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DATKO TYPE CHARACTERIZATIONS FOR UNIFORM DICHOTOMY IN MEAN WITH GROWTH RATES FOR REVERSIBLE STOCHASTIC SKEW-EVOLUTION SEMIFLOWS IN BANACH SPACES

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Abstract. The main aim of this paper is to give characterizations of Datko type for the uniform dichotomy in mean with growth rates concept for reversible stochastic skew-evolution semiflows in Banach spaces. As particular cases, we obtain integral characterizations for uniform exponential dichotomy in mean. The obtained results are generalizations of well-known theorems about uniform h-dichotomy of variational systems in deterministic case.

1. Introduction

Over the past few decades, uniform exponential behavior has emerged as one of the most prominent and widely debated subjects within the field of dynamical systems. O. Perron first introduced the concept (see [22]) during his exploration of the link between the conditional stability of the linear equation $\dot{x}(t) = A(t)x$ and the existence of bounded solutions to the nonlinear equation $\dot{x}(t) = A(t)x + f(t,x)$. The significance of exponential dichotomy for linear differential equations was solidified by two pivotal monographs: one by J. L. Massera and J. J. Schäffer in 1966 [15], and another by J. L. Daleckii and M. G. Krein in 1974 [8]. This topic has since been extensively studied, as evidenced by works such as [5], [9], [11], and [14].

Research into the asymptotic behavior of stochastic evolution equations within infinite-dimensional spaces has proven to be an area of considerable intensity and interest. Based on the stochastic equations studied in monographs

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by L. Arnold [1] and D. Prato and J. Zabczyk [23] were born important examples of stochastic evolution semiflows.

Several researchers have explored the concept of exponential dichotomy in a stochastic context, including A. M. Ateiwi [2] and T. Caraballo et al. [7].

The notion of skew-evolution semiflow became a front-line topic in the modern theory of dynamical system and differential equations. In the deterministic setting, this concept can be traced back to the works of M. Megan and C. Stoica in [17], where it extends and generalizes the notions of evolution operators, semigroups of operators, and skew-product semiflows (see [1, 11, 18, 16, 20, 19, 23]). The property of dichotomy for stochastic skew-evolution semiflows in Banach spaces is treated in [13, 24, 25, 26, 27, 29].

Over the years, an important extension of exponential and polynomial dichotomy was introduced by Pinto in his 1984 work [21], aimed at obtaining stability results for weakly stable systems under certain perturbations. This concept is known as dichotomy with growth rates, or h-dichotomy, where the growth rate refers to a bijective and non-decreasing function $h : \mathbb{R}_+ \to [1, \infty)$ with $\lim_{t\to\infty} h(t) = \infty$.

Datko's theorem served as the foundation for significant studies on the uniform exponential stability of evolution equations. Following Datko's seminal research [12], numerous papers have been dedicated to this subject (see [10], [28]). Extensions of Datko's results to polynomial behaviors are presented in [3, 4, 6].

The main aim of this paper is to adapt the proof methods from the deterministic case to the stochastic case. Specifically, we consider the case of reversible stochastic skew-evolution semiflows, using invariant projection families, and obtain two Datko-type characterizations for uniform dichotomy in mean with growth rates.

2. Definitions and notations

Let $(\Omega, \mathcal{B}, \mu)$ be a probability space. Let Δ be the set defined by $\Delta = \{(t, s) \in \mathbb{R}_+^2 : t \geq s \geq 0\}$ and let T be the set defined by $T = \{(t, s, t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}$. For a real or complex Banach space X we denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. We also denote by $L(\Omega, X, \mu)$ the Banach space of all Bochner-measurable functions $f : \Omega \to X$ such that $\int_\Omega \|f(\omega)\| d\mu(\omega) < \infty$.

Definition 2.1. A measurable random field $\varphi : \Delta \times \Omega \to \Omega$ is said to be a *stochastic evolution semiflow* on Ω if the following properties hold:

- (es_1) $\varphi(t, t, \omega) = \omega$, for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$,
- (es_2) $\varphi(t, s, \varphi(s, t_0, \omega)) = \varphi(t, t_0, \omega)$, for all $t \geq s \geq t_0 \geq 0$ and all $\omega \in \Omega$.

Definition 2.2. Let $\Phi : \Delta \times \Omega \to \mathcal{B}(X)$ be a measurable map. We say that Φ is a *stochastic evolution cocycle* associated to the stochastic evolution semiflow $\varphi : \Delta \times \Omega \to \Omega$ if the following conditions hold:

- (ec_1) $\Phi(t, t, \omega) = I$ (the identity operator on X), for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$,
- (ec_2) $\Phi(t, s, \varphi(s, t_0, \omega))\Phi(s, t_0, \omega) = \Phi(t, t_0, \omega)$, for all $t \geq s \geq t_0 \geq 0$ and all $\omega \in \Omega$.

If Φ represents a stochastic evolution cocycle over a stochastic evolution semiflow φ, then the pair $C = (\Phi, \varphi)$ is referred to as a *stochastic skew-evolution semiflow.*

Definition 2.3. The stochastic evolution cocycle $\Phi : \Delta \times \Omega \to \mathcal{B}(X)$ is said to be *reversible* if for all $(t, s, \omega) \in \Delta \times \Omega$, the map $\Phi(t, s, \omega)$ is bijective.

Definition 2.4. A map $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ with the property $P^2(s, \omega) = P(s, \omega)$ for all $(s, \omega) \in \mathbb{R}_+ \times \Omega$ is called *projections family* on X.

Remark 2.1. *If $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ is a projections family, then the map $Q : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ define as $Q(s, \omega) = I - P(s, \omega)$ also forms a projections family. This is referred to as the complementary projections family of P.*

Definition 2.5. A projections family $P : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ is said to be *invariant* to $C = (\Phi, \varphi)$ if

$$\Phi(t, s, \omega)P(s, \omega) = P(t, \varphi(t, s, \omega))\Phi(t, s, \omega),$$

for all $(t, s, \omega) \in \Delta \times \Omega$.

Remark 2.2. *If P remains invariant for $C = (\Phi, \varphi)$, we denote by*

$$\Phi_P(t, s, \omega) = \Phi(t, s, \omega)P(s, \omega)$$

for all $(t, s, \omega) \in \Delta \times \Omega$.

Proposition 2.1. *If the stochastic evolution cocycle $\Phi : \Delta \times \Omega \to \mathcal{B}(X)$ is reversible and the projection family P is invariant for $C = (\Phi, \varphi)$ then*

$$P(s, \omega)\Phi^{-1}(t, s, \omega) = \Phi^{-1}(t, s, \omega)P(t, \varphi(t, s, \omega)),$$

for all $(t, s, \omega) \in \Delta \times \Omega$.

Proof. It arises from Definition 2.3 and Remark 2.2. □

Proposition 2.2. *If $\Phi_P(t, s, \omega) : \Delta \times \Omega \to \mathcal{B}(X)$ and $\Phi_P^{-1}(t, s, \omega)$ is its inverse, then:*

(i) $\Phi(t, s, \omega)\Phi^{-1}(t, s, \omega)P(t, \varphi(t, s, \omega)) = P(t, \varphi(t, s, \omega))$, *for all $(t, s, \omega) \in \Delta \times \Omega$;*

(ii) $\Phi^{-1}(t, s, \omega)\Phi(t, s, \omega)P(s, \omega) = P(s, \omega)$, *for all $(t, s, \omega) \in \Delta \times \Omega$;*

(iii) $\Phi^{-1}(t, s, \omega)P(t, \varphi(t, s, \omega)) = P(s, \omega)\Phi^{-1}(t, s, \omega)P(t, \varphi(t, s, \omega))$, *for all $(t, s, \omega) \in \Delta \times \Omega$;*

Proof. It results from Definition 2.3 and Proposition 2.1. ∎

Definition 2.6. A nondecreasing map $h : \mathbb{R}_+ \to [1, \infty)$ with $\lim_{t\to\infty} h(t) = \infty$ is called a *growth rate.*

Definition 2.7. [27] The pair (C, P) is said to be *uniformly h-dichotomic in mean (u.h.d.m.)* if there are some constants $N > 1$ and $\nu > 0$ such that

$$(uhd_1m)\ \ h(t)^\nu \int_\Omega \|\Phi(t,t_0,\omega)P(t_0,\omega)x_0(\omega)\|d\mu(\omega) \leq N\cdot h(s)^\nu \int_\Omega \|\Phi(s,t_0,\omega)P(t_0,\omega)x_0(\omega)\|d\mu(\omega);$$

$$(uhd_2m)\ \ h(t)^\nu \int_\Omega \|\Phi(s,t_0,\omega))Q(t_0,\omega)x_0(\omega)\|d\mu(\omega) \leq N\cdot h(s)^\nu \int_\Omega \|\Phi(t,t_0,\omega))Q(t_0,\omega)x_0(\omega)\|d\mu(\omega),$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$;

When we examine the specific cases where $h(t) = e^t$ and $h(t) = t + 1$, we infer the concepts of *uniform exponential dichotomy in mean* and *uniform polynomial dichotomy in mean* respectively.

Remark 2.3. *The pair (C, P) is uniformly h-dichotomic in mean if and only if there exist $N > 1$ and $\nu > 0$ with*

$$(uhd_1'm)\ \ h(t)^\nu \int_\Omega \|\Phi(t,s,\omega)P(s,\omega)x(\omega)\|d\mu(\omega) \leq N\cdot h(s)^\nu \int_\Omega \|P(s,\omega)x(\omega)\|d\mu(\omega);$$

$$(uhd_2'm)\ \ h(t)^\nu \int_\Omega \|Q(s,\omega)x(\omega)\|d\mu(\omega) \leq N\cdot h(s)^\nu \int_\Omega \|\Phi(t,s,\omega)Q(s,\omega)x(\omega)\|d\mu(\omega),$$

for all $(t, s, \omega) \in \Delta \times \Omega$ and $x \in L(\Omega, X, \mu)$.

Theorem 2.1. *The pair (C, P) is uniformly h-dichotomic in mean with Φ reversible stochastic evolution cocycle if and only if there are $N > 1$ and $\nu > 0$ with:*

$$(uhd_1'''m)\ \ h(t)^\nu \int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq N\cdot h(s)^\nu \int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega);$$

$$(uhd_2'''m)\ \ h(t)^\nu \int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq N\cdot h(s)^\nu \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega),$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$;

Proof. It arises from Definition 2.7 and Proposition 2.1. ∎

Definition 2.8. [27] The pair (C, P) is said to be with *uniform h-growth in mean (u.h.g.m.)* if there exist constants $M \geq 1$ and $\alpha > 0$ such that:

$$(uhg_1m)\ \ h(s)^\alpha \int_\Omega \|\Phi(t,t_0,\omega)P(t_0,\omega)x_0(\omega)\|d\mu(\omega) \leq M\cdot h(t)^\alpha \int_\Omega \|\Phi(s,t_0,\omega)P(t_0,\omega)x_0(\omega)\|d\mu(\omega);$$

$$(uhg_2m)\ \ h(s)^\alpha \int_\Omega \|\Phi(s,t_0,\omega))Q(t_0,\omega)x_0(\omega)\|d\mu(\omega) \leq M\cdot h(t)^\alpha \int_\Omega \|\Phi(t,t_0,\omega))Q(t_0,\omega)x_0(\omega)\|d\mu(\omega),$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$;

As specific cases we note that when the growth rate is e^t, this establishes the concept of *uniform exponential growth in mean* and if the growth rate is $t+1$, then we arrive at the concept of *uniform polynomial growth in mean* respectively.

REMARK 2.4. *The pair (C, P) has uniform h-growth in mean if and only if there exist $M > 1$ and $\alpha > 0$ with*

$$(uhg_1^{'}m) \qquad h(s)^\alpha \int_\Omega \|\Phi(t,s,\omega)P(s,\omega)x(\omega)\| d\mu(\omega) \leq M \cdot h(t)^\alpha \int_\Omega \|P(s,\omega)x(\omega)\| d\mu(\omega);$$

$$(uhg_2^{'}m) \qquad h(s)^\alpha \int_\Omega \|Q(s,\omega)x(\omega)\| d\mu(\omega) \leq M \cdot h(t)^\alpha \int_\Omega \|\Phi(t,s,\omega)Q(s,\omega)x(\omega)\| d\mu(\omega),$$

for all $(t,s,\omega) \in \Delta \times \Omega$ and $x \in L(\Omega, X, \mu)$.

THEOREM 2.2. *The pair (C, P) is uniformly h-dichotomic in mean with Φ reversible stochastic evolution cocycle if and only if there exist $M > 1$ and $\alpha > 0$ with:*

$$(uhg_1^{'''}m) \qquad h(s)^\alpha \int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\| d\mu(\omega) \leq$$

$$\leq M \cdot h(t)^\alpha \int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\| d\mu(\omega);$$

$$(uhg_2^{'''}m) \qquad h(s)^\alpha \int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\| d\mu(\omega) \leq$$

$$\leq M \cdot h(t)^\alpha \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\| d\mu(\omega),$$

for all $(t,s,t_0,\omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$;

PROOF. The proof utilizes the exact same technique as demonstrated in Theorem 2.1. □

DEFINITION 2.9. Let $C = (\Phi, \varphi)$ be a stochastic skew-evolution semiflow. We say that C is *strongly measurable* if, for all $(t_0, x) \in \mathbb{R}_+ \times L(\Omega, X, \mu)$, the mapping

$s \mapsto \int_\Omega \|\Phi(s,t_0,\omega)x_0(\omega)\| d\mu(\omega)$, is measurable on $[t_0, \infty)$.

We denote by $\mathcal{H}$ the set of all growth rates $h : \mathbb{R}_+ \to [1, \infty)$ with the following properties:

- there exists $H > 1$ satisfying $h(t+1) \leq H h(t)$, $\forall\, t \geq 0$.
- for all $\beta < 0$ there exists $H_1 > 1$ with $\int_s^\infty h(t)^\beta dt \leq H_1\, h(s)^\beta$, $\forall\, s \geq 0$.
- for all $\beta > 0$ there exists $H_2 > 1$ with $\int_0^t h(s)^\beta ds \leq H_2\, h(t)^\beta$, $\forall\, t \geq 0$.

REMARK 2.5. *If $h(t) = e^t$, then $h \in \mathcal{H}$.*

3. Main results

Theorem 3.1. *We assume that $C=(\Phi,\varphi)$ is a strongly measurable stochastic skew-evolution semiflow, (C,P) with uniform h-growth in mean and $h\in\mathcal{H}$. The pair (C,P) is uniformly h-dichotomic in mean with Φ reversible stochastic evolution cocycle if and only if there exist constants $D\geq 1$ and $d\in(0,1)$ such that*

$$(uhD_1^1m)\ \int_s^\infty \frac{h(t)^d}{\int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)}dt \leq$$
$$\leq \frac{D\ h(s)^d}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)}$$

for all $(t,s,t_0,\omega)\in T\times\Omega$ and $x_0\in L(\Omega,X,\mu)$ with $P(s,\varphi(s,t_0,\omega))x_0(\omega)\neq 0$;

$$(uhD_2^1m)\ \int_s^\infty h(t)^d\left(\int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)\right)dt \leq$$
$$\leq D\ h(s)^d\int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega),$$

for all $(t,s,t_0,\omega)\in T\times\Omega$ and $x_0\in L(\Omega,X,\mu)$.

Proof. *Necessity.* To establish $(uhd_1^{'''}m)\implies(uhD_1^1m)$, we need to consider $d\in(0,\nu)$, resulting in:

$$\int_s^\infty \frac{h(t)^d}{\int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)}dt \leq$$
$$\leq N\int_s^\infty \left(\frac{h(t)}{h(s)}\right)^{-\nu}\frac{h(t)^d}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)}dt =$$
$$= \frac{N\ h(s)^\nu}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)}\int_s^\infty h(t)^{d-\nu}dt \leq$$
$$\leq \frac{N\ h(s)^\nu H_1 h(s)^{d-\nu}}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)\|d\mu(\omega)} =$$
$$= \frac{NH_1 h(s)^d}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)} \leq$$
$$\leq \frac{Dh(s)^d}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)\|d\mu(\omega)},$$

where $D=N\cdot H_1$

For $(uhd_2^{'''}m) \Longrightarrow (uhD_2^1 m)$, we have

$$\int_s^\infty \left(\int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)\right) dt \leq$$

$$\leq N \int_s^\infty \left(\frac{h(t)}{h(s)}\right)^{-\nu} h(t)^d \left(\int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)\right) dt \leq$$

$$\leq N\ h(s)^\nu \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega) \int_s^\infty h(t)^{d-\nu} dt \leq$$

$$\leq N\ h(s)^\nu \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega) h(s)^{d-\nu} H_1 \leq$$

$$\leq D h(s)^d \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega),$$

where $D = N \cdot H_1$

Sufficiency. For $(uhD_1^1 m) \Longrightarrow (uhd_1^{'''}m)$, firstly, we consider $(t,s,t_0,\omega) \in T \times \Omega$. There are two cases to be considered:
Case $I.1$. When $t \geq s+1$ we arrive at

$$\frac{h(t)^d}{\int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)} =$$

$$= \int_{t-1}^t \frac{h(t)^d}{\int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)} d\tau \leq$$

$$\leq M \int_{t-1}^t h(t)^d \left(\frac{h(t)}{h(\tau)}\right)^\alpha \frac{1}{\int_\Omega \|\Phi^{-1}(\tau,t_0,\omega)P(\tau,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)} d\tau =$$

$$= M \int_{t-1}^t \left(\frac{h(t)}{h(\tau)}\right)^{\alpha+d} \frac{h(\tau)^d}{\int_\Omega \|\Phi^{-1}(\tau,t_0,\omega)P(\tau,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)} d\tau \leq$$

$$\leq M \int_{t-1}^t \left(\frac{h(t)}{h(t-1)}\right)^{\alpha+d} \frac{h(\tau)^d}{\int_\Omega \|\Phi^{-1}(\tau,t_0,\omega)P(\tau,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)} d\tau \leq$$

$$\leq MH^{\alpha+d} \int_s^\infty \frac{h(\tau)^d}{\int_\Omega \|\Phi^{-1}(\tau,t_0,\omega)P(\tau,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)} d\tau \leq$$

$$\leq MH^{\alpha+d} D \frac{h(s)^d}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(t,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)}.$$

Case $I.2$. If $t \in [s, s+1)$ we obtain

$$h(t)^d \int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq Mh(t)^d \left(\frac{h(t)}{h(s)}\right)^\alpha \int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega) =$$
$$= M\left(\frac{h(t)}{h(s)}\right)^{\alpha+d} h(s)^d \int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq MH^{\alpha+d}h(s)^d \int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega).$$

It is a consequence of Case $I.1$. and Case $I.2$., that there exist $N = 1 + MH^{\alpha+d}D$ and $\nu = d$ such that $(uhd_1^{'''}m)$ holds for all $(t,s,t_0,\omega) \in T \times \Omega$ and all $x_0 \in L(\Omega, X, \mu)$.

To prove $(uhD_2^1m) \Longrightarrow (uhd_1^{'''}m)$ we take into account two cases as well: Case $II.1$. If $t \geq s+1$ we are provided with

$$h(t)^d \int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega) =$$
$$= h(t)^d \int_{t-1}^{t} \left(\int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)\right) d\tau \leq$$
$$\leq M \int_{t-1}^{t} h(t)^d \left(\frac{h(t)}{h(\tau)}\right)^\alpha \left(\int_\Omega \|\Phi^{-1}(\tau,t_0,\omega)Q(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\|d\mu(\omega)\right) d\tau =$$
$$= M \int_{t-1}^{t} h(\tau)^d \left(\frac{h(t)}{h(\tau)}\right)^{\alpha+d} \left(\int_\Omega \|\Phi^{-1}(\tau,t_0,\omega)Q(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\|d\mu(\omega)\right) d\tau \leq$$
$$\leq MH^{\alpha+d} \int_{s}^{\infty} h(\tau)^d \left(\int_\Omega \|\Phi^{-1}(\tau,t_0,\omega)Q(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\|d\mu(\omega)\right) d\tau \leq$$
$$\leq DMH^{\alpha+d}h(s)^d \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega).$$

Case $II.2$. If $t \in [s, s+1)$ we observe

$$h(t)^d \int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq Mh(t)^d \left(\frac{h(t)}{h(s)}\right)^\alpha \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega) =$$

$$= Mh(s)^d \left(\frac{h(t)}{h(s)}\right)^{\alpha+d} \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$

$$\leq Mh(s)^d \left(\frac{h(s+1)}{h(s)}\right)^{\alpha+d} \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$

$$\leq MH^{\alpha+d}h(s)^d \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(t,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega).$$

Combining Case II.1. with Case II.2., we can conclude that there exist $N = 1+MH^{\alpha+d}D$ and $\nu = d$ such that $(uhd_2^{'''}m)$ holds for all $(t,t_0,\omega) \in \Delta\times\Omega$ and all $x_0 \in L(\Omega,X,\mu)$. Hence, we have shown that (C,P) is u.h.d.m., completing the proof. □

Corollary 3.1. *We suppose that $C = (\Phi,\varphi)$ is a strongly measurable stochastic skew-evolution semiflow, (C,P) with uniform exponential growth in mean. The pair (C,P) is uniformly exponentially dichotomic in mean with Φ reversible stochastic evolution cocycle if and only if there exist constants $D \geq 1$ and $d \in (0,1)$ with*

$$(ueD_1^1m)\ \int_s^\infty \frac{e^{dt}}{\int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)}dt \leq$$

$$\leq \frac{D\ e^{ds}}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)};$$

for all $(t,s,t_0,\omega) \in T\times\Omega$ and $x_0 \in L(\Omega,X,\mu)$ with $P(s,\varphi(s,t_0,\omega))x_0(\omega) \neq 0$;

$$(ueD_2^1m)\ \int_s^\infty e^{dt}\left(\int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)\right)dt \leq$$

$$\leq D\ e^{ds}\int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega),$$

for all $(t,s,t_0,\omega) \in T\times\Omega$ and $x_0 \in L(\Omega,X,\mu)$.

Proof. It follows from Theorem 3.1 for $h(t) = e^t$. □

Theorem 3.2. *Consider $C = (\Phi,\varphi)$ as a strongly measurable stochastic skew-evolution semiflow, (C,P) has uniform h-growth in mean and $h \in \mathcal{H}$. The pair (C,P) is uniformly h-dichotomic in mean with Φ reversible stochastic evolution cocycle if and only if there exist constants $D \geq 1$ and $d \in (0,1)$ such that*

$$(uhD_1^2m)\ \int_{t_0}^t \frac{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)}{h(s)^d}ds \leq$$

$$\leq \frac{D\ \int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)}{h(t)^d};$$

for all $(t,s,t_0,\omega)\in T\times\Omega$ *and* $x_0\in L(\Omega,X,\mu)$*;*

(uhD_2^2m) $$\int\limits_{t_0}^{t}\frac{h(s)^{-d}}{\int_\Omega\|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)}ds\le$$
$$\le\frac{D\,h(t)^{-d}}{\int_\Omega\|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)},$$
for all $(t,s,t_0,\omega)\in T\times\Omega$ *and* $x_0\in L(\Omega,X,\mu)$ *with* $Q(t,\varphi(t,t_0,\omega))x_0(\omega)\neq 0$.

Proof. *Necessity.* For $(uhd_1^{'''}m)\Longrightarrow(uhD_1^2m)$, let $d\in(0,\nu)$ and we obtain:

$$\int\limits_{t_0}^{t}\frac{\int_\Omega\|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)}{h(s)^d}ds\le$$
$$\le\int\limits_{t_0}^{t}N\left(\frac{h(t)}{h(s)}\right)^{-\nu}\frac{\int_\Omega\|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)}{h(s)^d}ds\le$$
$$\le\frac{N\int_\Omega\|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)}{h(t)^\nu}\int\limits_{t_0}^{t}h(s)^{\nu-d}ds\le$$
$$\le\frac{NH_2\int_\Omega\|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)}{h(t)^\nu}h(t)^{\nu-d}\le$$
$$\le\frac{NH_2\int_\Omega\|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)}{h(t)^d}\le$$
$$\le\frac{D\int_\Omega\|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)}{h(t)^d},$$

where $D=N\cdot H_2$

Simillarly, for $(uhd_2^{'''}m)\Longrightarrow(uhD_2^2m)$, let $d\in(0,\nu)$ and we have:

$$\int\limits_{t_0}^{t}\frac{h(s)^{-d}}{\int_\Omega\|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)}ds\le$$
$$\le\int\limits_{t_0}^{t}N\left(\frac{h(t)}{h(s)}\right)^{-\nu}\frac{h(s)^{-d}}{\int_\Omega\|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)}ds=$$
$$=\frac{Nh(t)^{-\nu}}{\int_\Omega\|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)}\int\limits_{t_0}^{t}h(s)^{\nu-d}ds\le$$

$$\leq \frac{NH_2 h(t)^{-\nu}}{\int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)} h(t)^{\nu-d} \leq$$
$$\leq \frac{Dh(t)^{-d}}{\int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)},$$

where $D = NH_2$

Sufficiency. For $(uhD_2^1 m) \Longrightarrow (uhd_1^{'''} m)$, let $(t,s,t_0,\omega) \in T \times \Omega$. We can distinguish between two cases:

Case $I.1$. When $t \geq s+1$ we figure out

$$\frac{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)}{h(s)^d} =$$
$$= \int_s^{s+1} \frac{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)}{h(s)^d} d\tau \leq$$
$$\leq M \int_s^{s+1} \frac{\int_\Omega \|\Phi^{-1}(\tau,t_0,\omega)P(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\|d\mu(\omega)}{h(s)^d} \left(\frac{h(\tau)}{h(s)}\right)^\alpha d\tau =$$
$$= M \int_s^{s+1} \left(\frac{h(\tau)}{h(s)}\right)^{\alpha+d} \frac{\int_\Omega \|\Phi^{-1}(\tau,t_0,\omega)P(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\|d\mu(\omega)}{h(\tau)^d} d\tau \leq$$
$$\leq MH^{\alpha+d} \int_{t_0}^{t} \frac{\int_\Omega \|\Phi^{-1}(\tau,t_0,\omega)P(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\|d\mu(\omega)}{h(\tau)^d} d\tau \leq$$
$$\leq DMH^{\alpha+d} \frac{\int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)}{h(t)^d}.$$

Consequently, we obtain

$$h(t)^d \int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq DMH^{\alpha+d} h(s)^d \int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega).$$

Case $I.2$. If $t \in [s, s+1)$ we reach

$$h(t)^d \int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq Mh(s)^d \left(\frac{h(t)}{h(s)}\right)^{\alpha+d} \int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq Mh(s)^d \left(\frac{h(s+1)}{h(s)}\right)^{\alpha+d} \int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq MH^{\alpha+d} h(s)^d \int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega).$$

Accordingly, we derive

$$\frac{h(t)^d}{\int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)} \leq$$
$$\leq MH^\alpha \frac{h(t)^d}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)} =$$
$$= MH^\alpha \left(\frac{h(t)}{h(s)}\right)^d \frac{h(s)^d}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)} \leq$$
$$\leq MH^{\alpha+d} \frac{h(s)^d}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)}.$$

Hence, we arrive at

$$h(t)^d \int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq MH^{\alpha+d} h(s)^d \int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega).$$

From Case I.1. and Case I.2. the conclusion follows.
For the second relation, we initially consider $(t,s,t_0,\omega) \in T \times \Omega$. Moreover, we can distinguish between two cases:
Case $II.1$. When $t \geq s+1$ we deduce

$$\frac{h(s)^{-d}}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)} =$$
$$= \int_s^{s+1} \frac{h(s)^{-d}}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)} d\tau \leq$$
$$\leq M \int_s^{s+1} \left(\frac{h(\tau)}{h(s)}\right)^{\alpha+d} \frac{h(\tau)^{-d}}{\int_\Omega \|\Phi^{-1}(\tau,t_0,\omega)Q(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\|d\mu(\omega)} d\tau \leq$$
$$\leq M \int_s^{s+1} \left(\frac{h(s+1)}{h(s)}\right)^{\alpha+d} \frac{h(\tau)^{-d}}{\int_\Omega \|\Phi^{-1}(\tau,t_0,\omega)Q(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\|d\mu(\omega)} d\tau \leq$$
$$\leq MH^{\alpha+d} \int_s^{s+1} \frac{h(\tau)^{-d}}{\int_\Omega \|\Phi^{-1}(\tau,t_0,\omega)Q(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\|d\mu(\omega)} d\tau \leq$$
$$\leq DMH^{\alpha+d} \frac{h(t)^{-d}}{\int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)}.$$

Case $II.2$. If $t \in [s,s+1)$, we have

$$h(t)^d \int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$

$$\leq Mh(s)^d \left(\frac{h(t)}{h(s)}\right)^{\alpha+d} \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq Mh(s)^d \left(\frac{h(s+1)}{h(s)}\right)^{\alpha+d} \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq MH^{\alpha+d}h(s)^d \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega).$$

Accordingly, we derive

$$h(t)^d \int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq MH^\alpha \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq MH^\alpha \left(\frac{h(t)}{h(s)}\right)^d h(s)^d \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega) \leq$$
$$\leq MH^{\alpha+d}h(s)^d \int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega).$$

From Case II.1. and Case II.2. the conclusion follows. ∎

COROLLARY 3.2. *Let $C = (\Phi, \varphi)$ be a strongly measurable stochastic skew-evolution semiflow, (C, P) has uniform exponential growth in mean. The pair (C, P) is uniformly exponentially dichotomic in mean with Φ reversible stochastic evolution cocycle if and only if there exist some constants $D \geq 1$ and $d \in (0, 1)$ such that*

$$(ueD_1^2m) \int_{t_0}^{t} \frac{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)}{e^{ds}} ds \leq$$
$$\leq \frac{D \int_\Omega \|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)}{e^{dt}};$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$;

$$(ueD_2^2m) \int_{t_0}^{t} \frac{e^{-ds}}{\int_\Omega \|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\|d\mu(\omega)} ds \leq$$
$$\leq \frac{D\ e^{-dt}}{\int_\Omega \|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\|d\mu(\omega)},$$

for all $(t, s, t_0, \omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$ with $Q(t, \varphi(t, t_0, \omega))x_0(\omega) \neq 0$.

PROOF. It follows from Theorem 3.2 for $h(t) = e^t$. ∎

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