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REAL EQUATIONS FOR o-EXTREMAL RIEMANN SURFACES WITH ABELIAN AUTOMORPHISM GROUPS

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Abstract. It is well known that the fixed point set of a Riemann surface of genus g under the action of a symmetry is either empty or consists of a disjoint set of at most $g+1$ ovals. Bounds on the total number of fixed ovals given by a set of k non-conjugate symmetries are known. In this paper, for $k \geq 4$, we calculate all the possible topological types of symmetries in such a maximal configuration, provided that the symmetries commute. We also find real equations for the Riemann surfaces that achieve these bounds where the symmetries are expressed as complex conjugation.

1. Introduction

The study of Riemann surfaces that admit nontrivial groups of automorphisms has a long history. In general, an emphasis has been placed on determining maximal situations and we highlight two of these areas now. One line of investigation has been to find groups of automorphisms whose orders are maximal given the genus of the underlying surface. This follows from the work of Hurwitz who discovered the famous bound that a compact Riemann surface of genus $g \geq 2$ cannot admit a group of automorphisms of order greater than $84(g-1)$. One can also restrict consideration to particular groups of automorphisms, for example, cyclic or abelian groups, and determine the maximal order of such a group acting on a Riemann surface of genus g [14, 18]. A second line of research has been to determine the maximum number of fixed points admitted by particular automorphism groups of Riemann surfaces. It is well known that an automorphism of a Riemann surface of genus g can fix at most $2g+2$ points. In addition, a

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bound for the number of fixed points admitted by k commuting involutions of a Riemann surface was obtained in [6].

It is natural to extend this second line of research to symmetries that act on a Riemann surface. In this case, the fixed point set of a Riemann surface of genus g under the action of a symmetry is either empty or consists of a disjoint set of at most $g+1$ ovals [13]. A bound on the total number of ovals fixed by $k = 3$ or 4 non-conjugate symmetries acting on a Riemann surface was found in [19], for $k \geq 9$ a bound was found in [10], and for $5 \leq k \leq 8$ in [11]. A surface that admits this maximal number of fixed ovals is called an *o-extremal Riemann surface*, and we sometimes call this an *o-extremal configuration* of k symmetries. The structure of the 2-group generated by such a configuration of symmetries was studied and found to be isomorphic to a direct product of a dihedral group and some number of copies of cyclic groups of order 2, where non-abelian groups can only occur for $k = 4$ or 5 (see [4, 11, 12]). Given the extent of knowledge of *o*-extremal Riemann surfaces a natural next step is to determine the exact distribution of the topological types of their symmetries. These were already found for $k = 3$ or 4 in [15] and, in the non-abelian case, for $k = 5$ in [16]. In this paper we generalize these results for arbitrary $k \geq 4$, provided that the symmetries in question commute. One can also ask about real equations for *o*-extremal surfaces, which were only found for $k = 3$ symmetries in [17]. In the latter sections of this paper, we find real equations for all *o*-extremal Riemann surfaces with an abelian automorphism group expressed so that the $k \geq 4$ symmetries correspond to complex conjugation. Along with the previous results mentioned, our work yields a comprehensive analysis of *o*-extremal surfaces which have commuting symmetries.

2. Preliminaries

A symmetry of a Riemann surface $X = \mathcal{H}/\Gamma$ of genus $g \geq 2$, where Γ is a Fuchsian surface group and $\mathcal{H}$ is the hyperbolic plane, is an antiholomorphic involution $\tau \in G = \mathrm{Aut}^{\pm}(X)$, the group of conformal and anticonformal automorphisms of X. The set of points fixed by τ consists of no more than $g+1$ disjoint simple closed curves called *ovals*, see Harnack [13]. If the set $X \setminus \mathrm{Fix}(\tau)$ is disconnected, then we say that τ is *separating* and we call it *non-separating* in the other case. Moreover, we define the *topological type* of τ to be the symbol $\pm t$, where $t \geq 0$ denotes the number of ovals of τ, and the sign depends on the separability of τ: + for separating, − for a non-separating symmetry.

The main tools used in studying Riemann surfaces and their groups of conformal automorphisms and symmetries are provided by the Riemann uniformization theorem and the theory of Fuchsian and non-euclidean crystallographic groups (*NEC groups* for short). The latter are just the discrete and cocompact subgroups of the group $\mathcal{G}$ of all the isometries of the hyperbolic plane $\mathcal{H}$.

The algebraic structure of such a group Λ is determined by the *signature*:

(2.1)
$$s(\Lambda) = (h; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k}), (-)^l\}),$$

where the brackets $(n_{i1}, \ldots, n_{is_i})$ are called *the period cycles*, the integers n_{ij} are the *link periods*, m_i are the *proper periods* and finally h is the *orbit genus* of Λ. We shall also denote $s = s_1 + \ldots + s_k$. The algebraic presentation for the group Λ with signature (2.1) is as follows, where generators used are called *canonical*:

$x_1, \ldots, x_r, e_i,\ c_{ij}, 1 \leq i \leq k + l, 0 \leq j \leq s_i$ and $a_1, b_1, \ldots, a_h, b_h$ if the sign is + or $d_1, \ldots, d_h$ otherwise. Moreover, we have relators:

$x_i^{m_i}, i = 1, \ldots, r,\ c_{ij}^2, (c_{ij-1}c_{ij})^{n_{ij}}, c_{i0}e_i^{-1}c_{is_i}e_i,\ i = 1, \ldots, k + l, j = 0, \ldots, s_i$ and

$x_1 \ldots x_r e_1 \ldots e_{k+l} a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_h b_h a_h^{-1} b_h^{-1}$ or $x_1 \ldots x_r e_1 \ldots e_{k+l} d_1^2 \ldots d_h^2$,

according to whether the sign is + or −. Every element of finite order in Λ is conjugate either to a canonical reflection or to a power of some canonical elliptic element x_i or else to a power of the product of two consecutive canonical reflections. An abstract group with such a presentation can be realized as an NEC group Λ if and only if the value

$$2\pi \left(\varepsilon h + k + l - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right) \right),$$

where $\varepsilon = 2$ or 1 according to the sign being + or −, is positive. The value above is just the hyperbolic area $\mu(\Lambda)$ of any fundamental region for the group Λ and the Hurwitz-Riemann formula holds:

$$[\Lambda : \Lambda'] = \mu(\Lambda')/\mu(\Lambda),$$

where Λ' is a subgroup of finite index in an NEC group Λ.

Note that every Riemann surface can be represented as the orbit space $\mathcal{H}/\Gamma$ for some torsion free Fuchsian group with the complex structure inherited from the hyperbolic plane. Also, a group G

of automorphisms of the surface so represented can be seen as the factor group Λ/Γ for an NEC or Fuchsian group Λ, according to whether G contains anticonformal automorphisms or not. In particular, we will mainly be concerned with NEC groups Λ whose signatures have the form

$$(0; +; [-]; \{(2, \overset{s}{\ldots}, 2)\}). \tag{2.2}$$

The associated NEC groups are generated by reflections $c_0, c_1, \ldots, c_{s-1}$ which satisfy the relations

$$c_0^2 = c_1^2 = \cdots = c_{s-1}^2 = 1, (c_0c_1)^2 = (c_1c_2)^2 = \cdots = (c_{s-2}c_{s-1})^2 = (c_{s-1}c_0)^2 = 1. \tag{2.3}$$

The reader can find complete details about these NEC groups in the paper [15].

Now let us recall a few facts concerning defining equations of a Riemann surface and its symmetries. The field of meromorphic functions on a compact Riemann surface, which is an algebraic function field in one variable over $\mathbb{C}$, gives us a functorial equivalence between the following three categories: fields with $\mathbb{C}$-automorphisms, smooth projective irreducible complex algebraic curves with birational automorphisms, and compact Riemann surfaces with conformal automorphisms. Moreover, we have a bijective correspondence between real forms of a complex algebraic curve and symmetries of a compact Riemann surface [1, 2, 3, 8]. Any compact Riemann surface of genus $g \geq 2$ can be defined by an equation $F(x, y) = 0$ for some polynomial $F \in \mathbb{C}[x, y]$. If complex conjugation σ is an automorphism of X, then X can be defined by a polynomial with real coefficients, hence a symmetric Riemann surface can be defined by an equation $G(x, y) = 0$ where $G \in \mathbb{R}[x, y]$. As we are dealing with real and complex curves, we should note that equations which give non-isomorphic real curves can yield the same complex curve. This corresponds to different symmetries acting on the same Riemann surface which produce non-isomorphic orbit spaces, in other words, non-isomorphic Klein surfaces.

It is a difficult task, in general, to find the corresponding equation for the Riemann surface given by the form $\mathcal{H}/\Gamma$, unless we have more information about the automorphism group. For example, this problem was solved for the Accola-Maclachlan and Kulkarni surfaces (see for example [21]). Many useful techniques and facts concerning the problem of finding equations can be found in [22]. A procedure to count the number of ovals fixed by complex conjugation was found in [7] for n-cyclic covers of the sphere. We should also mention the important work [3], where all the possible automorphism groups, topological types of symmetries, explicit defining equations for the surface and its real forms are given in the case of hyperelliptic Riemann surfaces. Also, clearly the upper bound on the total number of ovals of two symmetries is $2g+2$ and it is realized for the pair of symmetries with $g+1$ ovals each, yielding a hyperelliptic Riemann surface and hence the underlying equations in this case are known.

Suppose that $F(x,y)$ is an irreducible polynomial and $F(x,y) = 0$ is the defining equation for a Riemann surface X. We can view x and y as elements of $\mathbb{C}(X)$, the field of meromorphic functions on X. In doing so, $\mathbb{C}(X) = \mathbb{C}(x,y)$ and we can view $F \in \mathbb{C}(x)[y]$ as a polynomial in y over the rational function field $\mathbb{C}(x)$. If, in addition F is monic (in y), then $F(x,y)$ is the minimal polynomial for y over $\mathbb{C}(x)$. Observe that we can also change from one equation to another when we determine functions $t, w \in \mathbb{C}(x,y)$ such that $\mathbb{C}(x,y) = \mathbb{C}(t,w)$. That is, we can determine $G(t,w)$, the minimal polynomial for w over $\mathbb{C}(t)$, and then $G(t,w) = 0$ defines the same Riemann surface as before, since the function fields are the same. One can think that these two equations give different views of the same surface, for example one equation might be better when dealing with some singular points induced by the other one. Now recall that a point (b,c) of X, being a solution of $F(x,y) = 0$, is nonsingular if at least one of $F_x(b,c)$ or $F_y(b,c)$ is non-zero. If $F_x(b,c) \neq 0$ then $y-c$ is a local parameter at (b,c) and similarly, if $F_y(b,c) \neq 0$ then $x-b$ is a local parameter at (b,c). This just means that the order ord at (b,c) is equal to 1 for $y-c$ or $x-b$ respectively. Also, if (b,c) is a singular point then there will be one or more points on the Riemann surface lying above (b,c) and possibly neither of $x-b$ nor $y-c$ will be a local parameter. In such a case, it is usually necessary to change the coordinates and find a function t with the property $ord(t) = 1$.

In this paper, we will primarily be interested in Riemann surfaces X defined by equations of the form

$$y_1^2 - f_1(x) = 0, y_2^2 - f_2(x) = 0, \dots, y_r^2 - f_r(x) = 0, \tag{2.4}$$

where each f_i is a squarefree polynomial. If b is a root of one of the polynomials, say $f_1(x)$, then for $i > 1$ we can make a change of variables by defining $y_i = y_i/y_1$ if f_i also has b as a root and leaving y_i unchanged if b is not a root of f_i. In this case the defining equations in (2.4) become

$$y_1^2 - f_1(x) = 0, y_2^2 - q_2(x) = 0, \dots, y_r^2 - q_r(x) = 0, \tag{2.5}$$

where each q_i either equals the polynomial f_i or it is the rational function f_i/f_1. In either case, q_i has neither a root nor a pole at b. Near $x = b$ a point on X has the coordinates $(x, y_1, q_2, \dots, q_r)$ and lying over $x = b$ there are the 2^{r-1} points $(b, 0, \pm c_2, \pm c_3, \dots, \pm c_r)$ where each c_i is nonzero. The ramification index of X over $x = b$ is 2 (since there are only 2^{r-1} points lying over it), so the order of $x - b$ at any point of X lying over b is 2. Since $y^2 = f_1(x)$, this means that y_1 has order 1 at points lying over $x = b$, and therefore y_1 is a local parameter at any point of X lying over $x = b$.

3. The topological types for $k \geq 4$ commuting symmetries on an extremal Riemann surface

In this section we find all the possibilities for epimorphisms $\theta : \Lambda \to G$, realizing an o-extremal configuration of symmetries. Actually we prove that the epimorphism must be of a special type, which makes it possible to determine all the possible topological types of the symmetries in the configuration.

Let us remember how we can determine the separability type of a symmetry. As we are dealing with abelian groups only, we can easily check if a symmetry is separating by using the *word algorithm* given below. Let Λ' be a normal subgroup of an NEC group Λ. A canonical generator of Λ is *proper* (with respect to Λ') if it does not belong to Λ'. The elements of Λ expressable as a composition of proper generators of Λ' are the *words* of Λ (with respect to Λ'). From [5] we have

Lemma 3.1 (c.f. Theorem 2.1.3). *Suppose that $[\Lambda : \Lambda']$ is even and Λ has sign $+$. Then Λ' has sign $+$ if and only if no orientation reversing word belongs to Λ'. If $[\Lambda : \Lambda']$ is even and Λ has the sign $-$, then Λ' has the sign $-$ if and only if either a glide reflection of the canonical generators of Λ or an orientation reversing word belongs to Λ'.*

Now to compute the number of ovals of symmetries, we use the following result from [9]. Let $C(G, g)$ denote the centralizer of an element g in G:

Theorem 3.2. *Let $X = \mathcal{H}/\Gamma$ be a Riemann surface and let $G = \mathrm{Aut}^{\pm}(X)$, $G = \Lambda/\Gamma$ for some NEC group Λ and let $\theta : \Lambda \to G$ be the canonical epimorphism. Then the number of fixed ovals of a symmetry τ of X equals*

$$\sum [C(G, \theta(c_i)) \, : \, \theta(C(\Lambda, c_i))],$$

where the sum is taken over a set of representatives of all conjugacy classes of canonical reflections whose images under θ are conjugate to τ. □

Remark 3.3. To apply Theorem 3.2 we need to know the order of the centralizer of a reflection in an NEC group. This was done by Singerman in [20]. In particular, for NEC groups with presentation (2.3), the centralizer of an element c_i in Λ is generated by c_{i-1}, c_i and c_{i+1}.

3.1. **Combinatorial lemma.** First, we give a Lemma, which is a simple adjustment of Lemma 3.1 in [11]. The proof is very similar, but we present it here for the convenience of the reader. Afterwards we discuss how this lemma can be applied to computing the number of ovals fixed by a symmetry.

Lemma 3.4. *Assume that $k \geq 3$ labels are used to label s points situated on a circle in such a way that no two consecutive points have the same label. Then at least $k-1$ points have neighbors with distinct labels. Moreover, if two points with distinct labels also have neighbors with distinct labels, then at least k points have neighbors with distinct labels.*

Proof. The first part was proved in [10] and we shall prove the second part using induction on s. Observe first that $s \geq k \geq 3$ and the cases $s = 3, s = 4$ are trivial.

We have two points with distinct labels such that their neighbors also have distinct labels. We may assume that, between these two points, there are no points that have neighbors with distinct labels. Assume first that we have a sequence of consecutive points $i-1, i, i+1, \ldots, i+\alpha+2, i+\alpha+3, i+\alpha+4$ with labels $1, 2, 3, \overset{\alpha}{\ldots}, 2, 3, 1$ respectively, where nothing is prescribed about the labels of the points in the positions $i+2, \ldots, i+\alpha+1$. Consider the induced configuration of $s-(\alpha+5)$ points $1, \ldots, i-1, i+\alpha+5, \ldots, s$. If none of the points in the new configuration has label 2 or 3, then by the first part of the lemma, at least $k-3$ points in the new configuration have neighbors with distinct labels and in addition points $i, i+\alpha+3, i+\alpha+4$ have neighbors with distinct labels, so in the former configuration we have k points that have neighbors with distinct labels.

If, in the new configuration, only one of the labels 2 or 3 is used, then by the first part of the lemma, at least $k-2$ points in the new configuration have neighbors with distinct labels. Now at least $k-3$ of these points (as the point $i-1$ can have distinct neighbors in the new configuration and the same neighbors in the original one) together with $i-1, i, i+\alpha+3$ or $i, i+\alpha+3, i+\alpha+4$ give k points with neighbors that have distinct labels.

If, in the new configuration, both of the labels 2 and 3 are used, then by the first part of the lemma, at least $k-1$ of the points in the new configuration have neighbors with distinct labels. Now at least $k-2$ of these points together with $i, i+\alpha+3$ give k points with neighbors that have distinct labels in the former configuration.

Assume now that the points $i-1, i, i+1, \ldots, i+\alpha+2, i+\alpha+3, i+\alpha+4$ have labels $1, 2, 3, \overset{\alpha}{\ldots}, 2, 3, 4$ and, again, nothing is known about the labels of the points in the positions $i+2, \ldots, i+\alpha+1$. Consider an induced configuration of $s-(\alpha+4)$ points $1, \ldots, i-1, i+\alpha+4, \ldots, s$. Now if, in the new configuration, the points $i-1, i+\alpha+4$ have neighbors with distinct labels, then we have two consecutive points with distinct neighbors in the new configuration. Therefore, by the inductive hypothesis, the number of points with distinct neighbors in the new configuration is greater than or equal to the number of labels used. So in the former configuration we have at least $k-2$ points with distinct neighbors coming from the new configuration. These points together with the points $i, i+\alpha+3$ give k points with distinct labels in the original configuration.

If at least one of $i-1, i+\alpha+4$ has neighbors with the same label in the new configuration, then at least $k-3$ of the points with distinct neighbors in the new configuration together with the points $i-1, i, i+\alpha+3$ or $i, i+\alpha+3, i+\alpha+4$ give k points with distinct neighbors in the former configuration. □

Now we can proceed to a Corollary, which is essential in our task of finding the epimorphisms realizing the maximal configuration of symmetries.

Corollary 3.5. *If exactly $k-1$ points on the circle have neighbors with distinct labels, then all these points have the same label and s is even.*

Proof. The first statement of the Corollary is obvious. Now if all the points with distinct neighbors have the same label, say 1, then necessarily exactly half of the points on a circle have this label and hence the length of the cycle is even. Indeed, if there are two consecutive points, say with labels $2, 3$, then they have neighbors with the same label and we obtain a sequence of alternating labels $2, 3$, which has to be finished. But then there is at least one point whose neighbors are labeled with distinct labels, a contradiction. □

Remark 3.6. To see how the above Lemma and Corollary are applied to determine the number of ovals fixed by a symmetry, recall that, by Remark 3.3 for NEC groups with presentation (2.3), the centralizer of an element c_i in Λ is generated by c_{i-1}, c_i and c_{i+1}. According to Theorem 3.2, we need to calculate the value of expressions such as

$$[C(G,\theta(c_i))\,:\,\theta(C(\Lambda,c_i))].$$

However, the finite groups generated by symmetries that we deal with will be abelian, so $C(G,\theta(c_i)) = G$. To calculate each $\theta(C(\Lambda,c_i))$, we imagine the symbols c_0, c_1, $\ldots$, c_{s-1} as s points on a circle and we imagine the images of each of these reflections in G under θ as a label at that point. If c_i has distinct neighbors, then $\theta(c_{i-1})$ and $\theta(c_{i+1})$ are distinct, which means that the group generated by $\theta(c_{i-1}), \theta(c_i)$ and $\theta(c_{i+1})$ has order 8 in G, and therefore

$$[C(G,\theta(c_i))\,:\,\theta(C(\Lambda,c_i))] = \frac{|G|}{8}.$$

On the other hand, if c_i does not have distinct neighbors, then $\theta(c_{i-1}) = \theta(c_{i+1})$, so the group generated by $\theta(c_{i-1}), \theta(c_i)$ and $\theta(c_{i+1})$ has order 4 in G, and therefore

$$[C(G,\theta(c_i))\,:\,\theta(C(\Lambda,c_i))] = \frac{|G|}{4}.$$

3.2. **Distribution of ovals.** In this subsection we determine the only types of epimorphisms $\theta : \Lambda \to G = \langle \tau_1, \ldots, \tau_k\rangle$, for which $X = \mathcal{H}/\Gamma$ is an o-extremal Riemann surface of genus g admitting k commuting symmetries $\tau_1, \ldots, \tau_k$. For each τ_i, we let $\|\tau_i\|$ denote the number of ovals fixed by τ_i and we let $\|X\|$ denote the total number of ovals fixed by all the symmetries of X.

The bound on the number of ovals fixed by such a Riemann surface of genus g is given by the following formula. First, if $4 \le k \le 8$, we define $r = k$ and if $k \ge 9$, then we define r to be the smallest integer such that $k \le 2^{r-1}$. Given this definition of r, the maximal number of ovals fixed by k commuting symmetries (which yields an o-extremal Riemann surface) is given by

$$\|X\| = 2g - 2 + 2^{r-3}(9-k). \tag{3.6}$$

In addition, if $k \ne 9$, there is a unique group G generated by the k commuting symmetries; it is $G = \mathbb{Z}_2^r$. If $k = 9$, then there are five groups which can occur in an o-extremal configuration: $G = \mathbb{Z}_2^5, G = \mathbb{Z}_2^6, \ldots, G = \mathbb{Z}_2^9$. Independent of the size of k, we define A so that the group G in an o-extremal configuration is $G = \mathbb{Z}_2^A$. With this notation, we can express the total number of fixed ovals as

(3.7)

$$\|X\| = 2g-2+2^{r-3}(9-k) = 2g-2+(9-k)2^{A-3} = 2g-2+(9-k)\frac{|G|}{8}.$$

The above results were proved in a series of papers. The case $k = 4$, without an assumption on commutativity, was considered in [15] and the case of 5 non-commuting symmetries was treated in [16]. Here we generalize this to arbitrary $k \geq 4$ in the commuting case. In [11, 12] we proved that if the $k \geq 6$ symmetries on a Riemann surface of genus g have the maximal total number of ovals, then they commute and generate the entire automorphism group. Therefore the assumption on commutativity here only concerns the cases $k = 4, 5$, for which the potential automorphism groups are $\mathcal{D}_n \times \mathbb{Z}_2^{k-2}$ in the case where the symmetries do not commute.

We previously stated that we would only be concerned with NEC groups with signature (2.2). We now explain why this is true. Let $X = \mathcal{H}/\Gamma$ be a Riemann surface of genus g admitting $k \geq 4$ commuting symmetries $\tau_1, \ldots, \tau_k$, which together realize the maximal total number of ovals. By the analysis from the proof of Theorem 4.1 in [10], we may assume that the group of automorphisms $G = \Lambda/\Gamma$ for some NEC group Λ with signature

$$(h; \pm; [2, \overset{v}{\ldots}, 2]; \{C_1, \ldots, C_p, (-)^l\})$$

where $C_i = (2, \overset{s_i}{\ldots}, 2)$ for $i = 1, \ldots, p$ with at least one non-empty period cycle. We may assume, as the symmetries commute, that all the proper and link periods in the signature of Λ are equal to 2, which follows from the results in Chapter 2 in [5]. Now we shall show that, in fact, we may assume that $p = 1$ and $l = 0$. If $l > 0$, we will now remove one empty period cycle to construct a surface with a larger number of fixed ovals, contradicting that X is o-extremal. If the reflection of the empty period cycle is mapped by the canonical epimorphism $\theta : \Lambda \to G$ to τ_i, then we adjust the first nonempty period cycle in such a way that it begins with a reflection mapped by θ to some $\tau_j \neq \tau_i$. This can be done by the usual cyclic permutation, which does not change the number of ovals of symmetries corresponding to the images of the canonical reflections of this cycle. Then we remove the empty period cycle in question and replace the cycle C_1 in the signature of Λ by a nonempty period cycle C_1' of length $s_1 + 4$. In such a way we obtain a new signature and the corresponding NEC group Λ'. The four additional consecutive reflections are placed in the beginning of the cycle and mapped to

$$\tau_j\,\tau_i\,\tau_j\,\tau_i$$

by a new epimorphism $\theta' : \Lambda' \to G$. Then the cycle continues in exactly the same way as the original cycle C_1, meaning that the images of consecutive reflections by θ' are the same as by θ. One might see this as "gluing" the empty cycle to the beginning of C_1. In such a way we obtained a new signature and the corresponding new NEC group Λ' has the same hyperbolic area as the original group Λ - we lost one cycle, but we obtained 4 link periods equal to 2. The new epimorphism θ' differs from θ only slightly, as one of the empty period cycles vanished and there are four new reflections in C_1'. Now when we look at the number of ovals we see, that τ_i had at most $|G|/2$ ovals from the vanishing empty cycle and now it has $|G|/2$ ovals from the second and fourth reflections of the cycle C_1'. As for the symmetry τ_j, it might have lost $|G|/8$ ovals in the process from the fifth reflection of C_1', but it gained at least $|G|/8 + |G|/4$ ovals from the first and third reflections in C_1'. Therefore we obtained a new Riemann surface $X' = \mathcal{H}/\ker\theta'$ of genus g whose total number of ovals is strictly larger than the one of X. This is a contradiction, as we assumed that X was o-extremal. Now let us assume that $l = 0$ and $p > 1$. We shall give a method of gluing two nonempty cycles together, which leads to a new surface with a strictly larger total number of ovals. As $k \geq 4$, we may assume that C_1 ends with a reflection mapped by θ to τ_j and the other non-empty period cycle C_2 begins with a canonical reflection mapped by θ to $\tau_i \neq \tau_1$. As in the previous case, we glue them together by taking C_1 first, then inserting a segment consisting of 4 link periods equal to 2 and then proceeding with C_2. We map the canonical reflections corresponding to the additional segment to

$$\tau_i\,\tau_j\,\tau_i\,\tau_j$$

obtaining a new epimorphism θ'. In this process, the last reflection of C_1 and the first reflection of C_2 might have lost $|G|/4$ ovals together, but the reflections of the additional segment contribute with $|G|$ ovals to the total number of ovals of the new surface X'. Therefore the total number of ovals is strictly greater for the new surface X', a contradiction again. Hence we may assume, indeed, that Λ has the signature

$$(h; [2, \overset{v}{\ldots}, 2], \{(2, \overset{s}{\ldots}, 2)\}).$$

However, if $h > 0$ or $v > 0$, then by the Hurwitz-Riemann formula

$$s \leq \frac{8(g-1)}{|G|} + 2. \tag{3.8}$$

Now the total number of ovals satisfies, by Lemma 3.4,

$$\|X\| \leq (k-1)\frac{|G|}{8} + (s-k+1)\frac{|G|}{4} \leq 2g - 2 + (5-k)\frac{|G|}{8},$$

which, by (3.7), contradicts our assumption that the total number of ovals is maximal. Notice that, in order to make the total number of ovals maximal, we want to minimize the number of terms above that are multiplied by $|G|/8$; Theorem 3.4 yields that there must be exactly $k-1$ canonical reflections that have neighbors with distinct images. Therefore we may assume that Λ has the signature

$$(0; +; [-]; \{(2, \overset{s}{\ldots}, 2)\}) \tag{3.9}$$

and for the epimorphism $\theta : \Lambda \to G$, exactly $k-1$ of the canonical reflections have neighbors with distinct images. By the Corollary 3.5, the length of the cycle is even, so $s = 2t$ for some integer $t \geq k-1$. The lower bound on k follows from the fact that we have k symmetries and exactly half of the reflections contribute to just one of them. Therefore

$$g = 2^{A-2}(t-2) + 1, \text{ so } \frac{g-1}{2^{A-2}} = t - 2, \text{ and } \frac{g-1}{2^{A-2}} \geq k - 3 \tag{3.10}$$

and we see that necessarily 2^{A-2} divides $g-1$ by the Hurwitz-Riemann formula.

We now look closely at the possible epimorphisms $\theta : \Lambda \to G$. By Corollary 3.5, we know that there is a single symmetry, say τ_1, for which there exist $k-1$ canonical reflections whose neighbors have distinct images under θ and half of the reflections in the cycle are mapped to τ_1. By Theorem 3.2, these $k-1$ reflections contribute $\frac{|G|}{8}$ ovals to τ_1, while all the remaining canonical reflections contribute $\frac{|G|}{4}$ ovals to τ_1. This basically means, that our epimorphism is of the form

(3.11)

$$\underbrace{\tau_1, \tau_2, \ldots, \tau_2}_{2\alpha_2}, \underbrace{\tau_1, \tau_3, \ldots, \tau_3}_{2\alpha_3}, \ldots, \underbrace{\tau_1, \tau_i, \ldots, \tau_i}_{2\alpha_i}, \ldots, \underbrace{\tau_1, \tau_k, \ldots, \tau_k}_{2\alpha_k}, \ \tau_1$$

where $s = \sum_{i=2}^{k} 2\alpha_i$ and all such distributions of α_i for $i = 2, \ldots, k$ can be realized. Since

$$\sum_{i=2}^{k} \alpha_i = 2 + \frac{g-1}{2^{A-2}}$$

it follows that

$$s = 2\sum_{i=2}^{k} \alpha_i = 4 + \frac{8(g-1)}{2^A} = 4 + \frac{8(g-1)}{|G|}$$

and so one can easily see that τ_1 has

$$\left(\frac{s}{2} - k + 1\right)\frac{|G|}{4} + (k-1)\frac{|G|}{8} = g - 1 + (5-k)\frac{|G|}{8} \tag{3.12}$$

ovals while for $i = 2, \ldots, k$ each symmetry has $\|\tau_i\| = \alpha_i \frac{|G|}{4}$ ovals, so that these $k-1$ symmetries yield a total of

$$\frac{s}{2} \cdot \frac{|G|}{4} = g - 1 + \frac{|G|}{2}$$

ovals. Therefore, the total number of ovals fixed by the k symmetries is

$$g - 1 + (5-k)\frac{|G|}{8} + g - 1 + \frac{|G|}{2} = 2g - 2 + (9-k)\frac{|G|}{8}. \tag{3.13}$$

Given the order of G determined previously in relation to k, we have proved the following theorem.

Theorem 3.7. *Let $\tau_1, \ldots, \tau_k$ for $k \geq 4$ be commuting symmetries that generate a group $\mathbb{Z}_2^A$ on a Riemann surface of genus g, which realize the bound on the maximal total number of ovals given in* (3.13)*. Then $\frac{g-1}{2^{A-2}} \geq k-3$ is an integer, one symmetry has $g - 1 + (5-k)2^{A-3}$ ovals and each of the remaining symmetries has $\alpha_i 2^{A-3}$ ovals, where $\sum \alpha_i = \frac{g-1}{2^{A-2}} + 2$. Conversely, for all sets of parameters g, k, α_i as above, there exists a Riemann surface which admits k commuting symmetries with the maximal total number of ovals, and the number of ovals for each symmetry satisfies the conditions announced above.*

3.3. **Distribution of separability.** In this subsection we shall determine the separability of the commuting symmetries constituting an *o*-extremal configuration of ovals on a Riemann surface of genus g. Our study splits naturally into three cases, depending on if the k symmetries are the minimal generating set for G or not, whereas the case $k = 9$ needs special attention as it allows several possibilities.

Let us first assume that $G = \mathbb{Z}_2^k$. This condition clearly concerns case $4 \leq k \leq 8$ and fulfills it. However, for $k = 9$ it is also possible that the group generated by the symmetries is $\mathbb{Z}_2^9$, although it is not the only possibility. First we assume that if $k = 9$, then $G = \mathbb{Z}_2^9$.

Proposition 3.8. *The $k \geq 4$ symmetries generating the group $G = \mathbb{Z}_2^k$, realize the maximal total number of ovals on a Riemann surface of genus g, if and only if all the symmetries are separating (in addition to the conditions of the Theorem 3.7).*

Proof. By Lemma 3.1, a symmetry is non-separating if it can be presented as a product of other symmetries that are images of some canonical reflections under the epimorphism $\theta : \Lambda \to G$. By looking carefully at the epimorphism given in (3.11) we see that the images of canonical reflections are solely and exactly the symmetries $\tau_1, \ldots, \tau_k$. Therefore all these symmetries in the construction must be separating, as any of them is independent from the others. Moreover, as the epimorphism was the only one possible, we see that, in fact, we have necessary and sufficient conditions here. □

The situation becomes dramatically different if the number of symmetries is greater than the minimal number of generators. Now let us assume that $G = \mathbb{Z}_2^r = \langle \tau_1, \ldots, \tau_r \rangle$, where r is the smallest integer such that $k \leq 2^{r-1}$. We may assume that $\tau_1, \ldots, \tau_r$ are among our k symmetries. We can view the symmetries in G as orientation reversing words in alphabet $\tau_1, \ldots, \tau_r$, where we omit the order of the letters as the symmetries commute. Now observe that the number of symmetries, which are words with an odd number of letters on an alphabet consisting of $r - 1$ letters, is 2^{r-2} by the properties of the Newton's symbol. Therefore, as $k \geq 2^{r-2} + 1$, we must in fact use all the r letters while defining our symmetries as the images of canonical reflections. Now every appearance of the word

$$\tau_{i_1} \ldots \tau_{i_v},$$

for some odd integer v, as an image of some canonical reflection of an NEC group Λ, clearly forces the symmetries used $\tau_{i_1}, \ldots, \tau_{i_v}$ to become non-separating. As actually all the letters must be used, also all the symmetries are in fact non-separating. We have proved the following result

Proposition 3.9. *The $k \geq 9$ symmetries generating the group $G = \mathbb{Z}_2^r$, where r is the smallest integer such that $k \leq 2^{r-1}$, realize the maximal total number of ovals on a Riemann surface of genus g, if and only if all the symmetries are non-separating (in addition to the conditions of the Theorem 3.7).*

Now the only case to be considered is the one concerning $k = 9$ symmetries, where $G = \mathbb{Z}_2^A$ for $A = 6, 7, 8$. Let first $A = 8$, so we have 8 generating symmetries in G and one additional symmetry $\tau = \tau_{i_1} \dots \tau_{i_v}$, where v is odd, and so can be equal to $3, 5$ or 7. Observe, as in the previous case, that the appearance of such a word as an image of the canonical reflection, makes symmetries $\tau_{i_1}, \dots, \tau_{i_v}$ non-separating and clearly symmetry τ is also non-separating, by Lemma 3.1. Hence, as v is odd, we have $4, 6$ or 8 non-separating symmetries in our set.

Similarly, if $A = 7$, then we use all the generating symmetries and in addition two more, say $\tau = \tau_{i_1} \dots \tau_{i_v}$ and $\tau' = \tau'_{i_1} \dots \tau'_{i_{v'}}$, where v, v' are odd. Clearly τ, τ' are non-separating. Now the lengths of τ, τ' as words can again be equal to $3, 5$ or 7. By choosing appropriately letters constituting τ and τ', we can make exactly 4, 5, 6 or 7 of the generating symmetries non-separating. Therefore the number of non-separating symmetries is between 6 and 9 and all these values are realized.

Finally, if $A = 6$, then we use all the generating symmetries and, in addition, three more. These three are obviously non-separating, again by Lemma 3.1. In an analogous way as in the previous cases, we may choose these three symmetries to use different letters of our choice in the alphabet $\tau_1, \dots, \tau_6$. Here at least 4 generating symmetries must become non-separating and hence the total number of non-separating symmetries is between 7 and 9 and all these values are in fact realized. Summing up, we have proved the following result.

Proposition 3.10. *The* 9 *symmetries in the group* $G = \mathbb{Z}_2^A$, *where* $A = 6, 7, 8$, *realize the maximal total number of ovals on a Riemann surface of genus* g, *if and only if:*
1. at least 7 *of them are non-separating for* $A = 6$*;*
2. at least 6 *of them are non-separating for* $A = 7$*;*
3. $4, 6$ *or* 8 *of them are non-separating for* $A = 8$. *All the possible values are realized (in addition to the conditions of the Theorem 3.7), that is for each of the values we can construct the appropriate Riemann surface.*

4. Real equations for an extremal Riemann surface admitting $k \geq 4$ commuting symmetries

We now find equations for the o-extremal Riemann surfaces and their real forms. Here we have $k \geq 4$ real forms, where the group generated is $G = \mathbb{Z}_2^A$, where A is as introduced in the previous section for various values of k. Observe that we actually proved that in the case of an extremal Riemann surface X, the orbit space X/G is a disk, that $G^+ = \langle \tau_1 \tau_i, i = 2, \dots, k \rangle \cong \mathbb{Z}_2^{A-1}$ where X/G^+

is the Riemann sphere, and that the projection $X \to X/G^+$ is ramified over $s = \frac{g-1}{2^{A-3}} + 4$ points $a_1, \ldots, a_s$ lying over the boundary of X/G, with respect to the canonical covering $X/G^+ \to X/G$. Now by using the appropriate conjugation by a Möbius transformation, we may assume that the boundary component of X/G is in fact the extended real line $\mathbb{R}^*$ and that complex conjugation is a symmetry of X.

By the above facts, we can choose the coordinates on the Riemann sphere such that $a_1, \ldots, a_s$ have the real coordinates respectively

(4.14)
$$x = b_1, x = b_2, \ldots, x = b_{s-2} = 0, x = b_{s-1} = 1, x = b_s = \infty,$$

where $b_1 < b_2 < \ldots < b_{s-3} < 0$. Observe that we can arbitrarily choose the coordinates for three points. Recall the definition of $\alpha_2, \ldots, \alpha_k$ defined in (3.11); in addition, we define $\alpha_1 = 0$. We can assume that $\tau_1\tau_i$, for $i = 2, \ldots, k$, fixes $a_{2(\alpha_1+\cdots+\alpha_{i-1})+1}, \ldots, a_{2(\alpha_1+\cdots+\alpha_i)}$. Recall that $k \geq A$; if this is a strict inequality, we rename the symmetries $\tau_2, \ldots \tau_{k-1}$, if necessary, so that the last $A-1$ elements $\tau_1\tau_{k-A+2}, \ldots, \tau_1\tau_k$ generate the entire group $G^+ = \mathbb{Z}_2^{A-1}$. To simplify subscripts, define $\gamma = k - A + 2$, so that $G^+ = \langle \tau_1\tau_i, i = \gamma, \ldots, k\rangle$, and define $\widehat{\gamma} = 2(\alpha_1 + \cdots + \alpha_{\gamma-1}) + 1$, which is the index of the smallest a_j that is fixed by $\tau_1\tau_\gamma$.

The function field of the Riemann surface X/G^+ is $\mathbb{C}(x)$, where

$$[\mathbb{C}(X) : \mathbb{C}(x)] = 2^{A-1}, \tag{4.15}$$

and the group G^+ acts on $\mathbb{C}(X)$ and yields $\mathbb{C}(x)$ as its fixed field. Since G^+ is not cyclic, we cannot obtain an equation of the form $z^{2^{A-1}} - f(x) = 0$. However, corresponding to each subgroup $G_i = \langle \tau_1\tau_\gamma, \ldots, \widehat{\tau_1\tau_i}, \ldots, \tau_1\tau_k\rangle$, of G^+, where $\widehat{\cdot}$ means that the corresponding generator is removed, there is a subfield $\mathbb{C}(X)^{G_i}$ of $\mathbb{C}(X)$ consisting of the elements fixed by each automorphism in G_i with the property that $[\mathbb{C}(X)^{G_i} : \mathbb{C}(x)] = 2$. Corresponding to each subgroup of G^+ there is a Fuchsian group and an associated Riemann surface. Since $[\mathbb{C}(X)^{G_i} : \mathbb{C}(x)] = 2$, for each i with $\gamma \leq i \leq k$, we obtain that $\mathbb{C}(X)^{G_i} \cong \mathbb{C}(x, y_i)$ where $y_i^2 - f_i(x) = 0$ for some polynomial $f_i(x)$. We denote the corresponding surface X/G_i by X_i. Obviously X_i is a double cover of the Riemann sphere. Since $\tau_1\tau_i \notin G_i$, the $2\alpha_i$ points $a_{2(\alpha_1+\cdots+\alpha_{i-1})+1}, \ldots, a_{2(\alpha_1+\cdots+\alpha_i)}$ are ramified in the field extension $\mathbb{C}(x, y_i)$ of $\mathbb{C}(x)$. Define $g_1(x) = 1$, and we define the polynomials

(4.16)
$g_i(x) = (x - b_{2(\alpha_1+\cdots+\alpha_{i-1})+1}) \cdots (x - b_{2(\alpha_1+\cdots+\alpha_i)})$, for $2 \leq i < k$ and

$$(4.17) \qquad g_k(x) = (x - b_{2(\alpha_1+\cdots+\alpha_{k-1})+1}) \cdots (x - b_{2(\alpha_1+\cdots+\alpha_k)-1}).$$

Note that for $i < k$ the degree of $g_i(x)$ is even and equals $2\alpha_i$, while the degree of g_k is odd since it lacks the root $b_{2(\alpha_1+\ldots+\alpha_k)} = b_s = \infty$. In the equation $y_i^2 - f_i(x) = 0$, we have that $g_i(x)$ divides $f_i(x)$ because the roots of $g_i(x)$ are fixed by $\tau_1\tau_i$. Clearly no point other than $b_1, \ldots, b_{s-1}$ can be a root of any $f_i(x)$ because ramification in the cover $X \to X/G^+$ only occurs at these points and at $b_s = \infty$.

In the case $4 \leq k \leq 8$, we have that $A = k$ and $\gamma = k-A+2 = 2$, so we obtain $k-1$ distinct double covers of the Riemann sphere with defining equations

$$(4.18) \qquad y_2^2 - f_2(x) = 0,\ \ y_3^2 - f_3(x) = 0, \ldots,\ \ y_k^2 - f_k(x) = 0,$$

and the Riemann surface X is defined by the common solutions to the equations (4.18). In this case, if $i \neq j$, then $\tau_1\tau_j \in G_i$ and $\tau_1\tau_i \in G_j$, which implies that $f_i(x)$ and $f_j(x)$ have no nontrivial common factors, and therefore each $f_i(x) = g_i(x)$.

We now deal with the case $k > A$. In this case we have $A-1$ equations

$$(4.19) \qquad y_\gamma^2 - f_\gamma(x) = 0, y_{\gamma+1}^2 - f_{\gamma+1}(x) = 0, \ldots, y_k^2 - f_k(x) = 0,$$

which yield $A-1$ double covers of the Riemann sphere. In addition, since $G^+ = \langle \tau_1\tau_i, i = \gamma, \ldots, k\rangle$, we see that $\mathbb{C}(X) = \mathbb{C}(x, y_\gamma, y_{\gamma+1}, \ldots, y_k)$. However, the points $a_1, a_2, \ldots, a_{\widehat{\gamma}-1}$ must also be ramified in the covering $X \to \mathbb{C}(x)$. Therefore for each root u of a polynomial $g_j(x)$ with $j < \gamma$, there must be an $f_i(x)$ with $\gamma \leq i$ which also has u as a root. To analyze this correctly, we make the following observations: since $G^+ \cong \mathbb{Z}_2^{A-1}$ is generated by the $A-1$ elements $\tau_1\tau_\gamma, \tau_1\tau_{\gamma+1}, \ldots, \tau_1\tau_k$, each element in G^+ can be expressed uniquely as a product of the $\tau_1\tau_\gamma, \tau_1\tau_{\gamma+1}, \ldots, \tau_1\tau_k$. Suppose now that $j < \gamma$ and when $\tau_1\tau_j$ is uniquely expressed in terms of these generators, the generator $\tau_1\tau_i$, with $i \geq \gamma$ appears. This means that $\tau_1\tau_j \notin G_i$ (because the generator $\tau_1\tau_i$ is not an element of G_i) and therefore in the cover of $\mathbb{C}(x)$ by $\mathbb{C}(x, y_i)$, ramification must occur at the points fixed by $\tau_1\tau_j$. This means that the polynomial $g_j(x)$ must divide the polynomial $f_i(x)$. In addition, it must do that for each generator $\tau_1\tau_\gamma, \tau_1\tau_{\gamma+1}, \ldots, \tau_1\tau_k$ that appears in the expression of $\tau_1\tau_j$ in terms of these generators. Therefore we have proved the form that the defining equations of X must possess.

Proposition 4.1. *In addition to the notation developed above, define $\mathcal{C} = \{g_1(x) = 1, g_2(x), \ldots, g_{\gamma-1}(x)\}$. Then the polynomials $f_\gamma(x), \ldots, f_k(x)$ which define the extremal surface X must have* *the following form:*

$$y_\gamma^2 - f_\gamma(x) = 0, \text{ where } f_\gamma(x) \text{ is a product of } g_\gamma(x) \text{ and some polynomials in } \mathcal{C},$$
$$y_{\gamma+1}^2 - f_{\gamma+1}(x) = 0, \text{ where } f_{\gamma+1}(x) \text{ is a product of } g_{\gamma+1}(x) \text{ and some polynomials in } \mathcal{C},$$
$$\vdots$$
$$y_k^2 - f_k(x) = 0, \text{ where } f_k(x) \text{ is a product of } g_k(x) \text{ and some polynomials in } \mathcal{C}.$$

Since ramification must occur at each of the points $b_1, \dots, b_{\widetilde{\gamma}-1}$, each polynomial $g_2(x), \dots, g_{\gamma-1}(x)$ must be a factor of some $f_i(x)$ with $\gamma \leq i$ in the above list. The list of restrictions on the polynomials above is given by the following proposition.

Proposition 4.2. *If g_j, with $j < \gamma$ divides some f_i, with $\gamma \leq i \leq k$, then there is some u, with $u \neq i$ and $\gamma \leq u \leq k$, for which f_u is also divisible by g_j. In addition, none of these polynomials f_i that are divisible by $g_j(x)$ are divisible by $g_{j+1}(x)$. Finally no other $g_{j'}$, divides precisely the same polynomials among the $f_\gamma, \dots, f_k$ that g_j divides.*

We will provide the justification for the proposition when we examine the action of the individual symmetries on the surface defined by the equations. A key feature will be that for $j = 2, \dots, k$, the symmetry τ_j has $\alpha_j 2^{A-3}$ ovals which are provided by the roots of g_j. This yields the restrictions stated in Proposition 4.2, since without these restrictions we could obtain symmetries with $(\alpha_i + \alpha_j)2^{A-3}$ ovals, for example, where $i \neq j$.

Finally, we note that the above list of polynomials in Proposition 4.1 subsumes the case $4 \leq k \leq 8$, where $A = k$, because we had previously defined $g_1(x) = 1$ and in this case, for $2 \leq i \leq k$, we have that $f_i(x) = g_i(x) \cdot g_1(x)$. Therefore, we may assume that the above list of polynomials holds independently of k.

We now examine the common real solutions to the equations in Proposition 4.1. For any polynomial equation $y^2 - f(x) = 0$ with real coefficients, a real solution (x, y) will be obtained if and only if x is to the left of an even number of roots of $f(x)$. In subsequent sections we will make changes of variables and obtain real equations of the form $y^2 + f(x) = 0$; they will have real solutions if and only if x is to the left of an odd number of roots of $f(x)$. A key feature of all of our arguments is that the degrees of g_k and f_k are odd while the degrees the remaining g_j's and f_i's are even. In addition, recall that if $j < i$, then all of the roots of $g_j(x)$ are smaller than any of the roots of $g_i(x)$.

Using the above facts, the following closed real intervals given by the roots of $g_k(x)$ contain solutions in common to all of the

equations defining X:
(4.20)
$$I_1 = [b_{s-2\alpha_k+1}, b_{s-2\alpha_k+2}], \dots, I_{\alpha_k-1} = [b_{s-3}, b_{s-2}] = [b_{s-3}, 0], I_{\alpha_k} = [1, \infty],$$

where $s = 2(\alpha_2 + \cdots + \alpha_k)$. More solutions would be obtained if it were the case that, for some $j < \gamma$, $g_j | f_k$ and g_j did not divide any f_i for $\gamma \leq i < k$. We will see below, when we consider the number of ovals fixed by complex conjugation, that this cannot happen. A generic point on X has the coordinates $(x, y_\gamma, \dots, y_k)$ and a point with real coordinates on X will have x in one of the closed intervals given in (4.20). This means that X possesses an automorphism group G^+ of order 2^{A-1} generated by $\langle \rho_\gamma, \dots, \rho_k \rangle$, where

$$(4.21) \qquad \rho_i(x) = x, \ \ \rho_i(y_i) = -y_i \text{ and } \rho_i(y_j) = y_j \text{ for } j \neq i.$$

Note that complex conjugation σ is clearly a symmetry of X. We now want to identify all of the symmetries of X and determine their number of fixed ovals. From our analysis in Section 3.2, we know that X has the symmetries $\{\tau_1, \dots, \tau_k\}$ (which are the only symmetries that contain fixed points), and the number of ovals fixed by each of these symmetries has been determined. On the other hand, using the defining equations of X, we have the symmetries $\{\rho\sigma \mid \rho \in G^+\}$. We will now determine which of these symmetries correspond to the symmetries τ_i by tracing fixed ovals on the surface X defined by the polynomials above. An example of this technique used in a simpler context is found in [7]. That paper also contains figures to help visualize the process of traversing a fixed oval on a Riemann surface defined by equations.

4.1. **Determining the symmetry corresponding to τ_k.** We first determine the number of ovals fixed by complex conjugation σ. Out of the α_k intervals in (4.20), the determination of the number of ovals corresponding to each of them is the same, except for the last. We will give an argument for the second to last (to simplify notation) and then give the argument for I_{α_k}. On the closed interval $I_{\alpha_k-1} = [b_{s-3}, 0]$, when $x = b_{s-3}$, $y_k = 0$ but $y_\gamma, \dots, y_{k-1}$ are all nonzero; assume a point on X lying over this point has coordinates $(b_{s-3}, q_\gamma, \dots, q_{k-1}, 0)$, where all but the last coordinate is nonzero. As x increases, the values of the q's do not change sign, and there are two choices for value of y_k; assume we choose $y_k > 0$. When x reaches the right endpoint of I_{α_k-1}, namely $x = 0$, y_k again returns to 0, all of the q's retain the sign they previously had. As we continue along the oval, x must decrease, since there are no real points with x between 0 and 1, and the value of y_k must become negative, because y_k is a local parameter at the right endpoint of I_{α_k-1}, and being locally analytic to X there, the fixed oval must pass through the point where $y_k = 0$ and proceed to where $y_k < 0$. Finally, as x decreases back to the left endpoint, y_k returns to 0 and all of the other coordinates return to their previous values, and the loop is closed. The key feature is that all of the points in I_{α_k-1} lie to the right of any of the roots of $f_\gamma(x), \ldots, f_{k-1}(x)$, therefore none of the corresponding y_i can change sign. Therefore we have one loop corresponding to each choice of $q_\gamma, \ldots, q_{k-1}$. Recall that $\gamma = k - A + 2$, this yields 2^{A-2} distinct ovals, each lying over I_{α_k-1}, given by alternating the signs of $q_\gamma, \ldots, q_{k-1}$. Since the same is true for the intervals $I_1, \ldots, I_{\alpha_k-1}$, this gives $(\alpha_k - 1)2^{A-2}$ distinct ovals so far.

We now determine the number of fixed ovals lying on I_{α_k} which is the interval $1 \leq x \leq \infty$. For $i = \gamma, \ldots, k-1$, define $d_i = \deg(f_i)/2$ and define $d_k = (\deg(f_k) + 1)/2$. For each y_i with $i = \gamma, \ldots, k$, we make the change of variables $t = 1/x, u_i = y_i/x^{d_i}$. Since the degree of each of $f_\gamma, \ldots, f_{k-1}$ is even, in the coordinates (t, u_i) there are two points over infinity: $(0, 1)$ and $(0, -1)$. Since the degree of f_k is odd, there is only one point lying over $x = \infty$, corresponding to $(t, u_k) = (0, 0)$, however in this case u_k is a local parameter at this point. Assume x starts at the left endpoint of I_{α_k}, namely $x = 1$. Lying over this point is a point $(x, y_\gamma, \ldots, y_k) = (1, q_\gamma, \ldots, q_{k-1}, 0)$. As x increases, we assume q_k is positive and as x approaches ∞, we switch coordinates and at $x = \infty$, we reach a point with $(0, u_2, u_3, \ldots, u_k) = (0, \pm 1, \pm 1, \ldots, 0)$. Note that the signs are the same as the signs of the $q_\gamma, \ldots, q_{k-1}$. As we cross this point, all of the $u_\gamma, \ldots, u_{k-1}$ keep the same sign, x decreases (because there are no solutions with x less than the negative number b_1, and u_k, being a local parameter, switches sign to become negative. This means that we have points with coordinates $(x, y_\gamma, \ldots, y_k)$, where $x > 1$ and $y_k < 0$. When x reaches $x = 1$, we return to the point $(1, q_\gamma, \ldots, q_{k-1}, 0)$ which closes the loop. Therefore, we obtain one loop for each choice of the signs of the $q_\gamma, \ldots, q_{k-1}$. This yields 2^{A-2} distinct ovals. Combining this with the $(\alpha_k - 1)2^{A-2}$ found above, we obtain that complex conjugation σ fixes $\alpha_k 2^{A-2}$ ovals, which means that σ *should correspond* to τ_k. However, to ensure that this happens, it must be true that if a g_j, with $j < \gamma$ divides f_k, then the roots corresponding to g_j do not yield intervals with real solutions. If they did yield real solutions, then the number of fixed ovals for complex conjugation would be a sum of $\alpha_k 2^{A-2}$ and terms of the form $\alpha_j 2^{A-2}$, for various j with $j < \gamma$. Since X does not have a symmetry with such a number of fixed ovals, this cannot occur. Since, for $\gamma \leq i < k$, the degree of each f_i is even, the only way that common real solutions will not be obtained is if there is an f_i, with $\gamma \leq i < k$, which is also divisible by g_j. Therefore, we obtain the following restriction on $f_k(x)$:

Lemma 4.3. *If g_j, with $j < \gamma$ divides f_k, then there is some f_i, with $\gamma \leq i < k$, that is also divisible by g_j.*

4.2. **Determining the symmetry corresponding to τ_j with $j \geq \gamma$.** Let $\gamma \leq j < k$; we determine the ovals of the symmetry $\rho_j\rho_k\sigma$. Note that if y_j is pure imaginary, say $y_j = ib$ where b is real, then $\rho_j\rho_k\sigma(y_j) = \rho_j\rho_k(-y_j) = y_j$, so y_j is fixed by $\rho_j\rho_k\sigma$. A similar result holds if y_k is pure imaginary. Therefore we make the change of coordinates $y_j = iy_j$ and $y_k = iy_k$, so that the defining equations for X are the same as those in Proposition 4.1 except that they contain

$$y_j^2 + f_j(x) \text{ and } y_k^2 + f_k(x) \tag{4.22}$$

instead of the original equations for indices j and k. With this change of variables, $\rho_j\rho_k\sigma$ is exhibited as complex conjugation. Note that the intervals

$$[b_{2(\alpha_1+\cdots+\alpha_{j-1})+1}, b_{2(\alpha_1+\cdots+\alpha_{j-1})+2)}], \ldots, [b_{2(\alpha_1+\cdots+\alpha_j)-1}, b_{2(\alpha_1+\cdots+\alpha_j)}], \tag{4.23}$$

yield a common set of real solutions; we are using the fact that none of the f_i for $\gamma \leq i \neq j$ have roots in the above intervals. Due to this fact, the determination of the number of ovals is analogous to the $(b_{s-3}, 0)$ case above. This yields 2^{A-2} fixed ovals corresponding to each interval, which yields $\alpha_j 2^{A-2}$ ovals in total corresponding to $\rho_j\rho_k\sigma$. Therefore, $\rho_j\rho_k\sigma$ should correspond to τ_j. However, to ensure that it does not possess more fixed ovals, we see that if a g_u divides f_j with $u < \gamma$, then in order for the roots of g_u to not yield more common real solutions, we must have that there is a $w \neq j$ with $w \geq \gamma$ for which f_w is also divisible by g_u. We are using that real solutions of $y^2 + f_j(x)$ and $y^2 + f_k(x)$ must occur to the left of an odd number of roots; real solutions of the remaining polynomials $y^2 - f_i(x)$ must occur to the left of an even number of roots and all of the g_u's and f_i's have even degree except for g_k and f_k. Therefore, we obtain the following restriction that extends Lemma 4.3:

Lemma 4.4. *If g_j, with $j < \gamma$ divides some f_i, with $\gamma \leq i \leq k$, then there is some u, with $u \neq i$ and $\gamma \leq u \leq k$, for which f_u is also divisible by g_j.*

4.3. **Determining the symmetry corresponding to τ_j with $2 \leq j < \gamma$.** Now assume $2 \leq j < \gamma$. In this case, there exists some f_n which is divisible by g_j. However, from Lemma 4.3 there exist at least two polynomials of the $f_\gamma, \dots, f_k$ that are divisible by g_j. To avoid this, choose n to be the first index with $\gamma \leq n \leq k$ for which $g_j(x)|f_n(x)$. We make the following change of variables: $\widehat{y}_i = y_i$, if $i = n$ or if g_j does not divide f_i and $\widehat{y}_i = y_i/y_n$ otherwise. Note that this is a real change of variables. When the polynomials in Proposition 4.1 are expressed in terms of the $\widehat{y}_i$ we obtain

$$\widehat{y}_\gamma^2 - q_\gamma(x) = 0,\ \widehat{y}_{\gamma+1}^2 - q_{\gamma+1}(x) = 0, \dots,\ \widehat{y}_k^2 - q_k(x) = 0, \tag{4.24}$$

where some of the $q_i(x)$ may be polynomials in x and others are rational functions, since $q_i(x)$ may have the form $f_i(x)/f_n(x)$ however, the important fact is that $q_n = f_n(x)$ is the only function out of the polynomials or rational functions appearing in the definition of X that has a root (or pole) appearing as an endpoint in the list of intervals in (4.23); recall that these endpoints correspond to the roots of $g_j(x)$. In addition, the total number of roots (or roots and poles in the case of rational functions) is even for each $q_i(x)$ except for $q_k(x)$. For $\gamma \leq i \leq k$, define the automorphisms $\widehat{\rho}_i$ which fix x and for which $\widehat{\rho}_i(\widehat{y}_i) = -\widehat{y}_i$ and $\widehat{\rho}_i(\widehat{y}_t) = \widehat{y}_t$ for $t \neq i$.

We continue to assume $j < \gamma$ and that $f_n(x)$ is the only polynomial or rational function that has roots (or poles) that correspond to the roots of $g_j(x)$. In this case, the argument mirrors that of Subsection 4.2. We determine the ovals of the symmetry $\widehat{\rho}_n\widehat{\rho}_k\sigma$. Note that if $\widehat{y}_n$ or $\widehat{y}_k$ is pure imaginary then this symmetry fixes it. Therefore we make the change of coordinates $\widehat{y}_n = i\widehat{y}_n$ and $\widehat{y}_k = i\widehat{y}_k$, so that the defining equations for X are the same as those in (4.24) except that they contain

$$\widehat{y}_n^2 + f_n(x) = 0 \text{ and } \widehat{y}_k^2 + q_k(x) = 0 \tag{4.25}$$

instead of the original equations for indices j and k. The common set of real solutions are the identical to the intervals listed above in (4.23) where we are using the fact that none of the q_t for $t \neq n$ have roots nor poles in the above intervals. Due to this fact, the determination of the number of ovals is the same as in the $(b_{s-3}, 0)$ case above. This yields 2^{A-2} fixed ovals corresponding to each interval, which yields $\alpha_j 2^{A-2}$ ovals in total corresponding to $\widehat{\rho}_n\widehat{\rho}_k\sigma$. Therefore, we see that $\widehat{\rho}_n\widehat{\rho}_k\sigma$ corresponds to τ_j. However, to ensure that this symmetry does not possess more fixed ovals, we see that if a g_u divides f_n with $j \neq u < \gamma$, then in order for the roots of g_u to not yield more common real solutions, we must have that there is a $w \neq n$ with $w \geq \gamma$ for which q_w is also divisible (in the sense that its numerator or denominator is divisible) by g_u. This means that either f_w is divisible by g_j but not g_u (so that q_w now has poles at the roots of g_u), or that g_u, but not g_j divides f_w. A more concise was of saying this is the following: No other g_u divides precisely the same polynomials among the $f_\gamma, \dots, f_k$ that g_j divides. Therefore, we obtain the following restriction that extends Lemma 4.4:

Lemma 4.5. *If g_j, with $j < \gamma$ divides some f_i, with $\gamma \le i \le k$, then there is some u, with $u \neq i$ and $\gamma \le u < k$, for which f_u is also divisible by g_j. In addition, no other $g_{j'}$, divides precisely the same polynomials of $f_\gamma, \dots, f_k$ that g_j divides.*

To express this symmetry in terms of the ρ's and σ, we note that

$$\widehat{\rho}_n = \prod_{g_j | f_i} \rho_i, \text{ and } \widehat{\rho}_k = \rho_k, \text{ so } \widehat{\rho}_n \widehat{\rho}_k \sigma = \rho_k \left(\prod_{g_j | f_i} \rho_i \right) \sigma.$$

4.4. **Determining the symmetry corresponding to τ_1.** We finally determine the symmetry associated with τ_1, whose number of ovals has a distinct form from the others. We make the change of variables $y_k = iy_k$ and note that this is fixed by the symmetry $\rho_k\sigma$. Under this change of variables, the defining equations for X are the same as those in (4.24) except that they contain

$$y_k^2 + f_k(x) = 0. \tag{4.26}$$

From (4.14), the common real solutions in this case are the intervals:

$$(-\infty, b_1), \dots, (b_{2n}, b_{2n+1}), \dots, (0, 1),$$

where $1 \le n \le (s-4)/2$. Out of these $s/2$ intervals, $k-1$ of them, specifically, $(-\infty, b_1)$ and the ones of the form $(b_{2(\alpha_1+\cdots+\alpha_i)}, b_{2(\alpha_1+\cdots+\alpha_i)+1)})$ occur when we skip from a root of a g_i to a root of g_{i+1}. This was not the case before and we shall see that each of these intervals yield 2^{A-3} ovals. We will see that the remaining $\alpha_j - 1$ intervals corresponding to the remaining roots of g_j will each yield 2^{A-2} ovals, analogous to what we have seen above for the $(b_{s-3}, 0)$ case. Summing up, we obtain $(s/2 - k + 1)2^{A-2} + (k-1)2^{A-3}$ ovals which, from (3.12), yields $g - 1 + 2^{A-3}(5-k)$ ovals, the correct number for τ_1.

We justify the above claims. If $\gamma \le j < k$, then in the equation $y_j^2 - f_j(x) = 0$, y_j is a local parameter at each root of g_j; see the discussion following (2.4) and (2.5) for details. Similarly y_k is a local parameter at each root of g_k. Complications arise only for roots of g_j, where $j < \gamma$. In this case, from Proposition 4.5 for each $j < \gamma$, g_j divides several f_i's with $\gamma \leq i \leq k$. For any root of g_j, we can take one of the polynomials f_i which is divisible by g_j and divide the remaining polynomials in $f_\gamma, \ldots, f_k$ which are divisible by g_j by f_i to obtain a change of variables for which y_i is a local parameter at each root of g_j and the resulting equations $y_u^2 - q_u(x)$ do not have a root of g_j as a root or pole of $q_u(x)$. Assume that none of the polynomials f_k divisible by g_j are divisible by g_{j+1}. We repeat the process for g_{j+1}: there exists an $f_{i'}$ which is divisible by g_{j+1} and we divide the remaining polynomials in $f_\gamma, \ldots, f_k$ which are divisible by g_{j+1} by $f_{i'}$ to obtain a change of variables for which $y_{i'}$ is a local parameter at each root of g_{j+1} and the resulting equations $y_u^2 - q'_u(x)$ do not have a root of g_{j+1} as a root or pole of $q'_u(x)$. The important point is that y_i is a local parameter at the left endpoint of the interval, but is nonzero at the right endpoint. Similarly $y_{i'}$ is a local parameter at the right endpoint and is nonzero at the left endpoint. Using the above changes of variables and the local parameters they create, each of the $\alpha_j - 1$ intervals that do not involve an interval connecting a root of g_j with a root of g_{j+1} can be analyzed as in the $(b_{s-3}, 0)$ case as above to yield 2^{A-2} ovals each. We now determine the number of ovals corresponding to intervals defined by the largest root of a g_j and the smallest root of g_{j+1}. We present here a calculation for one of these intervals, $(b_{2\alpha_2}, b_{2\alpha_2+1})$, to present the technique. For convenience of notation only, we assume $i < i'$. Observe that on the surface X, over $b_{2\alpha_2}$ we have points of the form $(x, y_\gamma, \ldots, y_i, \ldots, y_{i'}, \ldots, y_k) = (b_{2\alpha_2}, \pm, \ldots, \pm, 0_{y_i}, \pm, \ldots, \pm)$ (using the coordinates in which y_i is a local parameter) and above $b_{2\alpha_2+1}$ there are points $(x, y_\gamma, \ldots, y_i, \ldots, y_{i'}, \ldots, y_k) = (b_{2\alpha_2+1}, \pm, \ldots, \pm, 0_{y_{i'}}, \pm, \ldots, \pm)$ (where these coordinates have $y_{i'}$ as a local parameter). We start tracing the oval for $b_{2\alpha_2} < x < b_{2\alpha_2+1}$ by moving through the points of the form $(x, y_\gamma^+, \ldots, y_k^+)$. We reach the point $(b_{2\alpha_2+1}, +, \ldots, +, 0_{y_{i'}}, +, \ldots, +)$, where $y_{i'}$ changes sign but y_i does not. We now continue to the points of the form $(x, +, \ldots, +, -_{y_{i'}}, +, \ldots, +)$ for $b_{2\alpha_2} < x < b_{2\alpha_2+1}$. We reach the point $(b_{2\alpha_2}, +, \ldots, +, \ldots, 0_{y_i}, +, \ldots, +, -_{y_{i'}}, +, \ldots, +)$, where y_i changes sign but $y_{i'}$ does not. We let x increase again and see that now points have the form form $(b_{2\alpha_2}, +, \ldots, +, \ldots, -_{y_i}, +, \ldots, +, -_{y_{i'}}, +, \ldots, +)$. Now after reaching the right endpoint $y_{i'}$ changes sign again, and when x reaches the left endpoint again, y_i changes sign. Hence we obtain one oval for every combination of the signs of the functions $y_\gamma, \ldots, y_k$ that are not y_i or $y_{i'}$; this clearly yields 2^{A-3} combinations. Therefore we obtain 2^{A-3} ovals from each of the $k-1$ special intervals. Note that for the interval $(-\infty, b_1)$ we use the same change of coordinates as before in Section 4.1 for the points over ∞.

Recall that we assumed that none of the polynomials f_i divisible by a g_j with $j < \gamma$, are divisible by g_{j+1}. If this did not occur and f_i were divisible by both g_j and g_{j+1}, then y_i would be a local parameter at both the largest root of g_j and the smallest root of g_{j+1}, in other words, at both endpoints of one of the $k-1$ special intervals we were considering above. This would have yielded 2^{A-2} ovals for the interval, which does not match the number of ovals required for τ_1. This yields the final restriction on the g_j and completes the proof of Proposition 4.2.

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