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GROUPS $S_n \times S_m$ IN CONSTRUCTION OF FLAG-TRANSITIVE BLOCK DESIGNS

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Abstract. In this paper, we observe the possibility that the group $S_n \times S_m$ acts as a flag-transitive automorphism group of a block design with point set $\{1, \ldots, n\} \times \{1, \ldots, m\}, 4 \leq n \leq m \leq 70$. We prove the equivalence of that problem to the existence of an appropriately defined smaller flag-transitive incidence structure. By developing and applying several algorithms for the construction of the latter structure, we manage to solve the existence problem for the desired designs with nm points in the given range. In the vast majority of the cases with confirmed existence, we obtain all possible structures up to isomorphism.

1. Introduction and motivation

By definition, a flag-transitive block design is one that has an automorphism group acting transitively on the set of ordered pairs of incident points and blocks.

In the procedure of finding all nontrivial flag-transitive designs with the chosen point set Ω one must ensure that one has considered all subsets $B \subseteq \Omega$ as possible base blocks and all transitive subgroups of $Sym(\Omega)$ or $Alt(\Omega)$ as possible automorphism groups. To begin this work, which is challenging in many respects, one can use the well-known result of Cameron and Praeger [4, Proposition 1.1]. Namely, its equivalent form states that a flag-transitive block design can only occur as a substructure within an overstructure which is itself a flag-transitive design. This approves transitive maximal subgroups of $Sym(\Omega)$ or $Alt(\Omega)$ to be chosen as initial objects for consideration as possible

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automorphism groups. In [2] the designs with $\Omega = \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ and wreath product $S_n \wr S_2$ as an automorphism group were considered.
One of the maximal subgroups of $S_n \wr S_2$ is the group S_n^2, so we naturally want to explore which corresponding flag-transitive designs it yields. Moreover, it is interesting to see which designs have the group $S_n \times S_m$, $n \neq m$, as their full automorphism group, and that is the subject of this research. We modify and expand the techniques used in [2] to consider the group $S_n \times S_m$, for $n \leq m$, both in theory and in the construction algorithm, and obtain interesting theoretical results (Theorem 3.1 and the related remarks) as well as some new designs that correspond with group $S_n \times S_m$. A complete list of constructed designs can be found on our web page, [7].
For our construction, we use the software package MAGMA [1], and its libraries for transitive and primitive groups. We develop algorithms for construction in all cases of transitive group actions, whether that action is primitive or imprimitive. The range of our research is $4 \leq n \leq m \leq 70$, limited by the size of the MAGMA libraries.
In Section 2 we first review the definitions and known results from design theory. Section 3 contains detailed explanations of the theory behind our construction as well as the existence conditions for our designs, and Section 4 describes the construction. Section 5 contains some additional observations on constructed designs when $m = n$ and connects to the results in [2]. Finally, in Section 6 we present the table with the numbers of obtained designs, sorted by n and m.

2. Preliminaries

In this section, we overview some known concepts and definitions from design theory. Throughout the paper we use the standard notation of group theory [9].

An **incidence structure** is a triple $\Gamma = (\Omega, \mathcal{B}, \mathcal{I})$, where Ω and $\mathcal{B}$ are disjoint sets and $\mathcal{I}$ is a relation on $\Omega \times \mathcal{B}$. Elements of Ω are called **points** and elements of $\mathcal{B}$ are called **blocks**. Each pair $(p, B) \in \mathcal{I}$ is called a **flag**. If a pair (p, B) belongs to $\mathcal{I}$ we say that point p is **incident** with block B.

For $p \in \Omega$ and $B \in \mathcal{B}$ we denote

$$(p) = \{B \in \mathcal{B} : (p, B) \in \mathcal{I}\},$$
$$(B) = \{p \in \Omega : (p, B) \in \mathcal{I}\},$$

while cardinal numbers $|(p)|$ and $|(B)|$ are called the **degree of point** p and the **degree of block** B, respectively. If $|(p)| = 0$ ($|(B)| = 0$) then p is called an isolated point (B is called an isolated block).

An incidence structure is called **simple** if $B_i, B_j \in \mathcal{B}$, and $(B_i) = (B_j)$ imply $B_i = B_j$. If $\mathcal{B}$ is a set of subsets of nonempty, finite set Ω and $\mathcal{I}$ is the membership relation $\in$, then such an incidence structure is simple with $(B) = B$. It is shortly denoted by $(\Omega, \mathcal{B})$.

Incidence structure Γ can be presented by a bipartite graph whose vertices are points and blocks of Γ, and edges are incidences. This graph is called the **incidence graph** or **Levi graph** of Γ. If this graph is connected then we say that incidence structure Γ is connected. If $|\Omega| = |\mathcal{B}|$ holds for Γ, then we say that Γ is a **symmetric structure**. The **dual structure** of $\Gamma = (\Omega, \mathcal{B}, \mathcal{I})$ is the incidence structure $\Gamma^* = (\mathcal{B}, \Omega, \mathcal{I}^*)$, where $\mathcal{I}^* = \{(B, p) : (p, B) \in \mathcal{I}\} \subseteq \mathcal{B} \times \Omega$.

Two incidence structures $\Gamma_1 = (\Omega_1, \mathcal{B}_1, \mathcal{I}_1)$ and $\Gamma_2 = (\Omega_2, \mathcal{B}_2, \mathcal{I}_2)$ are **isomorphic** ih there exists a bijection $\phi : \Omega_1 \cup \mathcal{B}_1 \to \Omega_2 \cup \mathcal{B}_2$ such that $\phi(\Omega_1) = \Omega_2$, $\phi(\mathcal{B}_1) = \mathcal{B}_2$, and for each point $p \in \Omega_1$ and each block $B \in \mathcal{B}_1$ the equivalency $(p, B) \in \mathcal{I}_1 \iff (\phi(p), \phi(B)) \in \mathcal{I}_2$ holds. Function ϕ is an **isomorphism** from incidence structure Γ_1 to Γ_2. If Γ_1 and Γ_2 are isomorphic we write $\Gamma_1 \cong \Gamma_2$.

An **automorphism** of incidence structure $\Gamma = (\Omega, \mathcal{B}, \mathcal{I})$ is an isomorphism $\phi : \Gamma \to \Gamma$. **The Full automorphism group** of Γ, denoted by $Aut\Gamma$, is the group of all automorphisms of structure Γ. Any subgroup $H \leq Aut\Gamma$ is called an **automorphism group** of Γ.

A bijection $\psi : \Omega \cup \mathcal{B} \to \Omega \cup \mathcal{B}$ is called **duality** of $\Gamma = (\Omega, \mathcal{B}, \mathcal{I})$ if $\psi(\Omega) = \mathcal{B}, \psi(\mathcal{B}) = \Omega$ and for each $(p, B) \in \mathcal{I}$ it holds $(\psi(B), \psi(p)) \in \mathcal{I}$. An incidence structure admitting a duality is isomorphic to its dual structure and $|\Omega| = |\mathcal{B}|$ holds. By $Aut^*\Gamma$ we denote the group generated by automorphisms of Γ and its dualities if any exist.

We say that **group G acts on incidence structure** $\Gamma = (\Omega, \mathcal{B}, \mathcal{I})$ if G acts on sets Ω, $\mathcal{B}$ and for each $(p, B) \in \mathcal{I}$ and each $g \in G$ holds $g(p, B) = (g(p), g(B)) \in \mathcal{I}$. Group G **is flag-transitive** on Γ if it acts transitively on $\mathcal{I}$. For incidence structure Γ without isolated points or isolated blocks, we say that Γ is **flag-transitive** with regard to group $G \leq Aut\Gamma$, if G is flag-transitive on Γ.

The following characterization of flag-transitive structures will be used throughout the paper. Let $\Gamma = (\Omega, \mathcal{B}, \mathcal{I})$ be an incidence structure without isolated points and blocks with group G acting on it, let $p \in \Omega$ and $B \in \mathcal{B}$. Γ is flag-transitive if and only if

1. G is transitive on points and G_p is transitive on (p); or
2. G is transitive on blocks and G_B is transitive on (B).

Let $\Gamma = (\Omega, \mathcal{B}, \mathcal{I})$ be a flag-transitive incidence structure. It is easy to see that the degrees of all points are equal and the degrees of all blocks are equal, so we can denote

$$a := |\Omega|, \text{ the number of points;}$$
$$b := |\mathcal{B}|, \text{ the number of blocks;}$$
$$s := |(p)|, \text{ for each } p \in \Omega, \text{ the degree of point } p;$$
$$t := |(B)|, \text{ for each } B \in \mathcal{B}, \text{ the degree of block } B.$$

If this case, we introduce that Γ is **of type** $s^a t^b$. Obviously, $sa = tb$ holds. One can easily see that if Γ is flag-transitive and of type $s^a t^b$, then Γ^* is also flag-transitive and of type $t^b s^a$.

Besides dual structure, for $\alpha, \beta \in \mathbb{N}$ we make use of incidence structures $\alpha\Gamma$ and $\Gamma_{(\beta)}$. Namely, if $\Gamma_1 = (\Omega_1, \mathcal{B}_1, \mathcal{I}_1)$ and $\Gamma_2 = (\Omega_2, \mathcal{B}_2, \mathcal{I}_2)$ are incidence structures, then by $\Gamma_1 + \Gamma_2$ we denote incidence structure $(\Omega_1 \sqcup \Omega_2, \mathcal{B}_1 \sqcup \mathcal{B}_2, \mathcal{I}_1 \sqcup \mathcal{I}_2)$, where $\sqcup$ denotes disjoint union. If we add the same structure $\Gamma = (\Omega, \mathcal{B}, \mathcal{I})$ α times, the obtained structure is denoted by $\alpha\Gamma$.
Further, we define $\Gamma_{(\beta)} = (\Omega, \{1, ..., \beta\} \times \mathcal{B}, \mathcal{I}_{(\beta)})$, where $\mathcal{I}_{(\beta)}$ contains those and only those pairs $(p, (i, B)) \in \Omega \times (\{1, ..., \beta\} \times \mathcal{B})$ such that $(p, B) \in \mathcal{I}$, for all $i \in \{1, ..., \beta\}$. We say that blocks in $\Gamma_{(\beta)}$ are **multiples** of blocks in $\mathcal{B}$.
It can be easily proven that the following statements hold:

PROPOSITION 2.1. *If Γ is a flag-transitive incidence structure of type $s^a t^b$, then*

1. *$\alpha\Gamma$ is flag-transitive incidence structure of type $s^{\alpha a} t^{\alpha b}$, for each $\alpha \in \mathbb{N}$.*
2. *There exists $l \in \mathbb{N}$ and a connected flag-transitive incidence structure Φ such that $\Gamma \cong l\Phi$.*
3. *$\Gamma_{(\beta)}$ is flag-transitive incidence structure of type $(\beta s)^a t^{\beta b}$, for each $\beta \in \mathbb{N}$.*
4. *There exist $k \in \mathbb{N}$ and a simple flag-transitive incidence structure Ψ such that $\Gamma \cong \Psi_{(k)}$.*

Now let Ω be a set, $B \subseteq \Omega$, and G a subgroup of $Sym(\Omega)$ that acts transitively on Ω. Further, let $\mathcal{B} = \{g(B) : g \in G\}$ and let the group G_B, which is the stabilizer of set B, act transitively on B. The corresponding simple incidence structure $(\Omega, \mathcal{B})$ is flag-transitive and we denote it by $I(\Omega, G, B)$. In this way, each simple flag-transitive incidence structure can be described up to isomorphism. Moreover, according to Proposition 2.1, if Γ is flag-transitive incidence structure, there exists $k \in \mathbb{N}$ such that $\Gamma \cong I(\Omega, G, B)_{(k)}$, which we use in our construction.

Let group G act transitively on set X. A partition $\{\Delta_i \subset X : i \in \{1, ..., d\}\}$ of the set X is called a **block system** of this action if for each $i \in \{1, ..., d\}$ and for each $g \in G$ there exists $j \in \{1, ..., d\}$ such that $g(\Delta_i) = \Delta_j$. Each transitive action has at least two trivial block systems, a partition to singletons and the entire set X. A non-trivial block system is called a **system of imprimitivity** and its elements are called **blocks of imprimitivity**. They are of the same cardinality. If G-action preserves at least one non-trivial block system it is called **imprimitive on points**, and group G is called **imprimitive**. Otherwise, G is **primitive**.

Related to an imprimitive, flag-transitive action of group G on incidence structure $\Gamma = (\Omega, \mathcal{B}, \mathcal{I})$ one can observe substructures and factor structures. The notions were introduced for 2-designs in [5, 10]. We use the following definition.

DEFINITION 2.2. *Let group G act transitively on incidences and imprimitively on points of the incidence structure $\Gamma = (\Omega, \mathcal{B}, \mathcal{I})$. Let Σ be a system of imprimitivity of G on Ω and $\Delta \in \Sigma$. Incidence structure $\Gamma_\Delta = (\Delta, \mathcal{B}_\Delta, \mathcal{I}_\Delta)$, with $\mathcal{B}_\Delta = \{B \in \mathcal{B} : \Delta \cap (B) \neq \emptyset\}$ and $\mathcal{I}_\Delta = \mathcal{I} \cap (\Delta \times \mathcal{B}_\Delta)$, is called **incidence substructure** of Γ with respect to block of imprimitivity Δ. Incidence structure $\Gamma/\Sigma = (\Sigma, \mathcal{B}, \mathcal{I}_\Sigma)$, with $(\Delta, B) \in \mathcal{I}_\Sigma$ iff $\Delta \cap (B) \neq \emptyset$, is called **factor structure** of Γ with respect to the system of imprimitivity Σ.*

More details on imprimitive group action and systems of imprimitivity can be found, for instance, in [6]. The most important facts for this research are found in the following statement.

PROPOSITION 2.3. *Let Γ be a flag-transitive incidence structure of type $s^a t^b$ where $Aut\Gamma$ acts imprimitively on points with the system of imprimitivity Σ that has d blocks of imprimitivity of size c and let $|\mathcal{B}_\Delta| = r$. Then the following holds.*

1. There exists $\mu \in \mathbb{N}$ such that substructure Γ_Δ is a flag-transitive incidence structure of type $s^c \mu^r$;

2. There exists $\delta \in \mathbb{N}$ such that factor structure Γ/Σ is a flag-transitive incidence structure of type $r^d \delta^b$.

The parameters involved satisfy

$$cd = a, \mu\delta = t, r\mu = sc \text{ and } rd = \delta b. \tag{2.1}$$

In our construction procedures, which we describe in Section 4, we solve the system (2.1) with obvious inequalities $\mu \leq c$ and $\delta \leq d$.

Now let us turn to designs and some previous results. It is well-known that a finite incidence structure $\Gamma = (\Omega, \mathcal{B}, \mathcal{I})$ is a (v, k, λ) **block design** if $|\Omega| = v$, $|(B)| = k$ for all blocks $B \in \mathcal{B}$, and $|(p) \cap (q)| = \lambda$ for each two different points $p, q \in \Omega$.
The following two theorems are important results in design theory [4, 10].

THEOREM 2.4. *If $I(\Omega, H, B)$ is a flag-transitive block design and $H \leq G \leq Sym(\Omega)$, then $I(\Omega, G, B)$ is a flag-transitive block design.*

Theorem 2.5. *(Zieschang) Let group G act flag-transitively on an incidence structure $\Gamma = (\Omega, \mathcal{B})$, where $|\Omega| = v$ and $|B| = k$. Let $B \in \mathcal{B}$ be any block and $p \in \Omega$ any point of that block. Then Γ is a block design if and only if*

$$\frac{|B \cap O|}{|O|} = \frac{k-1}{v-1},$$

for each orbit O of the stabilizer G_p in $\Omega \setminus \{p\}$.

3. The existence of flag-transitive designs with automorphism group $S_n \times S_m$

In this paper, we are interested in imprimitive flag-transitive block designs with $\Omega = \{1, ..., n\} \times \{1, ..., m\}$, $n \leq m$. We will observe the group

$$G = S_n \times S_m.$$

The action of G is natural, $(\psi, \phi)(i, j) = (\psi(i), \phi(j))$, for each $\psi \in S_n, \phi \in S_m, (i, j) \in \Omega$.
Our goal is the construction of flag-transitive block designs

$$I(\Omega, G, B).$$

If $B \subseteq \Omega$ then by $\Gamma(B)$ we denote the incidence structure with point set $\{x : (x, y) \in B\}$, block set $\{y : (x, y) \in B\}$ and the set of flags B. We call this incidence structure an **incidence structure induced by** B.

From $B \subseteq \Omega$ it follows that $Aut\Gamma(B)$ is naturally inserted in G, because it is isomorphic to a subgroup of G. Let $\Gamma(B) = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ such that $|\mathcal{P}| = a$, $|\mathcal{B}| = b$. The group $Aut\Gamma(B)$ stabilizes B, so it holds $G_B \cong Aut\Gamma(B) \times S_{n-a} \times S_{m-b}$.

We seek to find such structures $\Gamma(B)$, induced by the set $B \subseteq \Omega$ that yield a design that has B as a base block. The existence condition of these designs is presented in the following theorem.

Theorem 3.1. *Let $m, n \in \mathbb{N} \setminus \{1\}$, $\Omega = \{1, ..., n\} \times \{1, ..., m\}$, $\mathcal{G} = S_n \times S_m$ and $B \subseteq \Omega$. Then $I(\Omega, G, B)$ is a block design on which G acts flag-transitively if and only if $\Gamma(B)$ is a flag-transitive incidence structure of type $s^a t^b$ where*

$$n = a + \frac{a-t}{s-1}, \ m = b + \frac{b-s}{t-1}. \tag{3.2}$$

Proof. Let $I(\Omega, G, B)$ be a block design on which G acts flag-transitively. We know that the action of block stabilizer G_B is transitive, so given that B is a set of flags for $\Gamma(B)$, it follows that $\Gamma(B)$ is flag transitive. Let a be the number of points of $\Gamma(B)$, b the number of blocks, s the degree of points and t the degree of blocks. Now we apply Theorem 2.5 on block design $I(\Omega, G, B)$ with nm points, taking, without loss of generality, $(1, 1) \in B$ and observing the action of its stabilizer. $G_{(1,1)}$ has four orbits, three of them non-trivial:

$$O_1 = \{(1,1)\};$$
$$O_2 = \{(x,1) : x = 2, ..., n\};$$
$$O_3 = \{(1,y) : y = 2, ..., m\};$$
$$O_4 = \{(x,y) : x = 2, ..., n; y = 2, ..., m\}.$$

It holds:

$|O_2| = n-1; |O_3| = m-1; |O_4| = (n-1)(m-1)$.

The number $|B| = sa = tb$ of flags in $\Gamma(B)$ is equal to the size k of blocks in $I(\Omega, G, B)$, thus Theorem 2.5 for orbits O_2 and O_3, respectively, gives:

$$\frac{t-1}{n-1} = \frac{tb-1}{nm-1}, \frac{s-1}{m-1} = \frac{sa-1}{nm-1}.$$

As $tb = sa$ holds, these equations imply

$$\frac{t-1}{n-1} = \frac{s-1}{m-1}.$$

By substituting $n = \dfrac{(m-1)(t-1)}{(s-1)} + 1$ into $\dfrac{t-1}{n-1} = \dfrac{tb-1}{nm-1}$ we come to the expression for m:

$$m = b + \frac{b-s}{t-1}.$$

Analogously we obtain

$$n = a + \frac{a-t}{s-1}.$$

For the reverse implication, let $\Gamma(B)$ be a flag-transitive structure of type $s^a t^b$ with numerical conditions (3.2) on n and m. We must prove that $I(\Omega, G, B)$ is a flag-transitive block design. Given that $\Gamma(B)$ is flag-transitive, it is transitive with regard to group $Aut\Gamma(B)$ and since G_B is isomorphic to $Aut\Gamma(B) \times S_{n-a} \times S_{m-b}$ it follows that G_B acts transitively on B, so $I(\Omega, G, B)$ is flag-transitive structure. The assumed numerical conditions (3.2) and Theorem 2.5 imply that it is also a block design. ∎

Remark 3.2. From

$$\frac{t-1}{n-1} = \frac{tb-1}{nm-1}, \quad \frac{s-1}{m-1} = \frac{sa-1}{nm-1},$$

and $tb = sa$ we have the following.

1. If $m = n$, then $t = s$ and $a = b$.
2. If $n < m$, then $t < s$ and $a < b$.

Remark 3.3. Let $B, B' \subseteq \Omega$ such that flag-transitive structures $\Gamma(B)$ and $\Gamma(B')$ are of the same type $s^a s^a$ and dually isomorphic, i.e. $\Gamma(B) \cong \Gamma(B')^*$. Then the corresponding block designs are isomorphic. This will be applied in the construction process described in Section 4.

4. Construction of flag-transitive block designs

In this section, we give the construction algorithms that produce flag-transitive incidence structures of type $s^a t^b$, whose parameters n, m, s, a, t, and b satisfy conditions (3.2) from Theorem 3.1. From the algorithms, it is clear that the constructed structures are isomorphic to the structures $\Gamma(B)$ introduced in Section 3, thus each structure obtained in this way corresponds to a flag-transitive block design with point set $\{1, ..., n\} \times \{1, ..., m\}$.

The starting point of the construction is the choice of $m \in \mathbb{N}$, $4 \leq m \leq 70$. In the construction course for each given pair (n, m), $n \leq m$, we first calculate all possible parameter sets $(n, m; (s, a, t, b))$. Parameter a is the number of points of $\Gamma(B)$, which means that with each admissible parameter a, for construction we need to have at disposal all transitive groups of degree a. In that sense, Magma library of transitive groups will cover the range $a \leq 31$. For $a > 31$ we have only Magma library of primitive groups at disposal. To obtain flag-transitive incidence structures of type $s^a t^b$ with $a > 31$ and imprimitive group action, we develop a construction method that involves substructures and factor structures (defined in Section 2). In this way, we reduce the degree of the necessary group action so that it suffices to use the Magma library of transitive groups.

The construction procedure denoted as *Algorithm FT-Simple* produces simple flag-transitive incidence structures of type $s^a t^b$.

Algorithm FT-Simple

(i) Given the quadruple of parameters (s, a, t, b), find all transitive groups G of degree a, whose order is divisible by tb.
(ii) For each group G selected in step (i) find all subgroups $H < G$ of index b. For each such H find H-orbits on the set $\{1, ..., a\}$.
(iii) Select and save H-orbits of size t (the degree of a prospective block).
(iv) Find orbits of t-sets from step (iii) under the action of group G. Save those of length b (number of blocks). Each such collection consisting of b t-subsets of $\{1, ..., a\}$ presents the block set of a simple flag-transitive incidence structure of type $s^a t^b$.
(v) Check every structure obtained in step (iv) for isomorphism with the previously obtained ones in that step or their duals, and rule out the surplus.

Structures with repeated blocks, if any, are constructed through the following algorithm.

Algorithm FT-Multi

(i) Given the quadruple of parameters (s, a, t, b), find common divisors of s and b greater than 1, if any.
(ii) For each common divisor β found in step (i) execute *Algorithm FT-Simple* with parameters $(\frac{s}{\beta}, a, t, \frac{b}{\beta})$.
(iii) For each flag-transitive structure Γ obtained in step (ii) construct structure $\Gamma_{(\beta)}$.
(iv) Check every structure $\Gamma_{(\beta)}$ of type $s^a t^b$ obtained in step (iii) for isomorphism with the previously obtained ones in that step or their duals and rule out the surplus.

In practice, due to the availability of the libraries of acting transitive groups, the above two algorithms are directly applicable in all cases with $a \leq 31$. If $a > 31$, then they can be used to obtain only flag-transitive incidence structures of type $s^a t^b$ with primitive group action. The procedure we developed to construct those with imprimitive group action is described in *Algorithm FT - Imprimitive*. The procedure includes substructures and factor structures related to the given parameters (s, a, t, b); we presume that systems of imprimitivity consist of d blocks of size c.

Algorithm FT - Imprimitive

(i) Given the quadruple of parameters (s, a, t, b), find all pairs $(c, d) \in \mathbb{N}^2$, $c, d > 1$ such that $cd = a$.
(ii) For each resulting pair (c, d) in step (i) solve the system (2.1) from Proposition 2.3 to obtain the related pairs $((s, c, \mu, r), (r, d, \delta, b))$ of quadruples determining substructure and factor structure, respectively.
If the system (2.1) does not have a solution for any pair (c, d), conclude that the structure of type $s^a t^b$ does not exist for that (c, d).
(iii) If the set of solutions in step (ii) is nonempty, with each component of a pair $((s, c, \mu, r), (r, d, \delta, b))$ that it contains, execute *Algorithm FT-Simple* and *Algorithm FT-Multi*.
If construction fails at one component, rule out the corresponding pair. If both structures related to a pair $((s, c, \mu, r), (r, d, \delta, b))$ are successfully constructed, through any of the two algorithms, proceed to use them to obtain a structure of type $s^a t^b$. For each pair of structures constructed proceed as follows.
(iv) Find the full automorphism group of both structures (by one command in Magma each) and select their subgroups with flag-transitive action thus forming the lists *SubG_List* and *FactG_List*, respectively.
(v) For each ordered pair (SG, FG), $SG \in SubG\_List$, $FG \in FactG\_List$, make wreath product $SG \wr FG$ with imprimitive action, [3].
(vi) For each group $SG \wr FG$ obtained in the previous step go down the lattice of its subgroups and select transitive subgroups H such that

$$(H_\Delta)^\Delta = SG \text{ and } H^\Sigma = FG.$$

(For a group G and a set X, the notation of G^X is explained in [3].) Save all obtained groups in the *Candidate_List*.

(vii) Proceed with *Algorithm FT-Simple*, from step (ii), using the parameters (s, a, t, b) and the *Candidate_List*.

(viii) Check every structure of type $s^a t^b$ obtained in step (vii) for isomorphism with the previously obtained ones in that step or their duals and rule out the surplus.

(ix) Proceed with *Algorithm FT-Multi*, from step (ii), using the parameters (s, a, t, b) and the *Candidate_List*.

(x) Check every structure of type $s^a t^b$ obtained in step (ix) for isomorphism with the previously obtained ones in that step or their duals and rule out the surplus.

Remark 4.1. In our constructions, we faced memory problems on different levels. In some cases, the number of transitive groups of degree a was too big. For instance, for parameters $(28, 46; (6, 24, 4, 36))$ there are 25000 transitive groups of degree 24. Likewise, sometimes the number of their subgroups of index b (potential block stabilizers) was too large. Performing the following three procedures proved beneficial to solving or circumventing the problem.

1. When searching for a structure Γ with parameter set $(n, m; (s, a, t, b))$, if parameters a and b have a common divisor α greater than 1 then Γ can be constructed as a not connected sum $\Gamma = \alpha\Gamma'$ of structures Γ' with smaller parameters $(n, m; (s, \frac{a}{\alpha}, t, \frac{b}{\alpha}))$. Applied on smaller parameters, the procedures *Algorithm FT-Simple* and *Algorithm FT-Multi* expectantly run easier.

2. An approach to construction through smaller structures (substructures and factor structures) is included in *Algorithm FT - Imprimitive*. Therefore it can be used when faced with a memory problem even if $a < 32$. For instance, we used it to solve the problems with parameters $(39, 58; (4, 30, 3, 40))$ and $(35, 69; (3, 24, 2, 36))$.

3. Instead of performing *Algorithm FT-Simple* with certain group G of degree a, we inductively construct the set R_t which contains all representatives of G-orbits of t-subsets of $\{1, ..., a\}$. We start with set R_1 consisting of only one element from $\{1, ..., a\}$, usually 1. Then we observe sets $\{1, x\}$, $2 \leq x \leq a$, so as to build set R_2 containing all representatives *of* G-orbits of 2-subsets of $\{1, ..., a\}$. Once we have constructed R_k, we obtain R_{k+1} observing unions $r \cup \{x\}$ of cardinality $k + 1$, where $r \in R_k$, $x \in \{1, ..., a\}$, $x \notin r$, and selecting representatives of G-orbits of $(k + 1)$-subsets of $\{1, ..., a\}$. Eventually, we attain R_t and select t-subsets $\Lambda \subseteq \{1, ..., a\}$ from R_t for which the stabilizer G_Λ acts transitively on Λ to be base blocks of desired flag-transitive incidence structures of type $s^a t^b$. This method proved efficient for relatively small values

of t. For instance, we used it to solve the problem for several groups related to parameters $((49, 55); 10, 45, 9, 50)$, $(t = 9)$.

Complete results are given in Section 6.

Example 4.2. We present a step-by-step construction procedure in case $m = 9$. It begins with calculating admissible parameter sets of the form $(n, m; (s, a, t, b))$ to be $(5, 9; (3, 4, 2, 6))$ and $(9, 9; (3, 7, 3, 7))$. First, let us proceed with the parameters $(5, 9; (3, 4, 2, 6))$ to determine the existence of flag-transitive incidence structures with 4 points and 6 blocks (type $3^4 2^6$). The application of *Algorithm FT-Simple* gives the following.

(i) There are 5 transitive groups of degree 4: $S_4, A_4, D_8, (C_2)^2$ and C_4. Only the order of groups A_4 and S_4 is divisible by $tb = 2 \cdot 6 = 12$.

(ii) Let us first observe A_4 and its subgroups of index 6, i.e. of order $\frac{12}{6} = 2$. Up to conjugation it is $Z_2 = \{(), (1, 2)(3, 4)\}$. The orbits of Z_2-action on the set $\{1, ..., 4\}$ are $\{1, 2\}$ and $\{3, 4\}$.

(iii) The length of all orbits of Z_2 on the set $\{1, ..., 4\}$ is equal to the desirable block size $t = 2$, so the orbits $\{1, 2\}$ and $\{3, 4\}$ are possible base blocks.

(iv) Next, we find the orbit of $\{1, 2\}$ under the action of A_4 to be

$$\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\},$$

which is obviously also the orbit of $\{3, 4\}$. This orbit proves to have desirable length $b = 6$, meaning that it presents the block set of a simple flag-transitive incidence structure of type $3^4 2^6$ with point set $\{1, ..., 4\}$ and block set $\{\Lambda_1 = \{1, 2\}, \Lambda_2 = \{2, 3\}, \Lambda_3 = \{1, 3\}, \Lambda_4 = \{2, 4\}, \Lambda_5 = \{3, 4\}, \Lambda_6 = \{1, 4\}\}$. The structure is shown in Figure 1 where the blocks are denoted by their indices, reflecting the isomorphism of this structure to $\Gamma(B)$ as introduced in Section 3.

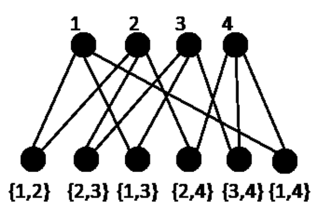

Figure 1. Levi graph of simple flag-transitive structure of type $3^4 2^6$

The same structure is obtained when executing steps (ii)-(iv) with group S_4.

The corresponding block design has 45 points, the elements of the set $\Omega = \{1, ..., 5\} \times \{1, ..., 9\}$. Its base block B consists of the flags of $\Gamma(B)$ and is of cardinality 12.

The application of the *Algorithm FT-Multi* for parameter set $(5, 9; (3, 4, 2, 6))$ yields one more flag-transitive structure of type $3^4 2^6$, this with repeated blocks. In step (i) a common divisor $\alpha = 3$ of $s = 3$ and $b = 6$ is found, so the procedure continues through executing *Algorithm FT-Simple* with parameters $(5, 9; (1, 4, 2, 2))$. Finally obtained structure is isomorphic to a structure with point set $\{1, 2, 3, 4\}$ and two different blocks, $\{1, 2\}$ and $\{3, 4\}$.

The procedure for parameter set $(9, 9; (3, 7, 3, 7))$ runs analogously and eventually yields one flag transitive incidence structure of type $3^7 3^7$.

5. Observations about constructed flag-transitive designs with $n = m$

In paper [2], one finds the presentation of construction of flag-transitive block designs with point set $\Omega = \{1, ..., n\} \times \{1, ..., n\}$, $n \in \mathbb{N} \setminus \{1\}$, and automorphism group $S_n \wr S_2 = (S_n \times S_n) \rtimes \langle \sigma \rangle$ which acts naturally. In the wreath product $S_n \wr S_2$ for all $\pi_1, \pi_2 \in S_n$ and $i, j \in \{1, ..., n\}$ one has $(\pi_1, \pi_2)(i, j) = (\pi_1(i), \pi_2(j))$, while $\sigma \in Sym(\Omega)$ is an involution, $\sigma(i, j) = (j, i)$. This action is primitive. For admissible choice of $B \subseteq \Omega$ the construction method, aimed at obtaining flag-transitive designs with base block B uses the idea of boiling the problem down to building smaller, appropriately defined "induced" incidence structures $\Gamma(B)$.

Here we take over the idea and proceed similarly. Although the observed structures $\Gamma(B)$ in the two pieces of research differ and the derived existence conditions for the aimed designs are different, on the grounds of Theorem 2.4 we can relate and compare our present results in case $n = m$ to the results obtained in [2].

In case $n = m$, S_n^2 is a subgroup of $S_n \wr S_2$ of index 2, while $S_n \wr S_2$ is a maximal subgroup of $Sym(\Omega)$ or $Alt(\Omega)$, for $n \geq 5$ [8]. According to Theorem 2.4, for a given $B \subseteq \Omega$ the existence of flag-transitive design $D_1 = I(\Omega, S_n^2, B)$ implies the existence of the flag-transitive design $D_2 = I(\Omega, S_n \wr S_2, B)$ and no design lies in between these two. Moreover, because we are interested in nontrivial designs only (which rules out groups $Sym(\Omega)$ and $Alt(\Omega)$), there are no bigger designs to be related to the existence of D_1.

Our attention in this section turns to the reverse problem: for the designs found in [2] we explore whether there exist their subdesigns isomorphic to the designs found in this research for $n = m$ and even whether some designs found in [2] are isomorphic to the designs found in this research for $n = m$. In that sense we pose a question: if $D_2 = I(\Omega, S_n \wr S_2, B)$ is a design, under which conditions $D_1 = I(\Omega, S_n^2, B)$ is also a design? Obviously, in terms of paper [2], D_2 must be a design of Class 1, as Class 2 and Class 3 do not refer to flag-transitive $\Gamma(B)$. Further, from [2] we know that for $\Gamma(B)$ of type $s^a t^b$ belonging to Class 1 holds $t \in \{s-1, s\}$. Now it is easily checked that only the possibility $t = s$ (which immediately gives $a = b$) ensures that conditions of Theorem 3.1 are satisfied and hence imply that D_2 is a design.

Now let us derive the conditions under which D_2 is isomorphic to D_1. If $D_2 \cong D_1$, then the action of σ does not produce new blocks in regard to the action of S_n^2. That is equivalent to the existence of $g \in S_n^2$ such that $g(B) = \sigma(B)$, so

$$g^{-1}\sigma(B) = B.$$

One easily verifies that $g^{-1}\sigma$ induces an automorphism of $\Gamma(B)$ which maps points to blocks and vice-versa, so $\Gamma(B)$ admits a duality. Finally, $D_1 \cong D_2$ is equivalent to $\Gamma(B) \cong \Gamma(B)^*$.

6. Results - number of designs

In this section, we present our results of the number of existing designs for each set of parameters obtained from the algorithm. The results are organized in two tables, separately for $n = m$ and for $n < m$. Related documentation as well as the constructed base blocks for each structure can be found in [7]. For $n = m$ the structures are of type $s^a s^a$, so in the table, we write: type s^a. For the number of designs, we shortly write $\#D$.

Table 1. The number of designs for $n = m$

n	type	$\#D$	n	type	$\#D$	n	type	$\#D$	n	type	$\#D$
4	2^{3}	1	28	7^{25}	0	44	22^{43}	0	60	2^{31}	1
6	2^{4}	2	28	14^{27}	0	45	3^{31}	1	60	3^{41}	0
6	3^{5}	0	30	2^{16}	4	45	5^{37}	0	60	4^{46}	1
8	2^{5}	1	30	3^{21}	3	45	9^{41}	0	60	5^{49}	0
8	4^{7}	1	30	5^{25}	2	45	15^{43}	0	60	6^{51}	1
9	3^{7}	1	30	6^{26}	4	46	2^{24}	7	60	10^{55}	6
10	2^{6}	3	30	10^{28}	0	46	23^{45}	0	60	12^{56}	≥ 64
10	5^{9}	0	30	15^{29}	0	48	2^{25}	2	60	15^{57}	0
12	2^{7}	1	32	2^{17}	1	48	3^{33}	1	60	20^{58}	0
12	3^{9}	2	32	4^{25}	4	48	4^{37}	1	60	30^{59}	0
12	4^{10}	4	32	8^{29}	0	48	6^{41}	0	62	2^{32}	5
12	6^{11}	1	32	16^{31}	1	48	8^{43}	0	62	31^{61}	0
14	2^{8}	3	33	3^{23}	0	48	12^{45}	15	63	3^{43}	1
14	7^{13}	0	33	11^{31}	0	48	16^{46}	0	63	7^{55}	0
15	3^{11}	0	34	2^{18}	5	48	24^{47}	0	63	9^{57}	4
15	5^{13}	0	34	17^{33}	0	49	7^{43}	1	63	21^{61}	0
16	2^{9}	2	35	5^{29}	0	50	2^{26}	3	64	2^{33}	3
16	4^{13}	2	35	7^{31}	0	50	5^{41}	1	64	4^{49}	2
16	8^{15}	2	36	2^{19}	1	50	10^{46}	0	64	8^{57}	1
18	2^{10}	3	36	3^{25}	1	50	25^{49}	0	64	16^{61}	0
18	3^{13}	1	36	4^{28}	10	51	3^{35}	1	64	32^{63}	4
18	6^{16}	6	36	6^{31}	2	51	17^{49}	0	65	5^{53}	0
18	9^{17}	0	36	9^{33}	0	52	2^{27}	3	65	13^{61}	0
20	2^{11}	1	36	12^{34}	0	52	4^{40}	40	66	2^{34}	3
20	4^{16}	9	36	18^{35}	1	52	13^{49}	0	66	3^{45}	4
20	5^{17}	0	38	2^{20}	5	52	26^{51}	0	66	6^{56}	≥ 59
20	10^{19}	0	38	19^{37}	0	54	2^{28}	5	66	11^{61}	0
21	3^{15}	2	39	3^{27}	4	54	3^{37}	1	66	22^{64}	0
21	7^{19}	0	39	13^{37}	0	54	6^{46}	0	66	33^{65}	0
22	2^{12}	5	40	2^{21}	3	54	9^{49}	1	68	2^{35}	3
22	11^{21}	0	40	4^{31}	0	54	18^{52}	6	68	4^{52}	18
24	2^{13}	1	40	5^{33}	1	54	27^{53}	0	68	17^{65}	0
24	3^{17}	0	40	8^{36}	≥ 31	55	5^{45}	1	68	34^{67}	0
24	4^{19}	0	40	10^{37}	0	55	11^{51}	0	69	3^{47}	0
24	6^{21}	6	40	20^{39}	0	56	2^{29}	1	69	23^{67}	0
24	8^{22}	0	42	2^{22}	3	56	4^{43}	0	70	2^{36}	8
24	12^{23}	0	42	3^{29}	0	56	7^{49}	2	70	5^{57}	1
25	5^{21}	1	42	6^{36}	≥ 42	56	8^{50}	≥ 19	70	7^{61}	0
26	2^{14}	3	42	7^{37}	0	56	14^{53}	0	70	10^{64}	≥ 17
26	13^{25}	0	42	14^{40}	1	56	28^{55}	0	70	14^{66}	0
27	3^{19}	1	42	21^{41}	0	57	3^{39}	3	70	35^{69}	0
27	9^{25}	0	44	2^{23}	1	57	19^{55}	0			
28	2^{15}	3	44	4^{34}	4	58	2^{30}	7			
28	4^{22}	1	44	11^{41}	0	58	29^{57}	0			

The results for $n < m$ are given in the following table.

TABLE 2. The number of designs for $n < m$

(n,m)	type	$\#D$	(n,m)	type	$\#D$	(n,m)	type	$\#D$
$(5,9)$	$3^4 2^6$	2	$(20,39)$	$3^{14} 2^{21}$	2	$(29,57)$	$3^{20} 2^{30}$	3
$(7,10)$	$4^6 3^8$	3	$(14,40)$	$4^{11} 2^{22}$	1	$(39,58)$	$4^{30} 3^{40}$	10
$(8,15)$	$3^6 2^9$	2	$(27,40)$	$4^{21} 3^{28}$	4	$(32,63)$	$3^{22} 2^{33}$	1
$(6,16)$	$4^5 2^{10}$	2	$(12,45)$	$5^{10} 2^{25}$	2	$(32,63)$	$7^{28} 4^{49}$	4
$(11,16)$	$4^9 3^{12}$	3	$(12,45)$	$9^{11} 3^{33}$	0	$(10,64)$	$8^9 2^{36}$	4
$(9,21)$	$6^8 3^{16}$	2	$(23,45)$	$3^{16} 2^{24}$	4	$(22,64)$	$4^{17} 2^{34}$	2
$(11,21)$	$3^8 2^{12}$	3	$(34,45)$	$5^{28} 4^{35}$	1	$(22,64)$	$16^{21} 6^{56}$	7
$(17,21)$	$6^{15} 5^{18}$	3	$(26,46)$	$10^{24} 6^{40}$	23	$(28,64)$	$8^{25} 4^{50}$	8
$(15,22)$	$4^{12} 3^{16}$	6	$(28,46)$	$6^{24} 4^{36}$	37	$(36,64)$	$10^{33} 6^{55}$	2
$(7,25)$	$5^6 2^{15}$	2	$(31,46)$	$4^{24} 3^{32}$	12	$(43,64)$	$4^{33} 3^{44}$	1
$(19,25)$	$5^{16} 4^{20}$	3	$(9,49)$	$7^8 2^{28}$	2	$(55,64)$	$8^{49} 7^{56}$	5
$(16,26)$	$6^{14} 4^{21}$	3	$(17,49)$	$7^{15} 3^{35}$	2	$(17,65)$	$5^{14} 2^{35}$	1
$(14,27)$	$3^{10} 2^{15}$	2	$(41,49)$	$7^{36} 6^{42}$	1	$(17,65)$	$13^{16} 4^{52}$	1
$(10,28)$	$4^8 2^{16}$	4	$(22,50)$	$15^{21} 7^{45}$	1	$(33,65)$	$5^{27} 3^{45}$	2
$(10,28)$	$7^9 3^{21}$	1	$(21,51)$	$6^{18} 3^{36}$	8	$(49,65)$	$5^{40} 4^{50}$	3
$(16,28)$	$10^{15} 6^{25}$	2	$(26,51)$	$3^{18} 2^{27}$	3	$(14,66)$	$16^{12} 2^{36}$	9
$(19,28)$	$4^{15} 3^{20}$	3	$(36,51)$	$21^{35} 15^{49}$	2	$(27,66)$	$6^{23} 3^{46}$	0
$(13,33)$	$9^{12} 4^{27}$	4	$(41,51)$	$6^{35} 5^{42}$	2	$(27,66)$	$11^{25} 5^{55}$	1
$(17,33)$	$3^{12} 2^{18}$	3	$(18,52)$	$4^{14} 2^{28}$	5	$(40,66)$	$6^{34} 4^{51}$	2
$(23,34)$	$4^{18} 3^{24}$	7	$(35,52)$	$4^{27} 3^{36}$	7	$(53,66)$	$6^{45} 5^{54}$	4
$(18,35)$	$5^{15} 3^{25}$	2	$(13,55)$	$10^{12} 3^{40}$	7	$(35,69)$	$3^{24} 2^{36}$	5
$(18,35)$	$7^{16} 4^{28}$	4	$(25,55)$	$10^{23} 5^{46}$	0	$(24,70)$	$7^{21} 3^{49}$	2
$(8,36)$	$6^7 2^{21}$	2	$(49,55)$	$10^{45} 9^{50}$	5	$(24,70)$	$10^{22} 4^{55}$	3
$(15,36)$	$6^{13} 3^{26}$	2	$(16,56)$	$12^{15} 4^{45}$	5	$(47,70)$	$4^{36} 3^{48}$	14
$(29,36)$	$6^{25} 5^{30}$	4	$(21,57)$	$15^{20} 6^{50}$	1			
$(31,36)$	$8^{28} 7^{32}$	6	$(22,57)$	$9^{20} 4^{45}$	3			

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