n_2-n_1}, \theta_{\beta,1}^{m_2-m_1}\right).$$

If $\Lambda < 0$, then we have the following inequality

$$0 < m_1 \frac{\log \theta_{\beta,1}}{\log \theta_{\alpha,1}} - n_1 + \frac{\log\left(c_{\beta,1}^{(p_1)}/c_{\alpha,1}^{(j_1)}\right)}{\log \theta_{\alpha,1}} < \frac{20.24\theta_{\alpha,1}}{\log \theta_{\alpha,1}} \times \max\left(\theta_{\alpha,1}^{n_2-n_1}, \theta_{\beta,1}^{m_2-m_1}\right).$$

In particular, for $\Lambda > 0$, $c_{\alpha,1}^{(j_1)} = \frac{1+\sqrt{3}}{2\sqrt{3}}$ and $c_{\beta,1}^{(p_1)} = \frac{3+\sqrt{21}}{2\sqrt{21}}$, we apply Lemma 2.8 with $\gamma = \dfrac{\log\theta_{\alpha,1}}{\log\theta_{\beta,1}} = \frac{\log(2+\sqrt{3})}{\log(5/2+\sqrt{21}/2)}$, $\kappa = \dfrac{1}{\log\theta_{\beta,1}}\log\left(\frac{c_{\alpha,1}^{(j_1)}}{c_{\beta,1}^{(p_1)}}\right)$, $A_1 = \dfrac{20.24\theta_{\alpha,1}}{\log\theta_{\beta,1}}$, $(A_2, k) = (\theta_{\alpha,1}, n_1-n_2)$ or $(\theta_{\beta,1}, m_1-m_2)$, and $M = 3.9\times10^{51}$. The continued fraction expansion of γ is $[0; 1, 5, 3, 1, 2, 4, 1, 29, 3, 1, \ldots]$. We consider the $105th$ convergent of the irrational γ such that $q = q_{105} > 6M$. This gives $\varepsilon > 0.35261$ and therefore, either $n_1 - n_2 \le \log(\frac{A_1 q}{0.35261})/\log\theta_{\alpha,1} \le 95$, or $m_1 - m_2 \le \log(\frac{A_1 q}{0.35261})/\log\theta_{\beta,1} \le 80$. For other choices of Λ, $c_{\alpha,1}^{(j_1)}$, and $c_{\beta,1}^{(p_1)}$, we summarize the results in Table 1. We note that all of these calculations were performed using SageMath.
If $|e^\Lambda - 1| \ge \frac{1}{4}$, then

$$\frac{1}{4} \le |e^\Lambda - 1| < 10.12\theta_{\alpha,1} \times \max\left(\frac{1}{\theta_{\alpha,1}^{n_1-n_2}}, \frac{1}{\theta_{\beta,1}^{m_1-m_2}}\right).$$

Therefore, $\min(m_1 - m_2, n_1 - n_2) \le 3$. Combining this with the results obtained in Table 1, we infer that either $n_1 - n_2 \le 96$ or $m_1 - m_2 \le 81$. Now, when $n_1 - n_2 \le 96$, we consider the inequality (4.34) and recall that $\Gamma_8 = \dfrac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)}}\theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1}$. Taking $\Lambda_1 := \log\Gamma_8$, we get

$$\Lambda_1 = n_1\log\theta_{\alpha,1} - m_1\log\theta_{\beta,1} + \log\left(\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)}}\right).$$

Assuming that $\left|e^{\Lambda_1} - 1\right| < \frac{1}{4}$, we get $|\Lambda_1| < \frac{1}{2}$. Since the inequality $|x| < 2|e^x - 1|$ is true for all nonzero $x \in (-\frac{1}{2}, \frac{1}{2})$, we get

$$|\Lambda_1| < \frac{4.42}{\theta_{\beta,1}^{m_1-m_2}}.$$

If $\Lambda_1 > 0$, then we have the following inequality

$$0 < n_1\frac{\log\theta_{\alpha,1}}{\log\theta_{\beta,1}} - m_1 + \frac{1}{\log\theta_{\beta,1}}\log\left(\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)}}\right) < \frac{4.42}{\log\theta_{\beta,1}} \times \frac{1}{\theta_{\beta,1}^{m_1-m_2}}.$$

If $\Lambda_1 < 0$, then we have the following inequality

$$0 < m_1\frac{\log\theta_{\beta,1}}{\log\theta_{\alpha,1}} - n_1 + \frac{1}{\log\theta_{\alpha,1}}\log\left(\frac{c_{\beta,1}^{(p_1)}}{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}\right) < \frac{4.42}{\log\theta_{\alpha,1}} \times \frac{1}{\theta_{\beta,1}^{m_1-m_2}}.$$

In particular, for $\Lambda_1 > 0$, $c_{\alpha,1}^{(j_1)} = \frac{1+\sqrt{3}}{2\sqrt{3}}$, $c_{\alpha,1}^{(j_2)} = \frac{1+\sqrt{3}}{2\sqrt{3}}$, and $c_{\beta,1}^{(p_1)} = \frac{3+\sqrt{21}}{2\sqrt{21}}$, we apply Lemma 2.8 with $\gamma = \dfrac{\log\theta_{\alpha,1}}{\log\theta_{\beta,1}}$, $\kappa_t = \dfrac{1}{\log\theta_{\beta,1}}\log\left(\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{-t}}{c_{\beta,1}^{(p_1)}}\right)$, $A_1 = \dfrac{4.42}{\log\theta_{\beta,1}}$, $A_2 = \theta_{\beta,1}$, $k = m_1 - m_2$, and $M = 3.9 \times 10^{51}$. We consider

the $109th$ convergent of the irrational γ such that $q = q_{109} > 6M$. For each possible value of $n_1 - n_2 = t = 1, \ldots, 96$, we get $\varepsilon > 0.00288$ and therefore, $m_1 - m_2 \leq \log(\frac{A_1 q}{0.00288})/\log\theta_{\beta,1} \leq 83$. For other choices of Λ_1, $c_{\alpha,1}^{(j_1)}, c_{\alpha,1}^{(j_2)}$, and $c_{\beta,1}^{(p_1)}$, we summarize the results in Table 2. As noted above, these calculations were performed using SageMath.
If $\left|e^{\Lambda_1} - 1\right| \geq \frac{1}{4}$, then

$$\frac{1}{4} \leq |e^{\Lambda_1} - 1| < \frac{2.21}{\theta_{\beta,1}^{m_1-m_2}}.$$

Therefore, $m_1 - m_2 \leq 1$. Finally, we conclude that if $n_1 - n_2 \leq 96$, then $m_1 - m_2 \leq 85$. Next, if $m_1 - m_2 \leq 81$, then we proceed in a similar manner and summarize the results in Table 3.
Finally, if $m_1 - m_2 \leq 81$, then $n_1 - n_2 \leq 101$. Combining the results obtained so far and taking the maximum of the bounds, we get $n_1 - n_2 \leq 101$ and $m_1 - m_2 \leq 85$. Now, we consider the inequality (4.36) and recall that $\Gamma_9 = \dfrac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}}\theta_{\alpha,1}^{n_1}\theta_{\beta,1}^{-m_1}$. Taking $\Lambda_3 := \log\Gamma_9$, we get

$$\Lambda_3 = n_1\log\theta_{\alpha,1} - m_1\log\theta_{\beta,1} + \log\left(\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}}\right).$$

Assuming that $\left|e^{\Lambda_3} - 1\right| < \frac{1}{4}$, we get $|\Lambda_3| < \frac{1}{2}$. Since the inequality $|x| < 2|e^x - 1|$ is true for all nonzero $x \in (-\frac{1}{2}, \frac{1}{2})$, we get

$$|\Lambda_3| < \frac{3.74}{\theta_{\beta,1}^{m_1}}.$$

If $\Lambda_3 > 0$, then we have the following inequality

$$0 < n_1\frac{\log\theta_{\alpha,1}}{\log\theta_{\beta,1}} - m_1 + \frac{1}{\log\theta_{\beta,1}}\log\left(\frac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}}\right) < \frac{3.74}{\log\theta_{\beta,1}} \times \frac{1}{\theta_{\beta,1}^{m_1}}.$$

If $\Lambda_3 < 0$, then we have the following inequality

$$0 < m_1\frac{\log\theta_{\beta,1}}{\log\theta_{\alpha,1}} - n_1 + \frac{1}{\log\theta_{\alpha,1}}\log\left(\frac{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{m_2-m_1}}{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{n_2-n_1}}\right) < \frac{3.74}{\log\theta_{\alpha,1}} \times \frac{1}{\theta_{\beta,1}^{m_1}}.$$

In particular, for $\Lambda_3 > 0$, $c_{\alpha,1}^{(j_1)} = \frac{1+\sqrt{3}}{2\sqrt{3}}$, $c_{\alpha,1}^{(j_2)} = \frac{1+\sqrt{3}}{2\sqrt{3}}$, $c_{\beta,1}^{(p_1)} = \frac{3+\sqrt{21}}{2\sqrt{21}}$, and $c_{\beta,1}^{(p_2)} = \frac{3+\sqrt{21}}{2\sqrt{21}}$, we apply Lemma 2.8 with $\gamma = \dfrac{\log\theta_{\alpha,1}}{\log\theta_{\beta,1}}$,
$\kappa_{t,t_1} = \dfrac{1}{\log\theta_{\beta,1}}\log\left(\dfrac{c_{\alpha,1}^{(j_1)} - c_{\alpha,1}^{(j_2)}\theta_{\alpha,1}^{-t}}{c_{\beta,1}^{(p_1)} - c_{\beta,1}^{(p_2)}\theta_{\beta,1}^{-t_1}}\right)$, $A_1 = \dfrac{3.74}{\log\theta_{\beta,1}}$, $A_2 = \theta_{\beta,1}$, $k = m_1$, and $M = 3.9 \times 10^{51}$. We consider the $112th$ convergent of the irrational γ

such that $q = q_{112} > 6M$. For each possible value of $n_1 - n_2 = t = 1, \ldots, 101$, and $m_1 - m_2 = t_1 = 1, \ldots, 85$, we get $\varepsilon > 0.00009$ and therefore, $m_1 \leq \log(\frac{A_1 q}{0.00009})/\log\theta_{\beta,1} \leq 87$. For other choices of Λ_3, $c_{\alpha,1}^{(j_1)}$, $c_{\alpha,1}^{(j_2)}$, $c_{\beta,1}^{(p_1)}$, and $c_{\beta,1}^{(p_2)}$, we summarize the results in Table 4. As noted previously, these calculations were also performed using SageMath.

For the values of $c_{\alpha,1}^{(j_1)}, c_{\alpha,1}^{(j_2)}, c_{\beta,1}^{(p_1)}$, and $c_{\beta,1}^{(p_2)}$, given in the last row of Table 4, the case where $n_1 - n_2 = 2$ and $m_1 - m_2 = 2$ is treated separately. This is because the value of ε turns out to be negative for all convergent denominators greater than $6M$.

When $n_1 - n_2 = 2$ and $m_1 - m_2 = 2$, we consider the inequality (4.35) and obtain the following

$$\left|\theta_{\beta,1}^{m_1} - \theta_{\alpha,1}^{n_1}\right| \leq 0.77.$$

Dividing by $\theta_{\alpha,1}^{n_1}$, we get $\left|\theta_{\beta,1}^{m_1}\theta_{\alpha,1}^{-n_1} - 1\right| \leq \frac{0.77}{\theta_{\alpha,1}^{n_1}}$.

If $\left|\theta_{\beta,1}^{m_1}\theta_{\alpha,1}^{-n_1} - 1\right| < \frac{1}{4}$, then $\left|\dfrac{\log\theta_{\alpha,1}}{\log\theta_{\beta,1}} - \dfrac{m_1}{n_1}\right| < \dfrac{1.54}{n_1\theta_{\alpha,1}^{n_1}\log\theta_{\beta,1}}$. For all $n_1 \in \mathbb{N}$, we have $\dfrac{1.54}{n_1\theta_{\alpha,1}^{n_1}\log\theta_{\beta,1}} < \dfrac{1}{2n_1^2}$, therefore,

$$\left|\frac{\log\theta_{\alpha,1}}{\log\theta_{\beta,1}} - \frac{m_1}{n_1}\right| < \frac{1}{2n_1^2},$$

and this implies that $\dfrac{m_1}{n_1}$ has to be a convergent of the irrational $\dfrac{\log\theta_{\alpha,1}}{\log\theta_{\beta,1}}$.

Recall that $n_1 \leq 3.9 \times 10^{51}$. Using this, the convergent with denominator less than or equal to 3.9×10^{51} closest to $\dfrac{\log\theta_{\alpha,1}}{\log\theta_{\beta,1}}$ could either be $\dfrac{p_{101}}{q_{101}}$ or $\dfrac{p_{102}}{q_{102}}$.

Since $\min\left(\left|\dfrac{\log\theta_{\alpha,1}}{\log\theta_{\beta,1}} - \dfrac{p_{101}}{q_{101}}\right|, \left|\dfrac{\log\theta_{\alpha,1}}{\log\theta_{\beta,1}} - \dfrac{p_{102}}{q_{102}}\right|\right) = \left|\dfrac{\log\theta_{\alpha,1}}{\log\theta_{\beta,1}} - \dfrac{p_{102}}{q_{102}}\right|$, we have $9 \times 10^{-104} \leq \left|\dfrac{\log\theta_{\alpha,1}}{\log\theta_{\beta,1}} - \dfrac{p_{102}}{q_{102}}\right| \leq \dfrac{1.54}{n_1\theta_{\alpha,1}^{n_1}\log\theta_{\beta,1}}$. This implies $n_1 \leq 176$.

If $\dfrac{1}{4} \leq \left|\theta_{\beta,1}^{m_1}\theta_{\alpha,1}^{-n_1} - 1\right|$, then $\dfrac{1}{4} \leq \dfrac{0.77}{\theta_{\alpha,1}^{n_1}}$ which gives $n_1 \leq 0.86$.

Therefore, when $n_1 - n_2 = 2 = m_1 - m_2$, we get $n_1 \leq 176$.

If $|e^{\Lambda_3} - 1| \geq \frac{1}{4}$, then

$$\frac{1}{4} \leq |e^{\Lambda_3} - 1| \leq \frac{1.87}{\theta_{\beta,1}^{m_1}}.$$

Therefore, $m_1 \leq 1$. Finally, combining all the bounds on m_1, we have $m_1 \leq 91$. Now, using inequality (4.28), we have $n_1 \leq 109$. Combining the bounds on n_1, we have $n_1 \leq 176$. Likewise, we proceed in the other three cases (Cases (i), (ii), and (iii)) and get a bound which is lower than $n_1 \leq 176$. Finally, we get $N_1 \leq 353$, which contradicts our assumption that $N_1 \geq 401$. Hence, Theorem 1.3.

6. Computational Results

Table 1. Computational results for Theorem 1.3.

$c_{\alpha,1}^{(j_1)}$	$c_{\beta,1}^{(p_1)}$	Λ	q	ε	$n_1-n_2\le$	$m_1-m_2\le$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	>0	q_{105}	0.35261	$n_1-n_2\le 95$	$m_1-m_2\le 80$
		<0	q_{104}	0.34567	$n_1-n_2\le 95$	$m_1-m_2\le 80$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	>0	q_{105}	0.08583	$n_1-n_2\le 96$	$m_1-m_2\le 81$
		<0	q_{104}	0.07888	$n_1-n_2\le 96$	$m_1-m_2\le 81$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	>0	q_{105}	0.41249	$n_1-n_2\le 95$	$m_1-m_2\le 80$
		<0	q_{104}	0.40555	$n_1-n_2\le 95$	$m_1-m_2\le 80$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	>0	q_{105}	0.24746	$n_1-n_2\le 95$	$m_1-m_2\le 80$
		<0	q_{104}	0.24051	$n_1-n_2\le 95$	$m_1-m_2\le 80$

Table 2. Computational results for Theorem 1.3.

$c_{\alpha,1}^{(j_1)}$	$c_{\alpha,1}^{(j_2)}$	$c_{\beta,1}^{(p_1)}$	Λ_1	q	ε	$m_1-m_2\le$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	>0	q_{109}	0.00288	$m_1-m_2\le 83$
			<0	q_{109}	0.00730	$m_1-m_2\le 83$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	>0	q_{109}	0.01266	$m_1-m_2\le 82$
			<0	q_{109}	0.00293	$m_1-m_2\le 84$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	>0	q_{109}	0.00719	$m_1-m_2\le 82$
			<0	q_{109}	0.00153	$m_1-m_2\le 84$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	>0	q_{109}	0.00362	$m_1-m_2\le 83$
			<0	q_{109}	0.00758	$m_1-m_2\le 83$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	>0	q_{110}	0.00043	$m_1-m_2\le 85$
			<0	q_{110}	0.00210	$m_1-m_2\le 84$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	>0	q_{109}	0.00725	$m_1-m_2\le 82$
			<0	q_{111}	0.00994	$m_1-m_2\le 84$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	>0	q_{109}	0.00153	$m_1-m_2\le 83$
			<0	q_{109}	0.00518	$m_1-m_2\le 84$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	>0	q_{109}	0.00304	$m_1-m_2\le 83$
			<0	q_{109}	0.00182	$m_1-m_2\le 84$

Table 3. Computational results for Theorem 1.3.

$c_{\alpha,1}^{(j_1)}$	$c_{\beta,1}^{(p_1)}$	$c_{\beta,1}^{(p_2)}$	Λ_2	q	ε	$n_1 - n_2 \leq$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	> 0	q_{109}	0.00573	$n_1 - n_2 \leq 99$
			< 0	q_{109}	0.00515	$n_1 - n_2 \leq 99$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	> 0	q_{110}	0.02362	$n_1 - n_2 \leq 99$
			< 0	q_{111}	0.02374	$n_1 - n_2 \leq 99$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	> 0	q_{109}	0.00093	$n_1 - n_2 \leq 101$
			< 0	q_{109}	0.00060	$n_1 - n_2 \leq 100$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	> 0	q_{109}	0.00069	$n_1 - n_2 \leq 101$
			< 0	q_{109}	0.00303	$n_1 - n_2 \leq 99$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	> 0	q_{109}	0.00118	$n_1 - n_2 \leq 101$
			< 0	q_{109}	0.00049	$n_1 - n_2 \leq 100$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	> 0	q_{111}	0.02121	$n_1 - n_2 \leq 99$
			< 0	q_{109}	0.00360	$n_1 - n_2 \leq 99$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	> 0	q_{109}	0.01133	$n_1 - n_2 \leq 99$
			< 0	q_{109}	0.02401	$n_1 - n_2 \leq 97$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	> 0	q_{109}	0.00226	$n_1 - n_2 \leq 100$
			< 0	q_{109}	0.00391	$n_1 - n_2 \leq 99$

Table 4. Computational results for Theorem 1.3.

$c_{\alpha,1}^{(j_1)}$	$c_{\alpha,1}^{(j_2)}$	$c_{\beta,1}^{(p_1)}$	$c_{\beta,1}^{(p_2)}$	Λ_3	q	ε	$m_1 \leq$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	> 0	q_{112}	0.00009	$m_1 \leq 87$
				< 0	q_{113}	0.00006	$m_1 \leq 90$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	> 0	q_{113}	0.00009	$m_1 \leq 89$
				< 0	q_{113}	0.00001	$m_1 \leq 90$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	> 0	q_{112}	0.00006	$m_1 \leq 87$
				< 0	q_{112}	6×10^{-6}	$m_1 \leq 90$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	> 0	q_{114}	0.00003	$m_1 \leq 90$
				< 0	q_{113}	0.00003	$m_1 \leq 90$

$c_{\alpha,1}^{(j_1)}$	$c_{\alpha,1}^{(j_2)}$	$c_{\beta,1}^{(p_1)}$	$c_{\beta,1}^{(p_2)}$	Λ_3	q	ε	$m_1 \leq$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	>0	q_{113}	5×10^{-6}	$m_1 \leq 90$
				<0	q_{113}	2×10^{-6}	$m_1 \leq 91$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	>0	q_{113}	0.00005	$m_1 \leq 89$
				<0	q_{113}	0.00010	$m_1 \leq 89$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	>0	q_{113}	0.00005	$m_1 \leq 89$
				<0	q_{113}	0.00003	$m_1 \leq 90$
$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	>0	q_{113}	8×10^{-6}	$m_1 \leq 90$
				<0	q_{113}	0.00016	$m_1 \leq 89$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	>0	q_{112}	0.00012	$m_1 \leq 86$
				<0	q_{112}	0.00004	$m_1 \leq 89$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	>0	q_{113}	0.00014	$m_1 \leq 88$
				<0	q_{113}	0.00002	$m_1 \leq 90$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	>0	q_{113}	0.00006	$m_1 \leq 89$
				<0	q_{113}	0.00001	$m_1 \leq 91$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	>0	q_{113}	0.00001	$m_1 \leq 90$
				<0	q_{113}	0.00001	$m_1 \leq 90$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	>0	q_{113}	0.00023	$m_1 \leq 88$
				<0	q_{113}	0.00018	$m_1 \leq 89$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{3+\sqrt{21}}{2\sqrt{21}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	>0	q_{113}	2×10^{-6}	$m_1 \leq 91$
				<0	q_{113}	0.00003	$m_1 \leq 90$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{1+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	>0	q_{113}	0.00001	$m_1 \leq 90$
				<0	q_{113}	9×10^{-6}	$m_1 \leq 91$
$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{2+\sqrt{3}}{2\sqrt{3}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	$\frac{5+\sqrt{21}}{2\sqrt{21}}$	>0	q_{114}	0.00002	$m_1 \leq 90$
				<0	q_{114}	0.00004	$m_1 \leq 90$

7. Appendix

(i) For $\alpha = [0; \overline{1,2}]$ and $\beta = [0; \overline{1,4}]$, we have

$$\mathcal{C} = \{-2, 5, 6, 10, 40\}.$$

The following list shows two distinct representations of each $c \in \mathcal{C}$:

$$q_{\alpha,3} - q_{\beta,3} = 4 - 6 = \boxed{-2} = 3 - 5 = q_{\alpha,2} - q_{\beta,2},$$
$$q_{\alpha,9} - q_{\beta,7} = 209 - 204 = \boxed{5} = 11 - 6 = q_{\alpha,4} - q_{\beta,3},$$
$$q_{\alpha,6} - q_{\beta,5} = 41 - 35 = \boxed{6} = 11 - 5 = q_{\alpha,4} - q_{\beta,2},$$
$$q_{\alpha,5} - q_{\beta,2} = 15 - 5 = \boxed{10} = 11 - 1 = q_{\alpha,4} - q_{\beta,1},$$
$$q_{\alpha,9} - q_{\beta,6} = 209 - 169 = \boxed{40} = 41 - 1 = q_{\alpha,6} - q_{\beta,1}.$$

(ii) For $\alpha = [0; \overline{1,2}]$ and $\beta = [0; \overline{1,5}]$, we have

$$\mathcal{C} = \{-37, -3, 0, 8\}.$$

The following list shows two distinct representations of each $c \in \mathcal{C}$:

$$q_{\alpha,4} - q_{\beta,5} = 11 - 48 = \boxed{-37} = 4 - 41 = q_{\alpha,3} - q_{\beta,4},$$
$$q_{\alpha,3} - q_{\beta,3} = 4 - 7 = \boxed{-3} = 3 - 6 = q_{\alpha,2} - q_{\beta,2},$$
$$q_{\alpha,6} - q_{\beta,4} = 41 - 41 = \boxed{0} = 1 - 1 = q_{\alpha,1} - q_{\beta,1},$$
$$q_{\alpha,7} - q_{\beta,5} = 56 - 48 = \boxed{8} = 15 - 7 = q_{\alpha,5} - q_{\beta,3}.$$

(iii) For $\alpha = [0; \overline{1,3}]$ and $\beta = [0; \overline{1,4}]$, we have

$$\mathcal{C} = \{-5, -1, 0, 18, 86\}.$$

The following list shows two distinct representations of each $c \in \mathcal{C}$:

$$q_{\alpha,5} - q_{\beta,4} = 24 - 29 = \boxed{-5} = 1 - 6 = q_{\alpha,1} - q_{\beta,3},$$
$$q_{\alpha,3} - q_{\beta,3} = 5 - 6 = \boxed{-1} = 4 - 5 = q_{\alpha,2} - q_{\beta,2},$$
$$q_{\alpha,3} - q_{\beta,2} = 5 - 5 = \boxed{0} = 1 - 1 = q_{\alpha,1} - q_{\beta,1},$$
$$q_{\alpha,5} - q_{\beta,3} = 24 - 6 = \boxed{18} = 19 - 1 = q_{\alpha,4} - q_{\beta,1},$$
$$q_{\alpha,7} - q_{\beta,4} = 115 - 29 = \boxed{86} = 91 - 5 = q_{\alpha,6} - q_{\beta,2}.$$

(iv) For $\alpha = [0; \overline{1,3}]$ and $\beta = [0; \overline{1,5}]$, we have

$$\mathcal{C} = \{-166, -2, 18\}.$$

The following list shows two distinct representations of each $c \in \mathcal{C}$:

$$q_{\alpha,10} - q_{\beta,9} = 2089 - 2255 = \boxed{-166} = 115 - 281 = q_{\alpha,7} - q_{\beta,6},$$
$$q_{\alpha,3} - q_{\beta,3} = 5 - 7 = \boxed{-2} = 4 - 6 = q_{\alpha,2} - q_{\beta,2},$$
$$q_{\alpha,5} - q_{\beta,2} = 24 - 6 = \boxed{18} = 19 - 1 = q_{\alpha,4} - q_{\beta,1}.$$

(v) For $\alpha = [0; \overline{1,4}]$ and $\beta = [0; \overline{1,5}]$, we have

$$\mathcal{C} = \{-6, -1, 0, 28, 163\}.$$

The following list shows two distinct representations of each $c \in \mathcal{C}$:

$$q_{\alpha,5} - q_{\beta,4} = 35 - 41 = \boxed{-6} = 1 - 7 = q_{\alpha,1} - q_{\beta,3},$$
$$q_{\alpha,3} - q_{\beta,3} = 6 - 7 = \boxed{-1} = 5 - 6 = q_{\alpha,2} - q_{\beta,2},$$
$$q_{\alpha,3} - q_{\beta,2} = 6 - 6 = \boxed{0} = 1 - 1 = q_{\alpha,1} - q_{\beta,1},$$
$$q_{\alpha,5} - q_{\beta,3} = 35 - 7 = \boxed{28} = 29 - 1 = q_{\alpha,4} - q_{\beta,1},$$
$$q_{\alpha,7} - q_{\beta,4} = 204 - 41 = \boxed{163} = 169 - 6 = q_{\alpha,6} - q_{\beta,2}.$$

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M. Mittal
Department of Mathematics
Birla Institute of Technology and Science, Pilani
333 031
India
E-mail: `mohit.mittal@pilani.bits-pilani.ac.in`