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SPHERICAL GENERALIZED HELICES IN 3-DIMENSIONAL LORENTZ-MINKOWSKI SPACE

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Abstract. In this paper we analyze generalized helices lying on the non-degenerated quadric surfaces in 3-dimensional Lorentz-Minkowski space, i.e. on a pseudosphere and in a hyperbolic plane. We provide their characterizations in terms of curvature and torsion and analyze their projections onto planes orthogonal to their axes. We show that these projections appear as Euclidean or Lorentzian cycloidal curves, so we also introduce natural equations and parametrizations of Lorentzian cycloidal curves.

1. Introduction

A curve of a constant slope or generalized helix is a space curve whose tangent vectors make a constant angle with a fixed straight line, called the axis of a generalized helix. In 1802 Lancret stated, and in 1845 de Saint Venant first proved that these curves are characterized by the constant ratio of its torsion to curvature. Thus, generalized helices are well known in classical differential geometry and it is noteworthy to study them with some additional property, such as lying on a quadratic surface. Since quadratic surfaces can be considered as spheres in a new ambient geometry, this analysis can be carried out as the analysis of spherical generalized helices. In particular, curves lying on non-degenerated quadratic surfaces can be regarded as spherical curves in either Euclidean or Lorentz-Minkowski geometry. This new ambient geometry, whether with positive definite, indefinite or degenerate metric, can be obtained by endowing a 3-dimensional affine space with a new metric in the sense of Cayley-Klein geometries.

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In this paper we discuss spherical generalized helices in 3-dimensional Lorentz-Minkowski space and their several properties – their characterization in terms of curvature and torsion (see Theorems 4.3 and 4.4), their plane projections and their natural equations. In Euclidean space, the orthogonal projection of a spherical generalized helix onto a plane normal to its axis is an epicycloid, [7]. This property is quite disregarded despite the fact that epicycloids, among other cycloidal curves, are widespread in physics (e.g. [2, 8, 10]), as well as in robotics (e.g. [12, 20]). Therefore, we were motivated to analyze Lorentzian analogues of the property, which is a much richer due to the various types of induced plane geometries. In Lorentz-Minkowski space the induced plane metric can be either positive definite, indefinite or isotropic and appears in spacelike, timelike or lightlike planes respectively.

The paper is organized as follows. First, we summarize conditions that characterize spherical curves in Lorentz-Minkowski space $\mathbb{R}^3_1$. Spherical curves in Lorentz-Minkowski space are those lying on a pseudosphere (Lorentzian spherical curves) or in a hyperbolic plane. In terms of their curvatures, results on spherical curves in 3-dimensional Lorentz-Minkowski space can be found in e.g. [16, 17, 18, 19].

Furthermore, we analyze spherical generalized helices in $\mathbb{R}^3_1$ and their planar projections in the direction of their axis. In Lorentz-Minkowski space, generalized helices are studied in e.g. [13], where their analogous characterization to the Euclidean counterparts is presented. Null helices are studied in [9]. Spherical generalized helices have many other properties, e.g. they appear as the tangent and binormal indicatrices of slant helices in Euclidean, [11], as well as in Minkowski space, [1]. A slant helix is a curve whose normal lines make a constant angle with a fixed direction. The tangent indicatrix of a curve of constant precession is also a spherical helix [15].

Projections of spherical generalized helices in 3-dimensional Lorentz-Minkowski space onto planes orthogonal to their axis were studied in the fourth section. It is shown that in the case when the axis is a timelike vector, the projection can be a Euclidean hypocycloid or Euclidean hyper- or paracycloid, depending on a causal character of a curve and the belonging sphere, while in the case when the axis is a spacelike vector, the projection can be Lorentzian hyper- or paracycloid or epi- or hypocycloid, depending on the same factors; see Theorems 5.1 and 5.2. Furthermore, the Lorentzian counterparts of cycloidal curves were analyzed in greater detail in the fourth section. The canonical examples of the planar projections of spherical generalized helices are given in Tables 5 and 6.

2. Preliminaries

Let $\mathbb{R}^3_1$ be Lorentz-Minkowski space, that is, the vector space $\mathbb{R}^3$ equipped with the indefinite symmetric bilinear form (a pseudoinner product)

$$x \cdot y = x_1 y_1 + x_2 y_2 - x_3 y_3.$$

A vector x in the Lorentz-Minkowski 3-space is called spacelike if $x \cdot x > 0$ or $x = 0$, timelike if $x \cdot x < 0$ and lightlike if $x \cdot x = 0$ and $x \neq 0$. The pseudonorm of a vector x is defined as the real number

$$\|x\| = \sqrt{|x \cdot x|} \geq 0. \tag{2.1}$$

In the following text we will abbreviate by $x^2 = x \cdot x$.

The Lorentzian cross-product of vectors x, y is defined by

$$x \times y = J(x \times_e y), \tag{2.2}$$

where on the right-hand side, $\times_e$ denotes the Euclidean cross product and

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

In $\mathbb{R}^3_1$ the following quadrics are introduced

$$\begin{aligned} \mathbb{S}^2_1(p,r) &= \{q \in \mathbb{R}^3_1 : (q-p)\cdot(q-p) = r^2\}, \\ \mathbb{H}^2(p,r) &= \{q \in \mathbb{R}^3_1 : (q-p)\cdot(q-p) = -r^2\}. \end{aligned}$$

We will consider these sets as counterparts of spheres in the Lorentz-Minkowski space $\mathbb{R}^3_1$. The set $\mathbb{S}^2_1(p,r)$ is called the Lorentzian sphere or a pseudosphere with center p and radius $r > 0$, the set $\mathbb{H}^2(p,r)$ a hyperbolic plane with center p and radius $r > 0$. For the unit spheres we put $\mathbb{S}^2_1 = \mathbb{S}^2_1(0,1)$, $\mathbb{H}^2 = \mathbb{H}^2(0,1)$. The set $\mathbb{S}^2_1(p,r)$ inherits a pseudo-Riemannian metric of index 1 from the ambient $\mathbb{R}^3_1$ (a pseudo-Euclidean space), $\mathbb{H}^2(p,r)$ a Riemannian metric (and it appears as the model for the hyperbolic plane, therefore the name).
Angle between two non-null vectors is introduced as follows in the Table 1.

Table 1. Angle between vectors in $\mathbb{R}^3_1$

x and y are timelike	x and y are different causal character	x and y are spacelike and span a spacelike (timelike) plane
$\cosh\varphi = -\dfrac{x \cdot y}{\|x\|\ \|y\|}$	$\sinh\varphi = \dfrac{\lvert x \cdot y\rvert}{\|x\|\ \|y\|}$	$\cos\varphi = \dfrac{x \cdot y}{\|x\|\ \|y\|}$ $\left(\cosh\varphi = \dfrac{\lvert x \cdot y\rvert}{\|x\|\ \|y\|}\right)$

The local theory of curves in Lorentz-Minkowski space can be found in e.g. [13]. The curvature κ and torsion τ of a curve are given by formulas

$$\kappa = \frac{\|\dot{c} \times \ddot{c}\|}{\|\dot{c}\|^3}, \quad \tau = \frac{\det(\dot{c}, \ddot{c}, \dddot{c})}{\|\dot{c} \times \ddot{c}\|^2},$$

where the norm is given by (2.1) and the cross-product by (2.2).
For a unit speed curve c whose vector fields T, N, B are all non-null, Frenet formulas are of the form, [13]

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\epsilon_0\epsilon_1\kappa & 0 & \tau \\ 0 & -\epsilon_1\epsilon_2\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where $\epsilon_0 = T^2 = \pm 1, \epsilon_1 = N^2 = \pm 1, \epsilon_2 = B^2 = \pm 1$ and $\epsilon_0\epsilon_1\epsilon_2 = -1$. For a spacelike curve c whose vector fields N and B are null (then c is the so-called pseudonull curve), Frenet formulas are of the form, [13]

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & \tau & 0 \\ -\kappa & 0 & -\tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where $\kappa = 0$ when c is a line or $\kappa = 1$, otherwise. The principal normal vector field is the null $N = c''$, whereas the binormal vector field B is defined as the null vector field orthogonal to T, satisfying $N \cdot B = 1$. Pseudotorsion τ is given by $\tau = N' \cdot B$.

3. Spherical curves

A pseudosphere $\mathbb{S}^2_1(p, r)$ is a (Euclidean) rotational hyperboloid of one sheet with the axis x_3. A curve $c(t)$ lying on it satisfies $(c(t) - p)^2 = r^2$. Therefore, a vector $c(t) - p$ is always spacelike and orthogonal vectors can either be timelike, spacelike or lightlike. Such curves can be characterized in terms of their torsion τ and radius of a curvature $\rho := 1/\kappa$.
In [16, 17, 18] it is shown:

Theorem 3.1. *A curve c lies on a pseudosphere $\mathbb{S}^2_1(p, r)$ with $r > 0$ and centered at p when*

1. *when c is a timelike curve if and only if*

$$\rho^2\tau^2 + (\rho')^2 = r^2\tau^2. \tag{3.3}$$

In particular, when $\tau \neq 0$, (3.3) is written as $\rho\tau + \left(\frac{\rho'}{\tau}\right)' = 0$;

2. *when c is a spacelike curve*
   (a) *with T' spacelike ($\epsilon = 1$) or timelike ($\epsilon = -1$), if and only if*

$$\rho^2\tau^2 - (\rho')^2 = \epsilon r^2\tau^2. \tag{3.4}$$

In particular, when $\tau \neq 0$, *(3.4) is written as* $\rho\tau - \left(\frac{\rho'}{\tau}\right)' = 0$;

(b) *with* T' *lightlike if and only if* c *is a circle in a lightlike plane (a Euclidean parabola with the axis parallel to the lightlike direction);*

3. *when* c *is a lightlike curve if and only if* c *is a generator line of a ruled surface* $\mathbb{S}^2_1(p, r)$.

Note that at every point of $\mathbb{S}^2_1(p, r)$, there exists two null lines contained in $\mathbb{S}^2_1(p, r)$, [9], which corrects the claim stated in [18] that there are no null (lightlike) curves lying on the Lorentzian sphere.

Remark 3.2. A curve in the statement *2(b)* of Theorem 3.1 is a pseudonull curve. It is known that such curves are planar, contained within a lightlike plane, [3, 6]. Therefore, if a pseudo-null curve c also lies on $\mathbb{S}^2_1(p, r)$, it is a Euclidean parabola with the lightlike axis (see Fig. 1, left).

A hyperbolic plane $\mathbb{H}^2(p, r)$ is a (Euclidean) rotational hyperboloid of two sheets with the axis x_3. A curve $c(t)$ lying on it satisfies $(c(t) - p)^2 = -r^2$. Therefore, a vector $c(t) - p$ is always timelike and the orthogonal vectors can only be spacelike. Analogously as in the previous situation it can be shown (see [19]):

Theorem 3.3. *A spacelike curve* c *lies on* $\mathbb{H}^2(p, r)$ *with* $r > 0$ *and centered at* p *when:*

1. T' *spacelike* $(\epsilon = 1)$ *or timelike* $(\epsilon = -1)$*, if and only if*

$$\rho^2\tau^2 - (\rho')^2 = -\epsilon r^2\tau^2. \tag{3.5}$$

In particular, when $\tau \neq 0$, *(3.5) is written as* $\rho\tau - \left(\frac{\rho'}{\tau}\right)' = 0$;

2. T' *lightlike if and only if* c *is a circle in a lightlike plane a Euclidean parabola with the axis parallel to the lightlike direction (see Fig. 1, right).*

4. Spherical generalized helices

A generalized helix is a regular unit speed curve defined by the property that its tangent vectors have constant pseudoinner product with constant vector $u \in \mathbb{R}^3_1$, $u \neq 0$. A line with the direction u is called the axis of a helix. We denote $\alpha := T \cdot u = const$. For a spacelike or a timelike generalized helix the following holds

$$\frac{\tau}{\kappa} = const =: A, \tag{4.6}$$

and conversely, for a curve c with non-lightlike normal vectors (Frenet curves, [13]). Furthermore, the relation between constants A and α is presented in

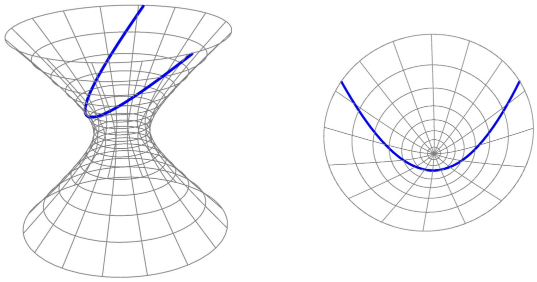

FIGURE 1. A circle in a lightlike plane (a Euclidean parabola with the axis parallel to the lightlike direction) on a pseudo-sphere $\mathbb{S}^2_1(p,r)$ and in a hyperbolic plane $\mathbb{H}^2(p,r)$

the Table 2. The proof is analogous to the Euclidean case, where $A = \cot\varphi$ and $\alpha = \cos\varphi$.

TABLE 2. Constant $A = \tau/\kappa$ for a generalized helix

T and u are timelike	T and u are different causal character	T and u are spacelike and span a spacelike (timelike) plane
$A = \coth\varphi$ where $\alpha = -\cosh\varphi$	$A = \tanh\varphi$ where $\alpha = \lvert\sinh\varphi\rvert$	$A = \cot\varphi$ ($A = \coth\varphi$) where $\alpha = \cos\varphi$ ($\alpha = \lvert\cosh\varphi\rvert$)

Let u be a unit spacelike or timelike vector, $\delta = u \cdot u = \pm 1$. Let $\tilde{c}$ be the projection of c on a plane orthogonal to u, that is

$$\tilde{c} = c - \delta(c \cdot u)u. \tag{4.7}$$

THEOREM 4.1. *Let c be a unit speed generalized helix with respect to a unit spacelike or timelike vector u. Then $\tilde{c}$ is:*

1. *spacelike if T and u are both timelike, T is spacelike and u timelike, or T and u are both spacelike and span a spacelike plane,*
2. *timelike if T is timelike and u spacelike, or T and u are both spacelike and span a timelike plane.*

The curve $\tilde{c}$ has a constant speed, and the curvature of c and of $\tilde{c}$ are related by

$$\tilde{\kappa}^2(T^2 - \delta\alpha^2)^2 = \kappa^2. \tag{4.8}$$

PROOF. Statements concerning a causal character of the curve $\tilde{c}$ follow from the analysis of the speed of a curve, $(\tilde{c}')^2 = T^2 - \delta\alpha^2$, and the fact that the constant α can be related to the angle between the Frenet curve c and u, see Table 1. Expressions for $(\tilde{c}')^2$ are given in the Table 3.

We prove the relation (4.8). Let s be the arc length parameter of c and $\tilde{s}$ of $\tilde{c}$. Differentiating (4.7) with respect to s yields

$$\tilde{T}\frac{d\tilde{s}}{ds} = T - \delta(T\cdot u)u \tag{4.9}$$

what gives

$$\left\|\frac{d\tilde{c}}{ds}\right\| = \left\|\tilde{T}\frac{d\tilde{s}}{ds}\right\| = \sqrt{|T^2 - \delta\alpha^2|} = const. \tag{4.10}$$

which implies that $\tilde{c}$ is of constant speed. Now, from the definition of a generalized helix we have $N\cdot u = 0$ and differentiating (4.9) again, we get $\kappa(s)N(s) = \tilde{\kappa}(\tilde{s})\tilde{N}(\tilde{s})(\frac{d\tilde{s}}{ds})^2$, where $\tilde{\kappa}$ is the curvature of $\tilde{c}$. Therefore, N, $\tilde{N}$ are of the same causal character, and formula (4.8) follows (see [14]).

□

Table 3. Causal character of $\tilde{c}$ depending on causal character of T and u

T and u are timelike	$(\tilde{c}')^2 = \sinh^2\varphi$	$\tilde{c}$ is spacelike
T is spacelike and u is timelike	$(\tilde{c}')^2 = \cosh^2\varphi$	
T and u are spacelike and span a spacelike plane	$(\tilde{c}')^2 = \sin^2\varphi$	
T is timelike and u is spacelike	$(\tilde{c}')^2 = -\cosh^2\varphi$	$\tilde{c}$ is timelike
T and u are spacelike and span a timelike plane	$(\tilde{c}')^2 = -\sinh^2\varphi$	

Remark 4.2. From the previous theorems we excluded the cases when either a curve c or a vector u is lightlike. The lightlike (null) generalized helix (parametrized by the pseudoarc length) is characterized by a constant pseudotorsion (lightlike curvature), [9, 13], and it is congruent to one of the following, $c(s) = \left(\frac{1}{a^2}\cos(as), \frac{1}{a^2}\sin(as), \frac{s}{a}\right)$, $c(s) = \left(-\frac{s}{a}, \frac{1}{a^2}\cosh(as), \frac{1}{a^2}\sinh(as)\right)$ or to so called null cubics $c(s) = \left(\frac{s^3}{4} - \frac{s}{3}, \frac{s^2}{2}, \frac{s^3}{4} + \frac{s}{3}\right)$. Their axes are timelike, spacelike and lightlike, respectively. None of them are spherical in Lorentz-Minkowski space.

Next, in the case when a projection plane is lightlike, i.e. when a vector u, the orthogonal projection of c is given by $\tilde{c} = c + (c\cdot u)v$, where v is a lightlike direction such that $u\cdot v = -1$, [4]. Differentiation yields $\tilde{T}\frac{d\tilde{s}}{ds} = T + \alpha v$,

$\tilde{T}'\left(\frac{d\tilde{s}}{ds}\right)^2 = T'$, hence $\tilde{T}', T'$ are of the same causal character. Furthermore, since $\tilde{T}\cdot u = 0$, $\tilde{T}'\cdot u = 0$, vectors $\tilde{T}$ and $\tilde{T}'$ are either lightlike or spacelike. If both $\tilde{T}'$ and T' are lightlike then c is a spacelike generalized helix with lightlike normal. It is spherical if and only if it is a Euclidean parabola in a lightlike plane. If both $\tilde{T}'$ and T' are spacelike, then $\tilde{T}$ and T are lightlike, i.e. c is a lightlike (null) generalized helix, which is never spherical, as discussed before.

4.1. *Generalized helices on a pseudosphere* $\mathbb{S}^2_1(p,r)$.

THEOREM 4.3. *Let c be a unit-speed generalized helix with respect to a unit vector u that lies on a pseudo-sphere $\mathbb{S}^2_1(p,r)$ and let $\tilde{c}$ be a projection of c on the plane orthogonal to u.*

1. *If c is a timelike curve, then the curvature and the torsion of c are given by*
$$\kappa^2(s) = \frac{1}{r^2 - A^2 s^2}, \quad \tau^2(s) = \frac{A^2}{r^2 - A^2 s^2}.$$
The radius of curvature $\tilde{\rho}(\tilde{s}) = 1/\tilde{\kappa}(\tilde{s})$ of $\tilde{c}$ satisfies
$$\frac{\tilde{s}^2}{a^2} + \frac{\tilde{\rho}^2}{b^2} = 1, \tag{4.11}$$
where $a^2 = \frac{r^2|1+\delta\alpha^2|}{A^2}$, $b^2 = r^2(1+\delta\alpha^2)^2$;
2. *if c is a spacelike curve, then its curvature and the torsion are given by*
$$\kappa^2(s) = \frac{1}{\epsilon r^2 + A^2 s^2}, \quad \tau^2(s) = \frac{A^2}{\epsilon r^2 + A^2 s^2},$$
where $\epsilon = N^2 = \pm 1$.
The radius of curvature $\tilde{\rho}(\tilde{s}) = 1/\tilde{\kappa}(\tilde{s})$ of $\tilde{c}$ satisfies
$$-\frac{\tilde{s}^2}{a^2} + \frac{\tilde{\rho}^2}{b^2} = \epsilon, \tag{4.12}$$
where $a^2 = \frac{r^2|1-\delta\alpha^2|}{A^2}$, $b^2 = r^2(1-\delta\alpha^2)^2$.

PROOF. Case 1. If c is a timelike spherical curve, then the curvature and torsion of c satisfy (3.3). From (4.6) we have $\tau = A\kappa$, $A = const.$, and therefore $\rho'^2 = A^2(r^2\kappa^2 - 1)$. Now,
$$\frac{\kappa'^2}{\kappa^4(r^2\kappa^2-1)} = A^2 \quad \text{or} \quad \frac{\kappa'}{\kappa^2\sqrt{r^2\kappa^2-1}} = \pm A.$$
Integration gives $\frac{1}{\kappa}\sqrt{r^2\kappa^2-1} = \pm As$ and therefore $\kappa^2 = \frac{1}{r^2 - A^2s^2}$, where r is a radius of a pseudosphere and s the arc-length parameter of c. The torsion of c is obtained from (4.6).

Since c is timelike, $T^2 = -1$, then (4.8) and (4.10) written as $\tilde{s}^2 = |1+\delta\alpha^2|s^2$ (up to a constant) imply

$$\tilde{\kappa}^2(\tilde{s}) = \frac{1}{(1+\delta\alpha^2)^2r^2 - |1+\delta\alpha^2|A^2\tilde{s}^2} \tag{4.13}$$

which implies the formula for the radius of curvature $\tilde{\rho}^2 = (1+\delta\alpha^2)^2r^2 - |1+\delta\alpha^2|A^2\tilde{s}^2$, and gives the formula (4.11).

Case 2. Analogously, if c is a spacelike spherical curve, then the curvature and torsion of c satisfy (3.4). Then (4.6) implies $\rho'^2 = A^2(1-\epsilon r^2\kappa^2)$, that is,

$$\frac{\kappa'^2}{\kappa^4(1-\epsilon r^2\kappa^2)} = A^2 \quad \text{or} \quad \frac{\kappa'}{\kappa^2\sqrt{1-\epsilon r^2\kappa^2}} = \pm A.$$

Integration gives $\frac{1}{\kappa}\sqrt{1-\epsilon r^2\kappa^2} = \pm As$ and therefore $\kappa^2 = \frac{1}{\epsilon r^2 + A^2s^2}$, where r is a radius of a pseudosphere, s the arc-length parameter of c and the constant A is given by (4.6). The curvature $\tilde{\kappa}$ of the projection is obtained from (4.8) with $T^2 = 1$ and the relation (4.10) between arc-length parameters (up to a constant) $\tilde{s}^2 = |1-\delta\alpha^2|s^2$. We have

$$\tilde{\kappa}^2(\tilde{s}) = \frac{1}{\epsilon r^2(1-\delta\alpha^2)^2 + |1-\delta\alpha^2|A^2\tilde{s}^2} \tag{4.14}$$

which implies the formula for the radius of curvature $\tilde{\rho}^2 = \epsilon r^2(1-\delta\alpha^2)^2 + |1-\delta\alpha^2|A^2\tilde{s}^2$, and gives the formula (4.12). ∎

4.2. *Generalized helices in a hyperbolic plane $\mathbb{H}^2(p,r)$.*

THEOREM 4.4. *Let c be a unit-speed generalized helix with respect to a unit vector u and let it lie in a hyperbolic plane $\mathbb{H}^2(p,r)$. Then the curvature and the torsion of c are given by*

$$\kappa^2(s) = \frac{1}{-\epsilon r^2 + A^2s^2}, \quad \tau^2(s) = \frac{A^2}{-\epsilon r^2 + A^2s^2}.$$

The radius of curvature $\tilde{\rho}(\tilde{s}) = 1/\tilde{\kappa}(\tilde{s})$ of a projection $\tilde{c}$ of c on the plane orthogonal to u satisfies

$$\frac{\tilde{s}^2}{a^2} - \frac{\tilde{\rho}^2}{b^2} = \epsilon, \tag{4.15}$$

where $a^2 = \frac{r^2|1-\delta\alpha^2|}{A^2}$, $b^2 = r^2(1-\delta\alpha^2)^2$.

PROOF. If c is a unit-speed in $\mathbb{H}^2(p,r)$, then c is a spacelike curve whose curvature and torsion satisfy (3.5). Then (4.6) implies $\rho'^2 = A^2(1+\epsilon r^2\kappa^2)$, that is,

$$\frac{\kappa'^2}{\kappa^4(1+\epsilon r^2\kappa^2)} = A^2 \quad \text{or} \quad \frac{\kappa'}{\kappa^2\sqrt{1+\epsilon r^2\kappa^2}} = \pm A.$$

Integration gives $\dfrac{1}{\kappa}\sqrt{1+\epsilon r^2\kappa^2} = \pm As$ and therefore $\kappa^2 = \dfrac{1}{-\epsilon r^2 + A^2 s^2}$. The curvature $\tilde{\kappa}$ of the projection is obtained from (4.8) with $T^2 = 1$ and from the relation (4.10) between arc-length parameters (up to a constant) $\tilde{s}^2 = |1-\delta\alpha^2|s^2$ imply

$$\tilde{\kappa}^2(\tilde{s}) = \frac{1}{-\epsilon r^2(1-\delta\alpha^2)^2 + |1-\delta\alpha^2|A^2\tilde{s}^2} \tag{4.16}$$

which implies the formula for the radius of curvature $\tilde{\rho}^2 = -\epsilon r^2(1-\delta\alpha^2)^2 + |1-\delta\alpha^2|A^2\tilde{s}^2$, and gives the formula (4.15).

□

5. Cycloidal curves in Lorentz-Minkowski plane

In Euclidean plane, cycloidal curves are curves traced by a generating point P of a circle which rolls without slipping on another circle. When a circle rolls inside (outside) a fixed circle, the traced curve is a hypocycloid (an epicycloid). In Euclidean plane, epicycloids and hypocycloids are given by their common parametrization

$$c(t) = \left((R+r)\cos t - r\cos\left(t + \frac{R}{r}t\right), (R+r)\sin t - r\sin\left(t + \frac{R}{r}t\right)\right)$$

where R, r are the radii of a fixed and moving circles respectively (the radius r of a moving circle is considered to be negative for hypocycloids). The natural equation of an epicycloid (hypocycloid) in Euclidean plane is given by

$$\frac{s^2}{c^2} + \frac{\rho^2}{d^2} = 1, \tag{5.17}$$

where $d < c$ ($d > c$). When such a rolling occurs, vertices of a cycloid lie a concentric circle $\bar{k}$ to the fixed circle k. For epi- and hypocycloids, both these circles are real. By letting k (resp. $\bar{k}$) be imaginary, the so called hypercycloids (resp. paracycloids) are defined, [5]. Their parametrizations can be written respectively as

$$c(t) = \left(a\cosh\left(\frac{bt}{a}\right)\sin t + b\sinh\left(\frac{bt}{a}\right)\cos t, -a\cosh\left(\frac{bt}{a}\right)\cos t + b\sinh\left(\frac{bt}{a}\right)\sin t\right)$$

or $$c(t) = \left(b\cosh\left(\frac{bt}{a}\right)\cos t + a\sinh\left(\frac{bt}{a}\right)\sin t, b\cosh\left(\frac{bt}{a}\right)\sin t - a\sinh\left(\frac{bt}{a}\right)\cos t\right).$$

The natural equation of hypercycloids (paracycloids) is of the form

$$\frac{s^2}{c^2} - \frac{\rho^2}{d^2} = -1, \quad \left(\frac{s^2}{c^2} - \frac{\rho^2}{d^2} = 1\right), \tag{5.18}$$

where $c^2 = \dfrac{(a^2+b^2)^2}{a^2}$, $d^2 = \dfrac{(a^2+b^2)^2}{b^2}$.

Now we consider cycloidal curves in a Lorentz-Minkowski plane with a pseudoinner product

$$x \cdot y = x_1 y_1 - x_2 y_2.$$

Notice that circles in Lorentz-Minkowski plane are Euclidean equilateral hyperbolas. Furthermore, since considered circles are tangent to one another at their point of contact, they need to be of the same causal character.

Let, for instance, a fixed circle be a timelike circle of radius R parametrized by $c_1(t) = (R\cosh t, R\sinh t)$ and a moving circle be a timelike circle of radius r that rolls inside a fixed circle. Its parametrization is given by $c_2(u) = ((R-r)+r\cosh u, r\sinh u)$. The generated curve will be a Lorentzian hypocycloid generated by a point P with starting position $P = c_1(0) = c_2(0) = (R, 0)$.

In order that a rolling without slipping occurs, we assume that that the arc-lengths from the starting position of P to the point of contact $c_1(t) = c_2(u)$ are the same along both curves, that is

$$\int_0^t \|\dot{c}_1(t)\| \, dt = \int_0^u \|\dot{c}_2(u)\| \, du$$

implying $u = \dfrac{R}{r} t$.

Next we introduce the moving coordinate system with the origin in a tangency point of two circles, and x-axis (y-axis) normal (resp. tangent) to the fixed circle in that point. The moving system is obtained from the fixed by translation for the vector $v = (R\cosh t, R\sinh t)$ and hyperbolic rotation by an angle t. Then coordinates (x, y) of a point T in a fixed system and its coordinates (x', y') in a translated system are related by $x = x' + R\cosh t$, $y = y' + R\sinh t$ and with coordinates (x'', y'') in a rotated system by t related by $x' = x''\cosh t + y''\sinh t$, $y' = y''\cosh t + x''\sinh t$.

Now, a relative position of a generating point P on a moving circle can be determined with respect to the moving system – its coordinates in the moving system are given by $\left(r\cosh\left(\frac{R}{r}t\right) - r, -r\sinh\left(\frac{R}{r}t\right)\right)$. Therefore, a Lorentzian hypocycloid can be parametrized by

$$c(t) = \Big((R-r)\cosh t + r\cosh\Big(t - \frac{R}{r}t\Big), (R-r)\sinh t + r\sinh\Big(t - \frac{R}{r}t\Big)\Big).$$

At the same time, for a path traced out by point of the other branch of a moving circle the coordinates of a generated point are $(-r\cosh\left(\frac{R}{r}t\right) - r, r\sinh\left(\frac{R}{r}t\right))$ and parametrization is

$$c(t) = \Big((R-r)\cosh t - r\cosh\Big(t - \frac{R}{r}t\Big), (R-r)\sinh t - r\sinh\Big(t - \frac{R}{r}t\Big)\Big).$$

Analogously, if the rolling outside of a fixed circle occurs, the obtained Lorentzian epicycloid is parametrized by

$$c(t) = \left((R+r)\cosh t - r\cosh\left(t+\frac{R}{r}t\right), (R+r)\sinh t - r\sinh\left(t+\frac{R}{r}t\right)\right), \tag{5.19}$$

$$\text{or} \quad c(t) = \left((R+r)\cosh t + r\cosh\left(t+\frac{R}{r}t\right), (R+r)\sinh t + r\sinh\left(t+\frac{R}{r}t\right)\right). \tag{5.20}$$

Similarly as in Euclidean space, both Lorentzian hypocycloid and epicycloid can be regarded either as given by (5.19) or by (5.20) respectively, by letting r be negative. The Lorentzian cycloidal curves of the type (5.19) are spacelike curves with the natural equation

$$\frac{s^2}{c^2} - \frac{\rho^2}{d^2} = 1, \tag{5.21}$$

while the curves of the type (5.20) are timelike curve with the natural equation

$$\frac{s^2}{c^2} - \frac{\rho^2}{d^2} = -1. \tag{5.22}$$

Parametrization of cycloidal curves generated by a spacelike circle which rolls along a spacelike circle are obtained by interchanging the x and y coordinates in previous parametrizations. Their causal character changes whereas their natural equations remain the same.

The Lorentzian counterparts of hyper- and paracycloids are given respectively by

$$c(t) = \left(a\cos\left(\frac{bt}{a}\right)\cosh t + b\sin\left(\frac{bt}{a}\right)\sinh t, a\cos\left(\frac{bt}{a}\right)\sinh t + b\sin\left(\frac{bt}{a}\right)\cosh t\right),$$

$$c(t) = \left(b\cos\left(\frac{bt}{a}\right)\cosh t - a\sin\left(\frac{bt}{a}\right)\sinh t, b\cos\left(\frac{bt}{a}\right)\sinh t - a\sin\left(\frac{bt}{a}\right)\cosh t\right).$$

Their natural equation is

$$\frac{s^2}{c^2} + \frac{\rho^2}{d^2} = 1, \tag{5.23}$$

with $c^2 = \dfrac{(a^2+b^2)^2}{a^2}$, $d^2 = \dfrac{(a^2+b^2)^2}{b^2}$.

A projection of a spherical generalized helix in $\mathbb{R}^3_1$ should be considered in the ambient geometry of a projection plane. From Theorems 4.3 and 4.4 we can conclude that in spacelike planes ($u^2 = -1$), projections are curves that satisfy (5.17) or (5.18), whereas in timelike planes ($u^2 = 1$) projections satisfy (5.21), (5.22) or (5.23) (see Table 4), that is, the following theorems hold.

TABLE 4. Projection $\tilde{c}$ of a spherical generalized helix c

Curve c	Projection $\tilde{c}$	
	on spacelike plane	on timelike plane
timelike on $\mathbb{S}^2_1(p,r)$	Euclidean hypocycloid $\dfrac{\tilde{\rho}^2}{b^2}+\dfrac{\tilde{s}^2}{a^2}=1$	Lorentzian hyper- or paracycloid $\dfrac{\tilde{\rho}^2}{b^2}+\dfrac{\tilde{s}^2}{a^2}=1$
spacelike on $\mathbb{S}^2_1(p,r)$ or $\mathbb{H}^2(p,r)$	Euclidean hyper- or paracycloid $\dfrac{\tilde{\rho}^2}{b^2}-\dfrac{\tilde{s}^2}{a^2}=\pm 1$	Lorentzian epi- or hypocycloid $\dfrac{\tilde{\rho}^2}{b^2}-\dfrac{\tilde{s}^2}{a^2}=\pm 1$

THEOREM 5.1. *Let c be a spherical generalized helix in $\mathbb{R}^3_1$.*

1. *The projection of c on a plane orthogonal to a vector u is a Euclidean hypocycloid if and only if the vector u is a timelike vector and c is a timelike curve that lies on $\mathbb{S}^2_1(p,r)$.*
2. *The projection of c on a plane orthogonal to a vector u is a Euclidean hyper- or paracycloid if and only if the vector u is a timelike vector and c is a spacelike curve that lies on $\mathbb{S}^2_1(p,r)$ or on $\mathbb{H}^2(p,r)$.*

THEOREM 5.2. *Let c be a spherical generalized helix in $\mathbb{R}^3_1$.*

1. *The projection c on a plane orthogonal to a vector u is a Lorentzian hyper- or paracycloid if and only if the vector u is a spacelike vector and c is a timelike curve that lies on $\mathbb{S}^2_1(p,r)$.*
2. *The projection c on a plane orthogonal to a vector u is a Lorentzian epicycloid or hypocycloid if and only if the vector u is a spacelike vector and c is a spacelike curve that lies on $\mathbb{S}^2_1(p,r)$ or on $\mathbb{H}^2(p,r)$.*

Our next goal is to analyze plane projections of generalized helices lying on pseudosphere $\mathbb{S}^2_1(p,r)$ or in a hyperbolic plane $\mathbb{H}^2(p,r)$ in Lorentz-Minkowski 3-space. Let c be a timelike curve on $\mathbb{S}^2_1(p,r)$ and u a unit timelike vector ($\delta=-1$). By inputting $A=\coth\varphi$ and $\alpha=-\cosh\varphi$ from Table 2 in (4.13), we can express the curvature $\tilde{\kappa}$ of $\tilde{c}$ in arc-length parametrization as

$$\tilde{\kappa}^2(\tilde{s})=\frac{1}{r^2\sinh^4\varphi-\tilde{s}^2\cosh^2\varphi}. \tag{5.24}$$

Then the natural equation of a projection has the form (4.11) with

$$a^2=\frac{r^2\sinh^4\varphi}{\cosh^2\varphi},\quad b^2=r^2\sinh^4\varphi.$$

Notice that $b \geq a > 0$. For $b \neq a$ equation (5.24) can be written as

$$\tilde{\kappa}(\tilde{s}) = \frac{a}{b\sqrt{a^2 - \tilde{s}^2}}. \tag{5.25}$$

Now, we can reconstruct the projection curve $\tilde{c}$ from its natural equation (5.25). Since $u = (0, 0, 1)$ is timelike the projection plane is spacelike xy-plane. Let $t\colon I \to \mathbb{R}$ be a function

$$t(\tilde{s}) = \int_0^{\tilde{s}} \tilde{\kappa}(\tilde{s})\mathrm{d}\tilde{s} = \frac{a}{b}\arcsin\left(\frac{\tilde{s}}{a}\right)$$

and therefore $\tilde{s} = a\sin\left(\frac{bt}{a}\right)$, $d\tilde{s} = b\cos\left(\frac{bt}{a}\right)dt$. The spacelike curve $\tilde{c}$ can be constructed as

$$\tilde{c}(\tilde{s}) = \left(\int_0^{\tilde{s}} \cos t(\tilde{s})d\tilde{s}, \int_0^{\tilde{s}} \sin t(\tilde{s})d\tilde{s}\right).$$

Therefore
$$\begin{aligned} x(t) &= \frac{ab}{b^2 - a^2}\left(b\sin\left(\frac{bt}{a}\right)\cos t - a\cos\left(\frac{bt}{a}\right)\sin t\right), \\ y(t) &= \frac{ab}{b^2 - a^2}\left(a\cos t\cos\left(\frac{bt}{a}\right) + b\sin t\sin\left(\frac{bt}{a}\right)\right). \end{aligned}$$

Since a curve c lies on a pseudosphere $\mathbb{S}^2_1(0, r)$, its last coordinate z is given by

$$z(t) = \frac{ab}{\sqrt{b^2 - a^2}}\sin\left(\frac{bt}{a}\right), \quad b^2 - a^2 > 0.$$

Furthermore, since $\dot{z}(t) = \dfrac{b^2}{\sqrt{b^2 - a^2}}\cos\left(\dfrac{bt}{a}\right)$, $\dot{c}(t)^2 = \dfrac{a^2b^2}{a^2 - b^2}\cos^2\left(\dfrac{bt}{a}\right)$, we can determine the angle between c and $u = (0, 0, 1)$ as $\alpha = \dfrac{\dot{c}(t)\cdot u}{\|\dot{c}(t)\|\,\|u\|} = \dfrac{\dot{z}(t)}{\|\dot{c}(t)\|} = \dfrac{b}{a}$.

Parametrization for projection curves that are Euclidean hyper- or para-cycloid or Lorentzian cycloidal curves can be obtained analogously. From expressions in Table 2 and formula (4.14) and (4.16), we obtain curvature $\tilde{\kappa}$ of $\tilde{c}$ and we can reconstruct $\tilde{c}$. If $\tilde{c}$ lies in timelike xz-plane and $\tilde{c}$ is timelike (spacelike) of the form

$$\tilde{c}(\tilde{s}) = \Big(\int_0^{\tilde{s}} \sinh t(\tilde{s})d\tilde{s}, \int_0^{\tilde{s}} \cosh t(\tilde{s})d\tilde{s}\Big) \quad \left(\tilde{c}(\tilde{s}) = \Big(\int_0^{\tilde{s}} \cosh t(\tilde{s})d\tilde{s}, \int_0^{\tilde{s}} \sinh t(\tilde{s})d\tilde{s}\Big)\right).$$

We finish this work with the canonical examples of plane projections of spherical generalized helices in Lorentz-Minkowski space presented in Table 5, respectively Table 6 (see Appendix A). Their natural equations, as well as parametrizations, are also given. The case when $a = b$ can be treated analogously. It occurs if and only if $\cosh\varphi = 1$ and then the projection onto

xy-plane or xz-plane is a Euclidean cycloid, respectively Lorentzian cycloid (see Table 7 in Appendix A).

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Appendix A.

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TABLE 5. The projection $\tilde{c}$ of a spherical generalized helix c onto a Euclidean plane

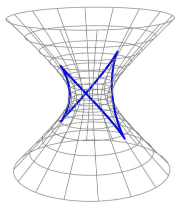
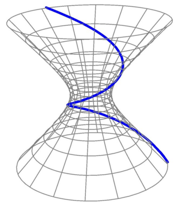
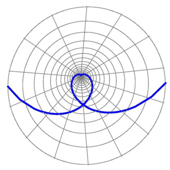
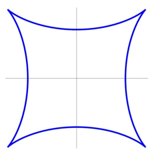
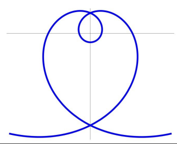
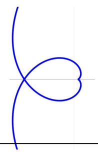
c is timelike	c is spacelike
	 
$\tilde{c}$ is a spacelike Euclidean hypocycloid 	$\tilde{c}$ is a spacelike Euclidean hyper- or paracycloid  
Natural equation of $\tilde{c}$	
$\dfrac{\tilde{s}^2}{a^2} + \dfrac{\tilde{\rho}^2}{b^2} = 1,$ $a^2 = \dfrac{r^2 \sinh^4 \varphi}{\cosh^2 \varphi},\ b^2 = r^2 \sinh^4 \varphi$	$\dfrac{\tilde{\rho}^2}{b^2} - \dfrac{\tilde{s}^2}{a^2} = \pm 1,$ $a^2 = \dfrac{r^2 \cosh^4 \varphi}{\sinh^2 \varphi},\ b^2 = r^2 \cosh^4 \varphi$
Parametrization of $\tilde{c}$	
$x(t) = \dfrac{ab}{b^2 - a^2}\big(b \cos t \sin(\dfrac{bt}{a}) - a \sin t \cos(\dfrac{bt}{a})\big)$ $y(t) = \dfrac{ab}{b^2 - a^2}\big(a \cos t \cos(\dfrac{bt}{a}) + b \sin t \sin(\dfrac{bt}{a})\big)$	Curve $\tilde{c}_1$ is the projection from $\mathbb{S}^2_1(p,r)$: $x(t) = \dfrac{ab}{a^2 + b^2}\big(a \sin t \cosh(\dfrac{bt}{a}) + b \cos t \sinh(\dfrac{bt}{a})\big)$ $y(t) = \dfrac{ab}{a^2 + b^2}\big(b \sin t \sinh(\dfrac{bt}{a}) - a \cos t \cosh(\dfrac{bt}{a})\big)$ Curve $\tilde{c}_2$ is the projection from $\mathbb{H}^2(p,r)$: $x(t) = \dfrac{ab}{a^2 + b^2}\big(b \cos t \cosh(\dfrac{bt}{a}) + a \sin t \sinh(\dfrac{bt}{a})\big)$ $y(t) = \dfrac{ab}{a^2 + b^2}\big(b \sin t \cosh(\dfrac{bt}{a}) - a \cos t \sinh(\dfrac{bt}{a})\big)$

Table 6. The projection $\tilde{c}$ of a spherical generalized helix c onto a Lorentz-Minkowski plane

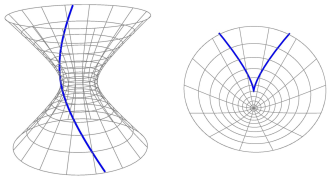
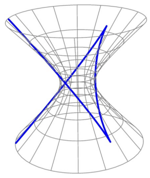
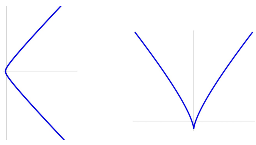
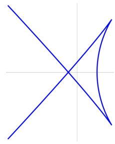
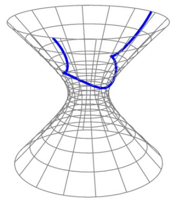
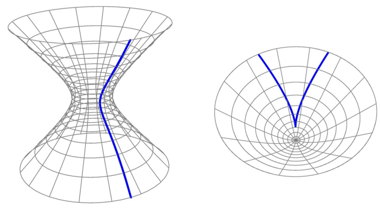
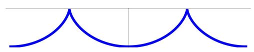
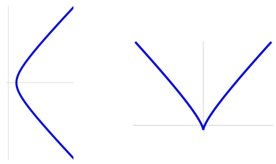
c is spacelike	c is timelike
	
$\tilde{c}$ is a Lorentzian epi- or hypocycloids	$\tilde{c}$ is a Lorentzian hypercycloid
	
Natural equation of $\tilde{c}$	
$\tilde{c}$ is timelike $\dfrac{\tilde{\rho}^2}{b^2} - \dfrac{\tilde{s}^2}{a^2} = \pm 1,\ a^2 = \dfrac{r^2 \sinh^4 \varphi}{\cosh^2 \varphi},\quad b^2 = r^2 \sinh^4 \varphi$ $\tilde{c}$ is spacelike $\dfrac{\tilde{\rho}^2}{b^2} - \dfrac{\tilde{s}^2}{a^2} = \pm 1,\ a^2 = \dfrac{r^2 \sin^4 \varphi}{\cos^2 \varphi},\quad b^2 = r^2 \sin^4 \varphi$	$\tilde{c}$ is timelike $\dfrac{\tilde{\rho}^2}{b^2} + \dfrac{\tilde{s}^2}{a^2} = 1,\ a^2 = \dfrac{r^2 \cosh^4 \varphi}{\sinh^2 \varphi},\ b^2 = r^2 \cosh^4 \varphi$
Parametrization of the projection curve $\tilde{c}$.	
<u>$\tilde{c}$ is the projection from $\mathbb{S}^2_1(p,r)$:</u> T and u span a timelike plane, timelike $\tilde{c}_1$ $x(t) = \dfrac{ab}{a^2-b^2}\big(a\cosh t\cosh(\dfrac{bt}{a}) - b\sinh t\sinh(\dfrac{bt}{a})\big)$ $z(t) = \dfrac{ab}{a^2-b^2}\big(a\sinh t\cosh(\dfrac{bt}{a}) - b\cosh t\sinh(\dfrac{bt}{a})\big)$ T and u span a spacelike plane, spacelike $\tilde{c}_2$ $x(t) = \dfrac{ab}{a^2-b^2}\big(a\sinh t\sinh(\dfrac{bt}{a}) - b\cosh t\cosh(\dfrac{bt}{a})\big)$ $z(t) = \dfrac{ab}{a^2-b^2}\big(a\cosh t\sinh(\dfrac{bt}{a}) - b\sinh t\cosh(\dfrac{bt}{a})\big)$ <u>$\tilde{c}$ is the projection from $\mathbb{H}^2(p,r)$:</u> When T and u span a timelike (spacelike) plane, the coordinates of $\tilde{c}$ are interchanged coordinates of $\tilde{c}_2$ ($\tilde{c}_1$).	$x(t) = \dfrac{ab}{a^2+b^2}\big(a\cosh t\cos(\dfrac{bt}{a}) + b\sinh t\sin(\dfrac{bt}{a})\big)$ $z(t) = \dfrac{ab}{a^2+b^2}\big(b\cosh t\sin(\dfrac{bt}{a}) + a\sinh t\cos(\dfrac{bt}{a})\big)$

TABLE 7. The projection $\tilde{c}$ of a spherical generalized helix c is a cycloid

<table>
<tr><td>c is timelike
</td><td>c is spacelike
</td></tr>
<tr><td>$\tilde{c}$ is a projection onto a Euclidean plane</td><td>$\tilde{c}$ is a projection onto a Lorentz-Minkowski plane</td></tr>
<tr><td>Euclidean cycloid is a spacelike curve
</td><td>Lorentzian cycloid is spacelike (timelike) when T and u span a spacelike (timelike) plane
</td></tr>
<tr><td colspan="2">Natural equation of the projection curve $\tilde{c}$.</td></tr>
<tr><td>$\frac{\tilde{s}^2}{a^2} + \frac{\tilde{\rho}^2}{a^2} = 1, \;\; a^2 = r^2 \sinh^4 \varphi$</td><td>$\frac{\tilde{s}^2}{a^2} - \frac{\tilde{\rho}^2}{a^2} = \pm 1, \;\; a^2 = r^2 \sinh^4 \varphi$</td></tr>
<tr><td colspan="2">Parametrization of the projection curve $\tilde{c}$.</td></tr>
<tr><td>$x(t) = \frac{1}{2} a (t + \frac{1}{2} \sin 2t)$
$y(t) = -\frac{1}{2} a \cos^2 t$</td><td><u>Curve is the projection from $\mathbb{S}^2_1(p, r)$:</u>
$(*) \;\; \begin{cases} x(t) = \frac{1}{2} a \cosh^2 t \\ z(t) = \frac{1}{2} a (\eta t + \frac{1}{2} \sinh 2t) \end{cases}$
$\eta = 1\,(-1)$ when T and u span a timelike (spacelike) plane.
If a cycloid $\tilde{c}$ is the projection of a curve c from $\mathbb{H}^2(p, r)$, its coordinates are interchanged coordinates of $(*)$.</td></tr>
</table>