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HILBERT C^*-MODULES IN WHICH ALL RELATIVELY STRICTLY CLOSED SUBMODULES ARE COMPLEMENTED

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Abstract. We give the characterization and description of all full Hilbert C^*-modules and associated C^*-algebras having the property that each relatively strictly closed submodule is orthogonally complemented. A strict topology is determined by an essential closed two-sided ideal in the associated algebra and a related ideal submodule. It is shown that these are some modules over hereditary C^*-algebras containing the essential ideal isomorphic to the algebra of (not necessarily all) compact operators on a Hilbert space. The characterization and description of that broader class of Hilbert modules and their associated C^*-algebras is given.

As auxiliary results we give properties of strict and relatively strict submodule closures, the characterization of orthogonal closedness and orthogonal complementing property for single submodules, relation of relative strict topology and projections, properties of outer direct sums with respect to the ideals in ℓ_∞ and isomorphisms of Hilbert C^*-modules, and we prove some properties of hereditary C^*-algebras and associated hereditary modules with respect to the multiplier C^*-algebras, multiplier Hilbert C^*-modules, corona algebras and corona modules.

1. Introduction and preliminaries

A (right) Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ is a right $\mathcal{A}$-module $\mathcal{X}$ equipped with an $\mathcal{A}$-valued inner product $\langle\cdot|\cdot\rangle$ which is $\mathcal{A}$-linear in the second and $*$-conjugate linear in the first variable such that $\mathcal{X}$ is a Banach space with the norm $\|x\| = \|\langle x|x\rangle\|^{\frac{1}{2}}$. $\mathcal{X}$ is a full Hilbert $\mathcal{A}$-module if $\mathcal{A} = \langle\mathcal{X}|\mathcal{X}\rangle$ where $\langle\mathcal{X}|\mathcal{X}\rangle$ is the closed linear span of all elements in the underlying C^*-algebra $\mathcal{A}$ of the form $\langle x|y\rangle, x, y \in \mathcal{X}$.

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For a submodule $\mathcal{F}$ of $\mathcal{X}$ we denote by $\mathcal{F}^\perp = \{x \in \mathcal{X}; \langle x|y\rangle = 0, \forall y \in \mathcal{F}\}$ the orthogonal complement of $\mathcal{F}$ in $\mathcal{X}$. Note that $\mathcal{F}^\perp$ is a norm-closed submodule of $\mathcal{X}$.

A submodule $\mathcal{F} \subseteq \mathcal{X}$ is said to be *orthogonally closed* if $\mathcal{F} = \mathcal{F}^{\perp\perp}$ and *orthogonally complemented* if $\mathcal{F} \oplus \mathcal{F}^\perp = \mathcal{X}$. Observe that each of these two properties implies that $\mathcal{F}$ is norm-closed.

If $\mathcal{F}, \mathcal{G}$ are submodules of $\mathcal{X}$ then, clearly, $\mathcal{F} \subseteq \mathcal{F}^{\perp\perp}$ and $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \mathcal{G}^\perp \subseteq \mathcal{F}^\perp$. These two facts imply that, for each submodule $\mathcal{F}$ of $\mathcal{X}$, we have $\mathcal{F}^\perp = \mathcal{F}^{\perp\perp\perp}$, i.e. $\mathcal{F}^\perp$ is orthogonally closed. It is proved in [9, Theorem 2] that a submodule $\mathcal{F}$ of $\mathcal{X}$ is orthogonally complemented if and only if $\mathcal{F} \oplus \mathcal{F}^\perp$ is orthogonally closed. Every orthogonally complemented submodule is orthogonally closed but the converse is not true in general (c.f. [14] or see Example 2.11).

Throughout the paper $\mathcal{A}$ is a C^*-algebra with an essential closed two-sided ideal $\mathcal{I}$ and $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module. The ideal submodule $\mathcal{X}_\mathcal{I}$ of $\mathcal{X}$ associated to $\mathcal{I}$ is $\mathcal{X}_\mathcal{I} = \mathcal{X}\mathcal{I} = \{x \in \mathcal{X} : \langle x|x\rangle \in \mathcal{I}\} = \{x \in \mathcal{X}; \langle x|y\rangle \in \mathcal{I}, \forall y \in \mathcal{X}\}$. If $\mathcal{X}$ is a full module, then $\mathcal{X}_\mathcal{I}$ is full as a Hilbert $\mathcal{I}$-module.

Recall that there exists a so called multiplier module $M(\mathcal{X}_\mathcal{I})$ of $\mathcal{X}_\mathcal{I}$ that is a (not necessarily full) Hilbert C^*-module over the multiplier algebra $M(\mathcal{I})$ and contains $\mathcal{X}_\mathcal{I}$. Besides the norm topology, $M(\mathcal{X}_\mathcal{I})$ is also endowed with the strict topology induced by $\mathcal{X}_\mathcal{I}$. This is the topology induced by two families of seminorms: $v \mapsto \|\langle v|y\rangle\|$, $(y \in \mathcal{X}_\mathcal{I})$, and $v \mapsto \|vb\|$, $(b \in \mathcal{I})$. The strict topology is Hausdorff since $\mathcal{I}$ is an essential ideal in $\mathcal{A}$. A net (v_λ) in $M(\mathcal{X}_\mathcal{I})$ converges strictly to $v \in M(\mathcal{X}_\mathcal{I})$, which is denoted by $v = \mathrm{st} - \lim_\lambda v_\lambda$, if and only if $\langle v|y\rangle = \lim_\lambda \langle v_\lambda|y\rangle$, $\forall y \in \mathcal{X}_\mathcal{I}$, and $vb = \lim_\lambda v_\lambda b$, $\forall b \in \mathcal{I}$. It is known that $\mathcal{X}_\mathcal{I}$ is strictly dense in $M(\mathcal{X}_\mathcal{I})$; moreover, it turns out that $M(\mathcal{X}_\mathcal{I})$ is the strict completion of $\mathcal{X}_\mathcal{I}$. Also, if $\mathcal{X}_\mathcal{I}$ is a full $\mathcal{I}$-module, we can look at $M(\mathcal{X}_\mathcal{I})$ as the largest Hilbert C^*-module over the C^*-algebra containing $\mathcal{I}$ as an essential ideal such that $\mathcal{X}_\mathcal{I}$ is its ideal submodule with respect to $\mathcal{I}$ (see Proposition 4.11). We denote by $C(\mathcal{I}) = M(\mathcal{I})/\mathcal{I}$ the corona algebra of $\mathcal{I}$, by $C(\mathcal{X}_\mathcal{I}) = M(\mathcal{X}_\mathcal{I})/\mathcal{X}_\mathcal{I}$ corona module of $\mathcal{X}_\mathcal{I}$. Let $\Pi : M(\mathcal{X}_\mathcal{I}) \to C(\mathcal{X}_\mathcal{I})$ be the canonical $\pi : M(\mathcal{I}) \to C(\mathcal{I})$-morphism of modules. For these and other facts concerning the strict topology and the multiplier module $M(\mathcal{X}_\mathcal{I})$ we refer the reader to [5, 6, 7].

Suppose now, that (e_λ) is any approximate unit for $\mathcal{I}$. For each $x \in \mathcal{X}$ we have $\forall b \in \mathcal{I}$, $\lim_\lambda xe_\lambda b = \lim_\lambda xbe_\lambda = xb \in \mathcal{X}_\mathcal{I}$ and $\forall y \in \mathcal{X}_\mathcal{I}$, $\lim_\lambda \langle y|xe_\lambda\rangle = \lim_\lambda \langle y|x\rangle e_\lambda = \langle y|x\rangle \in \mathcal{I}$. If we identify x with the strict limit of the net (xe_λ), we can regard $\mathcal{X}$ as a submodule of $M(\mathcal{X}_\mathcal{I})$. Thus, we have $\mathcal{X}_\mathcal{I} \subseteq \mathcal{X} \subseteq M(\mathcal{X}_\mathcal{I})$ and this allows us to analyze phenomena in $\mathcal{X}$ by working in the larger module $M(\mathcal{X}_\mathcal{I})$.

Suppose we are given a submodule $\mathcal{F}$ of $\mathcal{X}$. We denote by $\mathcal{F}^{\perp_{\mathcal{X}}}$ the orthogonal complement of $\mathcal{F}$ in $\mathcal{X}$ and by $\mathcal{F}^{\perp}$ the orthogonal complement of $\mathcal{F}$ in $M(\mathcal{X}_{\mathcal{I}})$. Obviously, $\mathcal{F}^{\perp_{\mathcal{X}}} = \mathcal{F}^{\perp} \cap \mathcal{X}$. We now say that $\mathcal{F}$ is orthogonally closed in $\mathcal{X}$ if $\mathcal{F} = \mathcal{F}^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}} = \mathcal{F}^{\perp\perp} \cap \mathcal{X}$.

In a similar fashion we define the concept of strict closedness in $\mathcal{X}$. First, we denote by $c\ell(\mathcal{F})$ and $c\ell^{st}(\mathcal{F})$ the closure of $\mathcal{F}$ in $M(\mathcal{X}_{\mathcal{I}})$ with respect to the norm topology and the strict topology, respectively. Obviously, since $\mathcal{F} \subseteq \mathcal{X}$ and $\mathcal{X}$ is norm-closed in $M(\mathcal{X}_{\mathcal{I}})$, we have $c\ell(\mathcal{F}) \subseteq \mathcal{X}$. On the other hand, since $\mathcal{X}$ is not closed in $M(\mathcal{X}_{\mathcal{I}})$ with the respect to the strict topology, $c\ell^{st}(\mathcal{F})$ is in general not contained in $\mathcal{X}$. The relative strict closure of $\mathcal{F}$ in $\mathcal{X}$ is denoted by $c\ell^{st}_{\mathcal{X}}(\mathcal{F})$. By definition, we have $c\ell^{st}_{\mathcal{X}}(\mathcal{F}) = c\ell^{st}(\mathcal{F}) \cap \mathcal{X}$. We now say that $\mathcal{F}$ is strictly closed in $\mathcal{X}$ if $\mathcal{F} = c\ell^{st}_{\mathcal{X}}(\mathcal{F})$, i.e. if $\mathcal{F}$ is closed in $\mathcal{X}$ with respect to the relative strict topology on $\mathcal{X}$.

These relative concepts of closedness of submodules of $\mathcal{X}$, when $\mathcal{X}$ is regarded as a submodule of $M(\mathcal{X}_{\mathcal{I}})$, are the main technical tools of the paper. Results similar to the ones presented in this article, but for the special case $\mathcal{A} = M(\mathcal{I})$, the multiplier algebra of a C^*-algebra $\mathcal{I}$, and $\mathcal{X} = M(\mathcal{X}_{\mathcal{I}})$, the strict completion of $\mathcal{X}_{\mathcal{I}}$, can be found in [9] and the special case $\mathcal{A} = \mathcal{I}$ and $\mathcal{X} = \mathcal{X}_{\mathcal{I}}$ can be found in [11] and [14]. They are extended here to an entire class of (not necessarily strictly complete) full Hilbert $\mathcal{A}$-modules where $\mathcal{I} \subseteq \mathcal{A} \subseteq M(\mathcal{I})$.

Further, by $\mathfrak{ha}(\mathcal{B})$ we denote the set of all *hereditary* C^*-subalgebras of some C^*-algebra $\mathcal{B}$, and these are C^*-subalgebras $\mathcal{A}$ having the property that if for $0 \leq b \in \mathcal{B}$ there exists $0 \leq a \in \mathcal{A}$ such that $b \leq a$ then $b \in \mathcal{A}$. The useful characterization of hereditary C^*-subalgebras $\mathcal{A} \in \mathfrak{ha}(\mathcal{B})$ is $\mathcal{A}\mathcal{B}\mathcal{A} = \mathcal{A}$ (cf. [12, 3.2.2. Theorem] and Remark 4.24). For a nonempty set $S \subset \mathcal{B}$ we denote by $\mathfrak{ha}_S(\mathcal{B})$ the set of all C^*-algebras from $\mathfrak{ha}(\mathcal{B})$ containing S. Note that each two-sided ideal is also a hereditary C^*-subalgebra, and in the case of commutative C^*-algebras the reverse is also true. Important for the construction is the fact that hereditary C^*-subalgebras of some C^*-algebra are connected by bijection to the closed left ideals in that C^*-algebra, i.e. every C^*-algebra $\mathcal{A} \in \mathfrak{ha}(\mathcal{B})$ is of the form $\mathcal{A} = \mathcal{L}^* \cap \mathcal{L}$, where $\mathcal{L}$ is an unique left ideal of $\mathcal{B}$ with this property (cf. [12, 3.2.1.Theorem]). Moreover, we have $\mathcal{A} = \mathcal{L}^*\mathcal{L}$ (see Remark 4.23), and this makes it easier to construct hereditary C^*-algebras from left ideals.

Analogously to the ideal submodules we expand the definition to the hereditary modules over hereditary subalgebras. For each full Hilbert $\mathcal{B}$-module $\mathcal{X}$ and $\mathcal{A} \in \mathfrak{ha}(\mathcal{B})$ we denote module $\mathcal{X}_{\mathcal{A}} = \mathcal{X}\mathcal{A} = \{x \in \mathcal{X} : \langle x|x\rangle \in \mathcal{A}\} = \{x \in \mathcal{X}; |\langle y|x\rangle| \in \mathcal{A}, \forall y \in \mathcal{X}\}$, a full Hilbert $\mathcal{A}$-module in $\mathcal{X}$ which is generally not a submodule of $\mathcal{X}$ (see Proposition 4.25), and we call it the hereditary $\mathcal{A}$-module of $\mathcal{X}$. We denote by $\mathfrak{hm}(\mathcal{X})$ the set of all hereditary C^*-modules of the Hilbert C^*-module $\mathcal{X}$ and for nonempty set $S \subset \mathcal{X}$ we denote by $\mathfrak{hm}_S(\mathcal{X})$ the set of all C^*-submodules in $\mathfrak{hm}(\mathcal{X})$ containing S.

If $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules, we denote by $\mathbf{B}_a(\mathcal{X},\mathcal{Y})$ the Banach space of all adjointable operators from $\mathcal{X}$ to $\mathcal{Y}$. When $\mathcal{X}=\mathcal{Y}$ we write $\mathbf{B}_a(\mathcal{X})$ instead of $\mathbf{B}_a(\mathcal{X},\mathcal{X})$, and this is a C^*-algebra. The Banach space of all "compact" operators from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\mathbf{K}(\mathcal{X},\mathcal{Y})$, respectively $\mathbf{K}(\mathcal{X})$, and they are generated by elementary "compact" operators $\Theta_{y,x}$, for all $x\in\mathcal{X}, y\in\mathcal{Y}$ acting as $\Theta_{y,x}z=y\langle x|z\rangle$, for all $z\in\mathcal{X}$.

Outer direct sums of families of C^*-algebras and Hilbert C^*-modules are essentially used in this paper. Let $\mathcal{J}$ be any nonempty set and let $\mathcal{C}_\infty=\ell_\infty(\mathcal{J})$ be the set of all bounded functions $c:\mathcal{J}\to\mathbb{C}$ (usually we write $c=(c_j)_{j\in\mathcal{J}}$, $c(j)=c_j, j\in\mathcal{J}$) which is a commutative C^*-algebra with the pointwise (componentwise) operations of addition, multiplication, conjugation and norm $\|c\|_\infty=\sup_{j\in\mathcal{J}}|c_j|$. Its C^*-subalgebra and essential ideal $\mathcal{C}_0$ is a set of all functions $c:\mathcal{J}\to\mathbb{C}$ that vanish at infinity, i.e. for every $\varepsilon>0$ is $|c_j|\geq\varepsilon$ for only a finite number of $j\in\mathcal{J}$, or equivalently, for any injective sequence $(j_n)_n$ in $\mathcal{J}$ the sequence $(c_{j_n})_n$ converges to 0 (we write $\lim_{j\in\mathcal{J}}c_j=0$).

Let $(\mathcal{B}_j,\|\cdot\|_j)_{j\in\mathcal{J}}$ be a family of Banach spaces. For any closed ideal $\mathcal{C}$ of $\mathcal{C}_\infty$ containing $\mathcal{C}_0$ we denote the outer direct sum

$$\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j=\{x=(x_j)_{j\in\mathcal{J}}\in\Pi_{j\in\mathcal{J}}\mathcal{B}_j;(\|x_j\|_j)_{j\in\mathcal{J}}\in\mathcal{C}\}. \tag{1.1}$$

The set $\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j$ is a Banach space with the norm $\|x\|_\infty=\sup_{j\in\mathcal{J}}\|x_j\|_j$ and componentwise operations.

In the case $\mathcal{B}_j$, $j\in\mathcal{J}$, are C^*-algebras or Hilbert C^*-modules we assume that $\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j$ is a C^*-algebra or Hilbert C^*-module with componentwise operations. If $\mathcal{B}=(\mathcal{B}_j,\|\cdot\|_j)_{j\in\mathcal{J}}$ with the norm $\|\cdot\|_\infty$ is a Banach space we have $\mathcal{C}_0\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j\subseteq\mathcal{B}\subseteq\mathcal{C}_\infty\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j$ (see Lemma 4.15). Also, sums defined by (1.1) naturally preserve the properties of ideals and hereditary C^*-algebras as well as ideal submodules and hereditary modules (see Lemma 4.34).

In what follows, by the C^*-algebra of compact operators on a Hilbert space we mean the C^*-algebra of not necessarily all compact operators on that space.

The first objective of this paper is to give different types of characterizations and description of all full Hilbert modules and their associated algebras containing the essential ideal isomorphic to some C^*-algebra of compact operators on a Hilbert space.

The second objective is to characterize and describe all full Hilbert modules and their associated algebras with the property that each relatively strictly closed submodule is orthogonally complemented. It turns out that this is a class of hereditary $\mathcal{A}$-modules of $M(\mathcal{X}_\mathcal{K})$, where $\mathcal{A}$ is a hereditary subalgebra of $M(\mathcal{K})$ containing $\mathcal{K}$ with $\mathcal{K}$ isomorphic to the C^*-algebra of compact operators on a Hilbert space.

Much of the article is filled with auxiliary results that may be interesting in a broader context as well. They contain results related to the properties of strict and relatively strict closures of submodules, we give the topological characterization of orthogonal closedness and orthogonal complementing property for individual submodules of Hilbert C^*-modules over C^*-algebras containing essential ideals, some results on strict orthogonal bases, connection of relative strict topology and projections, properties of outer direct sums with respect to ideals in $\mathcal{C}_\infty$ and isomorphisms of Hilbert C^*-modules and we prove some claims on hereditary C^*-algebras and corresponding hereditary modules.

For a better overview of the results, we present the main claims in the first three theorems and proposition. Other statements used in the proofs of these main results, as well as some additional results related to them, are formulated and proved in the last part of the paper.

2. The main results

First we give characterizations and descriptions of a class of full Hilbert C^*-modules over C^*-algebras containing an essential ideal isomorphic to some C^*-algebra of compact operators on a Hilbert space.

Theorem 2.1. *Let $\mathcal{A}$ be C^*-algebra with an essential ideal $\mathcal{I}$ and let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module. The following statements are equivalent:*

- (i) *$\mathcal{I}$ is isomorphic to some C^*-algebra of compact operators on a Hilbert space.*
- (ii) *Each relatively strictly closed submodule in $\mathcal{X}$ is orthogonally closed.*
- (iii) *For every relatively strictly closed submodule $\mathcal{F}$ in $\mathcal{X}$ the submodule $\mathcal{F} \oplus \mathcal{F}^{\perp_\mathcal{X}}$ is relatively strictly dense in $\mathcal{X}$, i.e. $\mathcal{X} = c\ell^{st}_\mathcal{X}(\mathcal{F} \oplus \mathcal{F}^{\perp_\mathcal{X}})$.*
- (iv) *There is a strict orthogonal bases for $\mathcal{X}$.*
- (v) *There are families of Hilbert spaces $(H_j)_{j\in\mathcal{J}}$, $(G_j)_{j\in\mathcal{J}}$, a family of C^*-algebras $\mathbf{A} = (\mathbf{A}_j)_{j\in\mathcal{J}}$,*

$$\mathbf{K}(H_j) \subseteq \mathbf{A}_j \subseteq \mathbf{B}(H_j),\ j \in \mathcal{J}, \tag{2.1}$$

and a family of Banach spaces of bounded linear operators $\mathbf{X} = (\mathbf{X}_j)_{j\in\mathcal{J}}$,

$$\mathbf{K}(H_j, G_j) \subseteq \mathbf{X}_j \subseteq \mathbf{B}(H_j, G_j),\ j \in \mathcal{J}, \tag{2.2}$$

such that $\mathcal{I}$ is isomorphic to the C^*-algebra c_0-$\oplus_j \mathbf{K}(H_j)$, the ideal submodule $\mathcal{X}_\mathcal{I}$ is isomorphic to the c_0-$\oplus_j \mathbf{K}(H_j, G_j)$ and the $\mathcal{A}$-module $\mathcal{X}$ is isomorphic to the $\mathbf{A}$-module $\mathbf{X}$.

Remark 2.2. In the case when $\mathcal{A} = \mathcal{I}$ and $\mathcal{X} = \mathcal{X}_\mathcal{I}$ in the previous theorem the relatively strict topology and the norm topology coincide on submodules (see Lemma 4.1). Then $(i) \Leftrightarrow (ii)$ coincide with the result by J. Schweizer [14, Theorem 1.], and considering that the orthogonal sum of closed submodules is closed, $(i) \Leftrightarrow (iii)$ coincide with the result by B. Magajna [11, Theorem 1.]. Also, in that case $(i) \Leftrightarrow (iv)$ is the same as by Lj. Arambašić [1, Corollary 7.]. Moreover, in the case $\mathcal{A} = M(\mathcal{I})$ and $\mathcal{X} = M(\mathcal{X}_\mathcal{I})$ result

$(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$, together with the fact that $\mathcal{F} \oplus \mathcal{F}^{\perp_{\mathcal{X}}}$ is strictly closed, is proved in [9, Theorems 3.4 and 3.5]. $\diamondsuit$

In what follows we say that a Hilbert C^*-module has the *complementing property* if each of its relatively strictly closed submodules is complemented.

In a class of full Hilbert C^*-modules over C^*-algebras containing an essential ideal isomorphic to some C^*-algebra of compact operators on a Hilbert space, one can characterize full Hilbert modules with complementing property by conditions that are not sufficient for the characterization of the complementing property in the class of all full Hilbert C^*-modules with arbitrary essential ideals, but they are certainly necessary.

THEOREM 2.3. *Let $\mathcal{A}$ be a C^*-algebra with an essential ideal $\mathcal{K}$ isomorphic to some C^*-algebra of compact operators on a Hilbert space and let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module. The following statements are equivalent:*

- (i) *The Hilbert $\mathcal{A}$-module $\mathcal{X}$ has the complementing property.*
- (ii) *For each relatively strictly closed submodule $\mathcal{F} \subseteq \mathcal{X}$ the orthogonal sum $\mathcal{F} \oplus \mathcal{F}^{\perp_{\mathcal{X}}}$ is relatively strictly closed in $\mathcal{X}$.*
- (iii) *For every relatively strictly closed submodules $\mathcal{F}, \mathcal{G} \subseteq \mathcal{X}, \mathcal{F} \perp \mathcal{G}$, the orthogonal sum $\mathcal{F} \oplus \mathcal{G}$ is relatively strictly closed in $\mathcal{X}$.*
- (iv) *The C^*-algebras $\mathbf{B}_a(\mathcal{X})$ and $\mathbf{B}_a(\mathcal{X}_{\mathcal{K}})$ of all adjointable operators on $\mathcal{X}$ and $\mathcal{X}_{\mathcal{K}}$, respectively, are isomorphic by isomorphism acting as restriction.*

REMARK 2.4. We show that condition (iv) in the previous theorem is not generally sufficient to characterize the complementing property of a full Hilbert modules over a C^*-algebra with some essential ideal. Let H be an non-separable Hilbert space and let $\mathbf{I}$ be an ideal of $\mathbf{B}(H)$ for which $\mathbf{K}(H) \subsetneqq \mathbf{I} \subsetneqq \mathbf{B}(H)$. Then $M(\mathbf{I}) = \mathbf{B}(H)$ (see Lemma 4.31) and let $\mathbf{X}$ be a strictly complete full $\mathbf{B}(H)$-module and $\mathbf{X}_{\mathbf{I}}$ its ideal submodule. Then $\mathbf{X} = M(\mathbf{X}_{\mathbf{I}})$ and $\mathbf{B}_a(M(\mathbf{X}_{\mathbf{I}})) = \mathbf{B}_a(\mathbf{X})$ is isomorphic to $\mathbf{B}_a(\mathbf{X}_{\mathbf{I}})$ with an isomorphism acting as a restriction (see [6, Theorem 2.3]). If $\mathbf{X}$ would have the complementing property then $\mathbf{I}$ would be isomorphic to some C^*-algebra of compact operators on Hilbert space (see [9, Theorem 3.4.]), which is contrary to the choice of the ideal $\mathbf{I}$. $\diamondsuit$

In the following theorem we characterize and describe the class of all full Hilbert C^*-modules which have the complementing property.

THEOREM 2.5. *Let $\mathcal{A}$ be C^*-algebra with an essential ideal $\mathcal{I}$ and let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module. The following statements are equivalent:*

- (i) *The Hilbert $\mathcal{A}$-module $\mathcal{X}$ has the complementing property.*
- (ii) *$\mathcal{I}$ is isomorphic to some C^*-algebra of compact operators on a Hilbert space, the C^*-algebra $\mathcal{A} \in \mathfrak{ha}_{\mathcal{I}}(M(\mathcal{I}))$ and the Hilbert $\mathcal{A}$-module $\mathcal{X} = M(\mathcal{X}_{\mathcal{I}})\mathcal{A} \in \mathfrak{hm}_{\mathcal{X}_{\mathcal{I}}}(M(\mathcal{X}_{\mathcal{I}}))$.*

(*iii*) *There are families of Hilbert spaces* $(H_j)_{j\in\mathcal{J}}$, $(G_j)_{j\in\mathcal{J}}$, *a* C^**-algebra* $\mathbf{A} = (\mathbf{A}_j)_{j\in\mathcal{J}} \in \mathfrak{ha}_{\mathbf{K}}(M(\mathbf{K}))$ *as in* (2.1) *and an* $\mathbf{A}$*-module* $\mathbf{X} = (\mathbf{X}_j)_{j\in\mathcal{J}} \in \mathfrak{hm}_{\mathbf{X}_{\mathbf{K}}}(M(\mathbf{X}_{\mathbf{K}}))$ *as in* (2.2), *such that* $\mathbf{K} = c_0\text{-}\oplus_j \mathbf{K}(H_j)$, *the ideal submodule* $\mathbf{X}_{\mathbf{K}} = c_0\text{-}\oplus_j \mathbf{K}(H_j, G_j)$, $\mathcal{I}$ *is isomorphic to* $\mathbf{K}$ *and the* $\mathcal{A}$*-module* $\mathcal{X}$ *is isomorphic to* $\mathbf{A}$*-module* $\mathbf{X}$.

REMARK 2.6. It turns out that the set of all C^*-algebras in Theorem 2.1 (v) is bijectively related to the set of all C^*-subalgebras in the corona algebras of all C^*-algebras of compact operators on Hilbert spaces, and the subset of all hereditary C^*-subalgebras in Theorem 2.5 is in bijective relation to the subset of all hereditary C^*-subalgebras of corona algebras of all C^*-algebras of compact operators in Hilbert spaces.

Also the set of all full Hilbert C^*-modules in Theorem 2.1 (v) is in bijective relation to the set of all full C^*-submodules in the corona modules of all C^*-modules over C^*-algebras of compact operators in Hilbert spaces, and the set of all C^*-modules with the complementing property in Theorem 2.5 is in a bijective relation to the set of all hereditary C^*-modules in the corona modules of all C^*-modules over C^*-algebras of compact operators in Hilbert spaces (see Proposition 4.30). $\diamondsuit$

REMARK 2.7. It follows from Theorem 2.5 that Hilbert C^*-modules with the complementing property are these and only these Hilbert C^*-modules which are hereditary modules of the multiplier modules of Hilbert C^*-modules over algebras isomorphic to some C^*-algebra of compact operators on a Hilbert space. With the intention to show that there are many such modules we give a way of constructing them.

Let's start from any Hilbert C^*-module over the algebra of compact operators $\mathbf{K}$. It is isomorphic to the $\mathbf{K}$-module $\mathbf{X}_{\mathbf{K}} = c_0\text{-}\oplus_{j\in\mathcal{J}} \mathbf{K}(H_j, G_j)$, where $\mathbf{K} = c_0\text{-}\oplus_{j\in\mathcal{J}} \mathbf{K}(H_j)$, its multiplier module is $M(\mathbf{X}_{\mathbf{K}}) = c_\infty\text{-}\oplus_j \mathbf{B}(H_j, G_j)$ and (H_j), (G_j) are families of Hilbert spaces (see Proposition 4.16).

Now the construction begins from the basic Hilbert C^*-module $\mathbf{K}(H,G)$ over the C^*-algebra $\mathbf{K}(H)$ and its multipliers $\mathbf{B}(H)$-module $\mathbf{B}(H,G)$. There is bijective connection between algebras in $\mathfrak{ha}_{\mathbf{K}(H)}(\mathbf{I}) \subseteq \mathfrak{ha}_{\mathbf{K}(H)}(\mathbf{B}(H))$ and modules in $\mathfrak{hm}_{\mathbf{K}(H,G)}(\mathbf{B}(H,G))$, where $\mathbf{I}$ is an ideal of $\mathbf{B}(H)$ generated by $\mathbf{B}(G,H)\mathbf{B}(H,G)$, i.e. each $\mathbf{A}$-module $\mathbf{X} = \mathbf{B}(H,G)\mathbf{A} \in \mathfrak{hm}_{\mathbf{K}(H,G)}(\mathbf{B}(H,G))$ is uniquely determined by a C^*-subalgebra $\mathbf{A} \in \mathfrak{ha}_{\mathbf{K}(H)}(\mathbf{I})$ (see Proposition 4.32).

The bijective correspondence between left ideals and hereditary subalgebras in every C^*-algebra is also well known. Therefore, for any set $S \subset \mathbf{B}(H) \setminus \mathbf{K}(H)$ we have a left ideal $\mathcal{L} = c\ell(\text{span}(\mathbf{B}(H)S + \mathbf{K}(H)))$ in $\mathbf{B}(H)$ containing $\mathbf{K}(H)$, and then $\mathbf{A} = \mathcal{L}^*\mathcal{L} = c\ell(\text{span}(S^*\mathbf{B}(H)S + \mathbf{K}(H))) \in \mathfrak{ha}_{\mathbf{K}(H)}(\mathbf{B}(H))$ is a corresponding hereditary subalgebra (see Remark 4.23).

Further, the construction of Hilbert C^*-modules with the complementing property can be continued as follows. Take any family of hereditary C^*-algebras $\mathbf{A} = (\mathbf{A}_j)_{j\in\mathcal{J}}$, any family $\mathbf{X} = (\mathbf{X}_j)_{j\in\mathcal{J}}$ of hereditary Hilbert $\mathbf{A}_j$-modules and an ideal $\mathcal{C}$ of $\mathcal{C}_\infty$ containing $\mathcal{C}_0$. Then the outer direct sum $\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}} \mathbf{A}_j$ is a hereditary C^*-algebra of $M(\mathbf{K})$ containing $\mathbf{K}$. Due to the fact that $\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}} \mathbf{X}_j$ is a Hilbert $\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}} \mathbf{A}_j$-module and that $\mathcal{C}$ is an ideal in $\mathcal{C}_\infty$, it follows that $\mathbf{X} = \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}} \mathbf{X}_j = M(\mathbf{X_K})\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}} \mathbf{A}_j$ is a hereditary module of $M(\mathbf{X_K})$ which contains $\mathbf{X_K}$ (see Lemma 4.34). ♢

Finally, we prove an important hereditary property of Hilbert C^*-modules having the complementing property. Namely we claim that any relatively strictly closed submodule of such a module possesses the complementing property.

Proposition 2.8. *Let $\mathcal{X} = (\mathcal{X}_j)_{j\in\mathcal{J}}$ be a full hereditary Hilbert $\mathcal{A}$-module of $M(\mathcal{X}_\mathcal{K})$ and $\mathcal{A} = (\mathcal{A}_j)_{j\in\mathcal{J}} \in \mathfrak{ha}_\mathcal{K}(M(\mathcal{K}))$, where $\mathcal{K} = c_0\text{-}\oplus_{j\in\mathcal{J}} \mathcal{K}_j$ and $\mathcal{K}_j$ is isomorphic to the C^*-algebra of all compact operators on some Hilbert space H_j, $j \in \mathcal{J}$. Then every relatively strictly closed submodule $\mathcal{Y}$ in $\mathcal{X}$ is a hereditary (not necessarily full) $\mathcal{A}$-module of $M(\mathcal{Y}_\mathcal{K})$.*

We have $\langle \mathcal{Y}_j | \mathcal{Y}_j \rangle_j = \mathcal{A}_j$ or $\langle \mathcal{Y}_j^{\perp_{\mathcal{X}_j}} | \mathcal{Y}_j^{\perp_{\mathcal{X}_j}} \rangle_j = \mathcal{A}_j$, for all $j \in \mathcal{J}$. Moreover, if $p = (p_j)_{j\in\mathcal{J}} \in \mathbf{B}_a(M(\mathcal{X}_\mathcal{K}))$ is a projection such that $\mathcal{Y} = p\mathcal{X}$, then for all $j \in \mathcal{J}$ we have $\langle \mathcal{Y}_j | \mathcal{Y}_j \rangle_j = \mathcal{I}_j$, where $\mathcal{I}_j$ is an ideal of $\mathcal{A}_j$ containing $\mathcal{K}_j$, if and only if $\mathcal{Y}_j = \mathcal{Y}_{j\mathcal{I}_j}$. If H_j is a separable Hilbert space then $\langle \mathcal{Y}_j | \mathcal{Y}_j \rangle_j = \mathcal{A}_j$ or $\langle \mathcal{Y}_j | \mathcal{Y}_j \rangle_j = \mathcal{K}_j$, for all $j \in \mathcal{J}$.

Remark 2.9. We draw attention to the fact that the strict and relatively strict topologies in the submodules $\mathcal{Y}$ and the Hilbert module $\mathcal{X}$ are not the same. Namely in submodules the strict topology is given by the smaller set of seminorms in relation to the smaller ideal submodule $\mathcal{Y}_\mathcal{K} \subseteq \mathcal{X}_\mathcal{K}$, i.e. the topology of the submodule is weaker than the topology of the module. ♢

Remark 2.10. We note that the characterization (iv) in Theorem 2.3, together with the existence of strict orthogonal bases, allows almost direct transfer of results on bounded linear operators on Hilbert spaces to adjointable modular operators on a full Hilbert C^*-modules with the complementing property and also a transfer of partial results on a full Hilbert C^*-module over algebras containing essential ideal isomorphic to a C^*-algebra of compact operators on a Hilbert space. These properties are also helpful when working with unbounded operators in previously mentioned Hilbert C^*-modules. ♢

In the following example we discuss some basic C^*-algebras and Hilbert C^*-modules with complementing property and those without it.

Example 2.11. Let H be infinite-dimensional separable Hilbert space, let $\mathbf{B}(H)$ be a C^*-algebra of all bounded operators on H and let $\mathbf{K}(H)$ be a C^*-algebra of all compact operators on H.

Then for a C^*-algebra $\mathbf{A}$, where $\mathbf{K}(H) \subset \mathbf{A} \subseteq \mathbf{B}(H)$, $\mathbf{K}(H)$ is an unique proper essential ideal in $\mathbf{A}$ and $\mathbf{B}(H)$. In $\mathbf{B}(H)$ we define inner product $\forall x, y \in \mathbf{B}(H)$, $\langle x|y\rangle = x^*y$ with which $\mathbf{K}(H)$, $\mathbf{A}$ and $\mathbf{B}(H)$ are full right Hilbert C^*-modules over $\mathbf{K}(H)$, $\mathbf{A}$ and $\mathbf{B}(H)$, respectively. The strict topology in these modules is the standard strict topology in $\mathbf{B}(H)$ generated by $\mathbf{K}(H)$. Submodules in these Hilbert modules are right ideals in corresponding algebras, and an ideal submodule of $\mathbf{A}$ and $\mathbf{B}(H)$ is $\mathbf{K}(H)$.

As for Hilbert modules in which each relatively strictly closed submodule is orthogonally complemented in this example we have basic cases $\mathbf{A} = \mathbf{K}(H)$, $\mathbf{X} = \mathbf{K}(H)$ and $\mathbf{A} = \mathbf{B}(H)$, $\mathbf{X} = \mathbf{B}(H)$. Also, for any projection $p \in \mathbf{B}(H)$ with infinite rank and kernel $\mathbf{A} = p\mathbf{B}(H)p + \mathbf{K}(H)$ is a hereditary C^*-subalgebra of $\mathbf{B}(H)$ containing $\mathbf{K}(H)$, and the associated hereditary module is $\mathbf{X} = \mathbf{B}(H)\mathbf{A} = \mathbf{B}(H)p + \mathbf{K}(H)$. By the way, the multiplier algebra of $\mathbf{A}$ is $M(\mathbf{A}) = p\mathbf{B}(H)p + (e-p)\mathbf{B}(H)(e-p) + \mathbf{K}(H)$, where e is the unit in $\mathbf{B}(H)$. Note that the C^*-algebra $\mathbf{Y} = \mathbf{A}$ is also a full Hilbert C^*-module over $\mathbf{A}$, but $\mathbf{Y} = p\mathbf{B}(H)p + \mathbf{K}(H) \subsetneqq \mathbf{X}$. This implies that some relatively strictly closed submodules in $\mathbf{Y}$ are not complemented. In order to determine which submodules are complemented we decompose the Hilbert space H as $H = R \oplus L$, where $R = pH$, $L = (e-p)H$. From that representation it follows that the projection p has a 2×2 matrix form $p = \begin{bmatrix} e_1 & 0_3 \\ 0_3^* & 0_2 \end{bmatrix}$ where $e_1 \in \mathbf{B}(R)$ is the unit, $0_3 \in \mathbf{B}(L, R)$, $0_2 \in \mathbf{B}(R)$, and therefore, $\mathbf{Y} = \begin{bmatrix} \mathbf{B}(R) & \mathbf{K}(L,R) \\ \mathbf{K}(R,L) & \mathbf{K}(L) \end{bmatrix}$. Any projection q from $\mathbf{B}(H)$ can be identified with the matrix $q = \begin{bmatrix} a & c \\ c^* & b \end{bmatrix}$, where $a \in \mathbf{B}(R)$, $b, e_2 \in \mathbf{B}(L)$, e_2 is the unit, and $c \in \mathbf{B}(L, R)$ such that $0 \le a \le e_1$, $0 \le b \le e_2$, $cc^* = a(e_1 - a)$, $c^*c = b(e_2 - b)$ and $ac = c(e_2 - b)$, i.e. $a = \frac{1}{2}(e_1 \pm (e_1 - 4cc^*)^{\frac{1}{2}})$ and $b = \frac{1}{2}(e_2 \mp (e_2 - 4c^*c)^{\frac{1}{2}})$. Then $q\mathbf{Y}$ is of the form $q\mathbf{Y} = \begin{bmatrix} a\mathbf{B}(R) + c\mathbf{K}(R,L) & a\mathbf{K}(L,R) + c\mathbf{K}(L) \\ c^*\mathbf{B}(R) + b\mathbf{K}(R,L) & c^*\mathbf{K}(L,R) + b\mathbf{K}(L) \end{bmatrix}$. From that it follows $q\mathbf{Y} \subseteq \mathbf{Y} \Leftrightarrow c^*\mathbf{B}(R) \subseteq \mathbf{K}(R,L) \Leftrightarrow c \in \mathbf{K}(L,R)$. Thus, according to the Proposition 4.13 below, complemented are those and only those submodules in $\mathbf{Y}$ of the form $q\mathbf{B}(H) \cap \mathbf{Y} = q\mathbf{Y}$ for which the c component of the projection q is a compact operator. Other submodules of the form $q\mathbf{B}(H) \cap \mathbf{Y}$ are not complemented in $\mathbf{Y}$, but they are relatively strictly closed in $\mathbf{Y}$ (see Proposition 4.14) and hence orthogonally closed in $\mathbf{Y}$ by Theorem 2.1.

Let's consider now $\mathbf{A} = \mathbb{C}e + \mathbf{K}(H)$, a minimal unitization of $\mathbf{K}(H)$, which is not a hereditary C^*-subalgebra of $\mathbf{B}(H)$. From proposition 4.14 we know that every submodule in $\mathbf{A}$ which is closed with respect to the relatively strict topology, which is also orthogonally closed in $\mathbf{A}$ by Corollary 4.6, is of the form $\mathcal{G} = (e-p)\mathbf{B}(H) \cap \mathbf{A}$ for some projection $p \in \mathbf{B}(H)$. Then every $t \in \mathcal{G}$ is of the form $t = (e-p)b = \alpha e + k$ for some $b \in \mathbf{B}(H)$, $k \in \mathbf{K}(H)$

and $\alpha \in \mathbb{C}$. This implies that the compact operator k can be represented as $k = (e-p)b - \alpha e$ for some $b \in \mathbf{B}(H)$ and $\alpha \in \mathbb{C}$. Then $pk = -\alpha p$, and this is possible if and only if either $\alpha = 0$ or the dimension of the range of p is finite.

Relatively strictly closed submodules in $\mathbf{A}$ defined by projections with infinite-dimensional range and kernel are closed submodules of $\mathbf{K}(H)$ (case $\alpha = 0$), i.e. $\mathcal{G} = (e-p)\mathbf{K}(H)$ and $\mathcal{G}^{\perp_\mathbf{A}} = \mathcal{G}^{\perp_{\mathbf{K}(H)}} = p\mathbf{K}(H)$. They are not complemented in $\mathbf{A}$, but they are complemented in $\mathbf{K}(H)$, i.e. $\mathcal{G} \oplus \mathcal{G}^{\perp_\mathbf{A}} = \mathbf{K}(H)$, so the orthogonal sum is not relatively strictly closed in $\mathbf{A}$, but it is relatively strictly dense in $\mathbf{A}$. This submodule $\mathcal{G}$ is also an example of a submodule that is orthogonally closed in $\mathbf{A}$ but not complemented in $\mathbf{A}$.

Consider a relatively strictly closed submodule in $\mathbf{A}$ that is defined by a projection with a finite-dimensional range or kernel. Then exactly one of the projections p or $e-p$ is in $\mathbf{K}(H)$. If this is p, then $p\mathbf{A} = \mathbb{C}p + p\mathbf{K}(H) \subset \mathbf{K}(H) \subset \mathbf{A}$, and then we have $(e-p)\mathbf{A} \subseteq \mathbf{A} - p\mathbf{A} \subseteq \mathbf{A}$, which gives $\mathbf{A} = (e-p)\mathbf{A} \oplus p\mathbf{A} = \mathcal{G} \oplus \mathcal{G}^{\perp_\mathbf{A}}$. Thus, submodules are orthogonally complemented in $\mathbf{A}$ if and only if the associated projection has a finite-dimensional range or kernel. ♢

3. Proof of the main results

Most of the claims and technical results needed in the following proofs of main results are found in Section 4.

Proof of Theorem 2.1.

[**(i) ⇔ (ii)**] Let $\mathcal{I}$ be isomorphic to some C^*-algebra of compact operators on a Hilbert space. Then for each relatively strictly closed submodule $\mathcal{F} \subseteq \mathcal{X}$ the associated submodule $\mathcal{F}\mathcal{I} \subseteq \mathcal{X}_\mathcal{I}$ is closed, so it follows from [14, Theorem 1] that it is orthogonally closed in $\mathcal{X}_\mathcal{I}$, i.e. $\mathcal{F}\mathcal{I} = (\mathcal{F}\mathcal{I})^{\perp_{\mathcal{X}_\mathcal{I}}\perp_{\mathcal{X}_\mathcal{I}}}$. Applying the first assertion of Theorem 4.5 we have that $\mathcal{F}$ is orthogonally closed in $\mathcal{X}$.

Suppose that each relatively strictly closed submodule $\mathcal{F} \subseteq \mathcal{X}$ is orthogonally closed in $\mathcal{X}$. Let's take any closed submodule $\mathcal{G} \subseteq \mathcal{X}_\mathcal{I}$. Then $c\ell^{st}_\mathcal{X}(\mathcal{G}) \subseteq \mathcal{X}$ is relatively strictly closed submodule and by assumption it is orthogonally closed in $\mathcal{X}$, i.e. $c\ell^{st}_\mathcal{X}(\mathcal{G}) = (c\ell^{st}_\mathcal{X}(\mathcal{G}))^{\perp_\mathcal{X}\perp_\mathcal{X}}$. Multiplying both sides of the equality by $\mathcal{I}$, using Lemma 4.1$(i),(vi)$ and Lemma 4.2(iii), we have $\mathcal{G} = \mathcal{G}\mathcal{I} = c\ell^{st}_\mathcal{X}(\mathcal{G})\mathcal{I} = (c\ell^{st}_\mathcal{X}(\mathcal{G}))^{\perp_\mathcal{X}\perp_\mathcal{X}}\mathcal{I} = \mathcal{G}^{\perp_{\mathcal{X}_\mathcal{I}}\perp_{\mathcal{X}_\mathcal{I}}}$, i.e. $\mathcal{G}$ is orthogonally closed in $\mathcal{X}_\mathcal{I}$. From [14, Theorem 1] it follows that $\mathcal{I}$ is isomorphic to some C^*-algebra of compact operators on a Hilbert space.

[**(i) ⇔ (iii)**] Suppose that for each relatively strictly closed submodule $\mathcal{F} \subseteq \mathcal{X}$ the submodule $\mathcal{F} \oplus \mathcal{F}^{\perp_\mathcal{X}}$ is relatively strictly dense in $\mathcal{X}$. Let's take any closed submodule $\mathcal{G} \subseteq \mathcal{X}_\mathcal{I}$. Then $c\ell^{st}_\mathcal{X}(\mathcal{G})$ is relatively strictly closed submodule in $\mathcal{X}$ and by assumption, using Lemma 4.2(ii) we get $c\ell^{st}_\mathcal{X}(c\ell^{st}_\mathcal{X}(\mathcal{G}) \oplus \mathcal{G}^{\perp_\mathcal{X}}) = \mathcal{X}$. Multiplying both sides of the equality by $\mathcal{I}$, using Lemma 4.1$(v),(vi)$ and Lemma 4.2$(ii),(iii)$, we have $\mathcal{G} \oplus \mathcal{G}^{\perp_{\mathcal{X}_\mathcal{I}}} = \mathcal{X}_\mathcal{I}$, i.e. $\mathcal{G}$ is orthogonally complemented in $\mathcal{X}_{\mathcal{I}}$. From [11, Theorem 1] we conclude that $\mathcal{I}$ is isomorphic to some C^*-algebra of compact operators on a Hilbert space.

If $\mathcal{I}$ is isomorphic to some C^*-algebra of compact operators on a Hilbert space, then for each relatively strictly closed submodule $\mathcal{F} \subseteq \mathcal{X}$ associated submodule $\mathcal{F}\mathcal{I} \subseteq \mathcal{X}_{\mathcal{I}}$ is closed and [11, Theorem 1] implies that it is orthogonally complemented in $\mathcal{X}_{\mathcal{I}}$. Now we have $(\mathcal{F} \oplus \mathcal{F}^{\perp_{\mathcal{X}}})\mathcal{I} = \mathcal{F}\mathcal{I} \oplus (\mathcal{F}\mathcal{I})^{\perp_{\mathcal{X}_{\mathcal{I}}}} = \mathcal{X}_{\mathcal{I}} = \mathcal{X}\mathcal{I}$ and by Lemma 4.1(iv) it follows $c\ell_{\mathcal{X}}^{st}(\mathcal{F} \oplus \mathcal{F}^{\perp_{\mathcal{X}}}) = c\ell_{\mathcal{X}}^{st}(\mathcal{X}) = \mathcal{X}$.

[$(i) \Leftrightarrow (iv)$] If $\mathcal{I}$ is an ideal isomorphic to some C^*-algebra of compact operators on a Hilbert space, then Theorem 4.9 ensures the existence of a strict orthogonal bases (SOB) for $\mathcal{X}$ (see Definition 4.7).

Conversely, if there is a SOB for $\mathcal{X}$, then it is an orthogonal basis for the full $\mathcal{I}$-module $\mathcal{X}_{\mathcal{I}}$. Namely for the SOB $(x_\lambda)_{\lambda\in\Lambda}$ from the Definition 4.7 we have $p_\lambda = p_\lambda^* = p_\lambda^2$, i.e. $\forall\lambda \in \Lambda$, p_λ is a minimal projection in $\mathcal{A}$. Actually, $p_\lambda \mathcal{A} p_\lambda = \mathbb{C}p_\lambda$ gives $p_\lambda \mathcal{I} p_\lambda = \mathbb{C}p_\lambda \subset \mathcal{I}$ (see Remark 4.8), so then $p_\lambda \in \mathcal{I}$ and $\langle x_\lambda, x_\lambda\rangle = p_\lambda \in \mathcal{I}$ imply $x_\lambda \in \mathcal{X}_{\mathcal{I}}$. Because the strict frame in $\mathcal{X}_{\mathcal{I}}$ converges in the norm topology in $\mathcal{X}_{\mathcal{I}}$ (cf. [2, Proposition 3.10.]), SOB is an orthogonal bases for $\mathcal{X}_{\mathcal{I}}$. From the characterization of Hilbert C^*-modules over compact algebras [1, Corollary 7] this is equivalent to the fact that $\mathcal{I}$ is an ideal isomorphic to some C^*-algebra of compact operators on a Hilbert space.

[$(i) \Leftrightarrow (v)$] Suppose that (i) holds true, i.e. let $\mathcal{J}$ be a nonempty set, $(H_j)_{j\in\mathcal{J}}$ is a family of Hilbert spaces and $\psi_0 : \mathcal{K} \to \mathbf{K}$ is isomorphism of algebras where $\mathbf{K}$ is of the form (4.1). By Proposition 4.10 we can extend this isomorphism to the isomorphism $\psi : M(\mathcal{K}) \to M(\mathbf{K})$ where $M(\mathbf{K}) = \mathbf{B}$ is of the form (4.3). Then, because $\mathcal{K} \subseteq \mathcal{A} \subseteq M(\mathcal{K})$, for $\psi(\mathcal{A}) = \mathbf{A}$ we have inclusions $\mathbf{K} = c_0\text{-}\oplus_j \mathbf{K}(H_j) \subseteq \mathbf{A} \subseteq c_\infty\text{-}\oplus_j \mathbf{B}(H_j) = \mathbf{B}$, and this gives $\mathbf{A} = (\mathbf{A}_j)_{j\in\mathcal{J}}$, with $\mathbf{K}(H_j) \subseteq \mathbf{A}_j \subseteq \mathbf{B}(H_j)$, $j \in \mathcal{J}$. The conclusion holds since the center is assumed to be atomic.

Furthermore, for the Hilbert module $\mathcal{X}_{\mathcal{K}}$ there is also a family of Hilbert spaces $(G_j)_{j\in\mathcal{J}}$ such that $\mathcal{X}_{\mathcal{K}}$ is isomorphic by ψ_0-isomorphism $\Psi_0 : \mathcal{X}_{\mathcal{K}} \to \mathbf{X}_{\mathbf{K}}$ to the Hilbert C^*-module $\mathbf{X}_{\mathbf{K}} = c_0\text{-} \oplus_j \mathbf{K}(H_j, G_j)$ (cf. [14, Theorem 1.]). Proposition 4.10 gives the possibility to extend that isomorphism of Hilbert modules to the ψ-isomorphism $\Psi : M(\mathcal{X}_{\mathcal{K}}) \to M(\mathbf{X}_{\mathbf{K}})$. By Proposition 4.17 we have $M(\mathbf{X}_{\mathbf{K}}) = c_\infty\text{-} \oplus_j \mathbf{B}(H_j, G_j)$. For Hilbert C^*-module $\Psi(\mathcal{X}) = \mathbf{X}$ we have $\mathbf{X}_{\mathbf{K}} \subseteq \mathbf{X} \subseteq M(\mathbf{X}_{\mathbf{K}})$, that is, we have inclusion $c_0\text{-}\oplus_j \mathbf{K}(H_j, G_j) \subseteq \mathbf{X} \subseteq c_\infty\text{-} \oplus_j \mathbf{B}(H_j, G_j)$. It follows that $\mathbf{X} = (\mathbf{X}_j)_{j\in\mathcal{J}}$, where $\mathbf{K}(H_j, G_j) \subseteq \mathbf{X}_j \subseteq \mathbf{B}(H_j, G_j)$, $j \in \mathcal{J}$.

Conclusion $(v) \Rightarrow (i)$ is obvious. ∎

Proof of Theorem 2.3.

[$(i) \Leftrightarrow (ii)$] The claim follows directly from the second claim in Corollary 4.6.

[(*i*) ⇒ (*iii*) ⇒ (*ii*)] Let (i) holds true. Take any two relatively strictly closed submodules $\mathcal{F}, \mathcal{G} \subseteq \mathcal{X}$ such that $\mathcal{F} \perp \mathcal{G}$. By assumption they are complemented in $\mathcal{X}$ and by Proposition 4.13 there are projections $P, Q \in \mathbf{B}_a(M(\mathcal{X}_\mathcal{K}))$ and $\widehat{P} = P|_\mathcal{X}, \widehat{Q} = Q|_\mathcal{X} \in \mathbf{B}_a(\mathcal{X})$ such that $\mathcal{F} = PM(\mathcal{X}_\mathcal{K}) \cap \mathcal{X} = P\mathcal{X} = \widehat{P}\mathcal{X}$, $\mathcal{G} = QM(\mathcal{X}_\mathcal{K}) \cap \mathcal{X} = Q\mathcal{X} = \widehat{Q}\mathcal{X}$ and $\mathcal{X}$ is invariant module for P and Q, where $\widehat{P}, \widehat{Q}$ are restrictions of P, Q on $\mathcal{X}$, respectively. Also, due to the mutual orthogonality of submodules $\mathcal{F}$ and $\mathcal{G}$ we have the orthogonality of their strict closures $c\ell^{st}(\mathcal{F}) = PM(\mathcal{X}_\mathcal{K}) \perp QM(\mathcal{X}_\mathcal{K}) = c\ell^{st}(\mathcal{G})$. This implies $PQ = QP = 0$, and then $P + Q$ is a projection in $\mathbf{B}_a(M(\mathcal{X}_\mathcal{K}))$ and $\mathcal{X}$ is invariant module for $P + Q$. From Proposition 4.13 we have that the submodule $(P + Q)M(\mathcal{X}_\mathcal{K}) \cap \mathcal{X} = (\widehat{P} + \widehat{Q})\mathcal{X}$ is orthogonally complemented in $\mathcal{X}$, therefore, it is also relatively strictly closed. The equality $(\widehat{P} + \widehat{Q})\mathcal{X} = \widehat{P}\mathcal{X} \oplus \widehat{Q}\mathcal{X}$ is required to prove the claim. It is clear that we have $(\widehat{P} + \widehat{Q})\mathcal{X} \subseteq \widehat{P}\mathcal{X} \oplus \widehat{Q}\mathcal{X}$. Let's take any $x \in ((\widehat{P} + \widehat{Q})\mathcal{X})^{\perp_\mathcal{X}} = \mathcal{N}(\widehat{P} + \widehat{Q})$ and we have $\widehat{P}x + \widehat{Q}x = 0$. Multiplying the previous equality by $\widehat{P}$ and $\widehat{Q}$ we get $\widehat{P}x = \widehat{Q}x = 0$, hence $x \in \mathcal{N}(\widehat{P}) \cap \mathcal{N}(\widehat{Q})$. Because of equalities $\mathcal{N}(\widehat{P}) \cap \mathcal{N}(\widehat{Q}) = (\widehat{P}\mathcal{X})^{\perp_\mathcal{X}} \cap (\widehat{Q}\mathcal{X})^{\perp_\mathcal{X}} = (\widehat{P}\mathcal{X} \oplus \widehat{Q}\mathcal{X})^{\perp_\mathcal{X}}$ (for the last one cf. [9, Lemma 1.4.]) we have $((\widehat{P} + \widehat{Q})\mathcal{X})^{\perp_\mathcal{X}} \subseteq (\widehat{P}\mathcal{X} \oplus \widehat{Q}\mathcal{X})^{\perp_\mathcal{X}}$. Now complementing the previous inclusion and because $(\widehat{P} + \widehat{Q})\mathcal{X}$ is orthogonally closed in $\mathcal{X}$ we have $\widehat{P}\mathcal{X} \oplus \widehat{Q}\mathcal{X} \subseteq (\widehat{P}\mathcal{X} \oplus \widehat{Q}\mathcal{X})^{\perp_\mathcal{X}\perp_\mathcal{X}} \subseteq ((\widehat{P} + \widehat{Q})\mathcal{X})^{\perp_\mathcal{X}\perp_\mathcal{X}} = (\widehat{P} + \widehat{Q})\mathcal{X}$, i.e. (iii) holds true.

Conclusion $(iii) \Rightarrow (ii)$ is obvious.

[(*i*) ⇔ (*iv*)] Suppose that (iv) holds true, i.e. we have that β defined in (4.16) is an isomorphism of C^*-algebras. Assuming that essential ideal $\mathcal{K}$ of $\mathcal{A}$ is isomorphic to some C^*-algebra of compact operators on a Hilbert space, it is well known that in $\mathcal{X}_\mathcal{K}$ all closed submodules are complemented (see [11, Theorem 1.]). Therefore, by Lemma 4.27 we have that all relatively strictly closed submodules in $\mathcal{X}$ are complemented, i.e. (i) holds true.

Vice versa, let us suppose that each relatively strictly closed submodule in $\mathcal{X}$ is orthogonally complemented in $\mathcal{X}$. In order to prove the surjective nature of morphism $\gamma = \alpha^{-1} \circ \beta$, where α and β are from (4.15) and (4.16), respectively, let's take any projection $P \in \mathbf{B}_a(M(\mathcal{X}_\mathcal{K}))$. Then by Proposition 4.14 submodule $\mathcal{G} = PM(\mathcal{X}_\mathcal{K}) \cap \mathcal{X}$ is relatively strictly closed in $\mathcal{X}$, and by assumption it is orthogonally complemented in $\mathcal{X}$. From Proposition 4.13 we have that $\mathcal{X}$ is invariant module for projection $P \in \mathbf{B}_a(M(\mathcal{X}_\mathcal{K}))$.

Consequently $\widehat{P} = P|_\mathcal{X} \in \mathbf{B}_a(\mathcal{X})$ is the projection such that $\gamma(\widehat{P}) = P \in \mathcal{R}(\gamma)$, i.e. restriction of monomorphism γ is a bijection from the set of all projections in $\mathbf{B}_a(\mathcal{X})$ onto the set of all projections in $\mathbf{B}_a(M(\mathcal{X}_\mathcal{K}))$. Moreover, because α is isomorphism of C^*-algebras $\mathbf{B}_a(M(\mathcal{X}_\mathcal{K}))$ and $\mathbf{B}_a(\mathcal{X}_\mathcal{K})$, we have that sets of projections in $\mathbf{B}_a(M(\mathcal{X}_\mathcal{K}))$, $\mathbf{B}_a(\mathcal{X})$ and $\mathbf{B}_a(\mathcal{X}_\mathcal{K})$ are successively in a bijective relation by functions acting as restrictions. Now let us take the advantage of the fact that $\mathcal{K}$ is an essential ideal isomorphic to the algebra of compact operators on a Hilbert space. By applying Proposition 4.22 to the ideal submodule $\mathcal{X}_{\mathcal{K}}$ we get that $\mathbf{B}_a(\mathcal{X}_{\mathcal{K}})$ is (weakly) generated by projections in $\mathbf{B}_a(\mathcal{X}_{\mathcal{K}})$. The fact that $\mathbf{B}_a(\mathcal{X}_{\mathcal{K}})$ and $\mathbf{B}_a(M(\mathcal{X}_{\mathcal{K}}))$ are isomorphic implies that $\mathbf{B}_a(M(\mathcal{X}_{\mathcal{K}}))$ is generated by projections in $\mathbf{B}_a(M(\mathcal{X}_{\mathcal{K}}))$.

Because $\mathcal{X}$ is an invariant module for all projections in $\mathbf{B}_a(M(\mathcal{X}_{\mathcal{K}}))$ it is also invariant module for all operators in $\mathbf{B}_a(M(\mathcal{X}_{\mathcal{K}}))$, i.e. for every operator $T \in \mathbf{B}_a(M(\mathcal{X}_{\mathcal{K}}))$ its restriction $\widehat{T} = T|_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ is a bounded operator, and so is a restriction of the operator $T^* \in \mathbf{B}_a(M(\mathcal{X}_{\mathcal{K}}))$, $\widehat{T^*} = T^*|_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$. Clearly, $(\widehat{T})^* = \widehat{T^*}$ implies $\widehat{T} \in \mathbf{B}_a(\mathcal{X})$, so we have proved that the function γ is surjective and $\gamma^{-1}(T) = \widehat{T}$, $\forall T \in \mathbf{B}_a(M(\mathcal{X}_{\mathcal{K}}))$. Now from $\beta = \alpha \circ \gamma$ we have (iv). ∎

Proof of Theorem 2.5.

[$(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$] If (i) holds true, i.e. every relatively strictly closed submodule $\mathcal{F}$ in $\mathcal{X}$ is orthogonally complemented in $\mathcal{X}$, then obviously $\mathcal{F} \oplus \mathcal{F}^{\perp_{\mathcal{X}}}$ is relatively strictly dense in $\mathcal{X}$. Therefore, by Theorem 2.1 (iii) the essential ideal $\mathcal{I}$ is isomorphic to some algebra of compact operators on a Hilbert space of the form (4.2) and the ideal submodule $\mathcal{X}_{\mathcal{I}}$ is of the form (4.6). Moreover, because of Theorem 2.3 (iv) and [6, Theorem 2.3] C^*-algebras $\mathbf{B}_a(M(\mathcal{X}_{\mathcal{I}}))$ and $\mathbf{B}_a(\mathcal{X})$ are isomorphic and the isomorphism from $\mathbf{B}_a(M(\mathcal{X}_{\mathcal{I}}))$ onto $\mathbf{B}_a(\mathcal{X})$ acts as restriction. In other words, the mapping β in Theorem 4.28 is an isometry and it follows that $\mathcal{A}$ is a hereditary C^*-subalgebra of $M(\mathcal{I})$ and $\mathcal{X} = M(\mathcal{X}_{\mathcal{I}})\mathcal{A}$ is the associated hereditary $\mathcal{A}$-module of $M(\mathcal{X}_{\mathcal{I}})$, i.e. (ii) holds true.

If (ii) holds true then from Theorem 2.1 (v) and Proposition 4.10 we have the existence of ψ-isomorphism of modules Ψ such that $\psi(\mathcal{I}) = \mathbf{K} = c_0\text{-}\oplus_{j\in\mathcal{J}} \mathbf{K}(H_j)$, $\psi(M(\mathcal{I})) = M(\mathbf{K}) = c_\infty\text{-}\oplus_{j\in\mathcal{J}} \mathbf{B}(H_j, G_j)$, $\psi(\mathcal{A}) = \mathbf{A} = (\mathbf{A}_j)_{j\in\mathcal{J}}$ as in (2.1), and the associated ideal submodule $\Psi(\mathcal{X}) = \mathbf{X} = (\mathbf{X}_j)_{j\in\mathcal{J}}$ as in (2.2). Then every $\mathbf{A}_j$ is a hereditary C^*-subalgebra of $\mathbf{B}(H_j)$ and $\mathbf{X}_j = \mathbf{B}(H_j, G_j)\mathbf{A}_j$, $j \in \mathcal{J}$. Because $\mathbf{A}M(\mathbf{K})\mathbf{A} = \psi(\mathcal{A}M(\mathcal{I})\mathcal{A}) = \psi(\mathcal{A}) = \mathbf{A}$ we have that $\mathbf{A}$ is a hereditary C^*-subalgebra of $M(\mathbf{K})$. Also we have $\mathbf{X} = \Psi(\mathcal{X}) = \Psi(M(\mathcal{X}_{\mathcal{I}})\mathcal{A}) = M(\mathbf{X}_{\mathbf{K}})\mathbf{A}$ and (iii) holds true.

Now suppose that (iii) hols true. Then the ψ^{-1}-isomorphism of modules $\Psi^{-1} : M(\mathbf{X}_{\mathbf{K}}) \to M(\mathcal{X}_{\mathcal{I}})$, where Ψ is as in the previous part of the proof, implies that $\mathcal{A}$ is a hereditary C^*-subalgebra of $M(\mathcal{I})$ and $\mathcal{X} = M(\mathcal{X}_{\mathcal{I}})\mathcal{A}$ is a hereditary module of $M(\mathcal{X}_{\mathcal{I}})$ with respect to $\mathcal{A}$, i.e. (ii) is true.

Let (ii) holds true and take any relatively strictly closed submodule $\mathcal{Y} \subseteq \mathcal{X}$. By Proposition 4.14 every such submodule is of the form $\mathcal{Y} = PM(\mathcal{X}_{\mathcal{I}}) \cap \mathcal{X}$, where $P = (P_j)_{j\in\mathcal{J}}$ is a projection in $M(\mathcal{X}_{\mathcal{I}})$. Because $\mathcal{X} = M(\mathcal{X}_{\mathcal{I}})\mathcal{A}$ is obviously invariant for all operators in $B_a(M(\mathcal{X}_{\mathcal{I}}))$ and particularly for the projection P, by Proposition 4.13 it follows that the submodule $\mathcal{Y}$ is complemented in $\mathcal{X}$. ∎

Proof of Proposition 2.8.

Let $X = M(\mathcal{X}_\mathcal{K})\mathcal{A}$ with $\mathcal{A} \in \mathfrak{ha}_\mathcal{K}(M(\mathcal{K}))$. Then every relatively strictly closed submodule $\mathcal{Y}$ is complemented by Theorem 2.5, so it is, by Proposition 4.13, of the form $\mathcal{Y} = p\mathcal{X}$, where p is a projection in $\mathbf{B}_a(M(\mathcal{X}_\mathcal{K}))$.

Let us show first that $M(\mathcal{Y}) = pM(\mathcal{X})$. It is well known that the $M(\mathcal{A})$-module $M(\mathcal{X})$ is isomorphic to the $\mathbf{B}_a(\mathcal{A})$-module $\mathbf{B}_a(\mathcal{A}, \mathcal{X})$, which consists of all functions $f : \mathcal{A} \to \mathcal{X}$ acting as $f(a) = xa, \forall a \in \mathcal{A}$, for some $x \in M(\mathcal{X})$. Now, any function $g : \mathcal{A} \to \mathcal{Y}$ which is acting as $g(a) = ya, \forall a \in \mathcal{A}$, for some $y = px$, where $x \in M(\mathcal{X})$, is of the form $g = pf$, for some $f \in \mathbf{B}_a(\mathcal{A}, \mathcal{X})$. Hence, the mapping $g \mapsto pf$ is obviously an isomorphism of Hilbert $M(\mathcal{K})$-modules $M(\mathcal{Y})$ and $pM(\mathcal{X})$. Further, we have $M(\mathcal{Y}_\mathcal{K})\mathcal{A} = M(p\mathcal{X}_\mathcal{K})\mathcal{A} = pM(\mathcal{X}_\mathcal{K})\mathcal{A} = p\mathcal{X} = \mathcal{Y}$, so Proposition 4.25 implies that $\mathcal{Y}$ is (not necessarily full) Hilbert $\mathcal{A}$-module, hence $\langle \mathcal{Y}|\mathcal{Y}\rangle$ is an ideal in $\mathcal{A}$. Because $\langle \mathcal{Y}|\mathcal{Y}\rangle$ is also a hereditary subalgebra of $\mathcal{A}$, it is a hereditary subalgebra of $M(\mathcal{K})$, so $\mathcal{Y}$ is a hereditary module of $M(\mathcal{Y}_\mathcal{K})$.

Also, for all $j \in \mathcal{J}$ closed ideals in $M(\mathcal{K}_j)$ form a chain between $\mathcal{K}_j \simeq \mathbf{K}(H_j)$ and $M(\mathcal{K}_j) \simeq \mathbf{B}(H_j)$ (see [10, Corollary 6.2] or [8, Theorem 3.3.]). Then all norm-closed ideals in a hereditary subalgebra also form an order-complete net between $\mathcal{K}_j$ and $\mathcal{A}_j$ (see [12, Theorem 3.2.7.]). Therefore, we have that $\mathcal{N} = \langle \mathcal{Y}_j|\mathcal{Y}_j\rangle$ and $\mathcal{M} = \langle \mathcal{Y}_j^\perp|\mathcal{Y}_j^\perp\rangle$ are ideals in $\mathcal{A}_j$ and suppose that $\mathcal{N} \subseteq \mathcal{M}$. Then $\mathcal{N} + \mathcal{M} \subseteq \mathcal{M} \subseteq \mathcal{N} + \mathcal{M}$. Also, for any $x \in \mathcal{X}$ we have $x = y \oplus z$ for some $y \in \mathcal{Y}_j$ and $z \in \mathcal{Y}_j^\perp$, i.e. $\langle x|x\rangle = \langle y|y\rangle + \langle z|z\rangle \in \mathcal{N} + \mathcal{M} = \mathcal{M}$, so we have $\mathcal{M} = \mathcal{A}_j$.

Finally, the equivalence $\langle \mathcal{Y}_j|\mathcal{Y}_j\rangle_j = \mathcal{I}_j$ holds if and only if $\mathcal{Y}_j = \mathcal{Y}_{j\mathcal{I}_j}$ follows from Proposition 4.25. □

4. Auxiliary Results

This section contains most of the results required in the proofs of the main results in this article. Many of them are expressed more broadly than is necessary for this purpose and are grouped into eight subsections by topic.

4.1. *Strict and relatively strict topology.* The following properties of strict and relatively strict closures are essential in proofs. Claims concerning Hilbert $\mathcal{I}$-modules $\mathcal{X}_\mathcal{I}$ and its strict completions Hilbert $M(\mathcal{I})$-module $M(\mathcal{X}_\mathcal{I})$ are from [9]. Therefore, we prove only those claims regarding Hilbert $\mathcal{A}$-module $\mathcal{X}$, where $\mathcal{X}_\mathcal{I} \subseteq \mathcal{X} \subseteq M(\mathcal{X}_\mathcal{I})$.

Lemma 4.1. *Let $\mathcal{A}$ be a C*-algebra with an essential ideal $\mathcal{I}$, let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module and let $\mathcal{F}, \mathcal{G} \subseteq M(\mathcal{X}_\mathcal{I})$ be submodules. Then*

(i) $c\ell(\mathcal{F}\mathcal{I}) = c\ell(\mathcal{F})\mathcal{I} = c\ell^{st}(\mathcal{F})\mathcal{I} = (c\ell^{st}(\mathcal{F}) \cap \mathcal{X})\mathcal{I}$, *and*

(ii) $c\ell^{st}(c\ell(\mathcal{F})) = c\ell^{st}(\mathcal{F}) = c\ell^{st}(c\ell^{st}(\mathcal{F})\mathcal{I}) = c\ell^{st}(\mathcal{F}\mathcal{I})$.

(iii) *If $\mathcal{F} \subseteq \mathcal{X}$ then* $c\ell^{st}_\mathcal{X}(c\ell(\mathcal{F})) = c\ell^{st}_\mathcal{X}(\mathcal{F}) = c\ell^{st}_\mathcal{X}(c\ell^{st}(\mathcal{F})\mathcal{I}) = c\ell^{st}_\mathcal{X}(\mathcal{F}\mathcal{I})$.

(*iv*) *Equalities* $c\ell(\mathcal{F})\mathcal{I} = c\ell(\mathcal{G})\mathcal{I}$, $c\ell^{st}(\mathcal{F}) \cap \mathcal{X} = c\ell^{st}(\mathcal{G}) \cap \mathcal{X}$ *and* $c\ell^{st}(\mathcal{F}) = c\ell^{st}(\mathcal{G})$ *are equivalent.*

(*v*) *If* $\mathcal{F} \perp \mathcal{G}$ *then* $(c\ell(\mathcal{F}) \oplus c\ell(\mathcal{G}))\mathcal{I} = c\ell(\mathcal{F})\mathcal{I} \oplus c\ell(\mathcal{G})\mathcal{I}$.

(*vi*) $\mathcal{F} \subseteq \mathcal{X}_\mathcal{I}$ *if and only if* $c\ell(\mathcal{F}) = c\ell(\mathcal{F})\mathcal{I}$.

(*vii*) $c\ell(\mathcal{F}) \cap \mathcal{X}_\mathcal{I} = c\ell(\mathcal{F})\mathcal{I}$.

(*viii*) *If* $\mathcal{F} \subseteq \mathcal{X}_\mathcal{I}$ *then* $c\ell^{st}_{\mathcal{X}_\mathcal{I}}(\mathcal{F}) = c\ell^{st}(\mathcal{F}) \cap \mathcal{X}_\mathcal{I} = c\ell(\mathcal{F})$.

PROOF. Claims (*i*) except for the last equality, (*ii*), (*iv*) except for the second equality, (*vi*) and (*vii*) are proved in [9, Lemma 3.1.]. The claim (*v*) is proved in [9, Lemma 3.3.].

In order to prove the last equality in (*i*) we note that second equality in (*i*) and (*vii*) give $c\ell^{st}(\mathcal{F})\mathcal{I} \overset{(i.2)}{=} c\ell(\mathcal{F})\mathcal{I} \overset{(vii)}{=} c\ell(\mathcal{F}) \cap \mathcal{X}_\mathcal{I} \subseteq c\ell^{st}(\mathcal{F}) \cap \mathcal{X} \subseteq c\ell^{st}(\mathcal{F})$. Now, by multiplying the previous inclusion by $\mathcal{I}$ and applying (*vi*) to $c\ell^{st}(\mathcal{F})\mathcal{I}$ we have $c\ell^{st}(\mathcal{F})\mathcal{I} \overset{(vi)}{=} (c\ell^{st}(\mathcal{F})\mathcal{I})\mathcal{I} \overset{(vii)}{=} (c\ell^{st}(\mathcal{F}) \cap \mathcal{X}_\mathcal{I})\mathcal{I} \subseteq (c\ell^{st}(\mathcal{F}) \cap \mathcal{X})\mathcal{I} \subseteq c\ell^{st}(\mathcal{F})\mathcal{I}$.

Equalities (*iii*) follow directly from (*ii*) intersecting with $\mathcal{X}$.

To prove the remaining equivalences in (*iv*), we note that the last equality implies the second equality simply intersecting both sides with $\mathcal{X}$. Also, by multiplying both sides in the second equality by $\mathcal{I}$ and applying (*i*) we get the first equality in (*iv*).

For the proof of (*viii*) we take any $x \in c\ell^{st}_{\mathcal{X}_\mathcal{I}}(\mathcal{F})$ and there is a net $(x_\lambda)_\lambda$ in $\mathcal{F}$ strictly convergent to $x \in \mathcal{X}_\mathcal{I}$, hence $\forall b \in \mathcal{I}$, $xb = \lim_\lambda x_\lambda b \in c\ell(\mathcal{F})$. Now, applying approximate unit $(e_\mu)_\mu$ in $\mathcal{I}$ we have $x = \lim_\mu xe_\mu \in c\ell(\mathcal{F})$, i.e. $c\ell^{st}_{\mathcal{X}_\mathcal{I}}(\mathcal{F}) \subseteq c\ell(\mathcal{F})$. Opposite inclusion is obvious. ∎

LEMMA 4.2. *Let* $\mathcal{A}$ *be a* C^**-algebra with an essential ideal* $\mathcal{I}$ *and let* $\mathcal{X}$ *be a full Hilbert* $\mathcal{A}$*-module. If* $\mathcal{F} \subseteq M(\mathcal{X}_\mathcal{I})$ *is a submodule then* $\mathcal{F}^\perp$ *is strictly closed and if* $\mathcal{F} \subseteq \mathcal{X}$ *then* $\mathcal{F}^{\perp_\mathcal{X}}$ *is relatively strictly closed in* $\mathcal{X}$*, hence* $c\ell^{st}(\mathcal{F}) \subseteq \mathcal{F}^{\perp\perp}$ *and* $c\ell^{st}_\mathcal{X}(\mathcal{F}) \subseteq \mathcal{F}^{\perp_\mathcal{X}\perp_\mathcal{X}} = \mathcal{F}^{\perp\perp} \cap \mathcal{X}$.

The following equalities apply:

(*i*) $(\mathcal{F}\mathcal{I})^\perp = (c\ell(\mathcal{F})\mathcal{I})^\perp = \mathcal{F}^\perp = c\ell(\mathcal{F})^\perp = (c\ell^{st}(\mathcal{F}))^\perp$
and if $\mathcal{F} \subseteq \mathcal{X}$ *then* $\mathcal{F}^\perp = (c\ell^{st}_\mathcal{X}(\mathcal{F}))^\perp$.

(*ii*) $(\mathcal{F}\mathcal{I})^{\perp_\mathcal{X}} = (c\ell(\mathcal{F})\mathcal{I})^{\perp_\mathcal{X}} = \mathcal{F}^{\perp_\mathcal{X}} = c\ell(\mathcal{F})^{\perp_\mathcal{X}} = (c\ell^{st}(\mathcal{F}))^{\perp_\mathcal{X}}$,
and if $\mathcal{F} \subseteq \mathcal{X}$ *then* $\mathcal{F}^{\perp_\mathcal{X}} = (c\ell^{st}_\mathcal{X}(\mathcal{F}))^{\perp_\mathcal{X}}$.

(*iii*) $(\mathcal{F}\mathcal{I})^{\perp_{\mathcal{X}_\mathcal{I}}} = \mathcal{F}^\perp\mathcal{I} = \mathcal{F}^{\perp_\mathcal{X}}\mathcal{I}$.

PROOF. We prove only the claims concerning the relative topology on $\mathcal{X}$ and the orthogonal complementing in $\mathcal{X}$. The rest of claims are proved in [9, Lemma 3.2.].

Because $c\ell^{st}_\mathcal{X}(\mathcal{F}^{\perp_\mathcal{X}}) = c\ell^{st}(\mathcal{F}^{\perp_\mathcal{X}}) \cap \mathcal{X} \subseteq c\ell^{st}(\mathcal{F}^\perp) \cap \mathcal{X} = \mathcal{F}^\perp \cap \mathcal{X} = \mathcal{F}^{\perp_\mathcal{X}}$ we have that $\mathcal{F}^{\perp_\mathcal{X}}$ is relatively strictly closed in $\mathcal{X}$.

Next, by putting $\mathcal{F}^{\perp}$ instead of $\mathcal{F}$ in (i) we have $(\mathcal{F}^{\perp}\mathcal{I})^{\perp} = \mathcal{F}^{\perp\perp}$ and this gives $\mathcal{F}^{\perp\perp} \cap \mathcal{X} \subseteq \mathcal{F}^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}} = (\mathcal{F}^{\perp} \cap \mathcal{X})^{\perp} \cap \mathcal{X} \subseteq (\mathcal{F}^{\perp} \cap \mathcal{X}_{\mathcal{I}})^{\perp} \cap \mathcal{X} = (\mathcal{F}^{\perp}\mathcal{I})^{\perp} \cap \mathcal{X} = \mathcal{F}^{\perp\perp} \cap \mathcal{X}$, i.e. $\mathcal{F}^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}} = \mathcal{F}^{\perp\perp} \cap \mathcal{X} = \mathcal{F}^{\perp\perp_{\mathcal{X}}}$.

Claim (ii) follows from (i) intersecting with $\mathcal{X}$.

For the proof of the last equality in (iii) we have $\mathcal{F}^{\perp}\mathcal{I} \subseteq \mathcal{F}^{\perp} \cap \mathcal{X}_{\mathcal{I}} = \mathcal{F}^{\perp_{\mathcal{X}_{\mathcal{I}}}} = \mathcal{F}^{\perp_{\mathcal{X}_{\mathcal{I}}}}\mathcal{I} \subseteq \mathcal{F}^{\perp_{\mathcal{X}}}\mathcal{I} \subseteq \mathcal{F}^{\perp}\mathcal{I}$. ◻

REMARK 4.3. It is clear from Lemma 4.2 that every orthogonally closed submodule in $M(\mathcal{X}_{\mathcal{I}})$ is strictly closed and that every orthogonally closed submodule in $\mathcal{X}$ is relatively strictly closed. ♢

REMARK 4.4. If a submodule $\mathcal{F} \oplus \mathcal{F}^{\perp_{\mathcal{X}}}$ is relatively strictly closed in $\mathcal{X}$ then $\mathcal{F}$ is relatively strictly closed in $\mathcal{X}$. Namely if the net $(x_{\lambda})_{\lambda}$ in $\mathcal{F} \subseteq \mathcal{F} \oplus \mathcal{F}^{\perp_{\mathcal{X}}}$ strictly converges to some $x_0 \in \mathcal{X}$ then $x_0 \in \mathcal{F} \oplus \mathcal{F}^{\perp_{\mathcal{X}}}$ is of the form $x_0 = u + v$ for some $u \in \mathcal{F}$ and $v \in \mathcal{F}^{\perp_{\mathcal{X}}}$. But $0 = \text{st-lim}_{\lambda}\langle x_{\lambda}|v\rangle = \langle x_0|v\rangle = \langle u|v\rangle + \langle v|v\rangle = \langle v|v\rangle$ implies $x_0 = u \in \mathcal{F}$. The opposite claim is not generally true, as it is shown in Example 2.11. ♢

4.2. *Relative strict topology and complementing.* The first results are on the characterization of orthogonal closedness and orthogonal complementing property for single submodules of the full Hilbert $\mathcal{A}$-module with an essential ideal, using the relative strict topology on this module.

THEOREM 4.5. *Let $\mathcal{A}$ be C^*-algebra with an essential ideal $\mathcal{I}$ and $\mathcal{X}$ a full Hilbert $\mathcal{A}$-module.*

A submodule $\mathcal{F} \subseteq \mathcal{X}$ is orthogonally closed in $\mathcal{X}$ if and only if $\mathcal{F}$ is relatively strictly closed in $\mathcal{X}$ and $\mathcal{F}\mathcal{I}$ is orthogonally closed in $\mathcal{X}_{\mathcal{I}}$.

A submodule $\mathcal{F} \subseteq \mathcal{X}$ is orthogonally complemented in $\mathcal{X}$ if and only if $\mathcal{F} \oplus \mathcal{F}^{\perp_{\mathcal{X}}}$ is relatively strictly closed in $\mathcal{X}$ and $\mathcal{F}\mathcal{I}$ is orthogonally complemented in $\mathcal{X}_{\mathcal{I}}$.

PROOF. Note that Lemma 4.2 $(ii), (iii)$ implies $(\mathcal{F}\mathcal{I})^{\perp_{\mathcal{X}_{\mathcal{I}}}\perp_{\mathcal{X}_{\mathcal{I}}}} = (\mathcal{F}^{\perp_{\mathcal{X}}}\mathcal{I})^{\perp_{\mathcal{X}_{\mathcal{I}}}} = (\mathcal{F}^{\perp_{\mathcal{X}}}\mathcal{I})^{\perp_{\mathcal{X}}}\mathcal{I} = \mathcal{F}^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}}\mathcal{I}$.

For the first assertion, suppose that the submodule $\mathcal{F}$ is orthogonally closed in $\mathcal{X}$, therefore, it is relatively strictly closed in $\mathcal{X}$. Then $\mathcal{F}\mathcal{I} = \mathcal{F}^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}}\mathcal{I} = (\mathcal{F}\mathcal{I})^{\perp_{\mathcal{X}_{\mathcal{I}}}\perp_{\mathcal{X}_{\mathcal{I}}}}$, i.e. $\mathcal{F}\mathcal{I}$ is orthogonally closed in $\mathcal{X}_{\mathcal{I}}$.

Conversely, if $\mathcal{F}$ is relatively strictly closed in $\mathcal{X}$ and if $\mathcal{F}\mathcal{I}$ is orthogonally closed in $\mathcal{X}_{\mathcal{I}}$, we have $\mathcal{F}\mathcal{I} = (\mathcal{F}\mathcal{I})^{\perp_{\mathcal{X}_{\mathcal{I}}}\perp_{\mathcal{X}_{\mathcal{I}}}} = \mathcal{F}^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}}\mathcal{I}$, and by Lemma 4.1$(iv)$ we conclude that $\mathcal{F}$ is orthogonally closed in $\mathcal{X}$.

For the second claim, suppose that the submodule $\mathcal{F}$ is orthogonally complemented in $\mathcal{X}$. Then [9, Theorem 2] implies that $\mathcal{F} \oplus \mathcal{F}^{\perp_{\mathcal{X}}}$ is orthogonally closed in $\mathcal{X}$, hence it is relatively strictly closed in $\mathcal{X}$ and from Remark 4.4 we have that $\mathcal{F}$ is relatively strictly closed in $\mathcal{X}$. Also, we have $\mathcal{F}\mathcal{I} \oplus (\mathcal{F}\mathcal{I})^{\perp_{\mathcal{X}_{\mathcal{I}}}} = (\mathcal{F} \oplus \mathcal{F}^{\perp_{\mathcal{X}}})\mathcal{I} = \mathcal{X}\mathcal{I} = \mathcal{X}_{\mathcal{I}}$.

If $\mathcal{F}\oplus\mathcal{F}^{\perp_{\mathcal{X}}}$ is relatively strictly closed in $\mathcal{X}$ and $\mathcal{F}\mathcal{I}$ is complemented in $\mathcal{X}_{\mathcal{I}}$ then by [9, Theorem 2] we have $\mathcal{F}\mathcal{I}\oplus(\mathcal{F}\mathcal{I})^{\perp_{\mathcal{X}_\mathcal{I}}} = (\mathcal{F}\mathcal{I}\oplus(\mathcal{F}\mathcal{I})^{\perp_{\mathcal{X}_\mathcal{I}}})^{\perp_{\mathcal{X}_\mathcal{I}}\perp_{\mathcal{X}_\mathcal{I}}}$ and $(\mathcal{F}\oplus\mathcal{F}^{\perp_{\mathcal{X}}})\mathcal{I} = (\mathcal{F}\oplus\mathcal{F}^{\perp_{\mathcal{X}}})^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}}\mathcal{I}$. Because $\mathcal{F}\oplus\mathcal{F}^{\perp_{\mathcal{X}}}$ is relatively strictly closed in $\mathcal{X}$, applying Lemma 4.1 (*iv*) we conclude that $\mathcal{F}\oplus\mathcal{F}^{\perp_{\mathcal{X}}} = (\mathcal{F}\oplus\mathcal{F}^{\perp_{\mathcal{X}}})^{\perp_{\mathcal{X}}\perp_{\mathcal{X}}}$, and [9, Theorem 2] gives that $\mathcal{F}$ is orthogonally complemented in $\mathcal{X}$. □

In the special case where $\mathcal{I}$ is isomorphic to some C^*-algebra of compact operators on a Hilbert space, we have the following characterization.

COROLLARY 4.6. *Let $\mathcal{A}$ be a C^*-algebra with an essential ideal $\mathcal{K}$ isomorphic to some C^*-algebra of compact operators on a Hilbert space and let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module.*

Submodule $\mathcal{F}\subseteq\mathcal{X}$ is orthogonally closed in $\mathcal{X}$ if and only if $\mathcal{F}$ is relatively strictly closed in $\mathcal{X}$.

Submodule $\mathcal{F}\subseteq\mathcal{X}$ is orthogonally complemented in $\mathcal{X}$ if and only if $\mathcal{F}\oplus\mathcal{F}^{\perp_{\mathcal{X}}}$ is relatively strictly closed in $\mathcal{X}$.

PROOF. The first corollary assertion follows from the first assertion of Theorem 4.5, [14, Theorem 1] and the fact that $\mathcal{X}_{\mathcal{I}}$ is a full Hilbert C^*-module over C^*-algebra isomorphic to some algebra of compact operators on a Hilbert space.

For the second claim, suppose $\mathcal{F}\oplus\mathcal{F}^{\perp_{\mathcal{X}}}$ is relatively strictly closed in $\mathcal{X}$. Then $\mathcal{F}\mathcal{I}$ is a closed submodule in $\mathcal{X}_{\mathcal{I}}$, therefore, it is orthogonally complemented in $\mathcal{X}_{\mathcal{I}}$ (cf. [11, Theorem 1]). By theorem 4.5, the submodule $\mathcal{F}$ is orthogonally complemented in $\mathcal{X}$.

Conversely, if $\mathcal{F}$ is orthogonally complemented in $\mathcal{X}$ then from [9, Theorem 2] we have that $\mathcal{F}\oplus\mathcal{F}^{\perp_{\mathcal{X}}}$ is orthogonally closed in $\mathcal{X}$, therefore, it is also relatively strictly closed in $\mathcal{X}$. □

4.3. *Strict orthogonal bases.*

DEFINITION 4.7. *Strict orthogonal bases (SOB) $(x_\lambda)_{\lambda\in\Lambda}$ for Hilbert $\mathcal{A}$-module $\mathcal{X}$ is a strict Parseval frame, that is $\langle x|x\rangle = st\text{-}\sum_{\lambda\in\Lambda}\langle x|x_\lambda\rangle\langle x_\lambda|x\rangle$, $\forall x\in\mathcal{X}$, such that $\forall\lambda,\mu\in\Lambda$, $\langle x_\lambda|x_\mu\rangle = \delta_{\lambda,\mu}p_\lambda$, where $\forall\lambda\in\Lambda$, $p_\lambda\in\mathcal{A}$ is orthogonal projection with the property $p_\lambda\mathcal{A}p_\lambda = \mathbb{C}p_\lambda$.*

REMARK 4.8. Note that if $\mathcal{I}$ is an essential ideal in $\mathcal{A}$ then for any projection $p\in\mathcal{A}$ we have $p\mathcal{A}p = \mathbb{C}p$ if and only if $p\mathcal{I}p = \mathbb{C}p$. If $p\mathcal{I}p = \mathbb{C}p$ then $\mathbb{C}p = p\mathcal{I}p\subseteq p\mathcal{A}p = p\mathcal{A}\mathbb{C}p = p\mathcal{A}p\mathcal{I}p\subseteq p\mathcal{I}p = \mathbb{C}p$ gives $p\mathcal{A}p = \mathbb{C}p$. Conversely, if $p\mathcal{A}p = \mathbb{C}p$ then $p\mathcal{I}p\subseteq p\mathcal{A}p = \mathbb{C}p$ and this implies $p\mathcal{I}p = \mathbb{C}p$. Because $\mathcal{I}$ is essential in $\mathcal{A}$ the equality $p\mathcal{I}p = \{0\}$ is possible if and only if $p = 0$.

The construction of a SOB for a full C^*-Hilbert $\mathcal{A}$-modules where $\mathcal{A}$ contains an essential ideal isomorphic to the C^*-algebra of compact operators on a Hilbert space is given in the following theorem.

Theorem 4.9. *Each full Hilbert module $\mathcal{X}$ over C^*-algebra $\mathcal{A}$ with the essential ideal $\mathcal{K}$ isomorphic to some C^*-algebra of compact operators on a Hilbert space has SOB and all SOBs for $\mathcal{X}$ have the same cardinality.*

Proof. Ideal submodule $\mathcal{X}_{\mathcal{K}}$ is a full $\mathcal{K}$-module and from [4, Theorem 2] we have the existence of orthogonal bases $(x_\lambda)_{\lambda\in\Lambda}$ for the Hilbert $\mathcal{K}$-module $\mathcal{X}_{\mathcal{K}}$ such that $\forall\lambda,\mu\in\Lambda$, $\langle x_\lambda|x_\mu\rangle = \delta_{\lambda,\mu}p_\lambda$, where $\forall\lambda\in\Lambda$, $p_\lambda\in\mathcal{A}$ is orthogonal projection with the property $p_\lambda\mathcal{K}p_\lambda = \mathbb{C}p_\lambda$. All such orthogonal bases have the same cardinality.

We claim that $(x_\lambda)_{\lambda\in\Lambda}$ is a SOB for $\mathcal{X}$. For any $b\in\mathcal{I}$ and $x\in\mathcal{X}$ we have $xb\in\mathcal{X}_{\mathcal{K}}$ and the fact that $(x_\lambda)_{\lambda\in\Lambda}$ is an orthogonal basis for $\mathcal{X}_{\mathcal{K}}$ gives $xb=\sum_\lambda x_\lambda\langle x_\lambda|xb\rangle$. Then $b^*\langle x|x\rangle b = \langle xb|xb\rangle = \sum_\lambda b^*\langle x|x_\lambda\rangle\langle x_\lambda|x\rangle b = b^*(\sum_\lambda\langle x|x_\lambda\rangle\langle x_\lambda|x\rangle)b$. Also from Remark 4.8 we have $p_\lambda\mathcal{A}p_\lambda = \mathbb{C}p_\lambda$ and this implies that all conditions for SOB in Definition 4.7 are fulfilled, i.e. $(x_\lambda)_{\lambda\in\Lambda}$ is a SOB for $\mathcal{X}$. □

4.4. *Another characterization of multiplier Hilbert modules.* The next result on the extension of the isomorphism of Hilbert C^*-modules (unitary operators) is from [6, Proposition 1.7] and is often useful.

Proposition 4.10. *Let $\mathcal{A}_i$ $(i=1,2)$ be C^*-algebras and $\mathcal{X}_i$ full Hilbert $\mathcal{A}_i$-modules $(i=1,2)$ and let $\Phi_0:\mathcal{X}_1\to\mathcal{X}_2$ be a ϕ_0-isomorphism of Hilbert C^*-modules. Then there exists a ϕ-isomorphism $\Phi: M(\mathcal{X}_1)\to M(\mathcal{X}_2)$ of Hilbert C^*-modules $M(\mathcal{X}_1)$ and $M(\mathcal{X}_2)$ such that $\Phi_0=\Phi|_{\mathcal{A}_1}$.*

In the sense of the property we prove in the following proposition, for a full $\mathcal{A}$-module $\mathcal{X}$ we say that $M(\mathcal{X})$ is the maximal Hilbert C^*-module over unital C^*-algebra containing $\mathcal{A}$ as an essential ideal such that $\mathcal{X}$ is its ideal submodule with respect to $\mathcal{A}$. This is analogous to the characterization of multiplier algebra as the maximal unitization of C^*-algebra.

Proposition 4.11. *Let $\mathcal{X}$ be a full $\mathcal{A}$-module and let $M(\mathcal{X})$ be its multiplier (not necessarily full) $M(\mathcal{A})$-module. For any injective j-morphism of Hilbert C^*-modules $J:\mathcal{X}\to\mathcal{Y}$, where (not necessarily full) $\mathcal{B}$-module $\mathcal{Y}$ is such that $j(\mathcal{A})$ is an essential ideal of an unital C^*-algebra $\mathcal{B}$ and $\mathcal{Y}\,j(\mathcal{A})\subseteq J(\mathcal{X})$, there exists an injective ϕ-morphism of Hilbert C^*-modules $\Phi:\mathcal{Y}\to M(\mathcal{X})$.*

Proof. Let's take any $y\in J(\mathcal{X})$ and put $\Phi_0(y)=J^{-1}(y)$. Then we have well defined ϕ_0-isomorphism of Hilbert C^*-modules $\Phi_0: J(\mathcal{X})\to\mathcal{X}$, where $\phi_0=j^{-1}: j(\mathcal{A})\to\mathcal{B}$ and $J(\mathcal{X})=\mathcal{Y}\,j(\mathcal{A})$ is an ideal submodule of $\mathcal{Y}$ with respect to the essential ideal $j(\mathcal{A})$ of $\mathcal{B}$.

Applying Proposition 4.10 we extend the ϕ_0-isomorphism Φ_0 by strict continuity to a ϕ_M-isomorphism $\Phi_M: M(J(\mathcal{X}))\to M(\mathcal{X})$ of Hilbert modules, where $\phi_M: M(j(\mathcal{A}))\to M(\mathcal{A})$ is the associated isomorphism of C^*-algebras. Because $J(\mathcal{X})=\mathcal{Y}\,j(\mathcal{A})\subseteq\mathcal{Y}\subseteq M(J(\mathcal{X}))$ it follows that $\Phi=\Phi_M|_{\mathcal{Y}}$ is the required injective morphism. □

The multiplier module of the full Hilbert C^*-module over some C^*-algebra is not necessarily a full module over the corresponding multiplier C^*-algebra, as we show in the following example.

EXAMPLE 4.12. Let H and G be a Hilbert spaces. Then $\mathbf{K}(H,G)$ is a full right $\mathbf{K}(H)$-module with inner product $\langle f|g\rangle = f^*g$, $f,g \in \mathbf{K}(H,G)$. The set $\mathbf{B}(H,G)$ is a maximal right $\mathbf{B}(H)$-module such that $\mathbf{K}(H)$ is an essential ideal in $\mathbf{B}(H)$ and $\mathbf{K}(H,G) = \mathbf{B}(H,G)\mathbf{K}(H)$, hence $M(\mathbf{K}(H,G)) = \mathbf{B}(H,G)$.

In case when H is infinitely dimensional and G is finitely dimensional space we have $\mathbf{B}(H,G) = \mathbf{K}(H,G)$ and $\mathbf{K}(H) \subsetneqq \mathbf{B}(H)$, so $\mathbf{B}(H,G)$ is not a full $\mathbf{B}(H)$-module. ♢

4.5. *Projections, relative closures and complementing.* The following result is a simple characterization of the orthogonal complementing property for submodules in $\mathcal{X}$ using (orthogonal) projections in $\mathbf{B}_a(M(\mathcal{X}_\mathcal{I}))$.

PROPOSITION 4.13. *Let $\mathcal{A}$ be a C^*-algebra with an essential ideal $\mathcal{I}$, $\mathcal{X}$ a full Hilbert $\mathcal{A}$-module and $\mathcal{G}$ a submodule in $\mathcal{X}$.*

The submodule $\mathcal{G} \subseteq \mathcal{X}$ is orthogonally complemented in $\mathcal{X}$ if and only if there exists a projection $P \in \mathbf{B}_a(M(\mathcal{X}_\mathcal{I}))$ such that $\mathcal{G} = PM(\mathcal{X}_\mathcal{I}) \cap \mathcal{X}$ and $\mathcal{X}$ is an invariant module for P.

Then $\mathcal{G} = P\mathcal{X} = \widehat{P}\mathcal{X}$ and $\mathcal{G}^{\perp_\mathcal{X}} = (I-P)\mathcal{X} = (I-\widehat{P})\mathcal{X}$, where $\widehat{P} = P|_\mathcal{X}$ is a projection in $\mathbf{B}_a(\mathcal{X})$.

PROOF. Let $\mathcal{G}$ be a submodule in $\mathcal{X}$ connected with the projection $P \in \mathbf{B}_a(M(\mathcal{X}_\mathcal{I}))$ in a way that $\mathcal{G} = PM(\mathcal{X}_\mathcal{I}) \cap \mathcal{X}$ and $P\mathcal{X} \subseteq \mathcal{X}$. Then $\mathcal{X} = P\mathcal{X} \oplus (I-P)\mathcal{X}$ and $\mathcal{X}$ is also an invariant module for $I-P$. To show that $\mathcal{G} = P\mathcal{X}$ we have $P\mathcal{X} \subseteq PM(\mathcal{X}_\mathcal{I}) \cap \mathcal{X} = \mathcal{G}$, and by the construction $\mathcal{G} = P\mathcal{G} \subseteq P\mathcal{X}$. Clearly, we also have $\mathcal{G}^{\perp_\mathcal{X}} = (I-P)\mathcal{X}$.

Now, let $\mathcal{G}$ be orthogonally complemented, i.e. $\mathcal{G} \oplus \mathcal{G}^{\perp_\mathcal{X}} = \mathcal{X}$. Then $\mathcal{G}$ is relatively strictly closed in $\mathcal{X}$ and there exists the unique projection $\widehat{P} \in M(\mathcal{X})$ such that $\mathcal{G} = \mathcal{R}(\widehat{P})$. It can be strictly extended to the projection $P \in \mathbf{B}_a(M(\mathcal{X}_\mathcal{I}))$ such that $\mathcal{R}(P) = c\ell^{st}(\mathcal{G})$ and $\widehat{P} = P|_\mathcal{X}$. Because of $c\ell^{st}(\mathcal{G}) = PM(\mathcal{X}_\mathcal{I})$ we have $\mathcal{G} = c\ell^{st}_\mathcal{X}(\mathcal{G}) = c\ell^{st}(\mathcal{G}) \cap \mathcal{X} = PM(\mathcal{X}_\mathcal{I}) \cap \mathcal{X}$, i.e. $\mathcal{G}$ is connected with the projection P and $P\mathcal{X} = \widehat{P}\mathcal{X} = \mathcal{G} \subseteq \mathcal{X}$. □

In the special case, when $\mathcal{A}$ is C^*-algebra with an essential ideal isomorphic to some C^*-algebra of compact operators on a Hilbert space, we have a simple characterization of the relative closedness for submodules in $\mathcal{X}$ by projections from $\mathbf{B}_a(M(\mathcal{X}_\mathcal{I})$.

PROPOSITION 4.14. *Let $\mathcal{A}$ be a C^*-algebra with an essential ideal $\mathcal{I}$ isomorphic to some C^*-algebra of compact operators on a Hilbert space, let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module and $\mathcal{G}$ a submodule in $\mathcal{X}$.*

The submodule $\mathcal{G} \subseteq \mathcal{X}$ is relatively strictly closed in $\mathcal{X}$ if and only if there exists projection $P \in \mathbf{B}_a(M(\mathcal{X}_\mathcal{I}))$ such that $\mathcal{G} = PM(\mathcal{X}_\mathcal{I}) \cap \mathcal{X}$.

Proof. Let $\mathcal{G} \subseteq \mathcal{X}$ be relatively strictly closed in $\mathcal{X}$, i.e. $\mathcal{G} = c\ell^{st}(\mathcal{G}) \cap \mathcal{X}$. From [9, Theorem 3.4] we have that $c\ell^{st}(\mathcal{G})$ is orthogonally complemented in $M(\mathcal{X}_\mathcal{I})$, so there exists projection $P \in \mathbf{B}_a(M(\mathcal{X}_\mathcal{I}))$ such that $c\ell^{st}(\mathcal{G}) = PM(\mathcal{X}_\mathcal{I})$. Consequently, it follows $\mathcal{G} = c\ell^{st}(\mathcal{G}) \cap \mathcal{X} = PM(\mathcal{X}_\mathcal{I}) \cap \mathcal{X}$.

Vice versa, let $\mathcal{G} = PM(\mathcal{X}_\mathcal{I}) \cap \mathcal{X}$ where P is a projection in $\mathbf{B}_a(M(\mathcal{X}_\mathcal{I}))$. Then $PM(\mathcal{X}_\mathcal{I})$ is strictly closed submodule in $M(\mathcal{X}_\mathcal{I})$, hence $\mathcal{G} = PM(\mathcal{X}_\mathcal{I}) \cap \mathcal{X}$ is relatively strictly closed in $\mathcal{X}$. ∎

4.6. *Outer direct sums and module isomorphisms.*

Lemma 4.15. *For any ideal $\mathcal{C}$ of $\mathcal{C}_\infty$ containing $\mathcal{C}_0$ and for each family $\mathcal{B} = (\mathcal{B}_j, \|\cdot\|_j)_{j\in\mathcal{J}}$ of Banach spaces, C^*-algebras or Hilbert C^*-modules the set $\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j$ defined by (1.1) is a Banach space, C^*-algebra or Hilbert C^*-module, respectively.*

If $(\mathcal{B}, \|\cdot\|_\infty)$ is a Banach space then $\mathcal{C}_0\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j \subseteq \mathcal{B} \subseteq \mathcal{C}_\infty\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j$.

Proof. Let $\lambda \in \mathbb{C}$ and $x = (x_j)_j, y = (y_j)_j \in \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j$. Then we have $(\|\lambda x_j + y_j\|_j)_{j\in\mathcal{J}} \leq |\lambda|(\|x_j\|_j)_{j\in\mathcal{J}} + (\|y_j\|_j)_{j\in\mathcal{J}} \in \mathcal{C}$ and therefore, $\lambda x + y \in \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j$. In the case of C^*-algebras, because of inequality $(\|x_jy_j\|_j)_{j\in\mathcal{J}} \leq (\|x_j\|_j)_{j\in\mathcal{J}}(\|y_j\|_j)_{j\in\mathcal{J}} \in \mathcal{C}$ and $(\|x_j^*\|_j)_{j\in\mathcal{J}} = (\|x_j\|_j)_{j\in\mathcal{J}} \in \mathcal{C}$ we have $xy, x^* \in \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j$. Further, let $\mathcal{B} = (\mathcal{B}_j, \|\cdot\|_j)_{j\in\mathcal{J}}$ and $\mathcal{A} = (\mathcal{A}_j, \|\cdot\|_j)_{j\in\mathcal{J}}$ be families where $\mathcal{B}_j$ are Hilbert $\mathcal{A}_j$-modules, $j \in \mathcal{J}$. Take any $a \in \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathcal{A}_j$ and $x \in \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j$ and we have $(\|x_ja_j\|_j)_{j\in\mathcal{J}} \leq (\|x_j\|_j)_{j\in\mathcal{J}}(\|a_j\|_j)_{j\in\mathcal{J}} \in \mathcal{C}$, i.e. $xa \in \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j$.

Now, let's prove that $\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j$ is a closed set in $\mathcal{C}_\infty\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j$. Take any sequence $(x^{(m)})_{m\in\mathbb{N}}$ in $\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j$ convergent to $x = (x_j)_{j\in\mathcal{J}} \in \mathcal{C}_\infty\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}$. Then for every $\varepsilon > 0$ there exists $m_\varepsilon \in \mathbb{N}$ such that $\|x^{(m)} - x\|_\infty < \varepsilon$ for all $m \geq m_\varepsilon$. Now, take any $x^{(m)}$, with $m \geq m_\varepsilon$. Then we have $\|x_j\|_j \leq \|x_j - x_j^{(m)}\|_j + \|x_j^{(m)}\|_j \leq \|x - x^{(m)}\|_\infty + \|x_j^{(m)}\|_j < \varepsilon + \|x_j^{(m)}\|_j$, for all $j \in \mathcal{J}$, i.e. $(\|x_j\|_j)_{j\in\mathcal{J}} \leq \varepsilon \cdot (1)_{j\in\mathcal{J}} + (\|x_j^{(m)}\|_j)_{j\in\mathcal{J}}$. From there, because of $(\|x_j^{(m)}\|_j)_{j\in\mathcal{J}} \in \mathcal{C}$ and the fact that $\mathcal{C}_\infty$ is an unital C^*-algebra it follows by [12, 3.2.6.Theorem] that $x \in \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j$.

To prove the last statement, we note that all elements in $\mathcal{C}_0\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}_j$ that have only one component different from 0 are in $\mathcal{B}$, and therefore, those elements that have only finitely many components different from 0 are in $\mathcal{B}$. Now, take any $x = (x_j)_{j\in\mathcal{J}} \in \mathcal{C}_0\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}$ and let for any $n \in \mathbb{N}$ be $x^{(n)} = (x_j^{(n)})_{j\in\mathcal{J}} \in \mathcal{C}_0\text{-}\oplus_{j\in\mathcal{J}}\mathcal{B}$ a member with components $x_j^{(n)} = x_j$ if $\|x_j\|_j \geq \frac{1}{n}$ (there are only finitely many of them), and other components are

0. Then $\forall n \in \mathbb{N}$, we have $x^{(n)} \in \mathcal{B}$ and $\|x - x^{(n)}\|_\infty \leq \frac{1}{n}$, i.e. $x = \lim_n x^{(n)}$, hence $x \in \mathcal{B}$. The right inclusion is clear. $\square$

Proposition 4.16. *Let $(\mathcal{A}_j)_{j\in\mathcal{J}}$ be a family of C^*-algebras and let $(\mathcal{X}_j)_{j\in\mathcal{J}}$ be a family of full Hilbert $\mathcal{A}_j$-modules, $j \in \mathcal{J}$. If $\mathcal{A} = \mathcal{C}_0\text{-}\oplus_{j\in\mathcal{J}} \mathcal{A}_j$ is a C^*-algebra and $\mathcal{X} = \mathcal{C}_0\text{-}\oplus_{j\in\mathcal{J}} \mathcal{X}_j$ is a Hilbert $\mathcal{A}$-module, then $M(\mathcal{X}) = \mathcal{C}_\infty\text{-}\oplus_{j\in\mathcal{J}} M(\mathcal{X}_j)$ is a strictly complete Hilbert $M(\mathcal{A})$-module.*

Proof. We construct an isomorphism (unitary operator) $\Phi : \mathbf{B}_a(\mathcal{A}, \mathcal{X}) \to M(\mathcal{X})$ and that will mean that $M(\mathcal{X})$ is strictly complete (cf. [6, Proposition 2.10]).

First we show that $\mathcal{X}$ is an ideal submodule in $M(\mathcal{X})$. Let us take any $x = (x_j)_j \in M(\mathcal{X})$ and $b = (b_j)_j \in \mathcal{A}$. Clearly, the component-wise product of bounded function and the function vanishing at infinity is a function vanishing at infinity, so $xb = (x_j b_j)_j \in \mathcal{X}$, i.e. $M(\mathcal{X})\mathcal{A} \subseteq \mathcal{X}$, and therefore, we have $M(\mathcal{X})\mathcal{A} = \mathcal{X}$.

In order to construct a proper unitary operator, we take any $T \in \mathbf{B}_a(\mathcal{A}, \mathcal{X})$. It acts on $\mathcal{A}$ as $T(a) = xa$ for the unique $x \in M(\mathcal{X})$ (cf. the proof of the Proposition 2.10 in [6]). Adjoint operator $T^* \in \mathbf{B}_a(\mathcal{X}, \mathcal{A})$ act as $T^*(y) = \langle x|y\rangle$, $\forall y \in \mathcal{X}$, namely $\langle y|T(a)\rangle_\mathcal{X} = \langle y|xa\rangle_\mathcal{X} = \langle y|x\rangle_\mathcal{X} a = \langle x|y\rangle_\mathcal{X}^* a = (T^*(y))^* a = \langle T^*(y)|a\rangle_\mathcal{A}$.

Now we define $\Phi(T) = x = (x_j)_j$. We have that $x = (x_j)_j$ is bounded, $\lim_{j\in\mathcal{J}} x_j a_j = 0$ and $\sup_{j\in\mathcal{J}} \|x_j\|_j = \|x\| \overset{[5]}{=} \sup_{\|a\|\leq 1} \|xa\| = \|T\|$ (cf. [5, Proposition 1.11.(1)]).

Let's show that Φ preserves the inner product, i.e. $\langle \Phi(T)|\Phi(T)\rangle = \langle T|T\rangle$ for all $T \in \mathbf{B}_a(\mathcal{A}, \mathcal{X})$. Namely for all $a \in \mathcal{A}$, $\langle T|T\rangle(a) = T^*T(a) = T^*(xa) = \langle x|xa\rangle = \langle x|x\rangle a = \langle \Phi(T)|\Phi(T)\rangle a$. So Φ is injective. Operator Φ is also surjective because for any $x \in M(\mathcal{X})$ the operator defined as $T(a) = xa$, $\forall a \in \mathcal{A}$ is in $\mathbf{B}_a(\mathcal{A}, \mathcal{X})$ and $\Phi(T) = x$. $\square$

The following proposition deals with the outer sums and isomorphisms of their components.

Proposition 4.17. *Let $(\mathcal{X}_j)_{j\in\mathcal{J}}$ and $(\mathcal{Y}_j)_{j\in\mathcal{J}}$ be families of Hilbert C^*-modules such that Hilbert C^*-modules $\mathcal{X}_j$ and $\mathcal{Y}_j$ are isomorphic for all $j \in \mathcal{J}$. Then for any ideal $\mathcal{C}$ of $\mathcal{C}_\infty$ containing $\mathcal{C}_0$ C^*-modules $\mathcal{X}_\mathcal{C} = \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}} \mathcal{X}_j$ and $\mathcal{Y}_\mathcal{C} = \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}} \mathcal{Y}_j$ are isomorphic and also Hilbert C^*-modules $M(\mathcal{X}_{\mathcal{C}_0}) = \mathcal{C}_\infty\text{-}\oplus_{j\in\mathcal{J}} M(\mathcal{X}_j)$ and $M(\mathcal{Y}_{\mathcal{C}_0}) = \mathcal{C}_\infty\text{-}\oplus_{j\in\mathcal{J}} M(\mathcal{Y}_j)$ are isomorphic.*

Proof. Let's suppose that for all $j \in \mathcal{J}$ there exists a ϕ_j-isomorphism Φ_j of $\mathcal{X}_j$ and $\mathcal{Y}_j$. Obviously, ϕ-isomorphism Φ, where $\phi = (\phi_j)_{j\in\mathcal{J}}$ and $\Phi = (\Phi_j)_{j\in\mathcal{J}}$, is an isomorphism of modules $\mathcal{X}_\mathcal{C}$ and $\mathcal{Y}_\mathcal{C}$. Namely because Φ_j, $j \in \mathcal{J}$, are isometries we have $(\|\Phi_j(x_j)\|_j)_{j\in\mathcal{J}} = (\|x_j\|_j)_{j\in\mathcal{J}}$, and the definition of $\mathcal{C}$-$\oplus$ sum implies $x \in \mathcal{X}_\mathcal{C}$ if and only if $\Phi(x) \in \mathcal{Y}_\mathcal{C}$, and the first claim follows.

For the proof on multiplier Hilbert modules recall that by Proposition 4.10 we are able to extend isomorphism of Hilbert modules $\mathcal{X}_j$ and $\mathcal{Y}_j$ to the isomorphism of Hilbert modules $M(\mathcal{X}_j)$ and $M(\mathcal{Y}_j)$. Therefore, Hilbert C^*-modules $\mathcal{X}_{M,\mathcal{C}} = \mathcal{C}\text{-} \oplus_{j\in\mathcal{J}} M(\mathcal{X}_j)$ and $\mathcal{Y}_{M,\mathcal{C}} = \mathcal{C}\text{-} \oplus_{j\in\mathcal{J}} M(\mathcal{X}_j)$ are isomorphic. Now from Proposition 4.16 follows that $\mathcal{X}_{M,\mathcal{C}_\infty} = M(\mathcal{X}_{\mathcal{C}_0})$ and $\mathcal{Y}_{M,\mathcal{C}_\infty} = M(\mathcal{Y}_{\mathcal{C}_0})$. □

If C^*-algebra $\mathcal{K}$ is isomorphic by isomorphism ψ_0 to some C^*-algebra of compact operators on Hilbert space it is well known (c.f. [3, Theorem 1.4.5]) that there is a family of Hilbert spaces $(H_j)_{j\in\mathcal{J}}$ such that

$$\psi_0(\mathcal{K}) = \mathcal{C}_0\text{-} \oplus_{j\in\mathcal{J}} \mathbf{K}(H_j) = \mathbf{K} \tag{4.1}$$

i.e. $\psi_0(\mathcal{K})$ is $\mathcal{C}_0$-direct sum of elementary C^*-algebras $\mathbf{K}(H_j)$ of all compact operators on the Hilbert space $H_j, j \in \mathcal{J}$. Let us denote $\mathbf{K}_i = \oplus_{j\in\mathcal{J}}\delta_{i,j}\mathbf{K}(H_j)$, $i \in \mathcal{J}$ ($\delta_{i,j}$ is the Kronecker symbol) and this is an ideal in $\mathbf{K}$. We define $\mathcal{K}_j = \psi_0^{-1}(\mathbf{K}_j)$, $j \in \mathcal{J}$, and from the Proposition 4.17 we have

$$\mathcal{K} = \mathcal{C}_0\text{-} \oplus_{j\in\mathcal{J}} \mathcal{K}_j. \tag{4.2}$$

Proposition 4.16 gives

$$M(\mathbf{K}) = \mathcal{C}_\infty\text{-} \oplus_{j\in\mathcal{J}} \mathbf{B}(H_j) = \mathbf{B} \tag{4.3}$$

and

$$M(\mathcal{K}) = \mathcal{C}_\infty\text{-} \oplus_{j\in\mathcal{J}} M(\mathcal{K}_j) = \mathcal{B}. \tag{4.4}$$

For the full $\mathcal{K}$-module $\mathcal{X}$ with C^*-algebra $\mathcal{K}$ as in (4.2) it is proved in [14] that there is another family of Hilbert spaces $(G_j)_{j\in\mathcal{J}}$ and ψ_0-isomorphism Ψ_0 such that

$$\Psi_0(\mathcal{X}) = \mathcal{C}_0\text{-} \oplus_{j\in\mathcal{J}} \mathbf{K}(H_j, G_j) = \mathbf{X}, \tag{4.5}$$

and

$$\mathcal{X} = \mathcal{C}_0\text{-} \oplus_{j\in\mathcal{J}} \mathcal{X}_j, \tag{4.6}$$

where $\mathcal{X}_j = \Psi_0^{-1}(\mathbf{X}_j)$, $\mathbf{X}_j = \oplus_{i\in\mathcal{J}}\delta_{i,j}\mathbf{K}(H_j, G_j)$, $j \in \mathcal{J}$. Submodule $\mathcal{X}_j = \mathcal{X}\mathcal{K}_j$ is full right Hilbert $\mathcal{K}_j$-module and the ideal submodule in $\mathcal{X}$, $j \in \mathcal{J}$. Moreover, assuming identification of $\mathcal{K}_j$ with $\mathbf{K}(H_j)$, we prove that it is possible to identify the Hilbert space G_j with $\mathcal{X}_j p_j$, where p_j is any minimal projection in $\mathcal{K}_j$, $j \in \mathcal{J}$.

For this purpose, let H be a Hilbert space with the scalar product $\langle\cdot,\cdot\rangle_H$ linear in the first argument. We look at $\mathbf{K}(H)$ and especially the rank 1 operators; for given $\eta, \xi \in H$ let $\theta_{\eta,\xi}$ be operator defined as $\theta_{\eta,\xi}(\nu) = \langle\nu, \eta\rangle_H \xi$. Note that for every operator $t \in \mathbf{B}(H)$ holds

$$t\theta_{\eta,\xi} = \theta_{\eta,t\xi} \text{ and } \theta_{\eta,\xi}t = \theta_{t^*\eta,\xi}. \tag{4.7}$$

Fix the unit vector $\varepsilon \in H$ and put $p = \theta_{\varepsilon,\varepsilon}$. We know that p is minimal projection.

Let X be right Hilbert $\mathbf{K}(H)$-module with inner product $\langle\cdot,\cdot\rangle_X$ linear in the second argument. Note that X is automatically full. Let $G = Xp$ and we know (c.f. [4, Remark 4]) that G is a Hilbert space with the scalar product $\langle\cdot,\cdot\rangle_G$ linear in the first argument given by the formula

$$\langle xp, yp\rangle_G = \operatorname{tr}\left(\langle yp, xp\rangle_X\right) = \operatorname{tr}\left(p\langle y, x\rangle_X p\right). \tag{4.8}$$

Now we note that for any operator $t \in \mathbf{B}(H)$ it is valid

$$ptp = \theta_{\varepsilon,\varepsilon} t \theta_{\varepsilon,\varepsilon} \overset{(4.7)}{=} \theta_{\varepsilon,\varepsilon}\theta_{\varepsilon,t\varepsilon} \overset{(4.7)}{=} \theta_{\varepsilon,\langle t\varepsilon,\varepsilon\rangle_H \varepsilon} = \langle t\varepsilon, \varepsilon\rangle_H p.$$

Therefore, we can write (4.8) in the form

$$\langle xp, yp\rangle_G = \langle\langle y, x\rangle_X \varepsilon, \varepsilon\rangle_H. \tag{4.9}$$

Lemma 4.18. *The mapping $L : X \to \mathbf{K}(H, G)$ defined by $x \mapsto L(x)$, $L(x)(\xi) = x\theta_{\varepsilon,\xi}$ is isometric isomorphism (i.e. unitary operator) of Hilbert $\mathbf{K}(H)$-modules.*

Proof. First we note that $\mathbf{K}(H, G)$ is really a Hilbert $\mathbf{K}(H)$-module with inner product $\langle k, l\rangle_{\mathbf{K}(H,G)} = k^* l$.

Second, note that $\theta_{\varepsilon,\xi} = \theta_{p\varepsilon,\xi} = \theta_{\varepsilon,\xi} p$ which allows us to write the formula for the operator $L(x)$ in a shape $L(x)(\xi) = x\theta_{\varepsilon,\xi} = x\theta_{\varepsilon,\xi} p$, whence we see that $L(x)$ really receives values in the space G. Obviously, $L(x)$ is linear. We will show that it is also bounded so that we will find its adjoint operator. Namely we claim it is

$$L(x)^*(yp) = \langle x, y\rangle_X(\varepsilon). \tag{4.10}$$

Indeed, if we take the arbitrary $\xi \in H$ and $yp \in G$ we have

$$\langle L(x)(\xi), yp\rangle_G = \langle x\theta_{\varepsilon,\xi} p, yp\rangle_G \overset{(4.9)}{=} \langle\langle y, x\theta_{\varepsilon,\xi}\rangle_X \varepsilon, \varepsilon\rangle_H = \langle\langle y, x\rangle_X \theta_{\varepsilon,\xi}\varepsilon, \varepsilon\rangle_H = \langle\langle y, x\rangle_X \xi, \varepsilon\rangle_H,$$

and also

$$\langle \xi, L(x)^*(yp)\rangle_H = \langle \xi, \langle x, y\rangle_X(\varepsilon)\rangle_H = \langle\langle y, x\rangle_X(\xi), \varepsilon\rangle_H.$$

So far, we have shown that L is a well-defined (obviously linear) mapping that receives values in $\mathbf{B}(H, G)$. We have argued that L receives values actually in $\mathbf{K}(H, G)$ and we will show this in the next step. Before that, show that L is a modular mapping, i.e. that

$$L(xk) = L(x)k, \ \forall x \in X, \forall k \in \mathbf{K}(H). \tag{4.11}$$

Let's take arbitrary $x \in X$ and $k \in \mathbf{K}(H)$. Now, for all $\xi \in H$ we have

$$L(xk)(\xi) = xk\theta_{\varepsilon,\xi} = x\theta_{\varepsilon,k\xi},$$

while on the other hand

$$L(x)k(\xi) = L(x)(k\xi) = x\theta_{\varepsilon,k\xi}.$$

now, let's take any $x \in X$. Using Cohen-Hewitt theorem on factorization (cf. [13, Proposition 2.31]) we can write $x = vk$ for some $v \in X$ and $k \in \mathbf{B}(H)$. Now we have $L(x) = L(vk) \overset{(4.11)}{=} L(v)k \in \mathbf{K}(H,G)$. Thus, we have shown that $L : X \to \mathbf{K}(H,G)$ is a modular operator that really takes values in $\mathbf{K}(H,G)$.

Now let's show that L preserves inner products, i.e.

$$\langle L(x), L(y)\rangle_{\mathbf{K}(H,G)} = \langle x, y\rangle_X, \ \forall x, y \in X. \tag{4.12}$$

Let's take arbitrary $x, y \in X$ and $\xi \in H$. Now, we have

$$\begin{aligned} \langle L(x), L(y)\rangle_{\mathbf{K}(H,G)}(\xi) &= L(x)^*(L(y)(\xi)) = \\ L(x)^*(y\theta_{\varepsilon,\xi}p) &\overset{(4.10)}{=} \langle x, y\theta_{\varepsilon,\xi}\rangle_X(\varepsilon) = \\ \langle x, y\rangle_X\theta_{\varepsilon,\xi}(\varepsilon) &= \langle x, y\rangle_X(\xi). \end{aligned}$$

This proves (4.12). It remains to show that L is an injection. Because L is isometry, so it has a closed image, it is sufficient to prove that in the image of L is every elementary operator $\Theta_{\xi,yp} \in \mathbf{K}(H,G)$ where for given $\xi \in H$ and $yp \in G$ the operator $\Theta_{\xi,yp}$ is defined by $\Theta_{\xi,yp}(\eta) = \langle \eta, \xi\rangle_H yp$.

Lets choose arbitrary $\xi \in H$ and $yp \in G$. Now for all $\eta \in H$ we have $L(y\theta_{\xi,\varepsilon})(\eta) = y\theta_{\xi,\varepsilon}\theta_{\varepsilon,\eta} = y\theta_{\varepsilon,\theta_{\xi,\varepsilon}(\eta)} = y\theta_{\varepsilon,\langle\eta,\xi\rangle_X\varepsilon} = \langle \eta, \xi\rangle_X y\theta_{\varepsilon,\varepsilon} = \langle \eta, \xi\rangle_X yp = \Theta_{\xi,yp}(\eta)$ which proves $L(y\theta_{\xi,\varepsilon}) = \Theta_{\xi,yp}$. □

PROPOSITION 4.19. *Let $\mathcal{K} = c_0\text{-}\oplus_{j\in\mathcal{J}} \mathcal{K}_j$ where for all $j \in \mathcal{J}$ C*-algebra $\mathcal{K}_j$ is isomorphic to C*-algebra of all compact operators on a Hilbert space and let $\mathcal{X}$ be a full Hilbert $\mathcal{K}$-module. Then C*-algebra $\mathbf{K}(\mathcal{X})$ of all "compact" operators on $\mathcal{X}$ is of the form*

$$\mathbf{K}(\mathcal{X}) = c_0\text{-}\oplus_{j\in\mathcal{J}} \mathbf{K}(\mathcal{X}_j), \tag{4.13}$$

and C-algebra $\mathbf{B}_a(\mathcal{X})$ of all adjointable operators on $\mathcal{X}$ is of the form*

$$\mathbf{B}_a(\mathcal{X}) = c_\infty\text{-}\oplus_{j\in\mathcal{J}} \mathbf{B}_a(\mathcal{X}_j) \tag{4.14}$$

and $\mathbf{B}_a(\mathcal{X}) = M(\mathbf{K}(\mathcal{X}))$.

PROOF. For the proof of (4.13), we take any elementary "compact" operator $\Theta_{x,y} \in \mathbf{K}(M(\mathcal{X}))$ with $x = (x_j)_j, y = (y_j)_j \in M(\mathcal{X})$. Then for all $z = (z_j)_j \in M(\mathcal{X})$ we have $\Theta_{x,y}z = x\langle y|z\rangle = (x_j\langle y_j|z_j\rangle)_{j\in\mathcal{J}} = (\Theta_{x_j,y_j}z_j)_{j\in\mathcal{J}}$ and from that follows $\|\Theta_{x_j,y_j}z_j\|_j^2 = \|\langle z_j|y_j\rangle\langle x_j|x_j\rangle\langle y_j|z_j\rangle\|_j$. Therefore, $\|\Theta_{x_j,x_j}\|_j = \|x_j\|_j^2$, $j \in \mathcal{J}$, so we have $\lim_{j\in\mathcal{J}} \|\Theta_{x_j,x_j}\|_j = 0$ if and only if $\lim_{j\in\mathcal{J}} \|x_j\|_j = 0$, i.e. $\Theta_{x,x} \in c_0\text{-}\oplus_{j\in\mathcal{J}} \mathbf{K}(\mathcal{X}_j)$ if and only if $\Theta_{x,x} \in \mathbf{K}(\mathcal{X})$ where $x \in \mathcal{X}$ and $\mathcal{X}$ is of the form (4.6). By applying polarization equality $\Theta_{x,y} = \frac{1}{4}\sum_{k=0}^{3} i^k\Theta_{x+i^k y, x+i^k y}$ the foregoing conclusions apply to all elementary "compact" operators $\Theta_{x,y}$, and we have (4.13).

The fact that each ideal submodule $\mathcal{X}_j$, $j \in \mathcal{J}$ is an invariant submodule for every bounded $\mathcal{K}$-linear operator on $\mathcal{X}$ and the fact that all bounded modular operators on C^*-Hilbert modules over C^*-algebras of compact operators are adjointable gives (4.14).

For all $j \in \mathcal{J}$ C^*-algebra $\mathbf{B}_a(\mathcal{X}_j)$ is isomorphic to the C^*-algebra $\mathbf{B}(G_j)$ and C^*-algebra $\mathbf{K}(\mathcal{X}_j)$ is isomorphic to C^*-algebra $\mathbf{K}(G_j)$ where $G_j = \mathcal{X}_j p_j$, for some minimal projection $p_j \in \mathbf{P}(\mathcal{K}_j)$, is a Hilbert space from Lemma 4.18, and the isomorphism acts as restriction and preserves the subalgebra of all compact operators (c.f. [4, Theorem 2]). This implies that $M(\mathbf{K}(\mathcal{X}_j)) = \mathbf{B}_a(\mathcal{X}_j)$ for all $j \in \mathcal{J}$. Now the last claim follows from (4.14) and Proposition 4.16. □

4.7. *Outer direct sums and von Neumann algebras.* For the proof of the fact that c_∞-sum of von Neumann algebras is a von Neumann algebra we are unable to provide a reference. Here, for the sake of completeness, we prove a slightly more general statement for the commutator of direct sum of the family of (possibly non-unital) $*$-algebras from which this immediately follows. First we list a few facts that are simple tasks.

For a nonempty subset $\mathcal{A} \subseteq \mathbf{B}(H)$, where H is Hilbert space, the equation $\bigcap_{a\in\mathcal{A}} \mathcal{N}(a) \oplus c\ell(\mathrm{span}(\bigcup_{a\in\mathcal{A}} \mathcal{R}(a^*))) = H$ holds true. This follows easily from the equalities $(\bigcup_{a\in\mathcal{A}} \mathcal{R}(a^*))^\perp = \bigcap_{a\in\mathcal{A}} \mathcal{N}(a)$ and $S^{\perp\perp} = c\ell(\mathrm{span}(S))$ for any nonempty $S \subseteq H$.

Further, for a nonempty subset $\mathcal{A} \subseteq \mathbf{B}(H)$ acting non-degenerately on H, i.e. for $h \in H$, $\mathcal{A}h = 0$ implies $h = 0$, we have $\bigcap_{a\in\mathcal{A}} \mathcal{N}(a) = \{0\}$, and $\mathrm{span}(\bigcup_{a\in\mathcal{A}} \mathcal{R}(a^*))$ is dense in H. In this case, for any $t \in \mathbf{B}(H, G)$ such that $ta^* = 0$ for all $a \in \mathcal{A}$ follows $t = 0$. Namely for any $h \in H$ we have a sequence $(h_n)_n$ in $\mathrm{span}(\bigcup_{a\in\mathcal{A}} \mathcal{R}(a^*))$ such that $h = \lim_n h_n$. For all $n \in \mathbb{N}$ we have $th_n = 0$, so $th = th - \lim_n th_n = 0$, hence $t = 0$.

LEMMA 4.20. *If $\mathcal{A} = (\mathcal{A}_j)_{j\in\mathcal{J}}$ is a family of $*$-algebras such that $\mathcal{A}_j \subseteq \mathbf{B}(H_j)$ are non-degenerately acting on H_j, $j \in \mathcal{J}$, where $(H_j)_{j\in\mathcal{J}}$ is a family of Hilbert spaces, then $\mathcal{A}' = c_\infty\text{-}\oplus_{j\in\mathcal{J}} \mathcal{A}'_j$ is its commutator algebra.*

PROOF. Let $H = \ell_2\text{-}\oplus_{j\in\mathcal{J}} H_j$ be a Hilbert space and $*$-algebra $\mathcal{A}$ can be identified with a subalgebra of $\mathbf{B}(H)$ such that for every $a \in \mathcal{A}$ invariant subspaces of a are H_j, $j \in \mathcal{J}$. If an operator $t = (t_{i,j}) \in \mathbf{B}(H)$ is from the commutator of $\mathcal{A}$, then for any $a = (\delta_{i,j} a_j)$ ($\delta_{i,j}$ is the Kronecker symbol) we have $t_{i,j} a_j = 0$ and $a_j t_{j,i} = 0$ for all $i, j \in \mathcal{J}$, $i \neq j$, and $a_j t_{j,j} = t_{j,j} a_j$ for all $j \in \mathcal{J}$. This implies $t_{i,j} = 0$ for all $i, j \in \mathcal{J}$, $i \neq j$, i.e. $\mathcal{A}' \subseteq c_\infty\text{-}\oplus_{j\in\mathcal{J}} \mathcal{A}'_j$. The opposite inclusion is obvious, so we have $\mathcal{A}' = c_\infty\text{-}\oplus_{j\in\mathcal{J}} \mathcal{A}'_j$. □

COROLLARY 4.21. *If $(\mathcal{A}_j)_{j\in\mathcal{J}}$ is a family of von Neumann algebras, then $\mathcal{A} = c_\infty\text{-}\oplus_{j\in\mathcal{J}} \mathcal{A}_j$ is a von Neumann algebra.*

PROOF. It follows from Lemma 4.20 that $\mathcal{A}'' = \mathcal{A}$. □

Proposition 4.22. *Let $\mathcal{K} = c_0\text{-}\oplus_{j\in\mathcal{J}} \mathcal{K}_j$ where for all $j \in \mathcal{J}$ C*-algebra $\mathcal{K}_j$ is isomorphic to C*-algebra of all compact operators on a Hilbert space and let $\mathcal{X}$ be a full Hilbert $\mathcal{K}$-module. Then C*-algebras $\mathbf{B}_a(\mathcal{X})$ from (4.14) and $\mathfrak{B} = c_\infty\text{-}\oplus_{j\in\mathcal{J}} \mathbf{B}(G_j)$ are isomorphic, where $(G_j)_{j\in\mathcal{J}}$ is a family of Hilbert spaces, $G_j = \mathcal{X}_j p_j$ and $p_j \in \mathcal{K}_j$ is a minimal projection, $j \in \mathcal{J}$. The C*-algebra $\mathfrak{B}$ is generated by its projections, so $\mathbf{B}_a(\mathcal{X})$ is generated by projections in $\mathbf{B}_a(\mathcal{X})$.*

Proof. Decomposition (4.13) from Proposition 4.19 and Proposition 4.17 imply that $\mathbf{B}_a(\mathcal{X})$ is isomorphic to $\mathfrak{B}$.

For all $j \in \mathcal{J}$, C*-algebra $\mathbf{B}(G_j)$ is von Neumann algebra and hence it is generated by projections in $\mathbf{B}(G_j)$. It follows from Corollary 4.21 that C*-algebra $\mathfrak{B}$ is von Neumann algebra and consequently it is generated by projections in $\mathfrak{B}$. Now the isomorphism of C*-algebras $\mathbf{B}_a(\mathcal{X})$ and $\mathfrak{B}$ implies that $\mathbf{B}_a(\mathcal{X})$ is generated by projections in $\mathbf{B}_a(\mathcal{X})$. □

4.8. *Hereditary C*-subalgebras and hereditary C*-modules.*

Remark 4.23. It is well known that every C*-algebra $\mathcal{A} \in \mathfrak{ha}(\mathcal{B})$ is of the form $\mathcal{A} = \mathcal{L}^* \cap \mathcal{L}$, where $\mathcal{L} = \{a \in \mathcal{B}; |a| = (a^*a)^{\frac{1}{2}} \in \mathcal{A}\}$ is a unique closed left ideal of $\mathcal{B}$ with this property (cf. [12, 3.2.1.Theorem]). Obviously, $\mathcal{A} \subseteq \mathcal{L}^*$ and $\mathcal{A} \subseteq \mathcal{L}$, as well as $\mathcal{L}^*\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{L}^*\mathcal{L} \subseteq \mathcal{L}$. By applying Cohen-Hewitt theorem on factorization this implies $\mathcal{A} = \mathcal{A}\mathcal{A} \subseteq \mathcal{L}^*\mathcal{L} \subseteq \mathcal{L}^* \cap \mathcal{L} = \mathcal{A}$, i.e. $\mathcal{A} = \mathcal{L}^*\mathcal{L}$. ♢

Remark 4.24. If $\mathcal{A}$ is a hereditary C*-subalgebra of C*-algebra $\mathcal{B}$, by applying Cohen-Hewitt theorem on factorization we have $\mathcal{A} \subseteq \mathcal{A}\mathcal{A}\mathcal{A} \subseteq \mathcal{A}\mathcal{B}\mathcal{A} \subseteq \mathcal{A}$, hence $\mathcal{A}\mathcal{B}\mathcal{A} = \mathcal{A}$.

Also, if $\mathcal{A}$ is a hereditary C*-subalgebra of C*-algebra $\mathcal{B}$ and $\mathcal{B}$ is a hereditary C*-subalgebra of C*-algebra $\mathcal{C}$, then $\mathcal{A}$ is a hereditary subalgebra of $\mathcal{C}$. Namely $\mathcal{A}\mathcal{C}\mathcal{A} = \mathcal{A}\mathcal{A}\mathcal{C}\mathcal{A}\mathcal{A} \subseteq \mathcal{A}\mathcal{B}\mathcal{C}\mathcal{B}\mathcal{A} = \mathcal{A}\mathcal{B}\mathcal{A} = \mathcal{A}$. ♢

Proposition 4.25. *Let $\mathcal{X}$ be a Hilbert $\mathcal{B}$-module, $\mathcal{A} \in \mathfrak{ha}(\mathcal{B})$ and let $\mathcal{X}_\mathcal{A} = c\ell(\mathrm{span}(\mathcal{X}\mathcal{A}))$, the closed linear span of the action of $\mathcal{A}$ on $\mathcal{X}$. Then $\mathcal{X}_\mathcal{A} = \mathcal{X}\mathcal{A} \in \mathfrak{hm}(\mathcal{X})$ is a hereditary $\mathcal{A}$-module of $\mathcal{X}$ and*

$$\mathcal{X}_\mathcal{A} = \{x \in \mathcal{X} : \langle x|x\rangle \in \mathcal{A}\} = \{x \in \mathcal{X}; |\langle y|x\rangle| \in \mathcal{A}, \forall y \in \mathcal{X}\}.$$

If $\mathcal{X}$ is a full $\mathcal{B}$-module, then $\mathcal{X}_\mathcal{A}$ is a full $\mathcal{A}$-module and it is a submodule of $\mathcal{X}$ if and only if $\mathcal{A}$ is an ideal in $\mathcal{B}$.

Proof. From the definition of $\mathcal{X}_\mathcal{A}$ and the fact that $\mathcal{A} \in \mathfrak{ha}(\mathcal{B})$ it follows that $\langle \mathcal{X}_\mathcal{A}|\mathcal{X}_\mathcal{A}\rangle$ is generated by elements of the form $a_i\langle x_i|x_j\rangle a_j \in \mathcal{A}\mathcal{B}\mathcal{A} = \mathcal{A}$, so $\mathcal{X}_\mathcal{A}$ is a Hilbert C*-module over $\mathcal{A}$. For all other statements except $\mathcal{X}_\mathcal{A} = \{x \in \mathcal{X}; |\langle y|x\rangle| \in \mathcal{A}, \forall y \in \mathcal{X}\}$ the proofs for ideal submodules from [5, Proposition 1.2, Proposition 1.3] are valid for hereditary modules.

To prove the above equality it is sufficient to prove that for $x \in \mathcal{X}$ such that $\langle x|x\rangle \in \mathcal{A}$ we have $|\langle y|x\rangle| \in \mathcal{A}, \forall y \in \mathcal{X}$, because the opposite inclusion is obviously true. For this purpose we use the fact that for each $x \in \mathcal{X}$ there exist $z \in \mathcal{X}$ such that $x = z\langle z|z\rangle$ (cf. [13, Proposition 2.31]). Then $\langle x|x\rangle = \langle z|z\rangle^3 \in \mathcal{A}$ and therefore, $a = \langle z|z\rangle \in \mathcal{A}$. Now for any $y \in \mathcal{X}$, because $b = \langle z|y\rangle\langle y|z\rangle \in \mathcal{B}$, we have $|\langle y|x\rangle|^2 = \langle x|y\rangle\langle y|x\rangle = \langle z|z\rangle(\langle z|y\rangle\langle y|z\rangle)\langle z|z\rangle = aba \in \mathcal{A}$. Then $|\langle y|x\rangle| \in \mathcal{A}$ implies the equality.

The last statement follows from the fact that in the case where $\mathcal{X}_\mathcal{A}$ is a submodule of $\mathcal{X}$, i.e. it is a $\mathcal{B}$-module, we have that $\langle \mathcal{X}_\mathcal{A}|\mathcal{X}_\mathcal{A}\rangle = \mathcal{A}$ is an ideal in $\mathcal{B}$ and vice versa. ∎

Recall from [6, Theorem 2.3] that for a full Hilbert $\mathcal{A}$-module $\mathcal{X}$ with an essential ideal $\mathcal{I}$ the mapping

$$\alpha : \mathbf{B}_a(M(\mathcal{X}_\mathcal{I})) \to \mathbf{B}_a(\mathcal{X}_\mathcal{I}),\ \alpha(T) = T|_{\mathcal{X}_\mathcal{I}}, \tag{4.15}$$

is an isomorphism of C^*-algebras. Also

$$\beta : \mathbf{B}_a(\mathcal{X}) \to \mathbf{B}_a(\mathcal{X}_\mathcal{I}),\ \beta(T) = T|_{\mathcal{X}_\mathcal{I}} \tag{4.16}$$

is an injective morphism of C^*-algebras (cf. [5, Theorem 1.12]), therefore, $\gamma = \alpha^{-1} \circ \beta : \mathbf{B}_a(\mathcal{X}) \to \mathbf{B}_a(M(\mathcal{X}_\mathcal{I}))$ is an injective morphism. Operators in $\mathbf{B}_a(M(\mathcal{X}_\mathcal{I}))$ and $\mathbf{B}_a(\mathcal{X})$ are extensions by the strict continuity of operators in $\mathbf{B}_a(\mathcal{X}_\mathcal{I})$ (cf. [6, Remark 2.4]).

Lemma 4.26. *Let $\mathcal{A}$ be a C^*-algebra with an essential ideal $\mathcal{I}$, $\mathcal{X}$ is a full Hilbert $\mathcal{A}$-module. If α and β are from (4.15) and (4.16), respectively, then for $\gamma = \alpha^{-1} \circ \beta$ we have its range $\mathcal{R}(\gamma) = \{T \in \mathbf{B}_a(M(\mathcal{X}_\mathcal{I}));\ T\mathcal{X} \subseteq \mathcal{X} \wedge T^*\mathcal{X} \subseteq\ \mathcal{X}\}$.*

Proof. Let's show first that $\mathcal{R}(\gamma) \subseteq \{T \in \mathbf{B}_a(M(\mathcal{X}_\mathcal{I})); T\mathcal{X} \subseteq \mathcal{X} \wedge T^*\mathcal{X} \subseteq \mathcal{X}\}$. Namly, if $T \in \mathcal{R}(\gamma)$ then $T^* \in \mathcal{R}(\gamma)$ and there exists $\widehat{T} \in \mathbf{B}_a(\mathcal{X})$ such that $T = \gamma(\widehat{T})$ and $T^* = \gamma(\widehat{T}^*)$. Therefore, we have $\alpha(T) = \beta(\widehat{T})$ and $\alpha(T^*) = \beta(\widehat{T}^*)$, i.e. $T|_{\mathcal{X}_\mathcal{I}} = \widehat{T}|_{\mathcal{X}_\mathcal{I}} = T_0 \in \mathbf{B}_a(\mathcal{X}_\mathcal{I})$. Now because $Tx = \text{st-lim}_\lambda T_0 x e_\lambda$ and $T^*x = \text{st-lim}_\lambda T_0^* x e_\lambda$ for all $x \in M(\mathcal{X}_\mathcal{I})$ it follows $T|_\mathcal{X} = \widehat{T}$ and $T^*|_\mathcal{X} = \widehat{T}^*$, so $T\mathcal{X} \subseteq \mathcal{X}$ and $T^*\mathcal{X} \subseteq \mathcal{X}$.

If $T\mathcal{X} \subseteq \mathcal{X}$ and $T^*\mathcal{X} \subseteq \mathcal{X}$ then $\widehat{T} = T|_\mathcal{X} \in \mathbf{B}_a(\mathcal{X})$. This implies $T|_{\mathcal{X}_\mathcal{I}} = \widehat{T}|_{\mathcal{X}_\mathcal{I}}$, i.e. $\alpha(T) = \beta(\widehat{T})$ or $T = \alpha^{-1} \circ \beta(\widehat{T})$, hence $T \in \mathcal{R}(\gamma)$. ∎

Lemma 4.27. *Let $\mathcal{A}$, $\mathcal{X}$, α, β be as in Lemma 4.26 and let $\mathcal{F} \subseteq \mathcal{X}$ be a relatively strictly closed submodule. If β is an isomorphism then $\mathcal{F}$ is complemented in $\mathcal{X}$ if and only if $\mathcal{F}\mathcal{I}$ is complemented in $\mathcal{X}_\mathcal{I}$.*

Proof. Let $\mathcal{F}$ be complemented in $\mathcal{X}$, i.e. $\mathcal{F} \oplus \mathcal{F}^{\perp_\mathcal{X}} = \mathcal{X}$. Multiplying by $\mathcal{I}$ and applying Lemma 4.1(v), Lemma 4.2(iii) gives $\mathcal{F}\mathcal{I} \oplus (\mathcal{F}\mathcal{I})^{\perp_{\mathcal{X}_\mathcal{I}}} = \mathcal{X}_\mathcal{I}$.

Now let $\mathcal{F}\mathcal{I}$ be complemented in $\mathcal{X}_\mathcal{I}$. Then there exists a projection $P_0 \in \mathbf{B}_a(\mathcal{X}_\mathcal{I})$ such that $\mathcal{F}\mathcal{I} = P_0\mathcal{X}_\mathcal{I}$. We have $c\ell^{st}(\mathcal{F}\mathcal{I}) = c\ell^{st}(\mathcal{F})$ and $P_d = \alpha^{-1}(P_0)$ is a projection such that $P_d M(\mathcal{X}_\mathcal{I}) = c\ell^{st}(\mathcal{F}\mathcal{I}) = c\ell^{st}(\mathcal{F})$ so $P_d M(\mathcal{X}_\mathcal{I}) \cap \mathcal{X} = c\ell^{st}_\mathcal{X}(\mathcal{F}) = \mathcal{F}$. Because β is an isomorphism $\mathcal{X}$ is an invariant module for all operators in $\mathbf{B}(M(\mathcal{X}_\mathcal{I}))$, and particularly for P_d. Now it follows from Proposition 4.13 that $\mathcal{F}$ is complemented in $\mathcal{X}$. □

The following theorem gives the characterization of hereditary C^*-algebras and the corresponding hereditary Hilbert modules via restriction as the morphism of C^*-algebras of all adjointable operators on the Hilbert C^*-module and its ideal submodule.

THEOREM 4.28. *Let $\mathcal{A}$ be a C^*-algebra with an essential ideal $\mathcal{I}$, $\mathcal{X}$ is a full Hilbert $\mathcal{A}$-module and let $\beta : \mathbf{B}_a(\mathcal{X}) \to \mathbf{B}_a(\mathcal{X}_\mathcal{I})$ acts as restriction. Mapping β is an isomorphism if and only if $\mathcal{A}$ is a hereditary subalgebra of $M(\mathcal{I})$ and $\mathcal{X}$ is a hereditary $\mathcal{A}$-module of $M(\mathcal{X}_\mathcal{I})$.*

PROOF. If $\mathcal{A}$ is a hereditary subalgebra of $M(\mathcal{I})$ and $\mathcal{X}$ is a hereditary $\mathcal{A}$-module of $M(\mathcal{X}_\mathcal{I})$, i.e. $\mathcal{X} = M(\mathcal{X}_\mathcal{I})\mathcal{A}$, then $T_d\mathcal{X} = T_d M(\mathcal{X}_\mathcal{I})\mathcal{A} \subseteq M(\mathcal{X}_\mathcal{I})\mathcal{A}$ for all $T_d \in \mathbf{B}_a(M(\mathcal{X}_\mathcal{I}))$. It follows from lemma 4.26 that $\mathcal{R}(\alpha^{-1} \circ \beta) = \mathbf{B}_a(M(\mathcal{X}_\mathcal{I}))$ so $\gamma = \alpha^{-1} \circ \beta$ is an isomorphism and therefore, β is also an isomorphism.

Conversely, let β be an isomorphism of C^*-algebras. It follows from lemma 4.26 that $\mathcal{X}$ is an invariant module for all operators in $\mathbf{B}_a(M(\mathcal{X}_\mathcal{I}))$. When we apply this fact to an elementary "compact" operator $\Theta_{x,y} \in \mathbf{K}(M(\mathcal{X}_\mathcal{I}))$ for any $x, y \in M(\mathcal{X}_\mathcal{I})$ and $z \in \mathcal{X}$ we have $\Theta_{x,y} z = x\langle y|z\rangle \in \mathcal{X}$. In particular, for all $x \in M(\mathcal{X}_\mathcal{I})$ and $z, w \in \mathcal{X}$ we have $x\langle z|w\rangle \in \mathcal{X}$. Because $\mathcal{X}$ and $\mathcal{A} = \langle \mathcal{X}|\mathcal{X}\rangle$ are closed we have $M(\mathcal{X}_\mathcal{I})\mathcal{A} \subseteq \mathcal{X}$. Applying the Cohen-Hewitt theorem of factorization (see [13, Proposition 2.31]) it follows $\mathcal{X} = \mathcal{X}\mathcal{A} \subseteq M(\mathcal{X}_\mathcal{I})\mathcal{A} \subseteq \mathcal{X}$, i.e. $M(\mathcal{X}_\mathcal{I})\mathcal{A} = \mathcal{X}$. From there it immediately follows $\mathcal{A}\mathcal{B}\mathcal{A} \subseteq \mathcal{A}$, where $\mathcal{B} = \langle M(\mathcal{X}_\mathcal{I})|M(\mathcal{X}_\mathcal{I})\rangle$ is an ideal in $M(\mathcal{I})$ and we have $\mathcal{A}\mathcal{B}\mathcal{A} = \mathcal{A}$ as above. Then $\mathcal{A}$ is a hereditary C^*-subalgebra of C^*-algebra $\mathcal{B}$ (see [12, Theorem 3.2.2.]), and because $\mathcal{B}$ is an ideal in $M(\mathcal{I})$ it follows that $\mathcal{A}$ is a hereditary C^*-subalgebra of $M(\mathcal{I})$. Now, from Proposition 4.25 we have that $M(\mathcal{X}_\mathcal{I})\mathcal{A}$ is a hereditary Hilbert $\mathcal{A}$-module. □

COROLLARY 4.29. *Let $\mathcal{A}$ be a C^*-algebra with an essential ideal $\mathcal{I}$. Then $\mathcal{A}$ is a hereditary subalgebra of $M(\mathcal{I})$ if and only if for any C^*-Hilbert module of the form $\mathcal{X} = M(\mathcal{X}_\mathcal{I})\mathcal{A}$ the mapping $\beta : \mathbf{B}_a(\mathcal{X}) \to \mathbf{B}_a(\mathcal{X}_\mathcal{I})$ which acts as restriction is an isomorphism.*

PROOF. If $\mathcal{A}$ is a C^*-algebra with an essential ideal $\mathcal{I}$ which is a hereditary subalgebra of $M(\mathcal{I})$ then any C^*-Hilbert module of the form $\mathcal{X} = M(\mathcal{X}_\mathcal{I})\mathcal{A}$ is an $\mathcal{A}$-module, namely $\langle \mathcal{X}|\mathcal{X}\rangle = \mathcal{A}\langle M(\mathcal{X}_\mathcal{I})|M(\mathcal{X}_\mathcal{I})\rangle\mathcal{A} \subseteq \mathcal{A}M(\mathcal{I})\mathcal{A} \subseteq \mathcal{A}$. Hence $\mathcal{X} \in \mathfrak{hm}_{\mathcal{X}_\mathcal{I}}(M(\mathcal{X}_\mathcal{I}))$ and by Theorem 4.28 β is an isomorphism. The opposite conclusion follows directly from Theorem 4.28. □

Denote by $\mathfrak{sa}(\mathcal{B})$ the set of all C^*-subalgebras of some C^*-algebra $\mathcal{B}$, and for a non-empty set $S \subset \mathcal{B}$ by $\mathfrak{sa}_S(\mathcal{B})$ denote the set of all C^*-algebras from $\mathfrak{sa}(\mathcal{B})$ containing S. We also denote by $\mathfrak{sm}(\mathcal{X})$ the set of all C^*-submodules of some Hilbert C^*-module $\mathcal{X}$, and for a non-empty set $S \subset \mathcal{X}$ by $\mathfrak{sm}_S(\mathcal{X})$ denote the set of all submodules from $\mathfrak{sm}(\mathcal{X})$ that contain S.

For the function $f : D \to E$ we denote with $f[A]$ the image of $A \subseteq D$ and with $f^{-1}[B]$ the inverse image of $B \subseteq E$. It is known that $A \subseteq f^{-1}[f[A]]$ and $B \supseteq f[f^{-1}[B]]$ are always valid. If f is a surjection then $B = f[f^{-1}[B]]$ and if it is injection then $A = f^{-1}[f[A]]$.

PROPOSITION 4.30. *Let $\mathcal{B}$ be a C^*-algebra with an essential ideal $\mathcal{I}$. The function $\omega : \mathfrak{sa}_{\mathcal{I}}(\mathcal{B}) \to \mathfrak{sa}(\mathcal{B}/\mathcal{I})$ defined by $\omega(\mathcal{A}) = \pi[\mathcal{A}]$ for all $\mathcal{A} \in \mathfrak{sa}_{\mathcal{I}}(\mathcal{B})$ is a bijection, and so is its restriction $\omega_h : \mathfrak{ha}_{\mathcal{I}}(\mathcal{B}) \to \mathfrak{ha}(\mathcal{B}/\mathcal{I})$.*

Let $\mathcal{X}$ be full Hilbert $\mathcal{B}$-module. The function $\Omega : \mathfrak{sm}_{\mathcal{X}_{\mathcal{I}}}(\mathcal{X}) \to \mathfrak{sa}(\mathcal{X}/\mathcal{X}_{\mathcal{I}})$ defined by $\Omega(\mathcal{Y}) = \Pi[\mathcal{Y}]$ for all $\mathcal{Y} \in \mathfrak{sm}_{\mathcal{X}_{\mathcal{I}}}(\mathcal{X})$ is a bijection, and so is its restriction $\Omega_h : \mathfrak{hm}_{\mathcal{X}_{\mathcal{I}}}(\mathcal{X}) \to \mathfrak{hm}(\mathcal{X}/\mathcal{X}_{\mathcal{I}})$.

PROOF. The morphism $\pi : \mathcal{B} \to \mathcal{B}/\mathcal{I}$ preserves structures of C^*-subalgebras, hereditary subalgebras and ideals, so the function ω is well defined.

Let's show that for every $A \in \mathfrak{sa}(\mathcal{B}/\mathcal{I})$ the inverse image $\mathcal{A} = \pi^{-1}[A] \in \mathfrak{sa}_{\mathcal{I}}(\mathcal{B})$. For $\lambda \in \mathbb{C}$ and $a_i \in \mathcal{A}$, $(i = 1, 2)$, we have $\pi(\lambda a_1 + a_2) = \lambda\pi(a_1) + \pi(a_2) \in A$, hence $\lambda a_1 + a_2 \in \pi^{-1}[A] = \mathcal{A}$. Also $\pi(a_1 a_2) = \pi(a_1)\pi(a_2) \in A$ and $\pi(a_1^*) = \pi(a_1)^* \in A$, i.e. $a_1 a_2, a_1^* \in \mathcal{A}$. For the proof of closedness in norm of the set $\mathcal{A}$ we take any sequence $(a_n)_n$ in $\mathcal{A}$, and $a = \lim_n a_n \in \mathcal{B}$. Then by closedness in norm of the C^*-algebra A we have $\pi(a) = \lim_n \pi(a_n) \in A$, so $a \in \mathcal{A}$. In order to prove that for $A \in \mathfrak{ha}(\mathcal{B}/\mathcal{I})$ the C^*-algebra $\mathcal{A}$ belongs to $\mathfrak{ha}_{\mathcal{I}}(\mathcal{B})$ we take arbitrary $a_i \in \mathcal{A}$, $(i = 1, 2)$, and $c \in \mathcal{B}$. Because A is a hereditary subalgebra of $\mathcal{B}/\mathcal{I}$ we have $\pi(a_1 c a_2) = \pi(a_1)\pi(c)\pi(a_2) \in A$, so $a_1 c a_2 \in \mathcal{A}$ and the C^*-algebra $\mathcal{A} \in \mathfrak{ha}_{\mathcal{I}}(\mathcal{B})$.

For the proof of surjectivity of the function ω we take any C^*-algebra $A \in \mathfrak{sa}(\mathcal{B}/\mathcal{I})$ and let $\mathcal{A} = \pi^{-1}[A] \in \mathfrak{sa}_{\mathcal{I}}(\mathcal{B})$. Now because of the surjectivity of π we have $\omega(\mathcal{A}) = \pi[\mathcal{A}] = \pi[\pi^{-1}[A]] = A$, and this also holds true for hereditary subalgebras.

For the proof of injectivity of the function ω we first show that for any C^*-subalgebra $\mathcal{A} \in \mathfrak{sa}_{\mathcal{I}}(\mathcal{B})$ we have $\mathcal{A} = \pi^{-1}[\pi[\mathcal{A}]]$. Otherwise, there would exist a $b \in \pi^{-1}[\pi[\mathcal{A}]] \setminus \mathcal{A}$ with $\pi(b) \in \pi[\pi^{-1}[\pi[\mathcal{A}]]] = \pi[\mathcal{A}]$ and there would exist a $c \in \mathcal{A}$ such that $\pi(b) = \pi(c)$, i.e. $b - c \in \mathcal{I}$. But then $b \in c + \mathcal{I} \subseteq \mathcal{A}$, which contradicts the choice of b.

Now let $\mathcal{A}, \mathcal{B} \in \mathfrak{sa}_{\mathcal{I}}(\mathcal{B})$ such that $\omega(\mathcal{A}) = \omega(\mathcal{B})$, i.e. $\pi[\mathcal{A}] = \pi[\mathcal{B}]$. It follows from the equation above that $\mathcal{A} = \pi^{-1}[\pi[\mathcal{A}]] = \pi^{-1}[\pi[\mathcal{B}]] = \mathcal{B}$, i.e. ω is an injection.

In the same manner as above we prove the claim about Hilbert C^*-modules. First observe that the π-morphism $\Pi : \mathcal{X} \to \mathcal{X}/\mathcal{X}_{\mathcal{I}}$ preserves structures of C^*-modules, hereditary modules and ideal submodules, so the functions Ω and Ω_h are well defined.

We show that for any full Hilbert C^*-module $Y \in \mathfrak{sm}(\mathcal{X}/\mathcal{X}_{\mathcal{I}})$ the inverse image $\mathcal{Y} = \Pi^{-1}[Y] \in \mathfrak{sm}_{\mathcal{X}_{\mathcal{I}}}(\mathcal{X})$ is a full $\pi^{-1}[\langle Y|Y\rangle]$-module. For any $\lambda \in \mathbb{C}$

and $x_i \in \mathcal{Y}$, $(i = 1, 2)$, we have $\Pi(\lambda x_1 + x_2) = \lambda\Pi(x_1) + \Pi(x_2) \in Y$, hence $\lambda x_1 + x_2 \in \Pi^{-1}[Y] = \mathcal{Y}$. Also for $a \in \mathcal{B}$ we have $\Pi(x_1 a) = \Pi(x_1)\pi(a) \in Y$, i.e. $x_1 a \in \mathcal{Y}$. For the proof of closedness in norm of the set $\mathcal{Y}$ we take any sequence $(x_n)_n$ in $\mathcal{Y}$, $n \in \mathbb{N}$, with $x = \lim_n x_n \in \mathcal{X}$. Then by closedness in norm of the C^*-module Y we have $\Pi(x) = \lim_n \Pi(x_n) \in Y$, so $x \in \mathcal{Y}$, i.e. $\mathcal{Y} = \Pi^{-1}[Y] \in \mathfrak{sm}_{\mathcal{X}_\mathcal{I}}(\mathcal{X})$. That $\mathcal{Y}$ is a full module follows from the fact that Y is a full module, the definition of the inner product for the quotient of the modules $\langle\Pi(\mathcal{Y})|\Pi(\mathcal{Y})\rangle = \pi(\langle\mathcal{Y}|\mathcal{Y}\rangle)$ and properties of π, so $\langle\mathcal{Y}|\mathcal{Y}\rangle = \pi^{-1}[\pi[\langle\mathcal{Y}|\mathcal{Y}\rangle]] = \pi^{-1}[\langle Y|Y\rangle]]$.

In order to prove that $Y \in \mathfrak{hm}(\mathcal{X}/\mathcal{X}_\mathcal{I})$ implies $\mathcal{Y} = \Pi^{-1}[Y] \in \mathfrak{hm}_{\mathcal{X}_\mathcal{I}}(\mathcal{X})$ first recall that $\langle\mathcal{Y}|\mathcal{Y}\rangle$ is a hereditary subalgebra of $\mathcal{B}$. For any $x \in \mathcal{X}$ and $a \in \langle\mathcal{Y}|\mathcal{Y}\rangle$ we have $\pi(a) \in \langle Y|Y\rangle$ and because Y is a hereditary module of $\mathcal{X}/\mathcal{X}_\mathcal{I}$ we have $\Pi(xa) = \Pi(x)\pi(a) \in Y$, so $xa \in \mathcal{Y}$.

For the proof of surjectivity of the function Ω we take any C^*-module $Y \in \mathfrak{sm}(\mathcal{X}/\mathcal{X}_\mathcal{I})$ and take $\mathcal{Y} = \Pi^{-1}[Y] \in \mathfrak{sm}_{\mathcal{X}_\mathcal{I}}(\mathcal{X})$. Now by surjectivity of Π we have $\Omega(\mathcal{Y}) = \Pi[\mathcal{Y}] = \Pi[\Pi^{-1}[Y]] = Y$, and the same is true for hereditary modules.

For the proof of injectivity of the function Ω we first show that for any C^*-submodule $\mathcal{Y}$ of $\mathcal{X}$ containing $\mathcal{X}_\mathcal{I}$ we have $\mathcal{Y} = \Pi^{-1}[\Pi[\mathcal{Y}]]$. Otherwise there would exist an $x \in \Pi^{-1}[\Pi[\mathcal{Y}]] \setminus \mathcal{Y}$ with $\Pi(x) \in \Pi[\Pi^{-1}[\Pi[\mathcal{Y}]]] = \Pi[\mathcal{Y}]$. Then there exists $y \in \mathcal{Y}$ such that $\Pi(x) = \Pi(y)$, i.e. $x - y \in \mathcal{X}_\mathcal{I}$. But then $x \in y + \mathcal{X}_\mathcal{I} \subseteq \mathcal{Y}$ contradicts with the choice of x.

Let $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathfrak{ha}_{\mathcal{X}_{\mathcal{X}_\mathcal{I}}}(\mathcal{X})$ are such that $\Omega(\mathcal{Y}_1) = \Omega(\mathcal{Y}_2)$, i.e. $\Pi[\mathcal{Y}_1] = \Pi[\mathcal{Y}_2]$. From the equality above it follows $\mathcal{Y}_1 = \Pi^{-1}[\Pi[\mathcal{Y}_1]] = \Pi^{-1}[\Pi[\mathcal{Y}_2]] = \mathcal{Y}_2$, i.e. Ω is injection. □

The following lemmas contain results on multiplier algebras of quotients of C^*-algebras and multiplier modules of quotients of Hilbert modules.

LEMMA 4.31. *Let $\mathcal{A}$ be a C^*-algebra containing an essential ideal $\mathcal{I}$ and let $\mathcal{X}$ be a full Hilbert $\mathcal{A}$-module. We have inclusions $\mathcal{I} \subseteq \mathcal{A} \subseteq M(\mathcal{A}) \cap M(\mathcal{I})$, $\mathcal{A}/\mathcal{I} \subseteq M(\mathcal{A})/\mathcal{I} \cap M(\mathcal{A}/\mathcal{I}) \cap C(\mathcal{I})$ and $\mathcal{X}_\mathcal{I} \subseteq \mathcal{X} \subseteq M(\mathcal{X}) \cap M(\mathcal{X}_\mathcal{I})$, $\mathcal{X}/\mathcal{X}_\mathcal{I} \subseteq M(\mathcal{X})/\mathcal{X}_\mathcal{I} \cap M(\mathcal{X}/\mathcal{X}_\mathcal{I}) \cap C(\mathcal{X}_\mathcal{I})$.*

$\mathcal{I}$ is essential in $M(\mathcal{A})$ if and only if $M(\mathcal{A}) \subseteq M(\mathcal{I})$, and then $M(\mathcal{X}) \subseteq M(\mathcal{X}_\mathcal{I})$, $M(\mathcal{X})/\mathcal{X}_\mathcal{I} \subseteq M(\mathcal{X}/\mathcal{X}_\mathcal{I}) \cap C(\mathcal{X}_\mathcal{I})$. In the previous case, $\mathcal{A}$ is an ideal of $M(\mathcal{I})$ if and only if $M(\mathcal{I}) = M(\mathcal{A})$, and then $M(\mathcal{X}) = M(\mathcal{X}_\mathcal{I})$, $C(\mathcal{X}_\mathcal{I}) \subseteq M(\mathcal{X}/\mathcal{X}_\mathcal{I})$.

PROOF. Because $\mathcal{I}$ is an essential ideal of $\mathcal{A}$ and $\mathcal{A}$ is an essential ideal of $M(\mathcal{A})$ we have that $\mathcal{I}$ is an ideal of $M(\mathcal{A})$. Then first inclusion follow directly from the definition of multiplier algebras.

If $\mathcal{I}$ is essential in $M(\mathcal{A})$ then $M(\mathcal{A})$ is an unitization of $\mathcal{I}$, hence $M(\mathcal{A}) \subseteq M(\mathcal{I})$. Conversely, if $M(\mathcal{A}) \subseteq M(\mathcal{I})$, because $\mathcal{I}$ is essential in $M(\mathcal{I})$ we have $\mathcal{I}^{\perp_{M(\mathcal{A})}} = \mathcal{I}^\perp \cap M(\mathcal{A}) = \{0\}$. In this case we immediately have $M(\mathcal{X})/\mathcal{X}_\mathcal{I} \subseteq$

$C(\mathcal{X}_\mathcal{I})$. In order to have inclusion $M(\mathcal{X})/\mathcal{X}_\mathcal{I} \subseteq M(\mathcal{X}/\mathcal{X}_\mathcal{I})$ we have to prove that $\pi(\mathcal{A})$, which is obviously an ideal of $\pi(M(\mathcal{A}))$, is essential. Therefore, let's take $b = \pi(c)$ for some $c \in M(\mathcal{A})$ and $\pi(\mathcal{A})b = 0$. Then $\pi(\mathcal{A}c) = 0$, hence $\mathcal{A}c \subseteq \mathcal{I}$, therefore, $c \in \mathcal{I}$ and $b = \pi(c) = 0$.

Furthermore in the previous case if $\mathcal{A}$ is an ideal in $M(\mathcal{I})$ then, as we did before, $\mathcal{A}^\perp \subseteq \mathcal{I}^\perp = \{0\}$, so $\mathcal{A}$ is essential and $M(\mathcal{I})$ is an unitization of $\mathcal{A}$, hence $M(\mathcal{I}) \subseteq M(\mathcal{A})$ and therefore, $M(\mathcal{I}) = M(\mathcal{A})$. The converse is obvious. Then $C(\mathcal{X}_\mathcal{I}) \subseteq M(\mathcal{X}/\mathcal{X}_\mathcal{I})$. ∎

In the following proposition we show the bijective connection of hereditary C^*-subalgebras and of corresponding hereditary modules.

PROPOSITION 4.32. *Let $\mathcal{I}$ be a non-unital C^*-algebra and let $\mathcal{X}_\mathcal{I}$ be a full $\mathcal{I}$-module.*

The mapping $\Upsilon : \mathfrak{ha}_\mathcal{I}(\langle M(\mathcal{X}_\mathcal{I})|M(\mathcal{X}_\mathcal{I})\rangle) \to \mathfrak{hm}_{\mathcal{X}_\mathcal{I}}(M(\mathcal{X}_\mathcal{I}))$, acting as $\Upsilon(\mathcal{A}) = M(\mathcal{X}_\mathcal{I})\mathcal{A}$, for any $\mathcal{A} \in \mathfrak{ha}_\mathcal{I}(\langle M(\mathcal{X}_\mathcal{I})|M(\mathcal{X}_\mathcal{I})\rangle)$, is a bijection.

PROOF. The surjectivity of the function Υ follows from the definition of the hereditary module because every hereditary Hilbert C^*-module of $M(\mathcal{X}_\mathcal{I})$ is of the form $M(\mathcal{X}_\mathcal{I})\mathcal{A}$ for some hereditary C^*-algebra $\mathcal{A}$ of $\langle M(\mathcal{X}_\mathcal{I})|M(\mathcal{X}_\mathcal{I})\rangle$ which is an ideal of $M(\mathcal{I})$, i.e. $M(\mathcal{X}_\mathcal{I})\mathcal{A} = \Upsilon(\mathcal{A})$.

For the proof of injectivity, suppose that $\Upsilon(\mathcal{A}) = \Upsilon(\mathcal{B})$ for some $\mathcal{A}, \mathcal{B} \in \mathfrak{ha}_\mathcal{I}(\langle M(\mathcal{X}_\mathcal{I})|M(\mathcal{X}_\mathcal{I})\rangle)$, so $\langle \Upsilon(\mathcal{A})|\Upsilon(\mathcal{A})\rangle = \langle \Upsilon(\mathcal{B})|\Upsilon(\mathcal{B})\rangle$. Because $M(\mathcal{X}_\mathcal{I})$ is a full $\langle M(\mathcal{X}_\mathcal{I})|M(\mathcal{X}_\mathcal{I})\rangle$-module, it follows from the Proposition 4.25 that $\mathcal{A} = \langle M(\mathcal{X}_\mathcal{I})\mathcal{A}|M(\mathcal{X}_\mathcal{I})\mathcal{A}\rangle = \langle \Upsilon(\mathcal{A})|\Upsilon(\mathcal{A})\rangle$ for any $\mathcal{A} \in \mathfrak{ha}_\mathcal{I}(\langle M(\mathcal{X}_\mathcal{I})|M(\mathcal{X}_\mathcal{I})\rangle)$, and we have $\mathcal{A} = \mathcal{B}$. ∎

REMARK 4.33. Because $\langle M(\mathcal{X}_\mathcal{I})|M(\mathcal{X}_\mathcal{I})\rangle$ is an ideal of $M(\mathcal{I})$, observe that from Remark 4.24 it follows $\mathfrak{ha}_\mathcal{I}(\langle M(\mathcal{X}_\mathcal{I})|M(\mathcal{X}_\mathcal{I})\rangle) \subseteq \mathfrak{ha}_\mathcal{I}(M(\mathcal{I}))$. However, it is not generally possible to extend the function Υ to $M(\mathcal{I})$ and preserve its injectivity because a multiplier module is not always a full $M(\mathcal{I})$-module (see Example 4.12). ♢

LEMMA 4.34. *Let $\mathbf{A} = (\mathbf{A}_j)_{j\in\mathcal{J}}$, $\mathbf{K}(H_j) \subseteq \mathbf{A}_j \subseteq \mathbf{B}(H_j), j \in \mathcal{J}$, be a family of hereditary C^*-algebras, let $\mathbf{X} = (\mathbf{X}_j)_{j\in\mathcal{J}}$, $\mathbf{K}(H_j, G_j) \subseteq \mathbf{X}_j \subseteq \mathbf{B}(H_j, G_j)$, be a family of hereditary Hilbert $\mathbf{A}_j$-modules of $\mathbf{B}(H_j, G_j)$, $j \in \mathcal{J}$, and let $\mathcal{C}$ be any ideal of $\mathcal{C}_\infty$ containing $\mathcal{C}_0$. Then $\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}} \mathbf{A}_j$ is a hereditary C^*-algebra of $M(\mathbf{K})$ containing $\mathbf{K} = \mathcal{C}_0\text{-}\oplus_{j\in\mathcal{J}} \mathbf{K}(H_j)$ and $\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}} \mathbf{X}_j = \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}} \mathbf{B}(H_j, G_j)\mathbf{A}_j$ is a hereditary $\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}} \mathbf{A}_j$-module of $M(\mathbf{X}_\mathbf{K})$ containing $\mathbf{X}_\mathbf{K}$.*

PROOF. Let $a_i = (a_j^{(i)}) \in \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}} \mathbf{A}_j$, $i = 1, 2$, and $m = (m_j) \in \mathcal{C}_\infty\text{-}\oplus_{j\in\mathcal{J}} \mathbf{B}(H_j)$ be such that for some $0 < \alpha \in \mathbb{R}$ we have $\|m_j\|_j \le \alpha$, $\forall j \in \mathcal{J}$. Then it follows $\|a_j^{(1)} m_j a_j^{(2)}\|_j \le \alpha \|a_j^{(1)}\|_j \|a_j^{(2)}\|_j$, for any $j \in \mathcal{J}$, i.e.

$\alpha(\|a_j^{(1)}\|_j)_j(\|a_j^{(2)}\|_j)_j \in \mathcal{C}$. Therefore, $a^{(1)}ma^{(2)} \in \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathbf{A}_j$, so is $\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathbf{A}_j$ a hereditary C^*-subalgebra of $M(\mathbf{K})$ containing $\mathbf{K}$.

Analogously we prove

$$\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathbf{X}_j\ \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathbf{A}_j \subseteq \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathbf{X}_j, \tag{4.17}$$

$$\mathcal{C}_\infty\text{-}\oplus_{j\in\mathcal{J}}\mathbf{B}(H_j,G_j)\ \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathbf{A}_j \subseteq \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathbf{X}_j. \tag{4.18}$$

Inclusion (4.17) implies that $\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathbf{X}_j$ is a Hilbert $\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathbf{A}_j$-module and from (4.18) we have

$$\begin{aligned} &M(\mathbf{X}_\mathbf{K})\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathbf{A}_j = \mathcal{C}_\infty\text{-}\oplus_{j\in\mathcal{J}}\mathbf{B}(H_j,G_j)\cdot\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathbf{A}_j \\ &\subseteq \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathbf{B}(H_j,G_j)\mathbf{A}_j = \mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathbf{X}_j. \end{aligned} \tag{4.19}$$

The opposite inclusions in (4.17), (4.18) and (4.19) follow by applying the Cohen-Hewitt theorem on factorization (cf. [13, Proposition 2.31]) to the Hilbert $\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathbf{A}_j$-module $\mathcal{C}\text{-}\oplus_{j\in\mathcal{J}}\mathbf{X}_j$. □

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