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SERIJA III

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Manuscript accepted
January 8, 2024.

ON THE BOUNDEDNESS OF EULER-STIELTJES CONSTANTS FOR THE RANKIN-SELBERG $L-$FUNCTION

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Abstract. Let E be a Galois extension of $\mathbb{Q}$ of finite degree and let π and π' be two irreducible automorphic unitary cuspidal representations of $GL_m(\mathbb{A}_E)$ and $GL_{m'}(\mathbb{A}_E)$, respectively. Let $\Lambda(s, \pi \times \widetilde{\pi}')$ be a Rankin-Selberg $L-$function attached to the product $\pi \times \widetilde{\pi}'$, where $\widetilde{\pi}'$ denotes the contragredient representation of π', and let its a finite part (excluding Archimedean factors) be $L(s, \pi \times \widetilde{\pi}')$. The Euler-Stieltjes constants of the Rankin-Selberg $L-$function are the coefficients in the Laurent (Taylor) series expansion around $s = 1 + it_0$ of the function $L(s, \pi \times \widetilde{\pi}')$. In this paper, we derive an upper bound of these constants.

1. Introduction

The classical Euler constant

$$\gamma = \gamma_0 = \lim_{x\to\infty}\left(\sum_{n<x}\frac{1}{n} - \log x\right) = 0.57721\ldots$$

discovered and computed correctly up to five decimal places by L. Euler [13] in 1731 is the constant term in the Laurent series expansion of the Riemann zeta function at $s = 1$

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty}\gamma_k(s-1)^k = \frac{1}{s-1} + \sum_{k=0}^{\infty}\gamma_k(s-1)^k.$$

In 1885, T. J. Stieltjes [17] pointed out that each γ_n can be obtained as

$$\gamma_k = \frac{(-1)^k}{k!}\lim_{x\to\infty}\left(\sum_{n<x}\frac{\log^k n}{n} - \frac{\log^{k+1} x}{k+1}\right). \tag{1.1}$$

2020 *Mathematics Subject Classification.* 11M26, 11S40.
Key words and phrases. Euler-Stieltjes constants, Rankin-Selberg $L-$function.

The proof of equation (1.1) can be found in [3] and [7]. Therefore, the constants γ_k ($k \geq 0$) are named the Stieltjes constants, the generalized Euler constants or the Euler-Stieltjes constants.

The Euler-Stieltjes constants γ_k are closely related (see e.g. [5]) to coefficients η_k of the Laurent series expansion of the logarithmic derivative of the Riemann zeta function at $s = 1$

$$\frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1} + \sum_{k=0}^{\infty} \eta_k (s-1)^k, \quad |s-1| < 3.$$

Constants η_k can be evaluated as (see e.g. [10])

$$\eta_k = \frac{(-1)^{k-1}}{k!} \lim_{x\to\infty} \left(\sum_{n<x} \frac{\Lambda(n)\log^k n}{n} - \frac{\log^{k+1} x}{k+1} \right),$$

where $\Lambda(n)$ is the von Mangoldt function [26, 40]. Usually, constants γ_k are called the Euler-Stieltjes constants of the first kind, while constants η_k are called the Euler-Stieltjes constants of the second kind.

The Euler-Stieltjes constants of the first and the second kind are important in both theoretical and computational analytic number theory since they appear in various estimations and as a result of asymptotic analysis. For example, the Euler-Stieltjes constants of the first kind can be used to determine a zero-free region of the Riemann zeta function near the real axis in the critical strip $0 < \mathrm{Re} s < 1$ [1]. The Euler-Stieltjes constants of the second kind are related to the Li positivity criterion for the Riemann hypothesis [5] since they appear in the arithmetic formula for the non-archimedean part of the Li coefficient. Numerical evaluation and estimations are given in [24].

The Euler-Stieltjes constants of the first and the second kind and their relation to the Li criterion for the Riemann hypothesis were further investigated by M. Coffey in [9] and [11] and by C. Knessl and M. Coffey in [21]. Some interesting formulas and bounds are recently derived in [31].

This concept is generalized in many different settings. Coefficients appearing in the Laurent (Taylor) series representation of a zeta or L function or its logarithmic derivative are called generalized Euler-Stieltjes constants of the first and the second kind. Different kinds of formulas, properties or bounds are derived.

Results related to the Hurwitz zeta function are given in [3], those for the Dedekind zeta function in [16] and [34], for the general setting of a non-co-compact Fuchsian group with unitary representation in [2], for a class of functions possessing an Euler product representation in [15], for a subclass $\mathcal{S}^\flat$ of the Selberg class in [39], for the extended Selberg class in [18] and for the Rankin-Selberg $L-$functions in [28] and [29]. Also, some investigations are done in the case of zeta functions with multiple variables, introducing multiple Stieltjes constants, for example, see [23] and [4]. A q-analogues of these coefficients are investigated in [8].

In this paper, we investigate generalized Euler-Stieltjes constants attached to the Rankin-Selberg $L-$functions associated with two representations. We precisely define coefficients under consideration in the sequel. Let E be a Galois extension of $\mathbb{Q}$ of finite degree and let π and π' be two irreducible automorphic unitary cuspidal representations (see e.g. [12]) of $GL_m(\mathbb{A}_E)$ and $GL_{m'}(\mathbb{A}_E)$, respectively. The generalized Euler-Stieltjes constants of the first kind $\gamma_{\pi,\pi'}(k)$ attached to the finite part of Rankin-Selberg $L-$function $L(s,\pi\times\widetilde{\pi}')$ (an analogue of classical ζ function) are defined as coefficients in the Laurent (Taylor) series representation of $L(s,\pi\times\widetilde{\pi}')$ at $s=1+it_0$:

$$L(s,\pi\times\widetilde{\pi}') = \sum_{k=-\delta(t_0)}^{\infty} \gamma_{\pi,\pi'}(k)(s-1-it_0)^k, \tag{1.2}$$

where $\delta(t_0)=1$ if and only if $m=m'$ and $\pi'\cong\pi\otimes|\det|^{it_0}$, for some $t_0\in\mathbb{R}$, where $\cong$ denotes isomorphic representations. Otherwise, $\delta(t_0)=0$.

In this paper, the finite part of Rankin-Selberg $L-$function we denote by $L(s,\pi\times\widetilde{\pi}')$ and call the Rankin-Selberg $L-$function, and its completed function (including Archimedean factors) we denote by $\Lambda(s,\pi\times\widetilde{\pi}')$.

The purpose of this paper is to derive an upper bound for coefficients $\gamma_{\pi,\pi'}(k)$ appearing in (1.2). The Rankin-Selberg $L-$functions attached to a convolution of two irreducible, unitary cuspidal representations of $GL_m(\mathbb{A}_E)$ and $GL_{m'}(\mathbb{A}_E)$ over number field E do not always belong to the extended Selberg class $\mathcal{S}^\sharp$, which is introduced in [20] (nor to the class of functions considered in [15]). In the case when $m=m'$ and $\pi'\cong\pi\otimes|\det|^{it_0}$, for some $t_0\in\mathbb{R}\setminus\{0\}$ the Rankin-Selberg $L-$function possesses pole at $s=1+it_0\neq 1$. Hence, they do not satisfy axiom (ii) of the class $\mathcal{S}^\sharp$. Furthermore, coefficients μ_j appearing in the functional equation for the Rankin-Selberg $L-$functions unconditionally satisfy the bound $\mathrm{Re}\mu_j>-1$, different from the bound $\mathrm{Re}\mu_j\geq 0$, posed in axiom (iii) of the class $\mathcal{S}^\sharp$.

The rest of the paper is organized as follows. In section 2 we give a complete overview of the setting we are dealing with, introduce necessary notation and recall some known results that will be used for the proofs. Section 3 contains some preliminary results about functions under consideration, while the main results are stated and proved in sections 4 and 5. In section 4 integral representation of coefficients under consideration is derived, while their bounds are proved in 5.

2. Preliminaries and notations

Let E be a Galois extension of $\mathbb{Q}$ of degree d, and let $\mathbb{A}_E$ denote the ring of adeles over E. For every place v, let E_v be the completion of a number field E at v, and let f_p denote the modular degree of E_v over the field of $p-$adic numbers $\mathbb{Q}_p$ for $v|p$, where p is a prime. Let S_∞ denotes a set of infinite places v of the number field E. The Rankin-Selberg L-function attached to the product $\pi \times \widetilde{\pi}'$ of irreducible cuspidal representations of $GL_m(\mathbb{A}_E)$ and $GL_{m'}(\mathbb{A}_E)$ with a unitary central character (see e.g. [12]), respectively, is given by absolutely convergent Euler product of local factors

$$L(s, \pi \times \widetilde{\pi}') = \prod_{v<\infty} L_v(s, \pi_v \times \widetilde{\pi}'_v),$$

for Re$s > 1$, see e.g. [19, Th. 5.3.], where $\widetilde{\pi}$ denotes the contragredient representation of π. For finite place v at which π_v and π'_v are unramified, the local factors of $L(s, \pi \times \pi')$ are given by

$$L_v(s, \pi \times \widetilde{\pi}') = \prod_{j=1}^{m} \prod_{k=1}^{m'} \left(1 - \alpha_\pi(v,j)\overline{\alpha_{\pi'}(v,k)} p^{-f_p s}\right)^{-1}, \tag{2.3}$$

where $\{\alpha_\pi(v,j)\}_{j=1}^m$ and $\{\alpha_{\pi'}(v,k)\}_{k=1}^{m'}$ are corresponding sets of Satake parameters associated to π and π', respectively. If π_v or $\pi_{v'}$ ramified, we can also write the local factors at ramified places v in the same form (2.3) with the convention that some of $\alpha_\pi(v,j)$ and $\alpha_{\pi'}(v,k)$ may be zero(see e.g. [28]).

The function $L(s, \pi \times \widetilde{\pi}')$ has a Dirichlet series expansion of the form

$$L(s, \pi \times \widetilde{\pi}') = \sum_{n=1}^{\infty} \frac{a_{\pi \times \widetilde{\pi}'}(n)}{n^s}, \tag{2.4}$$

that is valid for Re$s > 1$.

Similarly, at the infinite place $v \in S_\infty$, the archimedean local factor $L_v(s, \pi_v \times \widetilde{\pi}'_v)$ can be written as a product

$$L_v(s, \pi_v \times \widetilde{\pi}'_v) = \prod_{j=1}^{m} \prod_{k=1}^{m'} \Gamma_v(s + \mu_{\pi \times \widetilde{\pi}'}(v,j,k)),$$

where $\mu_{\pi \times \widetilde{\pi}'}(v,j,k) = \mu_\pi(v,j) + \overline{\mu_{\pi'}(v,k)}$, at the infinite places v unramified for both π and π', $\{\mu_\pi(v,j)\}_{j=1}^m$ and $\{\mu_{\pi'}(v,j)\}_{k=1}^{m'}$ are the Langlands parameters associated to π_v and π'_v respectively and $\Gamma_v(s) = \pi^{-s/2}\Gamma(s/2)$, if v is real and $\Gamma_v(s) = 2\,(2\pi)^{-s}\,\Gamma(s)$, if v is complex. In the case when infinite place v is ramified for π or π', parameters $\mu_{\pi \times \widetilde{\pi}'}(v,j,k)$ are described in [32, Appendix], where it is also proved that $\mu_{\pi \times \widetilde{\pi}'}(v,j,k)$, for all $j = 1, \ldots, m$ and $k = 1, \ldots, m'$ satisfy the trivial bound Re$\mu_{\pi \times \widetilde{\pi}'}(v,j,k) > -1$.

As proved in [12, Th. 9.1. and Th. 9.2.], the completed Rankin-Selberg L-function

$$\Lambda(s, \pi \times \widetilde{\pi}') = L(s, \pi \times \widetilde{\pi}') \prod_{v \in S_\infty} L_v(s, \pi_v \times \widetilde{\pi}'_v)$$

extends to a meromorphic function of order one on the whole complex plane, bounded (away from its possible poles) in the vertical strip. The functional equation, which is due to F. Shahidi ([36], [37], [38]),

$$\Lambda(s,\pi\times\widetilde{\pi}')=\varepsilon(\pi\times\widetilde{\pi}')Q_{\pi\times\widetilde{\pi}'}^{\frac{1}{2}-s}\Lambda(1-s,\widetilde{\pi}\times\pi') \tag{2.5}$$

is valid for all s, where $Q_{\pi\times\widetilde{\pi}'}>0$ is the arithmetic conductor and $\varepsilon(\pi\times\widetilde{\pi}')$ is a complex number of modulus 1. The function $\Lambda(s,\pi\times\widetilde{\pi}')$ has simple poles at $s=1+it_0$ and $s=it_0$, arising from $L(s,\pi\times\widetilde{\pi}')$ if and only if $m=m'$ and $\pi'\cong\pi\otimes|\det|^{it_0}$, for some $t_0\in\mathbb{R}$. Otherwise, it is an entire function.

Following [14] let us define

$$\delta(t_0)=\begin{cases}1\,, & m=m' \text{ and } \pi'\cong\pi\otimes|\det|^{it_0}\,, \quad \text{for some} \quad t\in\mathbb{R};\\ 0\,, & \text{otherwise,}\end{cases} \tag{2.6}$$

then the functional equation (2.5) can be written as

$$L(s,\pi\times\widetilde{\pi}')\Psi_{\pi,\pi'}(s)=\overline{L}(1-s,\pi\times\widetilde{\pi}'), \tag{2.7}$$

where $\overline{L}(s,\pi\times\widetilde{\pi}')=\overline{L(\overline{s},\pi\times\widetilde{\pi}')}$ and the factor $\Psi_{\pi,\pi'}(s)$ is given by

$$\Psi_{\pi,\pi'}(s)=\frac{Q_{\pi\times\widetilde{\pi}'}^{s-\frac{1}{2}}}{\varepsilon\left(\pi\times\widetilde{\pi}'\right)}\prod_{v\in S_\infty}\prod_{j=1}^{m}\prod_{k=1}^{m'}\frac{\Gamma_v(s+\mu_{\pi\times\widetilde{\pi}'}(v,j,k))}{\Gamma_v\left(1-s+\overline{\mu_{\pi\times\widetilde{\pi}'}(v,j,k)}\right)}. \tag{2.8}$$

As in [27], it follows that (2.8) can be written in more convenient form, as

$$\Psi_{\pi,\pi'}(s)=\frac{\left(Q_{\pi\times\widetilde{\pi}'}\pi^{-dmm'}\right)^{s-\frac{1}{2}}}{\epsilon\left(\pi\times\widetilde{\pi}'\right)}\prod_{l=1}^{dmm'}\frac{\Gamma\left(\frac{1}{2}\left(s+\mu_{\pi\times\widetilde{\pi}'}(l)\right)\right)}{\Gamma\left(\frac{1}{2}\left(1-s+\overline{\mu_{\pi\times\widetilde{\pi}'}(l)}\right)\right)}, \tag{2.9}$$

where $|\epsilon\left(\pi\times\widetilde{\pi}'\right)|=1$ and $\mu_{\pi\times\widetilde{\pi}'}(l)=\mu_{\pi\times\widetilde{\pi}'}(v,j,k)$, for r_1+r_2 places $v\in S_\infty$ and $\mu_{\pi\times\widetilde{\pi}'}(l)=\mu_{\pi\times\widetilde{\pi}'}(v,j,k)+1$, for the rest of r_2 places $v\in S_\infty$ ($j=1,\ \dots,\ m$, $k=1,\dots,m'$) and r_1 denotes number of real places $v\in S_\infty$ and r_2 denotes number of complex places $v\in S_\infty$.

The zeros of $\Lambda(s,\pi\times\widetilde{\pi}')$ are called non-trivial zeros of $L(s,\pi\times\widetilde{\pi}')$. They lie in the strip $0<\mathrm{Re}s<1$, see [35]. The function $L(s,\pi\times\widetilde{\pi}')$ may also have trivial zeros, which arise from the poles of the local $L-$factors at infinite places. There are finitely many of them inside the critical strip $0\le\mathrm{Re}s\le1$ at points $s=-\mu_{\pi\times\widetilde{\pi}'}(v,j,k)$, for those $v\in S_\infty$, $j\in\{1,\dots,m\}$ and $k\in\{1,\dots,m'\}$ such that $\mathrm{Re}\mu_{\pi\times\widetilde{\pi}'}(v,j,k)\le0$.

3. Some properties of the Rankin-Selberg $L-$functions

In the following proposition, we give some asymptotic bounds for the Rankin-Selberg $L-$ functions and the factor $\Psi_{\pi,\pi'}(s)$ of the functional equation. These results are used in proof of the main result of the paper.

Proposition 3.1. *Let E be a Galois extension of $\mathbb{Q}$ of finite degree d and let π and π' be two irreducible automorphic unitary cuspidal representations of $GL_m(\mathbb{A}_E)$ and $GL_{m'}(\mathbb{A}_E)$. The function $\Psi_{\pi,\pi'}(s)$ satisfies relation*

$$|\Psi_{\pi,\pi'}(\sigma+it)| \sim_\sigma \left(\frac{Q_{\pi\times\widetilde{\pi}'}}{(2\pi)^{dmm'}}\right)^{\sigma-\frac{1}{2}} |t|^{\left(\sigma-\frac{1}{2}\right)dmm'}, \tag{3.10}$$

as $|t|\to+\infty$. Further, for an arbitrary $\varepsilon>0$ the function $L(s,\pi\times\widetilde{\pi}')$ satisfies

$$L(\sigma+it,\pi\times\widetilde{\pi}') = \begin{cases} O_\varepsilon(1) & \text{if} \quad \sigma\geq 1+\varepsilon, \\ O_\varepsilon\left(|t|^{\frac{dmm'}{2}(1-\sigma+\varepsilon)}\right) & \text{if} \quad -\varepsilon\leq\sigma\leq 1+\varepsilon, \\ O_{\varepsilon,\sigma}\left(|t|^{\frac{dmm'}{2}(1-2\sigma)}\right) & \text{if} \quad \sigma\leq -\varepsilon. \end{cases} \tag{3.11}$$

Proof. The function $\Psi_{\pi,\pi'}(s)$ can be written as

$$\Psi_{\pi,\pi'}(s) = \frac{1}{\epsilon\left(\pi\times\widetilde{\pi}'\right)}\left(Q_{\pi\times\widetilde{\pi}'}\pi^{-dmm'}\right)^{s-\frac{1}{2}}$$
$$\times\exp\left[\sum_{l=1}^{dmm'}\left(\log\left[\Gamma\left(\frac{s+\mu_{\pi\times\widetilde{\pi}'}(l)}{2}\right)\right]-\log\left[\Gamma\left(\frac{1-s+\overline{\mu_{\pi\times\widetilde{\pi}'}(l)}}{2}\right)\right]\right)\right].$$

By applying the asymptotic series expansion of function $\log\Gamma(z+a)$ (see [22, Section 2.11, relation (4)]) on the functions $\log\left[\Gamma\left(\frac{s+\mu_{\pi\times\widetilde{\pi}'}(l)}{2}\right)\right]$ and $\log\left[\Gamma\left(\frac{1-s+\overline{\mu_{\pi\times\widetilde{\pi}'}(l)}}{2}\right)\right]$, with $z=\frac{it}{2}$ and $z=\frac{-it}{2}$ respectively, we obtain relation (3.10).

For $\mathrm{Re}s=\sigma\geq 1+\varepsilon>1$ the Rankin-Selberg L-function $L(s,\pi\times\widetilde{\pi}')$ is given by an absolutely convergent Euler product for $\mathrm{Re}s>1$, so

$$L(\sigma+it,\pi\times\widetilde{\pi}')=O_\varepsilon(1), \quad \text{for} \quad \sigma\geq 1+\varepsilon,$$

where O_ε denotes that a constant appearing in O notation depends on ε. For $\mathrm{Re}s=\sigma\leq-\varepsilon<0$, the functional equation for the Rankin-Selberg L-function given by (2.7) and relation (3.10) imply

$$L(\sigma+it,\pi\times\widetilde{\pi}')=O_{\varepsilon,\sigma}\left(|t|^{\frac{dmm'}{2}(1-2\sigma)}\right),$$

as $|t|\to+\infty$, where $O_{\varepsilon,\sigma}$ denotes that a constant appearing in O notation depends on σ and ε. In special case, if σ lies in a closed and bounded subset of $\mathbb{R}$, a constant in O notation is uniform in σ and depends on ε.

For σ such that $-\varepsilon\leq\sigma\leq 1+\varepsilon$, Phragmén-Lindelöf theorem for strip can be used to derive the desired result. Basically, since the function

$$(s-it_0)^{\delta(t_0)}(s-1-it_0)^{\delta(t_0)}L(s,\pi\times\widetilde{\pi}'),$$

where $\delta(t_0)$ is defined by (2.6), is an entire of finite order, the bound

$$|L(s,\pi\times\widetilde{\pi}')| = O\left(\exp(\exp(\delta|t|))\right),$$

holds true for sufficiently large $|t|$ and any $\delta>0$. Application of the result [30, Proposition 8.15] to the Rankin-Selberg L-function in the strip $-\varepsilon\le\sigma\le 1+\varepsilon$ implies

$$L(\sigma+it,\pi\times\widetilde{\pi}') = O_{\varepsilon}\left(|t|^{\frac{dmm'}{2}(1-\sigma+\varepsilon)}\right),$$

as $|t|\to+\infty$. The proof is complete. □

4. Integral representation of the generalized Euler-Stieltjes constants associated to the Rankin-Selberg $L-$function

In this section, we derive an integral representation for coefficients in the Laurent (Taylor) series expansion of the Rankin-Selberg $L-$function given by (1.2) using a classical method in the analytic number theory based on contour integrals (see e.g. [40, Section 4.14], [18]). A key idea in the method is to apply the Cauchy integral formula to obtain an integral expression for coefficients, and then deform the contour appearing in the integral expression to a line from $a-i\infty$ to $a+i\infty$. Cauchy integral formula implies

$$\gamma_{\pi,\pi'}(k) = \frac{1}{2\pi i}\int\limits_{C}\frac{L(s,\pi\times\widetilde{\pi}')}{(s-1-it_0)^{k+1}}ds, \tag{4.12}$$

where contour C is a positively oriented circle with centre $s=1+it_0$ and radius r such that it contains $s=1+it_0$ as the only singularity of the integrand[1]. If $\delta(t_0)=0$, for all $t_0\in\mathbb{R}$, then (1.2) gives Taylor series expansions of function $L(s,\pi\times\tilde{\pi}')$ and in that case, let $t_0=0$.

Proposition 4.1. *Let E be a Galois extension of $\mathbb{Q}$ of finite degree d and let $L(s,\pi\times\widetilde{\pi}')$ be Rankin-Selberg $L-$function attached to the product $\pi\times\widetilde{\pi}'$ be two irreducible automorphic unitary cuspidal representations of $GL_m(\mathbb{A}_E)$ and $GL_{m'}(\mathbb{A}_E)$. Let k be a positive integer and a be a real number such that $1<1+\varepsilon<a<\frac{k+1}{dmm'}+\frac{1}{2}$ and $\frac{1}{2}(1-a+\mathrm{Re}\mu_{\pi\times\widetilde{\pi}'}(l))\notin\mathbb{Z}$ for all $l=1,\dots,dmm'$. Then,*

$$\gamma_{\pi,\pi'}(k) = \frac{(-1)^k}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{\overline{L(\overline{s},\pi\times\widetilde{\pi}')G_L(s)}}{(s+it_0)^{k+1}}ds + \delta(t_0)(-1)^{k+1}\operatorname*{Res}_{s=it_0}L(s,\pi\times\widetilde{\pi}'), \tag{4.13}$$

[1]Since the function $L(s,\pi\times\tilde{\pi}')$ might have two poles $s=it_0$ and $s=1+it_0$, we can choose for radius r any positive number less than $\frac{1}{2}$.

where

$$G_L(s) = \frac{\epsilon(\pi\times\widetilde{\pi}')Q_{\pi\times\widetilde{\pi}'}^{s-\frac{1}{2}}}{(\pi^{dmm'})^{s+\frac{1}{2}}}\prod_{l=1}^{dmm'}\left[\Gamma\left(\frac{s+\overline{\mu_{\pi\times\widetilde{\pi}'}(l)}}{2}\right)\right.$$
$$\left.\times\,\Gamma\left(\frac{1+s-\mu_{\pi\times\widetilde{\pi}'}(l)}{2}\right)\sin\frac{\pi}{2}\left(1-s+\mu_{\pi\times\widetilde{\pi}'}(l)\right)\right]. \tag{4.14}$$

Proof. The proof is based on integral representation (4.12). The contour $\mathcal{C}$ is deformed to a suitable rectangular $\mathcal{R}_{a,A,T}$ and the integral is decomposed into integrals over its sides.

Let A and T be sufficiently large positive numbers. Let $\mathcal{R}_{a,A,T}$ be a positively oriented rectangle determined by vertices $-a+1-iT$, $A-iT$, $A+iT$ and $-a+1+iT$. Compared to the integral over C, the additional contribution can be from a simple pole $s=it_0$ of the function $L(s,\pi\times\widetilde{\pi}')$ if it exists. By the Cauchy's formula, we can write

$$\frac{1}{2\pi i}\int\limits_{\mathcal{R}_{a,A,T}}\frac{L(s,\pi\times\widetilde{\pi}')}{(s-1-it_0)^{k+1}}ds = \gamma_{\pi,\pi'}(k)+\delta(t_0)\operatorname*{Res}_{s=it_0}\frac{L(s,\pi\times\widetilde{\pi}')}{(s-1-it_0)^{k+1}}.$$

Therefore,

$$\gamma_{\pi,\pi'}(k) = \frac{1}{2\pi i}\int\limits_{\mathcal{R}_{a,A,T}}\frac{L(s,\pi\times\widetilde{\pi}')}{(s-1-it_0)^{k+1}}ds+\delta(t_0)(-1)^k\operatorname*{Res}_{s=it_0}L(s,\pi\times\widetilde{\pi}'). \tag{4.15}$$

Now, integral over $\mathcal{R}_{a,A,T}$ can be written as a sum of integrals over line segments $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$ joining $-a+1+iT$, $-a+1-iT$, $A-iT$, $A+iT$ and $-a+1+iT$, respectively.

For integral over $\mathcal{S}_2$, we have

$$\int\limits_{\mathcal{S}_2}\frac{L(s,\pi\times\widetilde{\pi}')}{(s-1-it_0)^{k+1}}ds = \int\limits_{-a+1-iT}^{A-iT}\frac{L(s,\pi\times\widetilde{\pi}')}{(s-1-it_0)^{k+1}}ds$$
$$= \left(\int\limits_{-a+1-iT}^{-\varepsilon-iT}+\int\limits_{-\varepsilon-iT}^{1+\varepsilon-iT}+\int\limits_{1+\varepsilon-iT}^{A-iT}\right)\frac{L(s,\pi\times\widetilde{\pi}')}{(s-1-it_0)^{k+1}}ds.$$

Using Proposition 3.1 we obtain following asymptotic bounds

$$\left|\int\limits_{-a+1-iT}^{-\varepsilon-iT}\frac{L(s,\pi\times\widetilde{\pi}')}{(s-1-it_0)^{k+1}}ds\right| = O_\varepsilon\left(\left|\frac{T}{T+t_0}\right|^{k+1}|T|^{\left(a-\frac{1}{2}\right)dmm'-k-1}\right),$$

$$\left|\int\limits_{-\varepsilon-iT}^{1+\varepsilon-iT}\frac{L(s,\pi\times\widetilde{\pi}')}{(s-1-it_0)^{k+1}}ds\right| = O_\varepsilon\left(\left|\frac{T}{T+t_0}\right|^{k+1}|T|^{\frac{dmm'}{2}(1+2\varepsilon)-k-1}\right),$$

and

$$\left| \int_{1+\varepsilon-iT}^{A-iT} \frac{L(s,\pi\times\widetilde{\pi}')}{(s-1-it_0)^{k+1}} ds \right| = O_\varepsilon\left(\frac{1}{|T+t_0|^{k+1}}\right),$$

where O_ε denotes that constants appearing in O notation are uniform in $\mathrm{Re}s=\sigma$, for $s\in\mathcal{S}_2$, and might depend on ε.

Hence, for $1+\varepsilon<a<\frac{k+1}{dmm'}+\frac{1}{2}$ and $k>-1$, we obtain

$$\int_{\mathcal{S}_2} \frac{L(s,\pi\times\widetilde{\pi}')}{(s-1-it_0)^{k+1}} ds \to 0, \quad \text{as} \quad |T|\to\infty. \tag{4.16}$$

Integral over $\mathcal{S}_4$ can be bounded completely analogously, i.e. we get

$$\int_{\mathcal{S}_4} \frac{L(s,\pi\times\widetilde{\pi}')}{(s-1-it_0)^{k+1}} ds \to 0, \quad \text{as} \quad |T|\to\infty. \tag{4.17}$$

Next, we consider the integral over $\mathcal{S}_3$. Here $s=A+it$, and by choice of A we are in the region of absolute convergence of the Rankin-Selberg $L-$function, thus from Proposition 3.1 and by substitution $u=t-t_0$ follows

$$\left| \int_{\mathcal{S}_3} \frac{L(s,\pi\times\widetilde{\pi}')}{(s-1-it_0)^{k+1}} ds \right| \le 2K \int_0^{+\infty} \frac{du}{((A-1)^2+u^2)^{\frac{k+1}{2}}},$$

where K is a positive constant such that $|L(A+it,\pi\times\tilde{\pi}'|\le K$. From Lebesgue's convergence theorem, when $A\to\infty$, it follows that the contribution of the integral over $\mathcal{S}_3$ tends to zero, as $|T|\to\infty$. Namely, for the integrand

$$f_A(t)=\frac{1}{((A-1)^2+t^2)^{\frac{k+1}{2}}},$$

and function

$$g(t)=\begin{cases} 1, & t\in[0,1]; \\ \frac{1}{t^{k+1}}, & t>1, \end{cases}$$

holds $f_A(t)\le g(t)$ on $[0,+\infty)$, for $k>0$ and $g(t)$ is integrable. Then, since $\lim\limits_{A\to+\infty} f_A(t)=0$, we have

$$\lim_{A\to+\infty}\lim_{T\to+\infty}\int_{\mathcal{S}_3} \frac{L(s,\pi\times\widetilde{\pi}')}{(s-1-it_0)^{k+1}} ds = 0.$$

Thus, the only contribution to the integral in (4.15), when $|T| \to \infty$, is from the integral over $\mathcal{S}_1$. So, for $k > \max\left\{0, \left(\frac{1}{2}+\varepsilon\right) dmm' - 1\right\}$, we have

$$\gamma_{\pi,\pi'}(k) = \frac{1}{2\pi i} \int\limits_{-a+1+i\infty}^{-a+1-i\infty} \frac{L(s,\pi\times\widetilde{\pi}')}{(s-1-it_0)^{k+1}} ds + \delta(t_0)(-1)^k \operatorname*{Res}_{s=it_0} L(s,\pi\times\widetilde{\pi}')$$
$$= \frac{(-1)^k}{2\pi i} \int\limits_{a-i\infty}^{a+i\infty} \frac{L(1-s,\pi\times\widetilde{\pi}')}{(s+it_0)^{k+1}} ds + \delta(t_0)(-1)^k \operatorname*{Res}_{s=it_0} L(s,\pi\times\widetilde{\pi}').$$

Functional equation (2.7) for the Rankin-Selberg $L-$function and definition (4.14) of the function $G_L(s)$, combined with formula $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$, which is valid for all $s \notin \mathbb{Z}$, applied to the gamma functions appearing in gamma factor of the functional equation imply

$$L(1-s,\pi\times\widetilde{\pi}') = \overline{L(\overline{s},\pi\times\widetilde{\pi}')} G_L(s),$$

for $\frac{1}{2}\left(1-s+\mu_{\pi\times\widetilde{\pi}'}(l)\right) \notin \mathbb{Z}$.

Hence, relation (4.13) holds true for all $k > \max\left\{0, \left(\frac{1}{2}+\varepsilon\right) dmm' - 1\right\}$, where $a \in \left(1+\varepsilon, \frac{k+1}{dmm'}+\frac{1}{2}\right)$ is chosen such that $\frac{1}{2}\left(1-a+\mathrm{Re}\mu_{\pi\times\widetilde{\pi}'}(l)\right) \notin \mathbb{Z}$ for all $l = 1, 2, \ldots, dmm'$. This completes the proof of Proposition 4.1. □

5. Bounds for the generalized Euler-Stieltjes constants associated to the Rankin-Selberg $L-$function

In this section, we prove the main result of the paper, the theorem that gives an upper bound for the Euler-Stieltjes coefficients $\gamma_{\pi,\pi'}(k)$ defined by (1.2). The proof is based on integral representation (4.13) derived in the previous section. Firstly, in the following lemma, we prove a bound for the function $G_L(s)$ appearing in the integrand in (4.13).

Lemma 5.1. *Let E be a Galois extension of $\mathbb{Q}$ of finite degree d and let $L(s,\pi\times\widetilde{\pi}')$ be Rankin-Selberg $L-$function attached to the product $\pi\times\widetilde{\pi}'$ two irreducible automorphic unitary cuspidal representations of $GL_m(\mathbb{A}_E)$ and $GL_{m'}(\mathbb{A}_E)$. Let $\mu_R = \max\limits_{l=1,\ldots,dmm'} |\mathrm{Re}\mu_{\pi\times\widetilde{\pi}'}(l)|$, $\mu_I = \max\limits_{l=1,\ldots,dmm'} |\mathrm{Im}\mu_{\pi\times\widetilde{\pi}'}(l)|$. For $a > \max\{1+\varepsilon, \mu_R\}$, where $\varepsilon > 0$, we have*

$$|G_L(a+it)| \le Q_{\pi\times\widetilde{\pi}'}^{a-\frac{1}{2}} C_L(a) \times \left[\left(\frac{1+a+\mu_R}{2}\right)^2 + \left(\frac{|t|+\mu_I}{2}\right)^2\right]^{dmm'\frac{2a-1}{4}}, \tag{5.18}$$

where constant $C_L(a)$ is given by

$$C_L(a) = \left(\frac{2}{\pi^{a-\frac{1}{2}}}\right)^{dmm'} \exp\left(\sum_{l=1}^{dmm'} \frac{2a+1}{6(a+\mathrm{Re}\mu_{\pi\times\widetilde{\pi}'}(l))(1+a-\mathrm{Re}\mu_{\pi\times\widetilde{\pi}'}(l))}\right).$$

PROOF. From definition (4.14) of function G_L for $s = a + it$, and having in mind that $\epsilon(\pi \times \widetilde{\pi}')$ is a complex number of modulus 1, one obtains

$$(5.19)\quad |G_L(a+it)| = \frac{Q_{\pi\times\widetilde{\pi}'}^{a-\frac{1}{2}}}{(\pi^{dmm'})^{a+\frac{1}{2}}} \prod_{l=1}^{dmm'} \left[\left| \sin\frac{\pi}{2}\left(1-a-it+\mu_{\pi\times\widetilde{\pi}'}(l)\right) \right| \right.$$
$$\left. \times \left| \Gamma\left(\frac{1+a+it-\mu_{\pi\times\widetilde{\pi}'}(l)}{2}\right) \Gamma\left(\frac{a+it+\overline{\mu_{\pi\times\widetilde{\pi}'}(l)}}{2}\right) \right| \right].$$

Factors containing sine function, we bound using a simple representation in terms of exponential functions, precisely for $z \in \mathbb{C}$,

$$(5.20)\qquad |\sin z| \le e^{|\mathrm{Im}z|}.$$

While bounds for the factors containing gamma functions will be based on Binet formula [41, p. 258]

$$(5.21)\quad \log|\Gamma(z)| = \left(\mathrm{Re}z - \frac{1}{2}\right)\log|z| - \mathrm{Im}z\arctan\frac{\mathrm{Im}z}{\mathrm{Re}z} - \mathrm{Re}z + \frac{1}{2}\log(2\pi)$$
$$+ \mathrm{Re}\left[\int_0^{+\infty}\left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-tz}}{t}dt\right],$$

valid for $\mathrm{Re}z > 0$. A simple calculation implies that the second term can be additionally simplified, i.e.

$$-\mathrm{Im}z\arctan\frac{\mathrm{Im}z}{\mathrm{Re}z} - \mathrm{Re}z \le -\frac{\pi}{2}|\mathrm{Im}z|.$$

The properties of the function $g(t) = \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{1}{t}$, specially, the fact that it attains its maximum $1/12$, at $t = 0$, gives us a bound

$$\mathrm{Re}\left[\int_0^{+\infty}\left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-tz}}{t}dt\right] \le \frac{1}{12\mathrm{Re}z}.$$

So, for $\mathrm{Re}z > 0$, relation (5.21) implies

$$(5.22)\qquad \log|\Gamma(z)| \le \left(\mathrm{Re}z - \frac{1}{2}\right)\log|z| - |\mathrm{Im}z|\frac{\pi}{2} + \frac{1}{2}\log(2\pi) + \frac{1}{12\mathrm{Re}z}.$$

For the arguments appearing in (5.19), bound (5.20) implies

$$(5.23)\qquad \left|\sin\frac{\pi}{2}\left(1-a-it+\mu_{\pi\times\widetilde{\pi}'}(l)\right)\right| \le \exp\left(\frac{\pi}{2}\left|t-\mathrm{Im}\mu_{\pi\times\widetilde{\pi}'}(l)\right|\right),$$

for all $l = 1,\ldots,dmm'$. Since, by the assumption, $a > \max\{1+\varepsilon, \mu_R\}$, and coefficients $\mu_{\pi\times\widetilde{\pi}'}(l)$ for the Rankin-Selberg $L-$function satisfy bound

$\mathrm{Re}\mu_{\pi\times\widetilde{\pi}'} > -1$, we have

$$\mathrm{Re}\left(\frac{a+it+\overline{\mu_{\pi\times\widetilde{\pi}'}(l)}}{2}\right) > 0 \quad \text{and} \quad \mathrm{Re}\left(\frac{1+a+it-\mu_{\pi\times\widetilde{\pi}'}(l)}{2}\right) > 0,$$

for all $l = 1, \ldots, dmm'$, thus inequality (5.22) may be applied for the gamma factors in (5.19).

In addition, definition of numbers μ_R and μ_I implies the following inequalities

$$\begin{aligned}
&(t - \mathrm{Im}\mu_{\pi\times\widetilde{\pi}'}(l))^2 \leq (|t| + \mu_I)^2,\\
&(a + \mathrm{Re}\mu_{\pi\times\widetilde{\pi}'}(l))^2 \leq (1 + a + \mu_R)^2,\\
&(1 + a - \mathrm{Re}\mu_{\pi\times\widetilde{\pi}'}(l))^2 \leq (1 + a + \mu_R)^2,
\end{aligned}$$

and from (5.22) we obtain

$$\begin{aligned}
&\log\left|\Gamma\left(\frac{a+it+\overline{\mu_{\pi\times\widetilde{\pi}'}(l)}}{2}\right)\right| + \log\left|\Gamma\left(\frac{1+a+it-\mu_{\pi\times\widetilde{\pi}'}(l)}{2}\right)\right|\\
&\leq \frac{2a-1}{4}\log\left(\left(\frac{1+a+\mu_R}{2}\right)^2 + \left(\frac{|t|+\mu_I}{2}\right)^2\right) - \frac{\pi}{2}\left|t - \mathrm{Im}\mu_{\pi\times\widetilde{\pi}'}(l)\right|\\
&+ \frac{1}{6}\frac{2a+1}{(a+\mathrm{Re}\mu_{\pi\times\widetilde{\pi}'}(l))(1+a-\mathrm{Re}\mu_{\pi\times\widetilde{\pi}'}(l))} + \log 2\pi,
\end{aligned}$$

for all $l = 1, \ldots, dmm'$. This bound combined with (5.23) implies

$$\begin{aligned}
&\left|\Gamma\left(\frac{a+it+\overline{\mu_{\pi\times\widetilde{\pi}'}(l)}}{2}\right)\Gamma\left(\frac{1+a+it-\mu_{\pi\times\widetilde{\pi}'}(l)}{2}\right)\right|\\
&\times\left|\sin\frac{\pi\left(1-a-it+\mu_{\pi\times\widetilde{\pi}'}(l)\right)}{2}\right|\\
&\leq \exp\left[\frac{2a-1}{4}\log\left(\left(\frac{1+a+\mu_R}{2}\right)^2 + \left(\frac{|t|+\mu_I}{2}\right)^2\right)\right.\\
&\left.+ \frac{2a+1}{6(a+\mathrm{Re}\mu_{\pi\times\widetilde{\pi}'}(l))(1+a-\mathrm{Re}\mu_{\pi\times\widetilde{\pi}'}(l))} + \log 2\pi\right].
\end{aligned}$$

Substituting it into (5.19), we obtain (5.18), and the proof is complete. □

The first explicit upper bound for coefficients in the Laurent series expansion of the Riemann zeta function about $s = 1$ has been given by Briggs [6]. Then, Matsuoka studied the asymptotic behaviour of these coefficients and he gave an excellent upper bound for its in [25]. Results related to upper bound for Stieltjes constants for the Dirichlet L-function when χ is a primitive character modulo q is given in [33], those for the Hurwitz zeta function in [3]. The investigation of Stieltjes constants for functions from the extended Selberg class $\mathcal{S}^\sharp$ is done and an upper bound for these coefficients is obtained in [18].

The following theorem is the main result of the paper, it gives a bound for the coefficients under consideration.

THEOREM 5.2. *Let E be a Galois extension of $\mathbb{Q}$ of finite degree d and let $L(s,\pi\times\widetilde{\pi}')$ be Rankin-Selberg $L-$function attached to the product $\pi\times\widetilde{\pi}'$ two irreducible automorphic unitary cuspidal representations of $GL_m(\mathbb{A}_E)$ and $GL_{m'}(\mathbb{A}_E)$ with pole at $s=1+it_0$ if $m=m'$ and $\pi'\cong\pi\otimes|det|^{it_0}$, otherwise $t_0=0$. Let $\mu_R=\max\limits_{l=1,\dots,dmm'}|\mathrm{Re}\mu_{\pi\times\widetilde{\pi}'}(l)|$, $\mu_I=\max\limits_{l=1,\dots,dmm'}|\mathrm{Im}\mu_{\pi\times\widetilde{\pi}'}(l)|$ and $\mu_{R,I}=\max\{\mu_R,\mu_I+t_0-1\}$. Let $a>\max\{1+\varepsilon,\mu_{R,I},|t_0|+\mu_I-\mu_{R,I}\}$ and $\frac{1}{2}(1-a+\mathrm{Re}\mu_{\pi\times\widetilde{\pi}'}(l))\notin\mathbb{Z}$ for all $l=1,\dots,dmm'$. For positive integer k such that $k>dmm'\left(a-\frac{1}{2}\right)$ we have*

$$|\gamma_{\pi,\pi'}(k)|\leq D_L(a)a^{-k}\left(2+\mu_{R,I}+\mu_I+\frac{4}{k-dmm'\frac{2a-1}{2}}\right)+\delta(t_0)\left|\operatorname*{Res}_{s=it_0}L(s,\pi\times\widetilde{\pi}')\right|, \tag{5.24}$$

where constant $D_L(a)$ is defined by

$$D_L(a)=\exp\left(\frac{2a+1}{6}\sum_{l=1}^{dmm'}\frac{1}{(a+\mathrm{Re}\mu_{\pi\times\widetilde{\pi}'}(l))(1+a-\mathrm{Re}\mu_{\pi\times\widetilde{\pi}'}(l))}\right)\times 2^{\frac{dmm'}{2}\left(3a+\frac{1}{2}\right)}\frac{Q_{\pi\times\widetilde{\pi}'}^{a-\frac{1}{2}}}{\pi}\left(\frac{a}{\pi}\right)^{dmm'\left(a-\frac{1}{2}\right)}\left(\sum_{n=1}^{+\infty}\frac{|a_{\pi\times\widetilde{\pi}'}(n)|}{n^a}\right).$$

PROOF. From the integral representation of generalized Euler-Stieltjes coefficients given in Proposition 4.1, and using the bound obtained in Lemma 5.1, we have

$$|\gamma_{\pi,\pi'}(k)|\leq C_L(a)\frac{Q_{\pi\times\widetilde{\pi}'}^{a-\frac{1}{2}}}{2\pi}\int_{-\infty}^{+\infty}\left[\left(\frac{1+a+\mu_R}{2}\right)^2+\left(\frac{|t|+\mu_I}{2}\right)^2\right]^{dmm'\frac{2a-1}{4}}\times\frac{\left|\overline{L(a-it,\pi\times\widetilde{\pi}')}\right|}{(a^2+(t+t_0)^2)^{\frac{k+1}{2}}}dt+\delta(t_0)\left|\operatorname*{Res}_{s=it_0}L(s,\pi\times\widetilde{\pi}')\right|,$$

where $C_L(a)$ is defined in Lemma 5.1.

Since the Rankin-Selberg $L-$function possesses a Dirichlet series representation (2.4) that converges absolutely for $\mathrm{Re}s>1$, for $a>1+\varepsilon>1$, one yields

$$\left|\overline{L(a-it,\pi\times\widetilde{\pi}')}\right|\leq\sum_{n=1}^{+\infty}\frac{|a_{\pi\times\widetilde{\pi}'}(n)|}{n^a}<+\infty,$$

hence

$$|\gamma_{\pi,\pi'}(k)| \leq C_L(a)\frac{Q_{\pi\times\widetilde{\pi}'}^{a-\frac{1}{2}}}{2\pi}\sum_{n=1}^{+\infty}\frac{|a_{\pi\times\widetilde{\pi}'}(n)|}{n^a}I + \delta(t_0)\left|\operatorname*{Res}_{s=it_0} L(s,\pi\times\widetilde{\pi}')\right|, \tag{5.25}$$

where

$$I = \int_{-\infty}^{+\infty}\left[\left(\frac{1+a+\mu_R}{2}\right)^2 + \left(\frac{|t|+\mu_I}{2}\right)^2\right]^{dmm'\frac{2a-1}{4}} \frac{dt}{(a^2+(t+t_0)^2)^{\frac{k+1}{2}}}.$$

Thus, it is left to derive a bound for the integral I. Depending on the value of t_0, we examine two cases.

(i) Let $t_0 \geq 0$. Then

$$I = \int_0^{+\infty}\left(\frac{1}{(a^2+(t+t_0)^2)^{\frac{k+1}{2}}} + \frac{1}{(a^2+(t-t_0)^2)^{\frac{k+1}{2}}}\right) \times \left[\left(\frac{1+a+\mu_R}{2}\right)^2 + \left(\frac{t+\mu_I}{2}\right)^2\right]^{dmm'\frac{2a-1}{4}} dt. \tag{5.26}$$

The interval of integration we derive into two parts. Denote by I_1 and I_2 integrals that correspond to intervals $(0,B)$ and $(B,+\infty)$, respectively, where $B = 1+a+\mu_{R,I}-\mu_I > t_0+1$.

For I_1 we have

$$I_1 \leq 2(2+\mu_{R,I}+\mu_I)8^{dmm'\frac{2a-1}{4}}a^{-k+\frac{2a-1}{2}dmm'}, \tag{5.27}$$

since $1+a+\mu_R \leq 1+a+\mu_{R,I} < 4a$ and $\frac{B}{a} \leq 2+\mu_{R,I}+\mu_I$, by assumptions of the theorem.

For integral I_2, we have $t \geq B$,

$$\left(\frac{1+a+\mu_R}{2}\right)^2 + \left(\frac{t+\mu_I}{2}\right)^2 \leq 2\left(\frac{t+\mu_I}{2}\right)^2,$$

and $(t+t_0)^2 \geq (t-t_0)^2$, so

$$\begin{aligned} I_2 &\leq \int_B^{+\infty}\frac{2}{(a^2+(t-t_0)^2)^{\frac{k+1}{2}}}\left[2\left(\frac{t+\mu_I}{2}\right)^2\right]^{dmm'\frac{2a-1}{4}} dt \\ &\leq \int_{B-t_0}^{+\infty}\left(\frac{t+t_0+\mu_I}{t}\right)^{k+1}\frac{2^{1-dmm'\frac{2a-1}{4}}}{(t+t_0+\mu_I)^{k+1}}(t+t_0+\mu_I)^{dmm'\frac{2a-1}{2}} dt. \end{aligned}$$

Furthermore, since the function $g(t) = \frac{t+t_0+\mu_I}{t}$ is monotonically decreasing for $t \geq B-t_0$, $g(t) > 1$ and $\lim\limits_{t\to+\infty} g(t) = 1$, it follows that maximal value of $g(t)$ is at point $t = B - t_0$ and it is equal to $\frac{B+\mu_I}{B-t_0}$. Hence,

$$I_2 \leq \left(\frac{B+\mu_I}{B-t_0}\right)^{k+1} 2^{1-dmm'\frac{2a-1}{4}} \times \int_{B-t_0}^{+\infty} (t+t_0+\mu_I)^{-(k+1)+dmm'\frac{2a-1}{2}}\, dt.$$

For constant a under consideration, we have $a < \frac{1}{2} + \frac{k}{dmm'}$, thus the above integral converges and yields

$$I_2 \leq \frac{2^{1-dmm'\frac{2a-1}{4}}}{k - dmm'\frac{2a-1}{2}} \frac{(1+a+\mu_{R,I})^{1+dmm'\frac{2a-1}{2}}}{(1+a+\mu_{R,I}-\mu_I-t_0)^{k+1}}.$$

Additionally, since $\mu_{R,I} = \max\{\mu_R, \mu_I + t_0 - 1\}$ inequalities $1 + a + \mu_{R,I} - \mu_I - t_0 > a > 1 + \varepsilon > 1$ hold true. Also, $1 + a + \mu_{R,I} < 4a$. Thus

$$I_2 \leq \frac{8^{1+dmm'\frac{2a-1}{4}}}{k - dmm'\frac{2a-1}{2}} a^{-k+dmm'\frac{2a-1}{2}}. \tag{5.28}$$

Substituting (5.27) and (5.28) into (5.26), combined with (5.25) implies (5.24).

(ii) The result for the case $t_0 < 0$ can be derived completely analogously as in (i) using simple substitution $-t_0 = t_1 > 0$.

The proof is complete. □

Acknowledgements.

I would like to thank the anonymous referee for his/her comments and suggestions that helped me to improve the paper.

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