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UNCOUNTABLE FAMILIES OF FANS THAT ADMIT TRANSITIVE HOMEOMORPHISMS

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Abstract. Recently, we constructed transitive homeomorphisms on the Cantor fan and the Lelek fan. In this paper, we construct a family of uncountably many pairwise non-homeomorphic smooth fans that admit transitive homeomorphisms. In order to do this, we use our recently developed techniques of combining Mahavier products of closed relations on intervals with quotients of dynamical systems. In addition, we show that the star of Cantor fans admits a transitive homeomorphism. At the end of the paper, we also construct a family of uncountably many pairwise non-homeomorphic non-smooth fans that admit transitive homeomorphisms.

1. Introduction

Many examples of continua that admit transitive homeomorphisms may be found in the literature, see [1, 2, 3, 4, 8, 10, 11, 14, 15, 16, 19], where more references may be found. Most of the known examples of such continua have a complicated topological structure, i.e., they are indecomposable or they are decomposable but have some other complicated topological property. However, smooth fans form a family of continua that have been considered not to be very complicated. In our previous papers, it is shown that the Cantor fan and the Lelek fan admit transitive homeomorphisms, see [1, 2]. We began to wonder if the Cantor fan and the Lelek fan were special in this regard among smooth fans. Gradually, we discovered more smooth fans that admit transitive homeomorphisms. We present in this paper a family

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of uncountably many pairwise non-homeomorphic smooth fans that admit transitive homeomorphisms. In order to do this, we use our recently developed techniques from [1, 2] of combining Mahavier products of closed relations on intervals with quotients of dynamical systems: we define an equivalence relation $\sim$ on the Mahavier product X_F of a closed relation F on an interval X, equipped with the shift map σ_F, to obtain the quotient $(X_F/_\sim, \sigma_F^\star)$ of the dynamical system (X_F, σ_F). The described technique is applied to our setting in such a way that the transitivity of the dynamical system (X_F, σ_F) is automatically transferred to the dynamical system $(X_F/_\sim, \sigma_F^\star)$. Also, the resulting quotient space $X_F/_\sim$ is a member of our family of smooth fans. At the end, we use these results to show that there are also examples of non-smooth fans that admit transitive homeomorphisms. Moreover, we show that there is a family of uncountably many pairwise non-homeomorphic non-smooth fans that admit transitive homeomorphisms.

We proceed as follows. In Section 2, we introduce the definitions, notation and the well-known results that will be used later in the paper. In Section 3, we show that the star of Cantor fans is another example of a smooth fan that admits a transitive homeomorphism, and then, in Section 4, a family of uncountably many pairwise non-homeomorphic smooth fans that admit a transitive homeomorphism is constructed. In Section 5, a family of uncountably many pairwise non-homeomorphic non-smooth fans that admit a transitive homeomorphism is constructed.

2. Definitions and Notation

The following definitions, notation and well-known results are needed in the paper.

Definition 2.1. *We use $\mathbb{N}$ to denote the set of positive integers and $\mathbb{Z}$ to denote the set of integers.*

Definition 2.2. *Let X and Y be metric spaces, and let $f : X \to Y$ be a function. We use $\Gamma(f) = \{(x, y) \in X \times Y \mid y = f(x)\}$ to denote* the graph of the function f.

Definition 2.3. *Let X be a metric space, $x \in X$ and $\varepsilon > 0$. We use $B(x, \varepsilon)$ to denote the open ball, centered at x with radius ε.*

Definition 2.4. *Let (X, d) be a compact metric space. Then we define 2^X by*

$$2^X = \{A \subseteq X \mid A \text{ is a non-empty closed subset of } X\}.$$

Let $\varepsilon > 0$ and let $A \in 2^X$. Then we define $N_d(\varepsilon, A) = \bigcup_{a\in A} B(a, \varepsilon)$. Let $A, B \in 2^X$. The function $H_d : 2^X \times 2^X \to \mathbb{R}$, defined by

$$H_d(A, B) = \inf\{\varepsilon > 0 \mid A \subseteq N_d(\varepsilon, B), B \subseteq N_d(\varepsilon, A)\},$$

is called the Hausdorff metric. *The Hausdorff metric is in fact a metric and the metric space* $(2^X, H_d)$ *is called* the hyperspace of the space (X, d).

REMARK 2.5. Let (X, d) be a compact metric space, let A be a non-empty closed subset of X, and let (A_n) be a sequence of non-empty closed subsets of X. When we say $A = \lim_{n\to\infty} A_n$ with respect to the Hausdorff metric, we mean $A = \lim_{n\to\infty} A_n$ in $(2^X, H_d)$.

DEFINITION 2.6. A continuum *is a non-empty compact connected metric space.* A subcontinuum *is a subspace of a continuum, which is itself a continuum.*

DEFINITION 2.7. *Let X be a continuum.*

1. *The continuum X is* unicoherent, *if for any subcontinua A and B of X such that $X = A \cup B$, the compactum $A \cap B$ is connected.*
2. *The continuum X is* hereditarily unicoherent *provided that each of its subcontinua is unicoherent.*
3. *The continuum X is a* dendroid, *if it is an arcwise connected, hereditarily unicoherent continuum.*
4. *Let X be a continuum. If X is homeomorphic to* $[0, 1]$, *then X is* an arc.
5. *A point x in an arc X is called* an end-point of the arc X, *if there is a homeomorphism* $\varphi : [0, 1] \to X$ *such that* $\varphi(0) = x$.
6. *Let X be a dendroid. A point $x \in X$ is called an* end-point of the dendroid X, *if for every arc A in X that contains x, x is an end-point of A. The set of all end-points of X will be denoted by $E(X)$.*
7. *A continuum X is* a simple triod, *if it is homeomorphic to* $([-1, 1] \times 0) \cup (\{0\} \times [0, 1])$.
8. *A point x in a simple triod X is called* the top-point *or just the* top of the simple triod X, *if there is a homeomorphism* $\varphi : ([-1, 1] \times 0) \cup (\{0\} \times [0, 1]) \to X$ *such that* $\varphi(0, 0) = x$.
9. *Let X be a dendroid. A point $x \in X$ is called* a ramification-point of the dendroid X, *if there is a simple triod T in X with the top x. The set of all ramification-points of X will be denoted by $R(X)$.*
10. *The continuum X is* a fan, *if it is a dendroid with at most one ramification point v, which is called the top of the fan X (if it exists).*
11. *Let X be a fan. For all points x and y in X, we define $A[x, y]$ to be the arc in X with end-points x and y, if $x \neq y$. If $x = y$, then we define $A[x, y] = \{x\}$.*
12. *Let X be a fan with the top v. We say that that the fan X is* smooth *if for any $x \in X$ and for any sequence (x_n) of points in X,*
$$\lim_{n\to\infty} x_n = x \Longrightarrow \lim_{n\to\infty} A[v, x_n] = A[v, x]$$
with respect to the Hausdorff metric.

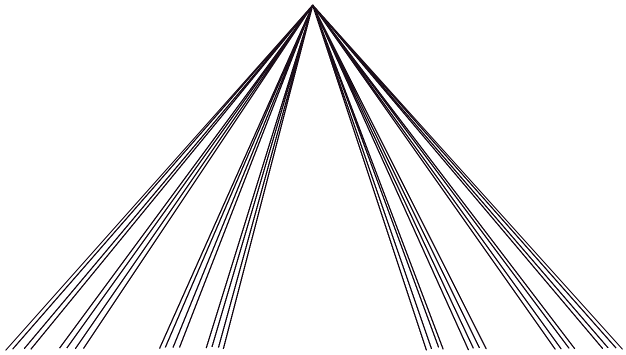

FIGURE 1. A Cantor fan

13. *Let X be a fan. We say that X is* a Cantor fan, *if X is homeomorphic to the continuum $\bigcup_{c\in C} S_c$, where $C \subseteq [0,1]$ is the standard Cantor set and for each $c \in C$, S_c is the straight line segment in the plane from $(0,0)$ to $(c,1)$. See Figure 1, where a Cantor fan is pictured.*
14. *Let X be a fan. We say that X is* a Lelek fan, *if it is smooth and $\mathrm{Cl}(E(X)) = X$. See Figure 2, where a Lelek fan is pictured.*

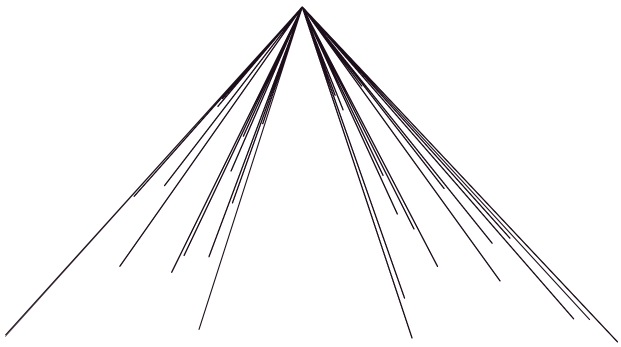

FIGURE 2. A Lelek fan

OBSERVATION 2.8. *It is a well-known fact that the Cantor fan is smooth and that any subcontinuum of a smooth fan is itself a smooth fan.*

An example of a Lelek fan was constructed by A. Lelek in [12]. He also showed that the set of the end-points of any Lelek fan is a dense one-dimensional set in it. Also, it is the only non-degenerate smooth fan with a dense set of end-points. This was proved independently by W. D. Bula and L. Oversteegen in [5] and by W. Charatonik in [6]. See [18] for more information about continua, fans and their properties.

DEFINITION 2.9. *Let X and Y be any continua and let $f : X \to Y$ be a continuous mapping. We say that f is* confluent*, if for every subcontinuum S of Y and for each component C of $f^{-1}(S)$, $f(C) = S$.*

The following is a well-known result.

THEOREM 2.10. *Let X and Y be any continua and let $f : X \to Y$ be a confluent surjection. If X is a smooth fan, then also Y is a smooth fan.*

PROOF. See [7, Theorem 13, page 33]. □

DEFINITION 2.11. *Let (X, f) be a dynamical system. We say that (X, f) is*

1. transitive*, if for all non-empty open sets U and V in X, there is a non-negative integer n such that $f^n(U) \cap V \neq \emptyset$.*
2. dense orbit transitive*, if there is a point $x \in X$ such that its trajectory $\{x, f(x), f^2(x), f^3(x), \ldots\}$ is dense in X. We call such a point x* a transitive point *in (X, f).*

We say that the mapping f is transitive*, if (X, f) is transitive.*

DEFINITION 2.12. *Let X be a compact metric space. We say that X* admits a transitive homeomorphism*, if there is a homeomorphism $f : X \to X$ such that (X, f) is transitive.*

DEFINITION 2.13. *For non-empty compact metric spaces X and Y, we use $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ to denote* the standard projections *defined by $p_1(s, t) = s$ and $p_2(s, t) = t$ for all $(s, t) \in X \times Y$.*

DEFINITION 2.14. *For an equivalence relation $\sim$ on a space X, we use*

1. *$[x]$ to denote the equivalence class $\{y \in X \mid y \sim x\}$ of an element $x \in X$ with respect to $\sim$,*
2. *$X/_\sim$ to denote the quotient space $\{[x] \mid x \in X\}$, which will always be equipped with the quotient topology.*

OBSERVATION 2.15. *Let X be a compact metric space, let $\sim$ be an equivalence relation on X, let $q : X \to X/_\sim$ be the quotient map that is defined by $q(x) = [x]$ for each $x \in X$, and let $U \subseteq X/_\sim$. Then*

$$U \text{ is open in } X/_\sim \iff q^{-1}(U) \text{ is open in } X.$$

DEFINITION 2.16. *Let X be a compact metric space, let $\sim$ be an equivalence relation on X, and let $f : X \to X$ be a function such that for all $x, y \in X$,*

$$x \sim y \iff f(x) \sim f(y).$$

Then we let $f^\star : X/_\sim \to X/_\sim$ be defined by $f^\star([x]) = [f(x)]$ for any $x \in X$.

The following proposition is a well-known result. To experts, it may be seen as an undergraduate topology textbook result. To our knowledge, the statement about transitivity is not explicitly given in earlier literature. However, the whole proof of the proposition can be found in [1, Theorem 3.4].

Proposition 2.17. *Let X be a compact metric space, let $\sim$ be an equivalence relation on X, and let $f : X \to X$ be a function such that for all $x, y \in X$,*

$$x \sim y \Longleftrightarrow f(x) \sim f(y).$$

Then the following hold.

1. *$f^\star$ is a well-defined function from $X/_\sim$ to $X/_\sim$.*
2. *If f is continuous, then $f^\star$ is continuous.*
3. *If f is a homeomorphism, then $f^\star$ is a homeomorphism.*
4. *If f is transitive, then $f^\star$ is transitive.*

3. A star of Cantor fans

In this section, we construct an example of a smooth fan, the star of Cantor fans, and show that it admits a transitive homeomorphism.

Definition 3.1. *Let F be a smooth fan and let (F_n) be a sequence of smooth fans in the plane such that*

1. *for each positive integer n, F_n is homeomorphic to F,*
2. *for each positive integer n, $\mathrm{diam}(F_n) \le \frac{1}{2^n}$,*
3. *for each positive integer n, $(0,0)$ is the top of F_n, and*
4. *for all positive integers m and n, $F_m \cap F_n = \{(0,0)\}$.*

Any space X that is homeomorphic to $\bigcup_{n=1}^{\infty} F_n$, is called a star of F's.

Observation 3.2. *Let F be a smooth fan and let X be a star of F's. Then X is also a smooth fan.*

Definition 3.3. *Let F be a smooth fan and let X be a star of F's. If F is a*

1. *Cantor fan, then X is called a star of Cantor fans, see Figure 3.*
2. *Lelek fan, then X is called a star of Lelek fans.*
3. *star of Cantor fans, then X is called a star of stars of Cantor fans.*

Observation 3.4. *Note that any star of Lelek fans is again a Lelek fan and that any star of stars of Cantor fans is again a star of Cantor fans. Also, note that any two stars of Cantor fans are homeomorphic.*

Definition 3.5. *We use*

1. *C to denote the standard middle-third Cantor set in $[0,1]$,*
2. *I to denote the closed interval $[0,1]$,*
3. *P to denote the topological product $P = C \times I$,*

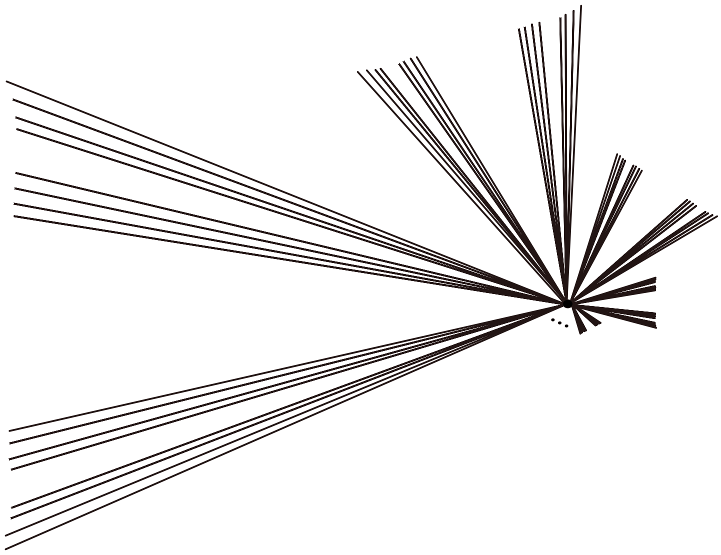

FIGURE 3. The star of Cantor fans

4. *R to denote the subspace $R = P \cap \{(s,t) \in I \times I \mid s \geq t\}$ of the space P, and*
5. *φ to denote the function $\varphi : P \to R$ that is defined by $\varphi(c,t) = (c, c \cdot t)$ for each $(c,t) \in P$.*

see Figure 4.

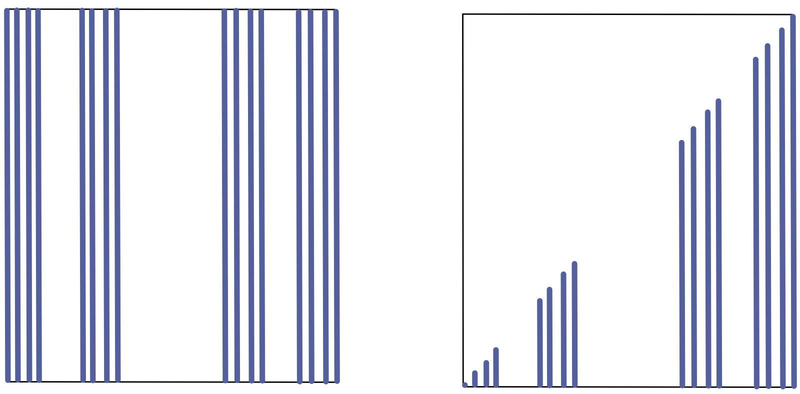

FIGURE 4. The spaces P and R

Observation 3.6. *Note that φ is a continuous surjection.*

Definition 3.7. *For any function $f : P \to P$, we denote by f_R the function $f_R : R \to R$, defined by*

$$f_R(c,t) = \begin{cases} (0,0); & c = 0 \\ \varphi(f(\varphi^{-1}(c,t))); & c \neq 0 \end{cases}$$

for any $(c,t) \in R$.

Proposition 3.8. *Let $f : P \to P$ be a function such that*

$$f(\{0\} \times I) \subseteq \{0\} \times I \quad \text{and} \quad f((C \setminus \{0\}) \times I) \subseteq (C \setminus \{0\}) \times I.$$

Then the following hold.

1. *If f is surjective, then f_R is surjective.*
2. *If f is injective, then f_R is injective.*
3. *If f is continuous, then f_R is continuous.*
4. *If f is a homeomorphism, then f_R is a homeomorphism.*
5. *If f is transitive, then f_R is transitive.*

Proof. First, suppose that f is surjective. To show that f_R is surjective, let $(c,t) \in R$. Also, let $(c_0, t_0) \in \varphi^{-1}(c,t)$. Since f is surjective, there is a point $(c_1, t_1) \in P$ such that $f(c_1, t_1) = (c_0, t_0)$. We treat the following possible cases.

1. $c_1 \neq 0$. It follows that

$$f_R(\varphi(c_1,t_1)) = \varphi(f(\varphi^{-1}(\varphi(c_1,t_1)))) = \varphi(f(c_1,t_1)) = \varphi(c_0,t_0) = (c,t).$$

2. $c_1 = 0$. Then $c_0 = 0$ and it follows that $(c,t) = (0,0)$. Therefore,

$$f_R(\varphi(c_1,t_1)) = f_R(\varphi(0,t_1)) = f_R(0,0) = (0,0) = (c,t).$$

It follows that f_R is surjective.

Next, suppose that f is injective. To see that f_R is injective, let $(c_1,t_1), (c_2,t_2) \in R$ be such points that $f_R(c_1,t_1) = f_R(c_2,t_2)$. To see that $(c_1,t_1) = (c_2,t_2)$, we consider the following possible cases.

1. $c_1 = 0$. It follows that $t_1 = 0$ and

$$f_R(c_1,t_1) = f_R(0,0) = (0,0).$$

Therefore, $f_R(c_2,t_2) = (0,0)$. Suppose that $c_2 \neq 0$. Then

$$f_R(c_2,t_2) = \varphi(f(\varphi^{-1}(c_2,t_2))) = \varphi\Big(f\Big(c_2, \frac{t_2}{c_2}\Big)\Big).$$

Since $c_2 \neq 0$ and $f((C \setminus \{0\}) \times I) \subseteq (C \setminus \{0\}) \times I$, it follows that $\varphi(f(c_2, \frac{t_2}{c_2})) \neq (0,0)$. Therefore, $f_R(c_2,t_2) \neq (0,0)$, which is a contradiction. Therefore, $c_2 = 0$ and it follows that also $t_2 = 0$. Hence, $(c_1,t_1) = (c_2,t_2)$.

2. $c_1 \neq 0$. If $c_2 = 0$, we obtain a contradiction similarly as in the previous case. Therefore, $c_2 \neq 0$. It follows from $f_R(c_1, t_1) = f_R(c_2, t_2)$ that $\varphi(f(\varphi^{-1}(c_1, t_1))) = \varphi(f(\varphi^{-1}(c_2, t_2)))$. Therefore, $\varphi(f(c_1, \frac{t_1}{c_1})) = \varphi(f(c_2, \frac{t_2}{c_2}))$. Since f is injective, since $f((C \setminus \{0\}) \times I) \subseteq (C \setminus \{0\}) \times I$, and since φ restricted to $(C \setminus \{0\}) \times I$ is injective, it follows that $(c_1, \frac{t_1}{c_1}) = (c_2, \frac{t_2}{c_2})$. Therefore, $(c_1, t_1) = (c_2, t_2)$.

Thus f_R is injective.

Next, suppose that f is continuous and let (c_n, t_n) be a sequence of points in R such that $\lim\limits_{n\to\infty} (c_n, t_n) = (0, 0)$ and such that for each positive integer n, $(c_n, t_n) \neq (0, 0)$. Then

$$\lim_{n\to\infty} f_R(c_n, t_n) = \lim_{n\to\infty} \varphi(f(\varphi^{-1}(c_n, t_n))) = \lim_{n\to\infty} \varphi\left(f\Big(c_n, \frac{t_n}{c_n}\Big)\right) =$$
$$\lim_{n\to\infty} \varphi\left(p_1\Big(f\Big(c_n, \frac{t_n}{c_n}\Big)\Big), p_2\Big(f\Big(c_n, \frac{t_n}{c_n}\Big)\Big)\right) =$$
$$\lim_{n\to\infty} \left(p_1\Big(f\Big(c_n, \frac{t_n}{c_n}\Big)\Big), p_1\Big(f\Big(c_n, \frac{t_n}{c_n}\Big)\Big) \cdot p_2\Big(f\Big(c_n, \frac{t_n}{c_n}\Big)\Big)\right) = (0, 0)$$

since $\lim\limits_{n\to\infty} p_1\Big(f\Big(c_n, \dfrac{t_n}{c_n}\Big)\Big) = 0$ and since the sequence $\Big(p_2\Big(f\Big(c_n, \frac{t_n}{c_n}\Big)\Big)\Big)$ is bounded (by 0 from below and by 1 from above). Note that φ is one to one everywhere except on $\{0\} \times [0, 1]$. Therefore, f_R is continuous also in (c, t) for each $(c, t) \in R \setminus \{(0, 0)\}$ (since for each such (c, t), $f_R(c, t) = \varphi(f(\varphi^{-1}(c, t)))$ and since the composition of continuous functions is continuous.)

It follows that f_R is continuous.

Next, suppose that f is a homeomorphism. It follows from the previous claims that also f_R is a homeomorphism.

Finally, suppose that f is transitive. Let U and V be non-empty open sets in R and let $U' = U \setminus \{(0, 0)\}$ and $V' = V \setminus \{(0, 0)\}$. Since R does not have any isolated points, it follows that U' and V' are also non-empty open sets in R. Since φ is continuous, it follows that $\varphi^{-1}(U')$ and $\varphi^{-1}(V')$ are open in P and it follows from the definition of φ that $\varphi^{-1}(U') \cap (\{0\} \times I) = \emptyset$ and $\varphi^{-1}(V') \cap (\{0\} \times I) = \emptyset$. Since f is transitive, there is a non-negative integer n such that $f^n(\varphi^{-1}(U')) \cap \varphi^{-1}(V') \neq \emptyset$. Let n be such a non-negative integer and let $(c, t) \in f^n(\varphi^{-1}(U')) \cap \varphi^{-1}(V')$. Since $(c, t) \in f^n(\varphi^{-1}(U')) \cap \varphi^{-1}(V')$, then $\varphi(c, t) \in \varphi(f^n(\varphi^{-1}(U'))) \cap V' = (\varphi \circ f \circ \varphi^{-1})^n(U') \cap V' = f_R^n(U') \cap V'$, so $f_R^n(U') \cap V' \neq \emptyset$. Since $f_R^n(U') \cap V' \subseteq f_R^n(U) \cap V$, it follows that $f_R^n(U) \cap V \neq \emptyset$. Therefore, f_R is transitive. ∎

Observation 3.9. *Note that there is a transitive homeomorphism $f : P \to P$ such that $f(\{0\} \times I) \subseteq \{0\} \times I$ and $f((C \setminus \{0\}) \times I) \subseteq (C \setminus \{0\}) \times I$. One such homeomorphism can be constructed using* [1, Theorem 3.32, page 17], *where a topological conjugacy of such a homeomorphism is obtained.* *In* [1, Theorem 3.32, page 17], *this homeomorphism is constructed by, first, defining homeomorphisms* $f_1, f_2, f_3 : [0,1] \to [0,1]$ *by*

$$f_1(t) = \sqrt{x},\ f_2(t) = \begin{cases} \frac{1}{2}t; & t \leq \frac{2}{3} \\ 2x-1; & t \geq \frac{2}{3} \end{cases} \text{ and } f_3(t) = f_1^{-1}(t)$$

for any $t \in [0,1]$, *and then, showing that the function* $(f_1, f_2, f_3)_{D_3} : D_3 \times [0,1] \to D_3 \times [0,1]$, *which is defined by*

$$(f_1, f_2, f_3)_{D_3}(\mathbf{x}, t) = (\tau_3(\mathbf{x}), f_{\mathbf{x}(1)}(t))$$

for any $(\mathbf{x}, t) \in D_3 \times [0,1]$, *is a transitive homeomorphism. Here,* D_3 *denotes the topological product* $D_3 = \prod_{k=-\infty}^{\infty}\{1,2,3\}$, *where the set* $\{1,2,3\}$ *is equipped with the discrete topology, and* τ_3 *denotes the shift map* $\tau_3 : D_3 \to D_3$, *defined by*

$$\tau_3(\mathbf{x}) = \tau_n(\ldots, \mathbf{x}(-1), \mathbf{x}(0); \mathbf{x}(1), \mathbf{x}(2), \ldots) = (\ldots, \mathbf{x}(-1), \mathbf{x}(0), \mathbf{x}(1); \mathbf{x}(2), \ldots)$$

for any $\mathbf{x} \in D_3$. *See* [1] *for more details.*

Definition 3.10. *We use*

1. $\sim$ *to denote the equivalence relation* $\sim$ *on* P, *which is defined by*

$$(c_1, t_1) \sim (c_2, t_2) \iff (c_1, t_1) = (c_2, t_2) \text{ or } t_1 = t_2 = 0$$

for all $(c_1, t_1), (c_2, t_2) \in P$.

2. q *to denote the quotient map* $q : P \to P/_\sim$, *defined by*

$$q(c, t) = [(c, t)]$$

for each $(c, t) \in P$.

3. $\sim_R$ *to denote the restriction of the relation* $\sim$ *to* R.

Observation 3.11. *Note that* $P/_\sim$ *is a Cantor fan, Also, note that* $R/_{\sim_R}$ *is a star of Cantor fans. To see this, let* $C_1 = [\frac{2}{3}, 1] \cap C$, $C_2 = [\frac{2}{9}, \frac{1}{3}] \cap C$, $C_3 = [\frac{2}{27}, \frac{1}{9}] \cap C$, $\ldots$. *Then* $C = \{0\} \cup \bigcup_{n=1}^{\infty} C_n$. *For each* n, *let* $\sim_n$ *be the restriction of* $\sim_R$ *to* $(C_n \times [0,1]) \cap R$ *and let* $R_n = ((C_n \times [0,1]) \cap R)/_{\sim_n}$. *Note that for each positive integer* n, R_n *is a Cantor fan and that* $R/_{\sim_R} = \{[(0,0)]\} \cup \bigcup_{n=1}^{\infty} R_n$, *where* $[(0,0)]$ *is an equivalent class of* $(0,0)$ *under* $\sim_R$. *Since* $\lim_{n\to\infty} \operatorname{diam}(R_n) = 0$, *it follows that* $R/_{\sim_R}$ *is a star of Cantor fans.*

Theorem 3.12. *The star of Cantor fans admits a transitive homeomorphism.*

Proof. By Observation 3.9, there is a transitive homeomorphism $f : P \to P$ such that

$$f(\{0\} \times I) \subseteq \{0\} \times I$$

and

$$f((C \setminus \{0\}) \times I) \subseteq (C \setminus \{0\}) \times I.$$

Fix such a homeomorphism f. By Proposition 3.8, f_R is a transitive homeomorphism from R to R. It follows from Proposition 2.17 that $f_R^\star$ is a transitive homeomorphism from $R/_{\sim_R}$ to $R/_{\sim_R}$. $\square$

Suppose that F is either a Cantor fan, a Lelek fan or a star of Cantor fans. Then, as seen above, a star of F's also admits a transitive homeomorphism (by Theorem 3.12 and Observation 3.4, since both the Cantor fan and the Lelek fan admit a transitive homeomorphism as seen in [1] and in [2]). Therefore, the following open problems are a good place to finish this section.

PROBLEM 3.1. *Let F be a smooth fan that admits a transitive homeomorphism. Does the star of F's also admit a transitive homeomorphism?*

PROBLEM 3.2. *Let F be a smooth fan. If a star of F's admits a transitive homeomorphism, then does F admit a transitive homeomorphism?*

4. An uncountable family of smooth fans that admit transitive homeomorphisms

In this section, we present our main result of the paper, an uncountable family of pairwise non-homeomorphic smooth fans that admit transitive homeomorphisms. First, we construct, using a Mahavier product of a closed relation, a space that is homeomorphic to a subspace of the product of a Cantor set and an interval in such a way that the shift map on it is a transitive homeomorphism. Second, with some identifications, we get uncountably many smooth fans while keeping the transitivity of the induced homeomorphisms. Great care must be taken to see that the model for the Mahavier product that we present is what we claim it is. Identifications must also be done with care to ensure the transitivity of the induced homeomorphisms.

We begin with the following definitions.

DEFINITION 4.1. *Let X be a non-empty compact metric space and let $F \subseteq X \times X$ be a relation on X. If F is closed in $X \times X$, then we say that F is* a closed relation on X.

DEFINITION 4.2. *Let X be a non-empty compact metric space and let F be a closed relation on X. For each positive integer m, we call*

$$X_F^m = \Big\{(x_1, x_2, x_3, \dots, x_{m+1}) \in \prod_{i=1}^{m+1} X \mid \text{for each } i \in \{1, 2, 3, \dots, m\}, (x_i, x_{i+1}) \in F\Big\}$$

the m-th Mahavier product of F, *and we call*

$$X_F^+ = \Big\{(x_1,x_2,x_3,\ldots) \in \prod_{i=1}^{\infty} X \mid \text{for each positive integer } i, (x_i,x_{i+1}) \in F\Big\}$$

the Mahavier product of F, *and*

$$X_F = \Big\{(\ldots,x_{-3},x_{-2},x_{-1},x_0;x_1,x_2,x_3,\ldots) \in \prod_{i=-\infty}^{\infty} X \mid \text{for each integer } i, (x_i,x_{i+1}) \in F\Big\}$$

the two-sided Mahavier product of F.

Definition 4.3. *Let X be a non-empty compact metric space and let F be a closed relation on X. The function $\sigma_F^+ : X_F^+ \to X_F^+$, defined by*

$$\sigma_F^+(x_1,x_2,x_3,x_4,\ldots) = (x_2,x_3,x_4,\ldots)$$

for each $(x_1,x_2,x_3,x_4,\ldots) \in X_F^+$, is called the shift map on X_F^+. *The function $\sigma_F : X_F \to X_F$, defined by*

$$\sigma_F(\ldots,x_{-3},x_{-2},x_{-1},x_0;x_1,x_2,x_3,\ldots) = (\ldots,x_{-3},x_{-2},x_{-1},x_0,x_1;x_2,x_3,\ldots)$$

for each $(\ldots,x_{-3},x_{-2},x_{-1},x_0;x_1,x_2,x_3,\ldots) \in X_F$, is called the shift map on X_F.

Observation 4.4. *Note that σ_F is always a homeomorphism while σ_F^+ may not be a homeomorphism.*

Definition 4.5. *Let X be a compact metric space, let F be a closed relation on X and let $x \in X$. Then we define*

$$\mathcal{U}_F^{\oplus}(x) = \{y \in X \mid \text{there are } n \in \mathbb{N} \text{ and } \mathbf{x} \in X_F^n \text{ such that } \mathbf{x}(1) = x, \mathbf{x}(n) = y\}$$

and we call it the forward impression of x by F.

Theorem 4.6. *Let X be a compact metric space, let F be a closed relation on X, let $\{f_\alpha \mid \alpha \in A\}$ be a non-empty collection of continuous functions from X to X such that $F^{-1} = \bigcup_{\alpha\in A}\Gamma(f_\alpha)$, and let $\{g_\beta \mid \beta \in B\}$ be a non-empty collection of continuous functions from X to X such that $F = \bigcup_{\beta\in B}\Gamma(g_\beta)$. If there is a dense set D in X such that for each $s \in D$, $\mathrm{Cl}(\mathcal{U}_F^{\oplus}(s)) = X$, then (X_F^+,σ_F^+) is transitive.*

Proof. See [1, Theorem 4.8, page 18]. ∎

Theorem 4.7. *Let X be a compact metric space and let F be a closed relation on X such that $p_1(F) = p_2(F) = X$. The following statements are equivalent.*

1. *The map σ_F^+ is transitive.*

2. *The homeomorphism σ_F is transitive.*

Proof. The theorem follows from [1, Theorem 4.5, page 17]. ☐

Next, we define a space $\mathbb{X}$, which will be used to construct our uncountable family of pairwise non-homeomorphic smooth fans that admit transitive homeomorphisms.

Definition 4.8. *We use $\mathbb{X}$ to denote the set*

$$\mathbb{X} = ([0,1] \cup [2,3] \cup [4,5] \cup [6,7] \cup \ldots) \cup \{\infty\}.$$

We equip $\mathbb{X}$ with the Alexandroff one-point compactification topology $\mathcal{T}$; i.e., $\mathcal{T}$ is obtained on $\mathbb{X}$ from the Alexandroff one-point compactification (also known as the Alexandroff extension) of the space $[0,1]\cup[2,3]\cup[4,5]\cup[6,7]\cup\ldots$ *(which is a subspace of the Euclidean line $\mathbb{R}$) with the point ∞. See* [9, pages 166–171] *or* [20, pages 135–145] *for more information on (one-point) compactifications.*

Note that this topology is precisely constructed below by defining a metric.

Observation 4.9. *For each non-negative integer k, let $q_k = 1 - \frac{1}{2^k}$ and let*

$$X = [q_0, q_1] \cup [q_2, q_3] \cup [q_4, q_5] \cup [q_6, q_7] \cup \ldots \{1\}$$

(we equip X with the usual topology). Note that the compacta $\mathbb{X}$ and X are homeomorphic.

Definition 4.10. *Let X be the compactum from Observation 4.9 and let $h : X \to \mathbb{X}$ be any homeomorphism such that for each non-negative integer k, $h(q_k) = k$. On the space $\mathbb{X}$, we always use the metric $\mathrm{d}_{\mathbb{X}}$ that is defined by*

$$\mathrm{d}_{\mathbb{X}}(x,y) = |h^{-1}(y) - h^{-1}(x)|$$

for all $x, y \in \mathbb{X}$.

Observation 4.11. *Note that the topology $\mathcal{T}_{\mathrm{d}_{\mathbb{X}}}$ on $\mathbb{X}$, that is obtained from the metric $\mathrm{d}_{\mathbb{X}}$, is exactly the one-point compactification topology $\mathcal{T}$ on $\mathbb{X}$. Also, note that (in this setting) for each non-negative integer k,*

$$\mathrm{diam}([2k, 2k+1]) = \frac{1}{2^{2k+1}}.$$

Definition 4.12. *For each non-negative integer k, we use I_{k+1} to denote*

$$I_{k+1} = [2k, 2k+1].$$

Observation 4.13. *Note that for each positive integer k,*

$$\mathrm{diam}(I_k) = \frac{1}{2^{2k-1}}.$$

Definition 4.14. *We use the product metric* $\mathrm{D}_{\mathbb{X}}$ *on the product* $\prod_{k=-\infty}^{\infty}\mathbb{X}$*, which is defined by*

$$\mathrm{D}_{\mathbb{X}}(\mathbf{x},\mathbf{y})=\sup\left\{\frac{\mathrm{d}_{\mathbb{X}}(\mathbf{x}(k),\mathbf{y}(k))}{2^{|k|}}\mid k \text{ is an integer}\right\}$$

for all $\mathbf{x},\mathbf{y}\in\prod_{k=-\infty}^{\infty}\mathbb{X}$.

Observation 4.15. *Since* $\mathbb{X}$ *is compact it follows that for all* $\mathbf{x},\mathbf{y}\in\prod_{k=-\infty}^{\infty}\mathbb{X}$,

$$\sup\left\{\frac{\mathrm{d}_{\mathbb{X}}(\mathbf{x}(k),\mathbf{y}(k))}{2^{|k|}}\mid k \text{ is an integer}\right\}=\max\left\{\frac{\mathrm{d}_{\mathbb{X}}(\mathbf{x}(k),\mathbf{y}(k))}{2^{|k|}}\mid k \text{ is an integer}\right\}$$

and, therefore, for all $\mathbf{x},\mathbf{y}\in\prod_{k=-\infty}^{\infty}\mathbb{X}$,

$$\mathrm{D}_{\mathbb{X}}(\mathbf{x},\mathbf{y})=\max\left\{\frac{\mathrm{d}_{\mathbb{X}}(\mathbf{x}(k),\mathbf{y}(k))}{2^{|k|}}\mid k \text{ is an integer}\right\}.$$

Next, we define the closed relation H on $\mathbb{X}$ that will play an importaint role in our construction of an uncountable family of pairwise non-homeomorphic smooth fans that admit transitive homeomorphisms.

Definition 4.16. *We use* H *to denote the closed relation on* $\mathbb{X}$ *that is defined as follows:*

$$\begin{aligned}H=&\left\{\left(t,t^{\frac{1}{3}}\right)\mid t\in I_1\right\}\cup\left\{\left(t,(t-2)^2+2\right)\mid t\in I_2\right\}\cup\\&\left\{(t,t+2)\mid t\in I_1\cup I_2\cup I_3\cup I_4\cup\ldots\right\}\cup\\&\left\{(t,t-2)\mid t\in I_2\cup I_3\cup I_4\cup I_5\cup\ldots\right\}\cup\\&\left\{(t,t)\mid t\in I_3\cup I_4\cup I_5\cup I_6\cup\ldots\right\}\cup\left\{(\infty,\infty)\right\};\end{aligned}$$

see Figure 5. We also use σ_H^+ *to denote the shift map on the Mahavier product* $\mathbb{X}_H^+$ *and* σ_H *to denote the shift map on the two-sided Mahavier product* $\mathbb{X}_H$.

First, we prove that the shift map σ_H is transitive (Theorem 4.17). To do that, we use Theorems 4.6 and 4.7.

Theorem 4.17. *The dynamical system* $(\mathbb{X}_H,\sigma_H)$ *is transitive.*

Proof. Since $p_1(H)=p_2(H)=\mathbb{X}$, it suffices to see that $(\mathbb{X}_H^+,\sigma_H^+)$ is transitive (by Theorem 4.7). We use Theorem 4.6 to do so. Let $f_1,f_2,f_3:\mathbb{X}\to\mathbb{X}$ be defined as follows. For each $x\in\mathbb{X}$, let

$$f_1(x)=\begin{cases}x^{\frac{1}{3}}; & x\in[0,1]\\(x-2)^2+2; & x\in[2,3]\\x; & x\in\mathbb{X}\setminus([0,1]\cup[2,3]),\end{cases}$$

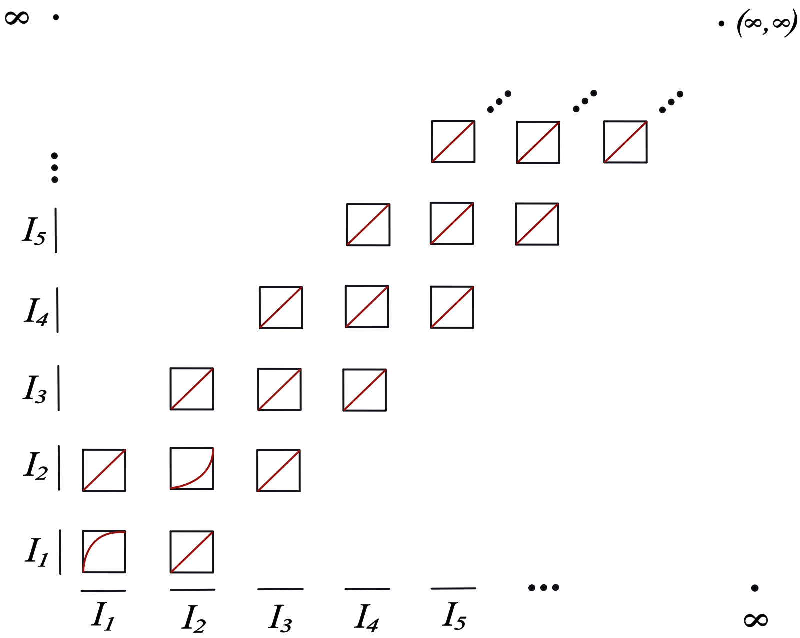

FIGURE 5. The relation H on $\mathbb{X}$

$$f_2(x) = \begin{cases} x+2; & x \in \bigcup_{k=0}^{\infty} I_{2k+1} \\ x-2; & x \in \bigcup_{k=0}^{\infty} I_{2k+2} \\ \infty; & x = \infty, \end{cases}$$

and

$$f_3(x) = \begin{cases} x^{\frac{1}{3}}; & x \in [0,1] \\ x+2; & x \in \bigcup_{k=0}^{\infty} I_{2k+2} \\ x-2; & x \in \bigcup_{k=0}^{\infty} I_{2k+3} \\ \infty; & x = \infty. \end{cases}$$

Note that f_1, f_2, and f_3 are homeomorphisms from $\mathbb{X}$ to $\mathbb{X}$ such that $H = \Gamma(f_1) \cup \Gamma(f_2) \cup \Gamma(f_3)$. Similarly, $H^{-1} = \Gamma(f_1^{-1}) \cup \Gamma(f_2^{-1}) \cup \Gamma(f_3^{-1})$. So, all the initial conditions from Theorem 4.6 are satisfied. To see that $(\mathbb{X}_H^+, \sigma_H^+)$ is transitive, we prove that there is a dense set D in $\mathbb{X}$ such that for each $s \in D$, $\mathrm{Cl}(\mathcal{U}_H^{\oplus}(s)) = \mathbb{X}$. Let $D = (0,1) \cup (2,3) \cup (4,5) \cup (6,7) \cup \ldots$ Then D is dense in $\mathbb{X}$. Let $s \in D$ be any point and let ℓ be a non-negative integer such that $s \in (2\ell, 2\ell+1)$. Note that

$$s, s-2, s-4, s-6, \ldots, s-2\ell \in \mathcal{U}_H^{\oplus}(s)$$

and let $t = s - 2\ell$. Then $t \in (0,1)$. It follows from the definition of H that for all non-negative integers m, n and k,

$$t^{\frac{2^m}{3^n}} + k \cdot 2 \in \mathcal{U}_H^{\oplus}(t) :$$

use n-times the cube-root function, m-times the squaring function and then do the translation for k times; note that in the definition of f_1, f_2 and f_3, the linear functions are used to jump between non-linear functions in any combination possible. It follows from Theorem [1, Lemma 4.9, page 19] that $\left\{t^{\frac{2^m}{3^n}} + k \cdot 2 \mid m,n,k \in \mathbb{N} \cup \{0\}\right\}$ is dense in $\mathbb{X}$. Since

$$\left\{t^{\frac{2^m}{3^n}} + k \cdot 2 \mid m,n,k \in \mathbb{N} \cup \{0\}\right\} \subseteq \mathcal{U}_H^{\oplus}(t) \subseteq \mathcal{U}_H^{\oplus}(s),$$

it follows that $\mathcal{U}_H^{\oplus}(s)$ is dense in $\mathbb{X}$. Therefore, by Theorem 4.6, $(\mathbb{X}_H^+, \sigma_H^+)$ is transitive. ∎

Next, we examine the space $\mathbb{X}_H$.

Definition 4.18. *For each positive integer k, we use L_k to denote*

$$L_k = \{(\ldots, t_{-2}, t_{-1}, t_0; t_1, t_2, \ldots) \in \mathbb{X}_H \mid t_0 \in I_k\}.$$

Observation 4.19. *Note that for each positive integer k, L_k is compact and that*

$$\mathbb{X}_H = \left(\bigcup_{k=1}^{\infty} L_k\right) \cup \{(\ldots, \infty, \infty; \infty, \ldots)\}$$

Proposition 4.20. *For each positive integer k,* $\operatorname{diam}(L_k) \le \frac{1}{2^{k-2}}$.

Proof. Let k be a positive integer and let n be an integer. We consider the following possible cases.

1. $|n| < k$. Then

$$\frac{\mathrm{d}_{\mathbb{X}}(\mathbf{x}(n), \mathbf{y}(n))}{2^{|n|}} = \frac{|h^{-1}(\mathbf{x}(n)) - h^{-1}(\mathbf{y}(n))|}{2^{|n|}} \le \frac{q_{2(k+|n|)-1} - q_{2(k-|n|)-2}}{2^{|n|}} =$$
$$\frac{(1 - \frac{1}{2^{2(k+|n|)-1}}) - (1 - \frac{1}{2^{2(k-|n|)-2}})}{2^{|n|}} = \frac{\frac{1}{2^{2(k-|n|)-2}} - \frac{1}{2^{2(k+|n|)-1}}}{2^{|n|}} =$$
$$\frac{1}{2^{2k-|n|-2}} - \frac{1}{2^{2k+3|n|-1}} = \frac{1}{2^{2k+1}}\left(2^{3+|n|} - \frac{1}{2^{3|n|}}\right) \le \frac{1}{2^{2k+1}} \cdot 2^{3+|n|} \le$$
$$\frac{1}{2^{2k+1}} \cdot 2^{3+k} = \frac{1}{2^{2k+1-3-k}} = \frac{1}{2^{k-2}}.$$

2. $|n| > k$. Then

$$\frac{\mathrm{d}_{\mathbb{X}}(\mathbf{x}(n), \mathbf{y}(n))}{2^{|n|}} \le \frac{1}{2^{|n|}} \le \frac{1}{2^k} \le \frac{1}{2^{k-2}}.$$

Therefore,

$$\operatorname{diam}(L_k) = \sup\{D_{\mathbb{X}}(\mathbf{x},\mathbf{y}) \mid \mathbf{x},\mathbf{y} \in L_k\} =$$
$$\sup\left\{\max\left\{\frac{d_{\mathbb{X}}(\mathbf{x}(n),\mathbf{y}(n))}{2^{|n|}} \mid n \text{ is an integer}\right\} \mid \mathbf{x},\mathbf{y} \in L_k\right\} \le$$
$$\sup\left\{\max\left\{\frac{1}{2^{k-2}} \mid k \text{ is an integer}\right\} \mid \mathbf{x},\mathbf{y} \in L_k\right\} = \frac{1}{2^{k-2}}.$$

∎

Definition 4.21. *We define the functions $f_{1,2}, f_{1,3} : I_1 \to \mathbb{X}$ as follows. For each $t \in I_1$, we define*

$$f_{1,2}(t) = t^{\frac{1}{3}} \text{ and } f_{1,3}(t) = t + 2.$$

We also define the functions $f_{2,1}, f_{2,2}, f_{2,3} : I_2 \to \mathbb{X}$ as follows. For each $t \in I_2$, we define

$$f_{2,1}(t) = t - 2,\ f_{2,2}(t) = (t-2)^2 + 2 \text{ and } f_{2,3}(t) = t + 2.$$

Also, for each positive integer k, we define the functions $f_{k,1}, f_{k,2}, f_{k,3} : I_k \to \mathbb{X}$ as follows. For each $t \in I_k$, we define

$$f_{k,1}(t) = t - 2,\ f_{k,2}(t) = t \text{ and } f_{k,3}(t) = t + 2.$$

We also use $\mathcal{H}$ to denote $\mathcal{H} = \{f_{1,2}, f_{1,3}\} \cup \bigcup_{k=2}^{\infty}\{f_{k,1}, f_{k,2}, f_{k,3}\}$ (see Figure 5 above – the relation H contains as a subset the union of the graphs of the defined functions).

Observation 4.22. *Let $\mathbf{x} \in \mathbb{X}_H \setminus \{(\ldots,\infty,\infty;\infty,\ldots)\}$. Then there is a unique point $\mathbf{h} = (\ldots, h_{-2}, h_{-1}, h_0; h_1, h_2, \ldots) \in \prod_{k=-\infty}^{\infty} \mathcal{H}$ such that for each integer k,*

$$\mathbf{x}(k+1) = h_k(\mathbf{x}(k)).$$

Let k be any integer. Then for each positive integer ℓ the following hold. If $\ell = 1$, then

1. *if $h_k = f_{\ell,2}$, then $h_{k+1} \in \{f_{\ell,2}, f_{\ell,3}\}$.*
2. *if $h_k = f_{\ell,3}$, then $h_{k+1} \in \{f_{\ell+1,1}, f_{\ell+1,2}, f_{\ell+1,3}\}$.*

If $\ell = 2$, then

1. *if $h_k = f_{\ell,1}$, then $h_{k+1} \in \{f_{\ell-1,2}, f_{\ell-1,3}\}$.*
2. *if $h_k = f_{\ell,2}$, then $h_{k+1} \in \{f_{\ell,1}, f_{\ell,2}, f_{\ell,3}\}$.*
3. *if $h_k = f_{\ell,3}$, then $h_{k+1} \in \{f_{\ell+1,1}, f_{\ell+1,2}, f_{\ell+1,3}\}$.*

If $\ell > 2$, then

1. *if $h_k = f_{\ell,1}$, then $h_{k+1} \in \{f_{\ell-1,1}, f_{\ell-1,2}, f_{\ell-1,3}\}$.*
2. *if $h_k = f_{\ell,2}$, then $h_{k+1} \in \{f_{\ell,1}, f_{\ell,2}, f_{\ell,3}\}$.*
3. *if $h_k = f_{\ell,3}$, then $h_{k+1} \in \{f_{\ell+1,1}, f_{\ell+1,2}, f_{\ell+1,3}\}$.*

Definition 4.23. *We define* $\mathbf{K}$ *to be the subset of the set* $\prod_{k=-\infty}^{\infty}\mathcal{H}$, *defined as follows. For any point* $\mathbf{h}=(\ldots,h_{-2},h_{-1},h_0;h_1,h_2,\ldots)\in\prod_{k=-\infty}^{\infty}\mathcal{H}$, $\mathbf{h}\in\mathbf{K}$ *if and only if for each integer* k *and for each positive integer* ℓ *the following hold.*

1. *If* $\ell=1$, *then*
   (a) *if* $h_k=f_{\ell,2}$, *then* $h_{k+1}\in\{f_{\ell,2},f_{\ell,3}\}$.
   (b) *if* $h_k=f_{\ell,3}$, *then* $h_{k+1}\in\{f_{\ell+1,1},f_{\ell+1,2},f_{\ell+1,3}\}$.
2. *If* $\ell=2$, *then*
   (a) *if* $h_k=f_{\ell,1}$, *then* $h_{k+1}\in\{f_{\ell-1,2},f_{\ell-1,3}\}$.
   (b) *if* $h_k=f_{\ell,2}$, *then* $h_{k+1}\in\{f_{\ell,1},f_{\ell,2},f_{\ell,3}\}$.
   (c) *if* $h_k=f_{\ell,3}$, *then* $h_{k+1}\in\{f_{\ell+1,1},f_{\ell+1,2},f_{\ell+1,3}\}$.
3. *If* $\ell>2$, *then*
   (a) *if* $h_k=f_{\ell,1}$, *then* $h_{k+1}\in\{f_{\ell-1,1},f_{\ell-1,2},f_{\ell-1,3}\}$.
   (b) *if* $h_k=f_{\ell,2}$, *then* $h_{k+1}\in\{f_{\ell,1},f_{\ell,2},f_{\ell,3}\}$.
   (c) *if* $h_k=f_{\ell,3}$, *then* $h_{k+1}\in\{f_{\ell+1,1},f_{\ell+1,2},f_{\ell+1,3}\}$.

We will also use $\mathbf{h}(j)$ *to denote* $\mathbf{h}(j)=h_j$.

Definition 4.24. *For each positive integer* k, *we define*

$$\mathbf{K}_k=\{\mathbf{h}\in\mathbf{K}\mid \mathbf{h}(0):I_k\to\mathbb{X}\}.$$

Observation 4.25. *Note that*

$$\mathbf{K}=\bigcup_{k=1}^{\infty}\mathbf{K}_k.$$

Observation 4.26. *For each* $\mathbf{x}\in\mathbb{X}_H$ *and for each positive integer* k, $\mathbf{x}\in L_k$ *if and only if there is a unique point* $\mathbf{h}=(\ldots,h_{-2},h_{-1},h_0;h_1,h_2,\ldots)\in\mathbf{K}_k$ *such that for each integer* j,

$$\mathbf{x}(j+1)=h_j(\mathbf{x}(j)).$$

Definition 4.27. *Let* $f_{1,1}=f_{1,2}$, *let* k *be a positive integer and let for each integer* ℓ,

$$A_\ell=\bigcup_{i=\max\{k-|\ell|,1\}}^{k+|\ell|}\{f_{i,1},f_{i,2},f_{i,3}\}.$$

We equip each A_ℓ *with the discrete topology. Then we use* $\mathbf{C}_k$ *to denote the set*

$$\mathbf{C}_k=\prod_{\ell=-\infty}^{\infty}A_\ell.$$

Observation 4.28. *For each positive integer* k, $\mathbf{C}_k$ *is a Cantor set and* $\mathbf{K}_k\subseteq\mathbf{C}_k$.

DEFINITION 4.29. *For each positive integer k and for each $\mathbf{h} \in \mathbf{K}_k$, we define*

$$A_{k,\mathbf{h}} = \{\mathbf{x} \in L_k \mid \text{for each integer } j, \mathbf{x}(j+1) = \mathbf{h}(j)(\mathbf{x}(j))\}.$$

OBSERVATION 4.30. *Note that for each positive integer k,*

$$L_k = \bigcup_{\mathbf{h} \in \mathbf{K}_k} A_{k,\mathbf{h}}.$$

Let k be a positive integer and let $\mathbf{h} \in \mathbf{K}_k$. Since for each integer j, $\mathbf{h}(j)$ is an increasing homeomorphism from an interval I_m to an interval I_n, it follows that $A_{k,\mathbf{h}}$ is an arc in L_k. For each of the end-points $\mathbf{e}$ of the arc $A_{k,\mathbf{h}}$, all coordinates of $\mathbf{e}$ are either all non-negative even integers or they are all non-negative odd integers. Also, note that for all $\mathbf{h}_1, \mathbf{h}_2 \in \mathbf{K}_k$,

$$\mathbf{h}_1 \neq \mathbf{h}_2 \implies A_{k,\mathbf{h}_1} \cap A_{k,\mathbf{h}_2} = \emptyset.$$

THEOREM 4.31. *Let k be a positive integer. Then $\mathbf{K}_k$ is a closed subset of $\mathbf{C}_k$ and L_k is homeomorphic to $\mathbf{K}_k \times [0, \frac{1}{2^{2k-1}}]$.*

PROOF. First, we show that L_k is homeomorphic to $\mathbf{K}_k \times [0, \frac{1}{2^{2k-1}}]$. Let $\varphi : \mathbf{K}_k \times [0, \frac{1}{2^{2k-1}}] \to L_k$ be defined by

$$\varphi(\mathbf{h}, t) = (\ldots, t_{-2}, t_{-1}, t_0; t_1, t_2, \ldots)$$

for any $(\mathbf{h}, t) \in \mathbf{K}_k \times [0, \frac{1}{2^{2k-1}}]$, where $t_0 = 2^{2k-1} \cdot t + 2k - 2$ and for each integer j,

$$t_{j+1} = \mathbf{h}(j)(t_j).$$

Then φ is a homeomorphism. Since L_k is compact, it follows that $\mathbf{K}_k \times [0, \frac{1}{2^{2k-1}}]$ is compact. Therefore, $\mathbf{K}_k$ is a closed subset of $\mathbf{C}_k$. □

OBSERVATION 4.32. *Note that in the proof of Theorem 4.31, the homeomorphism φ is constructed in such a way that for each $\mathbf{x} \in L_k$, if all the coordinates of $\mathbf{x}$ are even integers, then $p_2(\varphi^{-1}(\mathbf{x})) = 0$. In particular, if all the coordinates of $\mathbf{x}$ are even integers, then $\mathbf{x}(0)$ is an even integer. Since $\mathbf{x}(0) \in I_k$, it follows that $\mathbf{x}(0) = 2k - 2$. By the definition of the homeomorphism φ, $\mathbf{x}(0) = 2^{2k-1} \cdot t + 2k - 2$. Therefore, $2k - 2 = 2^{2k-1} \cdot t + 2k - 2$ and $t = 0$ follows.*

THEOREM 4.33. *For each positive integer k, $\mathbf{K}_k$ is a Cantor set.*

PROOF. Suppose that there is a positive integer k such that $\mathbf{K}_k$ is not a Cantor set. Note that $\mathbf{K}_k$ is a totally disconnected metric compactum since by Theorem 4.31 it is a closed subset of a Cantor set. Since $\mathbf{K}_k$ is not a Cantor set, it follows from [18, Theorem 7.14, page 109] there is an isolated point in $\mathbf{K}_k$. Let $\mathbf{h} \in \mathbf{K}_k$ be an isolated point of $\mathbf{K}_k$. First, we show that $\mathbf{h}$ is an isolated point in $\mathbf{K}$. Suppose that $\mathbf{h}$ is not an isolated point in $\mathbf{K}$. Then for each positive integer n, there is a positive integer i_n such that $i_n \neq k$ and there is $\mathbf{h}_n \in \mathbf{K}_{i_n}$ such that $D(\mathbf{h}_n, \mathbf{h}) < \frac{1}{n}$, where D is the product metric on $\mathbf{K}$, defined by

$$D(\mathbf{f}, \mathbf{g}) = \sup \left\{ \frac{d(\mathbf{f}(k), \mathbf{g}(k))}{2^{|k|}} \mid k \text{ is a positive integer} \right\}$$

for all $\mathbf{f}, \mathbf{g} \in \mathbf{K}$ (here, d is the discrete metric defined by $d(\mathbf{f}(k), \mathbf{g}(k)) = 0$ if $\mathbf{f}(k) = \mathbf{g}(k)$, and $d(\mathbf{f}(k), \mathbf{g}(k)) = 1$ if $\mathbf{f}(k) \neq \mathbf{g}(k)$). Therefore, for each positive integer n, $d(\mathbf{h}_n(0), \mathbf{h}(0)) < \frac{1}{n}$. This is a contradiction since for each positive integer n, $\mathbf{h}_n \neq \mathbf{h}$ and, therefore, $d(\mathbf{h}_n(0), \mathbf{h}(0)) = 1$ and $1 \not< \frac{1}{n}$. Therefore, $\mathbf{h}$ is an isolated point in $\mathbf{K}$. It follows that $A_{k,\mathbf{h}}$ is an isolated arc in $\mathbb{X}_H$ (meaning that there is an open set U in $\mathbb{X}_H$ such that $A_{k,\mathbf{h}} \subseteq U$ and $(\mathbb{X}_H \setminus A_{k,\mathbf{h}}) \cap U = \emptyset$).

Let $\mathbf{x} \in \mathbb{X}_H$ be any transitive point in $(\mathbb{X}_H, \sigma_H)$. If $\mathbf{x}$ is an element of an isolated arc A in $\mathbb{X}_H$, then (since by Theorem 4.17, σ_H is a transitive homeomorphism) there is a positive integer n such that $\sigma_H^n(\mathbf{x}) \in A$ and, therefore, $\sigma_H^n(A) = A$. It follows that $\mathbb{X}_H$ is the union of n mutually disjoint arcs: $\sigma_H(A)$, $\sigma_H^2(A)$, $\sigma_H^3(A)$, …, $\sigma_H^n(A)$, which is a contradiction. It follows that $\mathbf{x}$ is not an element of an isolated arc. Let A be an isolated arc in $\mathbb{X}_H$ and let U be an open set in $\mathbb{X}_H$ such that $A \subseteq U$ and $(\mathbb{X}_H \setminus A) \cap U = \emptyset$. Then for each non-negative integer k, $\sigma_H^k(\mathbf{x}) \notin U$ (note that if $\sigma_k^H(x) \in U$, then A would contain a point with a dense orbit, which we already concluded is not possible above). It follows that $\mathbf{x}$ is not a transitive point in $(\mathbb{X}_H, \sigma_H)$, which is a contradiction. ∎

Definition 4.34. *Let C be the standard middle-third Cantor set in $[0, 1]$. For each positive integer k, we use C_k to denote $C_k = C \cap [c_k, d_k]$, where $c_1 = 0$, $d_1 = \frac{1}{3}$, and for each positive integer k, $c_{k+1} = d_k + \frac{1}{3^k}$ and $d_{k+1} = c_{k+1} + \frac{1}{3^{k+1}}$.*

Observation 4.35. *Note that for each positive integer k, C_k is a Cantor set and that*

$$C = \left(\bigcup_{k=1}^{\infty} C_k \right) \cup \{1\}.$$

Also, note that for all positive integers k and ℓ,

$$k \neq \ell \implies C_k \cap C_\ell = \emptyset.$$

In the following theorem, we obtain a model for our two-sided Mahavier product $\mathbb{X}_H$. This will be used later in Theorem 4.46 where we show that the members of our uncountable family are in fact smooth fans.

Theorem 4.36. *There is a homeomorphism*

$$\varphi : \mathbb{X}_H \to \left(\bigcup_{k=1}^{\infty} \left(C_k \times \left[0, \frac{1}{2^{2k-1}}\right] \right) \right) \cup \{(1, 0)\}$$

such that for each $\mathbf{x} \in \mathbb{X}_H$*, if all the coordinates of* $\mathbf{x}$ *are even, then* $\varphi(\mathbf{x}) = (c, 0)$ *for some* $c \in C$*.*

Proof. For each positive integer k, let

$$f_k : \mathbf{K}_k \to C_k$$

be a homeomorphism. By Theorem 4.31, each L_k is homeomorphic to $\mathbf{K}_k \times [0, \frac{1}{2^{2k-1}}]$. For each positive integer k, let

$$v_k : L_k \to \mathbf{K}_k \times \Big[0, \frac{1}{2^{2k-1}}\Big]$$

be a homeomorphism such that for each $\mathbf{x} \in L_k$, if all the coordinates of $\mathbf{x}$ are even, then $p_2(v_k(\mathbf{x})) = 0$ (such a homeomorphism does exist by Observation 4.32). Then

$$\varphi : \mathbb{X}_H \to \Bigg(\bigcup_{k=1}^{\infty} \Big(C_k \times \Big[0, \frac{1}{2^{2k-1}}\Big]\Big)\Bigg) \cup \{(1,0)\},$$

defined by

$$\varphi(\mathbf{x}) = \begin{cases} (1,0); & \mathbf{x} = (\dots, \infty, \infty; \infty, \dots) \\ (f_k(p_1(v_k(\mathbf{x}))), p_2(v_k(\mathbf{x}))); & \text{there is } k \in \mathbb{N} \text{ such that } \mathbf{x} \in L_k \end{cases}$$

for each $\mathbf{x} \in \mathbb{X}_H$, is a homeomorphism such that for each $\mathbf{x} \in \mathbb{X}_H$, if all the coordinates of $\mathbf{x}$ are even, then $\varphi(\mathbf{x}) = (c, 0)$ for some $c \in C$. See Figure 6, where the space $\Big(\bigcup_{k=1}^{\infty} \Big(C_k \times \Big[0, \frac{1}{2^{2k-1}}\Big]\Big)\Big) \cup \{(1,0)\}$ is presented. □

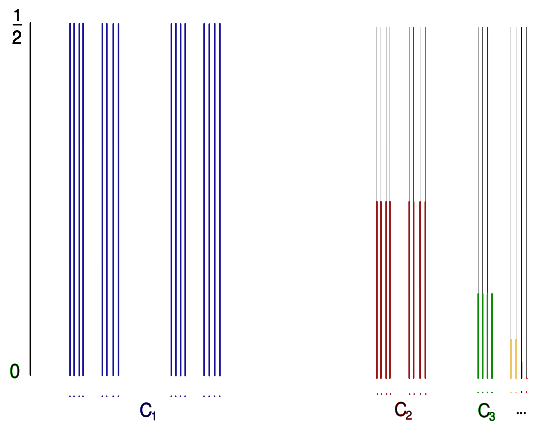

Figure 6. The space $\Bigg(\bigcup_{k=1}^{\infty} \Big(C_k \times \Big[0, \frac{1}{2^{2k-1}}\Big]\Big)\Bigg) \cup \{(1,0)\}$

Definition 4.37. *We choose and fix one of the homeomorphisms*

$$\varphi : \mathbb{X}_H \to \left(\bigcup_{k=1}^{\infty}\left(C_k \times \left[0, \frac{1}{2^{2k-1}}\right]\right)\right) \cup \{(1,0)\}$$

such that for each $\mathbf{x} \in \mathbb{X}_H$*, if all the coordinates of* $\mathbf{x}$ *are even, then* $\varphi(\mathbf{x}) = (c, 0)$ *for some* $c \in C$*, and we denote it by* φ_0*.*

The short intuitive description (suggested to us by one of the referees) of what the transitive homeomorphism σ_H now does to $\mathbb{X}_H$ in this homeomorphic representation $\varphi(\mathbb{X}_H)$, follows. Represent each point of the Cantor set $C \setminus \{(0,1)\} = \cup_k C_k$ by a sequence of positive integers, with restrictions that for any $k \geq 2$, k can be followed by $k-1$, k, or $k+1$, while $k = 1$ can be followed by 1 and 2. One can then order this space with a lexicographical ordering, and equip it with a product topology. Then σ_H maps each vertical arc of $\mathbb{X}_H$ over a point $(x_1, x_2, x_3, \ldots) \in C$ homeomorphically onto a vertical arc of $\mathbb{X}_H$ over $\sigma(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots) \in C$, where σ is a shift on the symbolic space. On vertical arcs, σ_H is just a linear mapping, except for arcs over symbolic sequences which start with $(1,1)$ (on which it acts as $x \mapsto \sqrt[3]{x}$), and arcs over symbolic sequences which start with $(2,2)$ (on which it acts as $x \mapsto (x-2)^2 + 2$). It should not be difficult to see transitivity of σ_H using this symbolic representation. We leave the details to the reader.

Next, we use the space $\mathbb{X}_H$ and the model that we obtained in Theorem 4.36 to construct a family of uncountable many pairwise non-homeomorphic smooth fans. First, we introduce the following definitions.

Definition 4.38. *We use* $\mathbb{A}$ *to denote the product*

$$\mathbb{A} = \{1,2\} \times \{3,4\} \times \{5,6\} \times \{7,8\} \times \{9,10\} \times \ldots$$

Observation 4.39. *Note that* $\mathbb{A}$ *is uncountable.*

Using the set $\mathbb{A}$, we define three relations on $\mathbb{X}_H$; see Definitions 4.40, 4.42 and 4.43.

Definition 4.40. *We define the relation* $\approx$ *on* $\mathbb{X}_H$ *as follows: for all* $\mathbf{x}, \mathbf{y} \in \mathbb{X}_H$*, we define* $\mathbf{x} \approx \mathbf{y}$ *if and only if one of the following holds:*

1. $\mathbf{x} = \mathbf{y}$*,*
2. $p_2(\varphi_0(\mathbf{x})) = p_2(\varphi_0(\mathbf{y})) = 0$*;*

see Figure 7.

Definition 4.41. *For each positive integer* k*,* $k \geq 3$*, we use* M_k *to be the following subspace of* L_k*:* $M_k = \{(\ldots, t, t; t, \ldots) \mid t \in I_k\}$*.*

Note that there are infinitely many of these M_k's, and that plays a necessary role in creating the uncountable family of smooth fans. Also, note that the shift map restricted to these M_k's is the identity, and that is crucial in

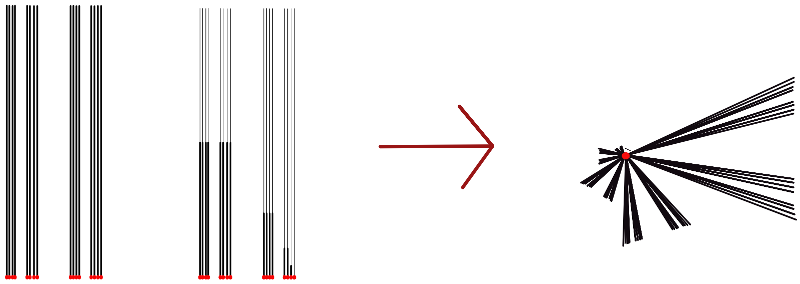

FIGURE 7. The relation $\approx$ from Definition 4.40

maintaining the transitivity of the induced map after identifications using the following relation.

DEFINITION 4.42. *Let* $\mathbf{a} = (a_1, a_2, a_3, \ldots) \in \mathbb{A}$. *Then we define the relation* $\approx_{\mathbf{a}}$ *on* $\mathbb{X}_H$ *as follows: for all* $\mathbf{x}, \mathbf{y} \in \mathbb{X}_H$, *we define* $\mathbf{x} \approx_{\mathbf{a}} \mathbf{y}$ *if and only if one of the following holds:*

1. $\mathbf{x} = \mathbf{y}$,
2. *there is a positive integer* k *and there is an* $i \in \{1, 2, 3, \ldots, a_k\}$ *such that either*
   (a) $\mathbf{x} \in M_{k^2+2}$ *and* $\mathbf{y} \in M_{k^2+2+i}$, *and*
   (b) $p_2(\varphi_0(\mathbf{x})) = p_2(\varphi_0(\mathbf{y}))$
   or
   (a) $\mathbf{y} \in M_{k^2+2}$ *and* $\mathbf{x} \in M_{k^2+2+i}$, *and*
   (b) $p_2(\varphi_0(\mathbf{x})) = p_2(\varphi_0(\mathbf{y}))$,
3. *there is a positive integer* k *and there are* $i, j \in \{1, 2, 3, \ldots, a_k\}$ *such that*
   (a) $\mathbf{x} \in M_{k^2+2+i}$ *and* $\mathbf{y} \in M_{k^2+2+j}$, *and*
   (b) $p_2(\varphi_0(\mathbf{x})) = p_2(\varphi_0(\mathbf{y}))$.

See Figure 8, which illustrates how the arcs M_{k^2+2+1}, M_{k^2+2+2}, M_{k^2+2+3}, $\ldots$, $M_{k^2+2+a_k}$ *are being glued to the arc* M_{k^2+2}.

DEFINITION 4.43. *For each* $\mathbf{a} = (a_1, a_2, a_3, \ldots) \in \mathbb{A}$, *we define the relation* $\sim_{\mathbf{a}}$ *on* $\mathbb{X}_H$ *by*

$$\mathbf{x} \sim_{\mathbf{a}} \mathbf{y} \iff \mathbf{x} \approx \mathbf{y} \text{ or there is } \mathbf{a} \in \mathbb{A} \text{ such that } \mathbf{x} \approx_{\mathbf{a}} \mathbf{y}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{X}_H$.

OBSERVATION 4.44. *Note that* $\sim_{\mathbf{a}}$ *is an equivalence relation on* $\mathbb{X}_H$.

DEFINITION 4.45. *For each* $\mathbf{a} \in \mathbb{A}$, *we use* $F_{\mathbf{a}}$ *to denote the quotient space*

$$F_{\mathbf{a}} = \mathbb{X}_H/_{\sim_{\mathbf{a}}}.$$

THEOREM 4.46. *For each* $\mathbf{a} \in \mathbb{A}$, $F_{\mathbf{a}}$ *is a smooth fan.*

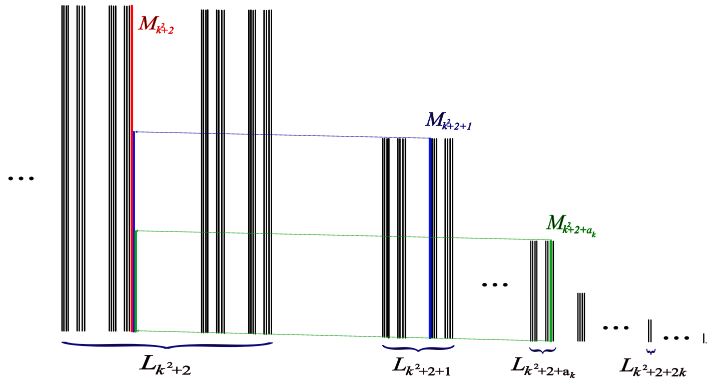

FIGURE 8. The relation $\approx_{\mathbf{a}}$ from Definition 4.40

PROOF. First, let $\mathbf{a} = (a_1, a_2, a_3, \dots) \in \mathbb{A}$ and let

$$i : \left(\bigcup_{k=1}^{\infty}\left(C_k \times \left[0, \frac{1}{2^{2k-1}}\right]\right)\right) \cup \{(1,0)\} \to C \times [0,1]$$

be the inclusion function. For each positive integer k and for each integer $i \in \{0, 1, 2, 3, \dots, a_k\}$, let $A_{k,i}$ be the connected component of $C \times [0,1]$ such that

$$i(M_{k^2+2+i}) \subseteq A_{k^2+2,i};$$

see Figure 9.

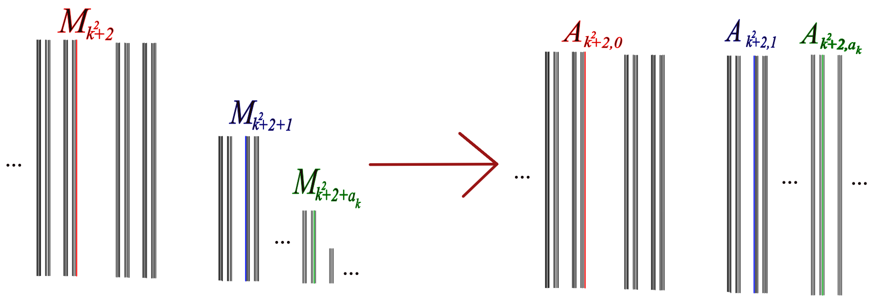

FIGURE 9. The arcs $A_{k,i}$ in $C \times [0,1]$

Next, let $\sim_1$ be the equivalence relation on $C \times [0,1]$, defined as follows. For all $(c_1, t_1), (c_2, t_2) \in C \times [0,1]$, we define that $(c_1, t_1) \sim_1 (c_2, t_2)$ if and only if one of the following holds.

1. $(c_1, t_1) = (c_2, t_2)$,

2. $t_1 = t_2 = 0$.

Note that $(C \times [0,1])/_{\sim_1}$ is a Cantor fan. We also define $\sim_2$ to be the equivalence relation on $C \times [0,1]$, defined as follows. For all $(c_1,t_1),(c_2,t_2) \in C \times [0,1]$, we define that $(c_1,t_1) \sim_2 (c_2,t_2)$ if and only if one of the following holds.

1. $(c_1,t_1) = (c_2,t_2)$,
2. there is a positive integer k and an integer $i \in \{1,2,3,\dots,a_k\}$ such that either
   (a) $(c_1,t_1) \in A_{k^2+2,0}$ and $(c_2,t_2) \in A_{k^2+2,i}$, and
   (b) $t_1 = t_2$,

   or
   (a) $(c_2,t_2) \in A_{k^2+2,0}$ and $(c_1,t_1) \in A_{k^2+2,i}$, and
   (b) $t_1 = t_2$.
3. there is a positive integer k and there are integers $i,j \in \{1,2,3,\dots,a_k\}$ such that
   (a) $(c_1,t_1) \in A_{k^2+2,i}$ and $(c_2,t_2) \in A_{k^2+2,j}$, and
   (b) $t_1 = t_2$.

Finally, we define the equivalence relation $\sim$ on $C \times [0,1]$ as follows. For all $(c_1,t_1),(c_2,t_2) \in C \times [0,1]$, we define

$$(c_1,t_1) \sim (c_2,t_2) \iff (c_1,t_1) \sim_1 (c_2,t_2) \text{ or } (c_1,t_1) \sim_2 (c_2,t_2).$$

Next, let

$$r : C \times [0,1] \to (C \times [0,1])/_{\sim_1}$$

be the quotient map defined by

$$r(c,t) = [(c,t)]_{\sim_1} = \{(d,s) \in C \times [0,1] \mid (d,s) \sim_1 (c,t)\}$$

for each $(c,t) \in C \times [0,1]$ and let

$$q : C \times [0,1] \to (C \times [0,1])/_{\sim}$$

be the quotient map defined by

$$q(c,t) = [(c,t)]_{\sim} = \{(d,s) \in C \times [0,1] \mid (d,s) \sim (c,t)\}$$

for each $(c,t) \in C \times [0,1]$. We use F to denote $F = (C \times [0,1])/_{\sim}$. Let

$$g : (C \times [0,1])/_{\sim_1} \to F$$

be defined by

$$g([(c,t)]_{\sim_1}) = q(r^{-1}([(c,t)]_{\sim_1}))$$

for any $(c,t) \in C \times [0,1]$. Note that g is a well-defined confluent surjection. Since $(C \times [0,1])/_{\sim_1}$ is a smooth fan (in fact, it is a Cantor fan), it follows from Theorem 2.10 that F is also a smooth fan. Finally, let

$$p : \mathbb{X}_H \to F_{\mathbf{a}}$$

be the quotient map defined by

$$p(\mathbf{x}) = [\mathbf{x}] = \{\mathbf{y} \in \mathbb{X}_H \mid \mathbf{y} \sim_{\mathbf{a}} \mathbf{x}\}$$

for each $\mathbf{x} \in \mathbb{X}_H$. Then

$$f : F_{\mathbf{a}} \to F,$$

defined by

$$f([\mathbf{x}]) = q(i(\varphi_0(p^{-1}([\mathbf{x}]))))$$

for any $\mathbf{x} \in \mathbb{X}_H$, is an embedding of $F_{\mathbf{a}}$ into the smooth fan F. Therefore, $F_{\mathbf{a}}$ is a smooth fan. □

In Theorem 4.48, we prove that each $F_{\mathbf{a}}$ admits a transitive homeomorphism. In it, we use the following observation.

Observation 4.47. *Let $\mathbf{a} \in \mathbb{A}$. For all $\mathbf{x}, \mathbf{y} \in \mathbb{X}_H$,*

$$\mathbf{x} \sim_{\mathbf{a}} \mathbf{y} \iff \sigma_H(\mathbf{x}) \sim_{\mathbf{a}} \sigma_H(\mathbf{y}).$$

Theorem 4.48. *Let $\mathbf{a} \in \mathbb{A}$. The mapping $\sigma_H^\star : F_{\mathbf{a}} \to F_{\mathbf{a}}$, defined by*

$$\sigma_H^\star([\mathbf{x}]) = [\sigma_H(\mathbf{x})]$$

for each $\mathbf{x} \in \mathbb{X}_H$, is a transitive homeomorphism.

Proof. By Theorem 4.17 and Observation 4.4, σ_H is a transitive homeomorphism. It follows from Observation 4.47 and from Proposition 2.17, that $\sigma_H^\star$ is a transitive homeomorphism. □

Observation 4.49. *Note that for each positive integer k, this transitive homeomorphism $\sigma_H^\star$, restricted to $M_k/_{\sim_{\mathbf{a}}} = \{[\mathbf{x}] \mid \mathbf{x} \in M_k\}$, is just the identity.*

Definition 4.50. *We use $\mathcal{F}$ to denote the family*

$$\mathcal{F} = \{F_{\mathbf{a}} \mid \mathbf{a} \in \mathbb{A}\}.$$

By Theorems 4.46 and 4.48, each member of $\mathcal{F}$ is a smooth fan that admits a transitive homeomorphism. Recall that by Observation 4.39, $\mathbb{A}$ is uncountable. So, if we show that for all $\mathbf{a}, \mathbf{b} \in \mathbb{A}$,

$$\mathbf{a} \neq \mathbf{b} \implies F_{\mathbf{a}} \text{ and } F_{\mathbf{b}} \text{ are not homeomorphic},$$

then this proves that $\mathcal{F}$ is a family of uncountably many pairwise non-homeomorphic smooth fans that admit transitive homeomorphisms. In the following definition, we define the new concept of JuMas, which will be used to prove this.

Definition 4.51. *Let X be a fan with the top o. We define the set* JuMa(X) *as follows:*

$$\mathrm{JuMa}(X) = \{x \in X \setminus \{o\} \mid \text{there is a sequence } (e_n) \text{ in } E(X) \text{ such that } \lim_{n\to\infty} e_n = x\}.$$

Definition 4.52. *Let X be a fan with the top o. For each $e \in E(X)$, we use $A_X[o,e]$ to denote the arc in X from o to e.*

Proposition 4.53. *Let X and Y be fans with tops o_X and o_Y, respectively, and let $f : X \to Y$ be a homeomorphism. Then for each $e \in E(X)$,*

$$|A_X[o_X,e] \cap \mathrm{JuMa}(X)| = |A_Y[o_Y,f(e)] \cap \mathrm{JuMa}(Y)|.$$

Here $|S|$ denotes the cardinality of S for any set S.

Proof. The lemma follows from the fact that for each $x \in X$,

$$x \in \mathrm{JuMa}(X) \implies f(x) \in \mathrm{JuMa}(Y),$$

which is easy to see and we leave the details to the reader. □

Corollary 4.54. *Let X and Y be fans with tops o_X and o_Y, respectively. If there is $e \in E(X)$ such that for each $e' \in E(Y)$,*

$$|A_Y[o_Y,e'] \cap \mathrm{JuMa}(Y)| \neq |A_X[o_X,e] \cap \mathrm{JuMa}(X)|,$$

then X and Y are not homeomorphic.

Proof. The corollary follows directly from Proposition 4.53. □

Theorem 4.55. *For all $\mathbf{a},\mathbf{b} \in \mathbb{A}$,*

$$\mathbf{a} \neq \mathbf{b} \implies F_{\mathbf{a}} \text{ and } F_{\mathbf{b}} \text{ are not homeomorphic.}$$

Proof. Let $\mathbf{a},\mathbf{b} \in \mathbb{A}$ be such that $\mathbf{a} \neq \mathbf{b}$. Let $\mathbf{o_a}$ and $\mathbf{o_b}$ be the tops of the fans $F_{\mathbf{a}}$ and $F_{\mathbf{b}}$, respectively. Since $\mathbf{a} \neq \mathbf{b}$, there is a positive integer k such that $\mathbf{a}(k) \neq \mathbf{b}(k)$. Then either $\mathbf{a}(k) = 2k-1$ and $\mathbf{b}(k) = 2k$ or $\mathbf{a}(k) = 2k$ and $\mathbf{b}(k) = 2k-1$. Without loss of generality we assume that $\mathbf{a}(k) = 2k-1$ and $\mathbf{b}(k) = 2k$. It follows from the definition of the relation $\sim_{\mathbf{a}}$ that in $F_{\mathbf{a}}$, there is an end-point $\mathbf{e} \in E(F_{\mathbf{a}})$ such that

$$|A_{F_{\mathbf{a}}}[\mathbf{o_a},\mathbf{e}] \cap \mathrm{JuMa}(F_{\mathbf{a}})| = 2k-1.$$

Note that it follows from the definition of the relation $\sim_{\mathbf{b}}$ that for each $\mathbf{e}' \in E(F_{\mathbf{b}})$,

$$|A_{F_{\mathbf{b}}}[\mathbf{o_b},\mathbf{e}'] \cap \mathrm{JuMa}(F_{\mathbf{b}})| \neq 2k-1.$$

Therefore, by Corollary 4.54, $F_{\mathbf{a}}$ and $F_{\mathbf{b}}$ are not homeomorphic. □

Finally, we state and prove Theorem 4.56 – the main theorem of the paper.

Theorem 4.56. *There is a family of uncountably many pairwise non-homeomorphic smooth fans that admit transitive homeomorphisms.*

Proof. The collection $\mathcal{F}$ is such a family. By Theorem 4.55, $\mathcal{F}$ is uncountable, since $\mathbb{A}$ is uncountable. By Theorem 4.46, for each $\mathbf{a} \in \mathbb{A}$, $F_{\mathbf{a}}$ is a smooth fan and by Theorem 4.48, σ_H^* is a transitive homeomorphism on $F_{\mathbf{a}}$. This completes the proof. □

The following open problem is a good place to finish the section.

PROBLEM 4.1. *Is there a smooth fan X with the top o that has the following properties?*

1. *X does not admit a transitive homeomorphism.*
2. *For each $\varepsilon > 0$, for each $e \in E(X)$ and for each $x \in A_X[o,e]$, there is $e' \in E(X) \setminus \{e\}$ such that*

$$B(x,\varepsilon) \cap A_X[o,e'] \neq \emptyset.$$

5. An uncountable family of non-smooth fans that admit transitive homeomorphisms

In this section, we construct a family of uncountably many non-smooth fans that admit transitive homeomorphisms.

DEFINITION 5.1. *Let F be a fan with top o. For each end-point $e \in E(F)$ of the fan F, the arc in F from o to e is called* a leg of F. *The set of all legs of F is denoted by $\mathcal{L}(F)$.*

In Definition 4.50, a family $\mathcal{F}$ of uncountably many pairwise non-homeomorphic smooth fans that admit transitive homeomorphisms was constructed. Also in this section, we continue working with the family $\mathcal{F}$.

DEFINITION 5.2. *Let F be the Cantor fan, defined by $F = \bigcup_{c\in C} S_c$, where $C \subseteq [0,1]$ is the standard Cantor set and for each $c \in C$, S_c is the straight line segment in the plane from $(0,0)$ to $(c,1)$. For each $X \in \mathcal{F}$, let $\Psi_X : X \to F$ be an embedding. We use $\mathcal{E}$ to denote the family*

$$\mathcal{E} = \{\Psi_X(X) \mid X \in \mathcal{F}\}.$$

We denote the members of the family $\mathcal{E}$ by F_λ:

$$\mathcal{E} = \{F_\lambda \mid \lambda \in \Lambda\}.$$

OBSERVATION 5.3. *Note that for each $\lambda \in \Lambda$, F_λ is a smooth fan such that $F_\lambda \subseteq F$. Also, for each $\lambda \in \Lambda$,*

1. *there is a transitive homeomorphisms $\varphi_\lambda : F_\lambda \to F_\lambda$*
2. *there is a leg $A_\lambda \in \mathcal{L}(F_\lambda)$*

such that

1. *for each $x \in A_\lambda$, $\varphi_\lambda(x) = x$, and*
2. *$A_\lambda \cap \mathrm{JuMa}(F_\lambda) = \emptyset$.*

For each $\lambda \in \Lambda$, we choose and fix such a homeomorphism φ_λ and such a leg A_λ. We also assume for the rest of the paper that for all $\lambda_1, \lambda_2 \in \Lambda$,

$$\lambda_1 \neq \lambda_2 \implies F_{\lambda_1} \text{ is not homeomorphic to } F_{\lambda_2}.$$

DEFINITION 5.4. *For each $\lambda \in \Lambda$, we use a_λ to denote the end-point of F_λ that is defined as follows:*

$$E(F_\lambda) \cap A_\lambda = \{a_\lambda\}.$$

Observation 5.5. *Note that for each* $\lambda \in \Lambda$,

$$A_\lambda = \{t \cdot a_\lambda \mid t \in [0,1]\}.$$

Definition 5.6. *For each* $\lambda \in \Lambda$, *we define the relation* $\sim_\lambda$ *on* F_λ *as follows. For each* $\lambda \in \Lambda$ *and for all* $x, y \in F_\lambda$, *we define that*

$$x \sim_\lambda y \iff x = y \text{ or there is } t \in [0,1] \text{ such that } x = t \cdot a_\lambda \text{ and } y = (1-t) \cdot a_\lambda.$$

Proposition 5.7. *For each* $\lambda \in \Lambda$, $\sim_\lambda$ *is an equivalence relation on* F_λ *such that for all* $x, y \in F_\lambda$,

$$x \sim_\lambda y \Longleftrightarrow \varphi_\lambda(x) \sim_\lambda \varphi_\lambda(y).$$

Proof. The proof is straight forward. We leave it to the reader. □

Definition 5.8. *For each* $\lambda \in \Lambda$ *and for each* $x \in F_\lambda$, *we define* $[x]_\lambda$ *to be the equivalence class of* x *with respect to the relation* $\sim_\lambda$:

$$[x]_\lambda = \{y \in F_\lambda \mid y \sim_\lambda x\}.$$

We also define J_λ *to be the quotient space*

$$J_\lambda = F_\lambda /_{\sim_\lambda}$$

and we use q_λ *to denote the quotient map* $q_\lambda : F_\lambda \to J_\lambda$, *defined by*

$$q_\lambda(x) = [x]_\lambda$$

for each $x \in F_\lambda$.

Observation 5.9. *It follows from* [9, Theorem 4.2.13] *that for each* $\lambda \in \Lambda$, J_λ *is metrizable. Since for each* $\lambda \in \Lambda$, F_λ *is connected and compact and since* q_λ *is continuous, it follows from* $J_\lambda = q_\lambda(F_\lambda)$ *that also* J_λ *is connected and compact. Therefore, for each* $\lambda \in \Lambda$, J_λ *is a continuum.*

Theorem 5.10. *For each* $\lambda \in \Lambda$, $\varphi_\lambda^\star : J_\lambda \to J_\lambda$, *defined by* $\varphi_\lambda^\star([x]_\lambda) = [\varphi_\lambda(x)]_\lambda$ *for each* $x \in F_\lambda$, *is a transitive homeomorphism.*

Proof. For each $\lambda \in \Lambda$, φ_λ is (by Proposition 5.7) a transitive homeomorphism, such that for all $x, y \in F_\lambda$,

$$x \sim_\lambda y \Longleftrightarrow \varphi_\lambda(x) \sim_\lambda \varphi_\lambda(y).$$

It follows from Proposition 2.17 that for each $\lambda \in \Lambda$, $\varphi_\lambda^\star$ is a transitive homeomorphism. □

Definition 5.11. *For each* $\lambda \in \Lambda$ *and for each* $e \in E(F_\lambda)$, *we define the subsets* $K_e \subseteq F_\lambda$ *and* $L_e \subseteq J_\lambda$ *by*

$$K_e = \{e \cdot t \mid t \in [0,1]\} \text{ and } L_e = \{[e \cdot t]_\lambda \mid t \in [0,1]\}.$$

Observation 5.12. *Let* $\lambda \in \Lambda$. *Note that for each* $e \in E(F_\lambda)$, $K_e \in \mathcal{L}(F_\lambda)$, *and that* $K_{a_\lambda} = A_\lambda$.

Proposition 5.13. *For each $\lambda \in \Lambda$, and for each $e \in E(F_\lambda)$, L_e is an arc in J_λ.*

Proof. Let $\lambda \in \Lambda$ and let $e \in E(F_\lambda)$. We consider the following cases.

1. $e = a_\lambda$. Note that
$$L_{a_\lambda} = \Big\{[a_\lambda \cdot t]_\lambda \mid t \in \Big[0, \frac{1}{2}\Big]\Big\} = \Big\{\{t \cdot a_\lambda, (1-t) \cdot a_\lambda\} \mid t \in \Big[0, \frac{1}{2}\Big]\Big\}.$$
Let $h : L_{a_\lambda} \to [0,1]$ be defined by
$$h([a_\lambda \cdot t]_\lambda) = 2t$$
for each $t \in [0, \frac{1}{2}]$. We show that h is a homeomorphism. Note that h is a bijection. To show that h is continuous, let U be any open set in $[0,1]$. We show that $h^{-1}(U)$ is open in L_{a_λ} by using Observation 2.15. Note that
$$h^{-1}(U) = \Big\{\Big[\frac{1}{2}t \cdot a_\lambda\Big]_\lambda \mid t \in U\Big\} = \Big\{\Big\{\frac{1}{2}t \cdot a_\lambda, \Big(1 - \frac{1}{2}t\Big) \cdot a_\lambda\Big\} \mid t \in U\Big\}$$
and that
$$q_\lambda^{-1}(h^{-1}(U)) = \Big\{\frac{1}{2}t \cdot a_\lambda \mid t \in U\Big\} \cup \Big\{\Big(1 - \frac{1}{2}t\Big) \cdot a_\lambda \mid t \in U\Big\}.$$
Then $q_\lambda^{-1}(h^{-1}(U))$ is a union of two open sets in A_λ, therefore, $q_\lambda^{-1}(h^{-1}(U))$ is open in A_λ. It follows that $h^{-1}(U)$ is open in L_{a_λ}. This proves that h is continuous. Note that A_λ is compact (since it is an arc), therefore, since $L_{a_\lambda} = q_\lambda(A_\lambda)$, it follows that L_{a_λ} is compact. Hence, h is a continuous surjection from a compact space to a metric space. It follows that h is a homeomorphism. This proves that L_{a_λ} is an arc in J_λ.
2. $e \neq a_\lambda$. We show that L_e is an arc by showing that it is homeomorphic to K_e. Let $H : L_e \to K_e$ be defined by
$$H([t \cdot e]_\lambda) = t \cdot e$$
for each $t \in [0,1]$. We show that H is a homeomorphism. Note that H is a bijection and that for each $t \in [0,1]$, $[t \cdot e]_\lambda = \{t \cdot e\}$. Next, we show that H is continuous. Let U be any open set in K_e. Note that $q_\lambda^{-1}(H^{-1}(U)) = U$. It follows that $q_\lambda^{-1}(H^{-1}(U))$ is open in K_e. Therefore, $H^{-1}(U)$ is open in L_e by Observation 2.15. This proves that H is continuous. Note that K_e is compact (since it is an arc), therefore, since $L_e = q_\lambda(K_e)$, it follows that L_e is compact. Hence, H is a continuous surjection from a compact space to a metric space. It follows that H is a homeomorphism. Therefore, L_e is an arc in J_λ.

Therefore, for each $e \in E(F_\lambda)$, L_e is an arc in J_λ. □

Observation 5.14. *Let $\lambda \in \Lambda$. Note that L_{a_λ} is an arc with end-points $[(0,0)]_\lambda$ and $[\frac{1}{2} \cdot a_\lambda]_\lambda$.*

In Theorem 5.18, we prove that each J_λ is a fan. In its proof, we use Lemma 5.15.

LEMMA 5.15. *Let $\lambda \in \Lambda$ and let A be a subcontinuum of J_λ. The following statements are equivalent.*

1. *The preimage $q_\lambda^{-1}(A)$ is not connected.*
2. *$A \cap L_{a_\lambda} = \{[(0,0)]_\lambda\}$ or there is $s \in \left(0, \frac{1}{2}\right)$ such that $A \cap L_{a_\lambda} = \{[t \cdot a_\lambda]_\lambda \mid t \in [0,s]\}$.*

PROOF. To prove the implication from 1. to 2., suppose that $A \cap L_{a_\lambda} \neq \{[(0,0)]_\lambda\}$ and that for each $s \in \left(0, \frac{1}{2}\right)$, $A \cap L_{a_\lambda} \neq \{[t \cdot a_\lambda]_\lambda \mid t \in [0,s]\}$. Then $A \cap L_{a_\lambda} = \emptyset$ or $A \cap L_{a_\lambda} = L_{a_\lambda}$. In both cases, $q_\lambda^{-1}(A)$ is connected. Next, we prove the implication from 2. to 1.. If $A \cap L_\lambda = \{[(0,0)]_\lambda\}$, then a_λ is an isolated point of $q_\lambda^{-1}(A)$. Therefore, in this case, $q_\lambda^{-1}(A)$ is not connected. Next, suppose that there is $s \in \left(0, \frac{1}{2}\right)$ such that $A \cap L_\lambda = \{[t \cdot a_\lambda]_\lambda \mid t \in [0,s]\}$. Choose and fix such an s. Let $U = \{(1-t) \cdot a_\lambda \mid t \in [0,s]\}$. Then U is clopen in $q_\lambda^{-1}(A)$. Since $U \neq q_\lambda^{-1}(A)$, it follows that also in this case, $q_\lambda^{-1}(A)$ is not connected. □

OBSERVATION 5.16. *Let $\lambda \in \Lambda$ and let A be a subcontinuum of J_λ such that the preimage $q_\lambda^{-1}(A)$ is not connected. Note that*

1. *if $A \cap L_{a_\lambda} = \{[(0,0)]_\lambda\}$, then $q_\lambda^{-1}(A)$ has exactly two components, $C_1 = \{a_\lambda\}$ and $C_2 = q_\lambda^{-1}(A) \setminus C_1$.*
2. *if there is $s \in \left(0, \frac{1}{2}\right)$ such that $A \cap L_{a_\lambda} = \{[t \cdot a_\lambda]_\lambda \mid t \in [0,s]\}$, then $q_\lambda^{-1}(A)$ has exactly two components, $C_1 = \{(1-t) \cdot a_\lambda \mid t \in [0,s]\}$ and $C_2 = q_\lambda^{-1}(A) \setminus C_1$.*

DEFINITION 5.17. *Let X and Y be continua and let $f : X \to Y$ be a continuous function. We say that f is semi-confluent, if for any subcontinuum C of Y and for all components C_1 and C_2 of $f^{-1}(C)$, $f(C_1) \subseteq f(C_2)$ or $f(C_2) \subseteq f(C_1)$.*

THEOREM 5.18. *For each $\lambda \in \Lambda$, J_λ is a fan.*

PROOF. Let $\lambda \in \Lambda$. We show that the quotient map $q_\lambda : F_\lambda \to J_\lambda$ is a semi-confluent surjection. Since q_λ is a quotient map, it is a surjection. To see that it is semi-confluent, let C be any subcontinuum of J_λ. We consider the following cases.

1. $q_\lambda^{-1}(C)$ is connected. Then for all components C_1 and C_2 of $q_\lambda^{-1}(C)$, $q_\lambda(C_1) \subseteq q_\lambda(C_2)$ or $q_\lambda(C_2) \subseteq q_\lambda(C_1)$ (since $C_1 = C_2 = q_\lambda^{-1}(C)$).
2. $q_\lambda^{-1}(C)$ is not connected. By Lemma 5.15, $q_\lambda^{-1}(C) \cap L_{a_\lambda} = \{[(0,0)]_\lambda\}$ or there is $s \in \left(0, \frac{1}{2}\right)$ such that $q_\lambda^{-1}(C) \cap L_{a_\lambda} = \{[t \cdot a_\lambda]_\lambda \mid t \in [0,s]\}$. According to this, we consider the following two cases.

(a) $q_\lambda^{-1}(C) \cap L_{a_\lambda} = \{[(0,0)]_\lambda\}$. By Observation 5.16, $q_\lambda^{-1}(C)$ has exactly two components: $C_1 = \{a_\lambda\}$ and $C_2 = q_\lambda^{-1}(C) \setminus C_1$. Note that $q_\lambda(C_1) \subseteq q_\lambda(C_2)$.

(b) There is $s \in \left(0, \frac{1}{2}\right)$ such that $q_\lambda^{-1}(C) \cap L_{a_\lambda} = \{[t \cdot a_\lambda]_\lambda \mid t \in [0,s]\}$. By Observation 5.16, $q_\lambda^{-1}(C)$ has exactly two components, $C_1 = \{(1-t) \cdot a_\lambda \mid t \in [0,s]\}$ and $C_2 = q_\lambda^{-1}(C) \setminus C_1$. Note that also in this case, $q_\lambda(C_1) \subseteq q_\lambda(C_2)$.

It follows that $q_\lambda : F_\lambda \to J_\lambda$ is a semi-confluent surjection. Now it follows from Maćkowiak's result [13, Theorem 5.6], which says that a semi-confluent image of a fan is a fan, that J_λ is a fan. ∎

THEOREM 5.19. *For each $\lambda \in \Lambda$, the fan J_λ is not smooth.*

PROOF. Let $\lambda \in \Lambda$ and let (x_k) be a sequence of points in F_λ such that

1. for each positive integer k, there is $e_k \in E(F_\lambda) \setminus \{a_\lambda\}$ such that
$$x_k \in K_{e_k} \setminus \{(0,0), e_k\},$$
2. for all positive integers k, ℓ and for all $e, f \in E(F_\lambda)$ such that $x_k \in K_e \setminus \{(0,0)\}$ and $x_\ell \in K_f \setminus \{(0,0)\}$,
$$k \neq \ell \implies e \neq f.$$
3. $\lim_{k\to\infty} x_k = \frac{3}{4} \cdot a_\lambda$.

It follows from the construction of the family $\mathcal{F}$ that such a sequence exists. Since q_λ is continuous, it follows that

$$\lim_{k\to\infty} q_\lambda(x_k) = q_\lambda\Big(\frac{3}{4} \cdot a_\lambda\Big)$$

Note that for each positive integer k,

$$q_\lambda(x_k) = [x_k]_\lambda = \{x_\lambda\}$$

and that

$$q_\lambda\Big(\frac{3}{4} \cdot a_\lambda\Big) = \Big[\frac{3}{4} \cdot a_\lambda\Big]_\lambda = \Big\{\frac{1}{4} \cdot a_\lambda, \frac{3}{4} \cdot a_\lambda\Big\} = \Big[\frac{1}{4} \cdot a_\lambda\Big]_\lambda \in L_{a_\lambda}.$$

Note that L_{a_λ} is an arc in J_λ with end-points $[(0,0)]_\lambda$ and $[\frac{1}{2} \cdot a_\lambda]_\lambda$. Therefore, $q_\lambda\left(\frac{3}{4} \cdot a_\lambda\right)$ is a point in the interior of the arc L_{a_λ}. Next, for each positive integer k, let $e_k \in E(F_\lambda) \setminus \{a_\lambda\}$ be such that

$$x_k \in K_{e_k} \setminus \{(0,0), e_k\},$$

let $s_k \in (0,1)$ be such that $x_k = s_k \cdot e_k$ and let

$$A_k = \{t \cdot e_k \mid t \in [0, s_k]\} \text{ and } B_k = \{[t \cdot e_k]_\lambda \mid t \in [0, s_k]\}.$$

Also, let

$$A = \left\{ t \cdot a_\lambda \mid t \in \left[0, \frac{3}{4}\right] \right\} \text{ and } B = \left\{ [t \cdot a_\lambda]_\lambda \mid t \in \left[0, \frac{1}{4}\right] \right\}.$$

Note that for each positive integer k, x_k is an end-point of A_k, and that $\frac{3}{4} \cdot a_\lambda$ is an end-point of A. Since $\lim_{k\to\infty} x_k = \dfrac{3}{4} \cdot a_\lambda$ and since F_λ is smooth, it follows that

$$\lim_{k\to\infty} A_k = A.$$

On the other hand, note that

1. for each positive integer k, $q_\lambda(x_k)$ is an end-point of B_k,
2. $q_\lambda(\frac{3}{4} \cdot a_\lambda) = [\frac{1}{4} \cdot a_\lambda]_\lambda$ is an end-point of B,
3. $\lim_{k\to\infty} q_\lambda(x_k) = q_\lambda\left(\dfrac{3}{4} \cdot a_\lambda\right)$, and
4. $\lim_{k\to\infty} B_k = L_{a_\lambda}$ but $L_{a_\lambda} \neq B$.

This proves that the fan J_λ is not smooth. □

THEOREM 5.20. *For all $\lambda_1, \lambda_2 \in \Lambda$,*

$$\lambda_1 \neq \lambda_2 \implies F_{\lambda_1} \text{ and } F_{\lambda_2} \text{ are not homeomorphic.}$$

PROOF. Let $\lambda_1, \lambda_2 \in \Lambda$ be such that $\lambda_1 \neq \lambda_2$. Then F_{λ_1} is not homeomorphic to F_{λ_2} since there is $e \in E(F_{\lambda_1}) \setminus \{a_{\lambda_1}\}$ such that for each $f \in E(F_{\lambda_2})\}$,

$$|A_{F_{\lambda_2}}[(0,0), f] \cap \mathrm{JuMa}(F_{\lambda_2})| \neq |A_{F_{\lambda_1}}[(0,0), e] \cap \mathrm{JuMa}(F_{\lambda_1})|.$$

Choose and fix such a point $e \in E(F_{\lambda_1}) \setminus \{a_{\lambda_1}\}$. Note that

$$A_{F_{\lambda_1}}[(0,0), a_{\lambda_1}] \cap \mathrm{JuMa}(F_{\lambda_1}) = \emptyset.$$

It follows from the definition of the relations $\sim_{\lambda_1}$ and $\sim_{\lambda_2}$ that for each element $[f]_{\lambda_2} \in E(J_{\lambda_2})$,

$$|A_{J_{\lambda_2}}[[(0,0)]_{\lambda_2}, [f]_{\lambda_2}] \cap \mathrm{JuMa}(J_{\lambda_2})| \neq |A_{J_{\lambda_1}}[[(0,0)]_{\lambda_1}, [e]_{\lambda_1}] \cap \mathrm{JuMa}(J_{\lambda_1})|.$$

Therefore, by Corollary 4.54, J_{λ_1} and J_{λ_2} are not homeomorphic. □

Finally, we prove the main result of the paper.

THEOREM 5.21. *There is a family $\mathcal{H}$ of uncountably many pairwise non-homeomorphic fans that are not smooth and that admit transitive homeomorphisms.*

PROOF. Let

$$\mathcal{H} = \{J_\lambda \mid \lambda \in \lambda\}.$$

It follows from Theorem 5.20 that $|\mathcal{H}| = |\mathcal{E}|$. Since the family $\mathcal{F}$ is uncountable and since $|\mathcal{E}| = |\mathcal{F}|$, it follows from Theorem 5.19 that $\mathcal{H}$ is a family of uncountably many pairwise non-homeomorphic non-smooth fans. By Theorem 5.10, for each $X \in \mathcal{H}$, X admits a transitive homeomorphism. □

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