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ON H-SMOOTH AND H-CONVEX SETS IN LINEAR SPACES

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Abstract. In this paper the properties of H-smooth and H-convex sets are investigated. It is shown that any H-convex set is convex. The centric, balanced and convex hulls of an H-smooth set, as well as its radial frontier are studied. A necessary and sufficient condition is given for an H-convex set to be strictly convex.

1. Let X denote a linear space over the field of all real or complex numbers.

If $M \subset X$ is an absorbent set, then the functional $p_M : X \to \mathbf{R}$ defined by

$$p_M(x) := \inf\{a > 0 : x \in aM\}, \quad x \in X$$

is called the Minkowski functional of M.

The notion of H-smooth and H-convex set in a linear space was introduced by T. Precupanu in [3]. Such sets are of interest because the Minkowski functional p_M corresponding to an absorbent and H-smooth or H-convex set $M \subset X$ is a Hilbertian semi-norm, that is a semi-norm which satisfies the parallelogram law:

$$p_M(x+y)^2 + p_M(x-y)^2 = 2p_M(x)^2 + 2p_M(y)^2, \; x, y \in X$$

(see [2] and [3]).

We modify slightly the definition of an H-smooth set in comparison with those occurring in [1] and [2].

Definition 1. A non-empty subset M of a linear space X is called H-smooth if and only if for any $a, \beta \in \mathbf{R}$, $a > 0, \beta > 0$ and each $x \in aM$, $y \in \beta M$ there exist $a_0, \beta_0 \in \mathbf{R}$, $a_0 \geqslant 0, \beta_0 \geqslant 0$ such that

$$a_0^2 + \beta_0^2 \leqslant 2(a^2 + \beta^2); \tag{1}$$

$$x + y \in a_0 M; \tag{2}$$

$$x - y \in \beta_0 M. \tag{3}$$

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The conditions of the definition just proposed are easier to check in concrete cases. We shall prove that our definition is actually equivalent to the one in [1]. This fact is useful in the proofs of some theorems concerning properties of H-smooth sets.

LEMMA 1. *Let $M \subset X$ be an H-smooth set. Then for any $x \in M$ there exists a $\lambda \in (0, 1]$ such that $-x \in \lambda M$.*

Proof. Let us fix an $x \in M$. If $x = 0$, we can put $\lambda = 1$. Suppose that $x \neq 0$. We write $a := \inf\{\lambda > 0 : x \in \lambda M\}$. Since $x \in M$, we have $a \leqslant 1$. There exists a sequence $(a_n)_{n \in \mathbf{N}}$ of positive numbers such that $a_n \to a$ and $x \in a_n M$ for each $n \in \mathbf{N}$. Put $\beta_n := 2a_n$. Then $2x \in \beta_n M$ for $n \in \mathbf{N}$. From the H-smoothness of the set M it follows that for each $n \in \mathbf{N}$ there exist $a_{0,n} \geqslant 0$ and $\beta_{0,n} \geqslant 0$ fulfilling the following conditions:

$$a_{0,n}^2 + \beta_{0,n}^2 \leqslant 2(a_n^2 + \beta_n^2) = 10\,a_n^2; \tag{4}$$

$$3x = x + 2x \in a_{0,n} M; \tag{5}$$

$$-x = x - 2x \in \beta_{0,n} M. \tag{6}$$

If it were $a_{0,n} = 0$ or $\beta_{0,n} = 0$, we would have $x = 0$, opposite to our hypothesis. So we have $a_{0,n} > 0$ and $\beta_{0,n} > 0$. Hence and from (5) it follows that $a_{0,n} \geqslant 3a$. In view of (4) we obtain

$$\beta_{0,n}^2 \leqslant 10\,a_n^2 - a_{0,n}^2 \leqslant 10\,a_n^2 - 9a^2 \quad \text{for } n \in \mathbf{N},$$

that is

$$\beta_{0,n} \leqslant \sqrt{10\,a_n^2 - 9a^2}, \quad \text{for each } n \in \mathbf{N}.$$

Letting now n tend to infinity we deduce that $\liminf \beta_{0,n} \leqslant a$. Hence and from (6):

$$\inf\{\lambda > 0 : -x \in \lambda M\} \leqslant a \leqslant 1.$$

If $a < 1$, we have $\inf\{\lambda > 0 : -x \in \lambda M\} < 1$. In such a case there exists an $\lambda \in (0, 1)$ for which $-x \in \lambda M$.

Suppose now that $a = 1$. Since $x \in M$, $2x \in 2M$, it follows from the H-smoothness of the set M that there exist numbers $a_0 \geqslant 0$ and $\beta_0 \geqslant 0$ fulfilling the conditions:

$$3x = x + 2x \in a_0 M; \tag{7}$$

$$-x = x - 2x \in \beta_0 M; \tag{8}$$

$$a_0^2 + \beta_0^2 \leqslant 2(1^2 + 2^2) = 10.$$

Since $x \neq 0$ and conditions (7) and (8) hold, we deduce that $a_0 > 0$ and $\beta_0 > 0$. Hence, by (7) we get $a_0 \geqslant 3a = 3$.

Consequently,

$$\beta_0^2 \leqslant 10 - a_0^2 \leqslant 10 - 9 = 1 \quad \text{whence } \beta_0 \leqslant 1 \text{ and } -x \in \beta_0 M.$$

This ends the proof of our lemma.

Example 1. An H-smooth set need not be symmetric. For, take $x_0 \in (-1, 1) \setminus \{0\}$, $M := (-1, 1) \setminus \{x_0\}$. If $\alpha > 0$, $\beta > 0$, $x \in \alpha M$, $y \in \beta M$, then $|x| < \alpha$, $|y| < \beta$. Hence

$$|x + y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2) < 2(\alpha^2 + \beta^2).$$

One can find numbers $\alpha_0 > 0, \beta_0 > 0$ such that $|x + y| < \alpha_0$, $|x - y| < \beta_0$, $\alpha_0^2 + \beta_0^2 < 2(\alpha^2 + \beta^2)$ and $x + y \neq \alpha_0 x_0$, $x - y \neq \beta_0 x_0$. Thus $x + y \in \alpha_0 M$, $x - y \in \beta_0 M$, which shows that M is an H-smooth set. If $x_0 \neq 0$, the set M is not symmetric.

PROPOSITION 1. *A non-empty set $M \subset X$ is H-smooth if and only if for each $\alpha \geqslant 0, \beta \geqslant 0$ and each $x \in \alpha M$, $y \in \beta M$ there exist numbers $\alpha_0 \geqslant 0$, $\beta_0 \geqslant 0$ fulfilling conditions* (1), (2) *and* (3).

Proof. We have to prove necessity only. Suppose that M is an H-smooth set and take $\alpha \geqslant 0$, $\beta \geqslant 0$, $x \in \alpha M, y \in \beta M$. Let us consider the following four cases:

1. $\alpha > 0$ and $\beta > 0$. Then the existence of numbers $\alpha_0 \geqslant 0$, $\beta_0 \geqslant 0$ with properties (1), (2) and (3) follows from the definition of H-smoothness.

2. $\alpha = 0$ and $\beta = 0$. In such a case $x = y = 0$ and we can put $\alpha_0 = \beta_0 = 0$.

3. $\alpha > 0, \beta = 0$. Then $y = 0$ and putting $\alpha_0 = \beta_0 := \alpha$ we obtain $x + y = x \in \alpha_0 M$, $x - y = x \in \beta_0 M$, $\alpha_0^2 + \beta_0^2 = 2\alpha^2 = 2(\alpha^2 + \beta^2)$.

4. $\alpha = 0, \beta > 0$. Then $x = 0$ and in view of Lemma 1 there exists $\beta_0 \in (0, \beta]$ such that $-y \in \beta_0 M$. Setting $\alpha_0 := \beta$ we have $x + y = y \in \alpha_0 M$, $x - y = -y \in \beta_0 M$, $\alpha_0^2 + \beta_0^2 \leqslant 2\beta^2 = 2(\alpha^2 + \beta^2)$.

The above cases exhaust all the possibilities and the proof is completed.

Remark 1. Zero need not belong to an H-smooth set. The set $M := (-1, 1) \setminus \{0\}$ may be used as an example.

PROPOSITION 2. *If $M \subset X$ is an H-smooth set, then the set $M_0 := \{0\} \cup M$ is H-smooth.*

Proof. Let us take $\alpha > 0$, $\beta > 0$, $x \in \alpha M_0$, $y \in \beta M_0$. Then the following cases are possible:

1. $x \in \alpha M$, $y \in \beta M$;

2. $x = 0 \in 0 \cdot M$, $y \in \beta M$;

3. $x \in \alpha M$, $y = 0 \in 0 \cdot M$;

4. $x = 0 \in 0 \cdot M$, $y = 0 \in 0 \cdot M$.

On account of Proposition 1 in each of the above cases there exist $\alpha_0 \geqslant 0$, $\beta_0 \geqslant 0$ such that

$$\alpha_0^2 + \beta_0^2 \leqslant 2(\alpha^2 + \beta^2), x + y \in \alpha_0 M \subset \alpha_0 M_0, x - y \in \beta_0 M \subset \beta_0 M_0$$

which proves that M_0 is an H-smooth set.

2. *Definition* 2. An H-smooth and balanced subset of a linear space is said to be H-convex.

In [1] we find the definition of the so called strictly H-convex set. We shall show that this definition does not distinguish any new class of sets. Every H-convex set satisfies the condition which appears in the definition. In the present paper the notion »strictly H-convex set« will be used in another sense.

THEOREM 1. *If $M \subset X$ is an H-smooth and absorbent set, then for any $\alpha \geqslant 0, \beta \geqslant 0$ and any $x \in \alpha M, y \in \beta M$ there exist $\alpha_0 \geqslant 0, \beta_0 \geqslant 0$ fulfilling conditions* (1), (2), (3) *and the following condition*:

$$\max(\alpha_0, \beta_0) \leqslant \alpha + \beta. \tag{9}$$

Proof. The Minkowski functional p_M of the set M is a Hilbertian semi-norm. If $\alpha \geqslant 0$, $\beta \geqslant 0$, $x \in \alpha M$, $y \in \beta M$, then $p_M(x) \leqslant \alpha$ and $p_M(y) \leqslant \beta$. Suppose first that $p_M(x) < \alpha$ or $p_M(y) < \beta$. In such a case we have

$$p_M(x+y)^2 + p_M(x-y)^2 = 2(p_M(x)^2 + p_M(y)^2) < 2(\alpha^2 + \beta^2)$$

and

$$p_M(x+y) \leqslant p_M(x) + p_M(y) < \alpha + \beta,$$

$$p_M(x-y) \leqslant p_M(x) + p_M(-y) < \alpha + \beta.$$

Then we can find numbers $\alpha_1 > 0$ and $\beta_1 > 0$ such that

$$p_M(x+y) < \alpha_1 < \alpha + \beta, \; p_M(x-y) < \beta_1 < \alpha + \beta$$

and

$$\alpha_1^2 + \beta_1^2 < 2(\alpha^2 + \beta^2).$$

Hence it follows that there exist numbers $\alpha_0 > 0$, $\beta_0 > 0$, $\alpha_0 < \alpha_1$, $\beta_0 < \beta_1$ for which $x + y \in \alpha_0 M$ and $x - y \in \beta_0 M$. Moreover,

$$\alpha_0^2 + \beta_0^2 < 2(\alpha^2 + \beta^2), \; \alpha_0 < \alpha + \beta \text{ and } \beta_0 < \alpha + \beta.$$

It remains to consider the case where $p_M(x) = \alpha$ and $p_M(y) = \beta$. From the H-smoothness of the set M it follows that there exist $\alpha_1 \geqslant 0$, $\beta_1 \geqslant 0$, such that

$$\begin{gathered} \alpha_1^2 + \beta_1^2 \leqslant 2(\alpha^2 + \beta^2), \\ x + y \in \alpha_1 M, \; x - y \in \beta_1 M. \end{gathered} \tag{10}$$

Put $\alpha_0 := p_M(x+y)$, $\beta_0 := p_M(x-y)$. From the definition of the Minkowski functional we get $\alpha_0 \leqslant \alpha_1$ and $\beta_0 \leqslant \beta_1$. If it were $\alpha_0 < \alpha_1$ or $\beta_0 < \beta_1$, we would have

$$\alpha_1^2 + \beta_1^2 > \alpha_0^2 + \beta_0^2 = p_M(x+y)^2 + p_M(x-y)^2 =$$
$$= 2(p_M(x)^2 + p_M(y)^2) = 2(\alpha^2 + \beta^2),$$

contrary to (10). Thus $\alpha_0 = \alpha_1, \beta_0 = \beta_1$, $x + y \in \alpha_0 M$ and $x - y \in \in \beta_0 M$. Moreover,

$$\alpha_0^2 + \beta_0^2 = 2(\alpha^2 + \beta^2) \text{ and}$$

$$\alpha_0 = p_M(x+y) \leqslant p_M(x) + p_M(y) = \alpha + \beta,$$

$$\beta_0 = p_M(x-y) \leqslant p_M(x) + p_M(-y) = \alpha + \beta.$$

This completes our proof.

The previous theorem remains true in the case where $M \subset X$ is an arbitrary H-smooth set (not necessarily absorbent). Namely, we have the following:

THEOREM 2. *If $M \subset X$ is an H-smooth set, then for any $\alpha \geqslant 0$, $\beta \geqslant 0$ and any $x \in \alpha M$, $y \in \beta M$ there exist numbers $\alpha_0 \geqslant 0$, $\beta_0 \geqslant 0$ such that conditions* (1), (2), (3) *and* (9) *are fulfilled.*

Proof. Put $M_0 := \{0\} \cup M$. On account of Proposition 2, M_0 is an H-smooth set. Let Y be the set of all points $x \in X$ for which there exists $\alpha > 0$ such that $x \in \alpha M_0$. Since M_0 is an H-smooth set and $0 \in M$, in view of Lemma 1, one can easily check that Y is a linear subspace of the space X in which M_0 is an absorbent set. From Theorem 1 it follows that for any $\alpha \geqslant 0$, $\beta \geqslant 0$, $x \in \alpha M$, $y \in \beta M$ there exist $\alpha_1 \geqslant 0$, $\beta_1 \geqslant 0$ fulfilling condition (10) and

$$x + y \in \alpha_1 M_0,\ x - y \in \beta_1 M_0,\ \max(\alpha_1, \beta_1) \leqslant \alpha + \beta.$$

Put

$$\alpha_0 := \begin{cases} \alpha_1, & \text{for } x + y \in \alpha_1 M \\ 0, & \text{for } x + y \notin \alpha_1 M \quad (\text{i. e. } x + y = 0) \end{cases}$$

$$\beta_0 := \begin{cases} \beta_1, & \text{for } x - y \in \beta_1 M \\ 0, & \text{for } x - y \notin \beta_1 M \quad (\text{i. e. } x - y = 0). \end{cases}$$

The numbers $\alpha_0 \geqslant 0$, $\beta_0 \geqslant 0$ fulfil conditions (1), (2), (3) and (9).

The example of an H-convex but not convex set, which was given by E. Kramar in [1], and Muntean and Precupanu in [2], is improper. Namely, we have the following

THEOREM 3. *Every H-convex set $M \subset X$ is convex.*

Proof. Take $x, y \in M$ and $t \in [0, 1]$. Then $tx \in tM$ and $(1 - t)y \in (1 - t)M$. From Theorem 2 it follows, in particular, that there exists an $\alpha_0 \geqslant 0$ such that $tx + (1 - t)y \in \alpha_0 M$ and $\alpha_0 \leqslant t + (1 - t) = 1$. Since M is balanced, we have $tx + (1 - t)y \in M$. This ends the proof.

In the proofs of Theorems 1 and 2 we have made use of the fact that the Minkowski functional of an H-smooth set is a Hilbertian seminorm. Now we shall give another quite elementary proof of the convexity of an H-convex set. Having such a proof one is able to obtain immediately the subadditivity of the Minkowski functional corresponding to an absorbent H-convex set. Now, we proceed with the

Proof. Let $M \subset X$ be an H-convex set. If $x, y \in M$, $t \in (0, 1)$, then $tx \in tM$, $(1 - t)y \in (1 - t)M$. From the H-convexity of the set M it follows that there exists an $\alpha_0 \geqslant 0$ such that

$$tx + (1 - t)y \in \alpha_0 M \text{ and } \alpha_0^2 \leqslant 2(t^2 + (1 - t)^2).$$

Since the set M is balanced we obtain

$$tx + (1 - t)y \in \sqrt{2(t^2 + (1 - t)^2)}\, M.$$

Putting $t = \frac{1}{2}$ we have $\frac{x + y}{2} \in M$. By induction one can prove that $\frac{k}{2^n} x + \left(1 - \frac{k}{2^n}\right) y \in M$ for any $k, n \in \mathbf{N}$, $k < 2^n$.

Fix $x, y \in M$, $t \in (0, 1)$, put $z = tx + (1 - t)y$ and take arbitrary numbers $r, s \in (0, 1)$ such that $r < t < s$ and $t = \frac{r + s}{2}$.

Since the set $A := \left\{\frac{k}{2^n} \in (0, 1) : k, n \in \mathbf{N},\ k < 2^n\right\}$ is dense in the interval $(0, 1)$ we can choose two sequences $(r_n)_{n \in N}$ and $(s_n)_{n \in N}$ such that $r_n, s_n \in A$, $r_n \leqslant r$, $s \leqslant s_n$ for each $n \in \mathbf{N}$ and $r_n \to r$, $s_n \to s$.

Defining $t_n := \frac{s_n - t}{s_n - r_n}$ we have $t_n \in (0, 1)$, $t = t_n r_n + (1 - t_n) s_n$ for each $n \in \mathbf{N}$ and $t_n \to \frac{s - t}{s - r} = \frac{1}{2}$. Hence:

$$z = tx + (1 - t)y = [t_n r_n + (1 - t_n) s_n]x + [1 - t_n r_n - (1 - t_n) s_n] y = t_n [r_n x + (1 - r_n) y] + (1 - t_n)[s_n x + (1 - s_n) y].$$

Since $r_n x + (1 - r_n) y \in M$ and $s_n x + (1 - s_n) y \in M$ the relation $z \in \sqrt{2(t_n^2 + (1 - t_n)^2)}\, M$ holds for each $n \in \mathbf{N}$. As a consequence of the fact that $\sqrt{2(t_n^2 + (1 - t_n)^2)} \to 1$ we obtain $\inf\{\alpha > 0 : z \in \alpha M\} \leqslant 1$, whence $\lambda z \in M$ follows for each $\lambda \in [0, 1)$ because M

is a balanced set. We define $T(x,y) := \{u \in X : u = \alpha x + \beta y, \alpha, \beta \in [0, 1), \alpha + \beta < 1\}$. Obviously $T(x,y) = T(y,x)$ and

$$T(x,y) = \{u \in X : u = \lambda(tx + (1-t)y), \lambda \in [0,1), t \in [0,1]\}.$$

Hence $T(x,y) \subset M$ for any $x, y \in M$.

Now we are going to prove that for any $x, y \in M$ we have

$$\overline{(x,y)} := \{tx + (1-t)y \in X : t \in (0,1)\} \subset M.$$

If x and y are linearly dependent over $\mathbf{R}$, then the fact that M is a balanced set implies $(x,y) \subset M$. Suppose further on that x and y are linearly independent over $\mathbf{R}$. Put

$$P := \{u \in X : u = ax + by, a, b \in \mathbf{R}, a + b \leqslant 1\},$$

$$S := \{u \in X : u = ax + by, a, b \in \mathbf{R}, a + b > 1\},$$

$P \cap S = \emptyset$, $P \cup S = \mathrm{Lin}_R\{x, y\}$ and consider two cases.

Case 1. There exists a $v \in S \cap M$. We shall show that $(x,y) \subset T(x,y) \cup T(y,v)$. There exist $a, b \in \mathbf{R}$ such that $a + b > 1$ and $v = ax + by$. At least one of the numbers a and b has to be positive. Suppose e. g. that $a > 0$.

For $t \in \left(0, \dfrac{a}{a+b}\right]$ we define $\alpha := \dfrac{t}{a}$, $\beta := \dfrac{a - t(a+b)}{a}$.

Then $\alpha \in (0,1)$, $\beta \in [0,1)$, $\alpha + \beta < \alpha(a+b) + \beta = 1$,

$$t = \alpha a,\ 1 - t = \alpha b + \beta$$

and

$$tx + (1-t)y = \alpha ax + (\alpha b + \beta)y = \alpha(ax + by) + \beta y =$$
$$= \alpha v + \beta y \in T(v,y).$$

Hence it follows that $(x,y) \subset T(v,y)$ provided $b \leqslant 0$.

If $b > 0$, then for $t \in \left[\dfrac{a}{a+b}, 1\right)$ we define

$$\alpha := \frac{t(a+b) - a}{a},\quad \beta := \frac{1-t}{b}.$$

Then $\alpha \in [0,1)$, $\beta \in (0,1)$, $\alpha + \beta < \alpha + \beta(a+b) = 1$, $t = \alpha + \beta a$, $1 - t = \beta b$ and

$$tx + (1-t)y = (\alpha + \beta a)x + \beta by = \alpha x + \beta(ax + by) =$$
$$= \alpha x + \beta v \in T(x,v).$$

Consequently $(x, y) \subset T(v, y) \cup T(x, v)$ which ends the proof of the inclusion announced. Since $T(x, v) \subset M$ and $T(v, y) \subset M$ we obtain $(x, y) \subset M$.

Case 2. $S \cap M = \emptyset$. Take $\alpha > 0$, $\beta > 0$, $\alpha \geqslant \beta$. Then $\alpha x \in \alpha M$, $\beta y \in \beta M$ and from the H-convexity of M it follows that there exist $\alpha_0 \geqslant 0$, $\beta_0 \geqslant 0$ such that $\alpha_0^2 + \beta_0^2 \leqslant 2(\alpha^2 + \beta^2)$ and $\alpha x + \beta y \in \alpha_0 M$, $\alpha x - \beta y \in \beta_0 M$. Since $M \cap \mathrm{Lin}_R\{x, y\} \subset P$ we have:

$$\alpha x + \beta y = \alpha_0 (ax + by) \text{ for some } a, b \in \mathbf{R},\ a + b \leqslant 1,$$

$$\alpha x - \beta y = \beta_0 (cx + dy) \text{ for some } c, d \in \mathbf{R},\ c + d \leqslant 1.$$

From the linear independence (over $\mathbf{R}$) of the vectors x and y we obtain $\alpha = \alpha_0 a$, $\beta = \beta_0 b$, $\alpha = \beta_0 c$, $-\beta = \beta_0 d$. Hence

$$0 < \alpha + \beta = \alpha_0 (a + b) \leqslant \alpha_0,\ 0 \leqslant \alpha - \beta = \beta_0 (c + d) \leqslant \beta_0.$$

If it were $\alpha + \beta < \alpha_0$ or $\alpha - \beta < \beta_0$, we would have

$$\alpha_0^2 + \beta_0^2 > (\alpha + \beta)^2 + (\alpha - \beta)^2 = 2(\alpha^2 + \beta^2),$$

which leads to a contradiciton. So, we have $\alpha_0 = \alpha + \beta$, $\beta_0 = \alpha - \beta$ and, in particular, $\alpha x + \beta y \in (\alpha + \beta) M$, that is

$$\frac{\alpha x + \beta y}{\alpha + \beta} \in M \text{ for } \alpha > 0,\ \beta > 0,\ \alpha \geqslant \beta.$$

Interchanging the roles of x and y we obtain the analogous relation for $\alpha > 0$, $\beta > 0$, $\alpha \leqslant \beta$. Hence, for $t \in (0, 1)$, we have $tx + (1 - t) y \in M$. This ends the proof.

3. *Definition* 3. The set $M \subset X$ is called centric if $\lambda M \subset M$ for each $\lambda \in [0, 1]$.

Any centric and symmetric set is balanced.

PROPOSITION 3. *If $M \subset X$ is an H-smooth and centric set, then M is H-convex.*

Proof. We shall prove that M is symmetric. Take an $x \in M$. On account of Lemma 1, there exists a $\lambda \in (0, 1]$ such that $-x \in \lambda M \subset M$, which ends the proof.

THEOREM 4. *The set $M \subset X$ is H-convex if and only if it is H-smooth and convex.*

Proof. In view of Theorem 3 one has only to prove that the condition is sufficient. For, suppose that M is an H-smooth and convex set and take an arbitrary $x \in M$. From Lemma 1 it follows, in particular, that $-x \in \lambda M$ for some $\lambda > 0$. Hence $-\frac{1}{\lambda} x \in M$ and from the convexity of the set M we obtain

$$\left[-\frac{1}{\lambda}x, x\right] := \left\{t\left(-\frac{1}{\lambda}x\right) + (1-t)x \in X : t \in [0, 1]\right\} \subset M.$$

Consequently, $0 \in M$. So we have $\lambda x = \lambda x + (1-\lambda) \cdot 0 \in M$ for each $x \in M$, $\lambda \in [0, 1]$. This shows that the set M is centric and we can use the previous proposition to complete the proof.

Now we shall investigate some connections between an H-smooth set $M \subset X$ and its centric, balanced and convex hulls i. e. the smallest sets containing M which are centric, balanced or convex, respectively. These results are complementary to those presented in [1] and [2]. A centric hull of the set M will be denoted by $\mathrm{Cn}\, M$, whereas the symbols $\mathrm{Bn}\, M$ and $\mathrm{Conv}\, M$ will stand for its balanced and convex hull, respectively.

LEMMA 2. *The centric hull of an H-smooth set is H-smooth.*

Proof. If $\alpha > 0$, $\beta > 0$, $x \in \alpha\, \mathrm{Cn}\, M$, $y \in \beta\, \mathrm{Cn}\, M$ then, according to definition of a centric hull, there exist $\lambda, \mu \in [0, 1]$ such that $x \in \alpha\lambda M$, $y \in \beta\mu M$. We have $\alpha\lambda \geqslant 0$, $\beta\mu \geqslant 0$ and from Proposition 1 it follows that there exist numbers $\alpha_0 \geqslant 0$, $\beta_0 \geqslant 0$, fulfilling the conditions

$$\alpha_0^2 + \beta_0^2 \leqslant 2(\alpha\lambda)^2 + 2(\beta\mu)^2 \leqslant 2\alpha^2 + 2\beta^2,$$

$$x + y \in \alpha_0 M \subset \alpha_0\, \mathrm{Cn}\, M, \quad x - y \in \beta_0 M \subset \beta_0\, \mathrm{Cn}\, M.$$

Hence $\mathrm{Cn}\, M$ is an H-smooth set.

From Lemma 2 and Proposition 3 it follows:

THEOREM 5. *If* $M \subset X$ *is an H-smooth set, then* $\mathrm{Cn}\, M = \mathrm{Bn}\, M$. *In particular, the balanced hull of an H-smooth set is an H-convex set.*

THEOREM 6. *If* $M \subset X$ *is an H-smooth set, then* $\mathrm{Conv}\, M = \mathrm{Bn}\, M$. *In particular, the convex hull of an H-smooth set is an H-convex set.*

Proof. The set $\mathrm{Bn}\, M = \mathrm{Cn}\, M$ is H-convex and so it is convex. Hence $\mathrm{Conv}\, M \subset \mathrm{Bn}\, M = \mathrm{Cn}\, M$. If $x \in M$ then, in view of Lemma 1, we have $\frac{-x}{\lambda} \in M$ for some $\lambda > 0$. Thus $\left[\frac{-x}{\lambda}, x\right] \subset \mathrm{Conv}\, M$ and, in particular, $0 \in \mathrm{Conv}\, M$.

If $x \in \mathrm{Cn}\, M$, then there exist $\lambda \in [0, 1]$ and $y \in M$ such that $x = \lambda y = \lambda y + (1-\lambda) \cdot 0 \in \mathrm{Conv}\, M$. Consequently, $\mathrm{Cn}\, M \subset \mathrm{Conv}\, M$ whence $\mathrm{Conv}\, M = \mathrm{Cn}\, M = \mathrm{Bn}\, M$.

Following T. Precupanu, by the radial frontier of a set $M \subset X$ we mean the collection of all points $x \in X \setminus \{0\}$ such that $(x, \rightarrow) \cap M = \emptyset$ and $[x_1, x] \cap M \neq \emptyset$ for each $x_1 \in (0, x)$, where $(0, x) :=$

$= \{t x \in X : 0 < t < 1\}$, $[x_1, x] := \{t x_1 + (1-t) x \in X : 0 \leqslant t \leqslant 1\}$, $(x, \rightarrow) := \{tx \in X : t > 1\}$.

The radial frontier of M will be denoted by $\operatorname{Fr} M$.

LEMMA 3. *For any* $M \subset X$ *equality* $\operatorname{Fr} M = \operatorname{Fr} \operatorname{Cn} M$ *holds.*

Proof. Let us first fix an $x \in \operatorname{Fr} M$. Then $x \neq 0$ and $(x, \rightarrow) \cap \cap M = \emptyset$. If it were $(x, \rightarrow) \cap \operatorname{Cn} M \neq \emptyset$, it would exist a $t > 1$ such that $tx \in \operatorname{Cn} M$. Then $tx \in \lambda M$ for some $\lambda \in (0, 1)$ and so we would have $\frac{t}{\lambda} > 1$ and $\frac{t}{\lambda} x \in M$, contrary to our hypothesis. Consequently, $(x, \rightarrow) \cap \operatorname{Cn} M = \emptyset$. For any $x_1 \in (0, x)$, we have $[x_1, x] \cap \cap M \neq \emptyset$. Since $M \subset \operatorname{Cn} M$, we get $[x_1, x] \cap \operatorname{Cn} M \neq \emptyset$; hence $x \in \operatorname{Fr} \operatorname{Cn} M$. Now, suppose that $x \in \operatorname{Fr} \operatorname{Cn} M$. Then $x \neq 0$ and $(x, \rightarrow) \cap M = \emptyset$, since $M \subset \operatorname{Cn} M$. For any $x_1 \in (0, x)$ one has $[x_1, x] \cap \operatorname{Cn} M \neq \emptyset$. If $x_1 = t_1 x$, $t_1 \in (0, 1)$ and $t x \in \operatorname{Cn} M$ for some $t \in [t_1, 1]$, then there exists a $\lambda \in (0, 1]$ such that $tx \in \lambda M$ i. e. $\frac{t}{\lambda} x \in M$. Since $(x, \rightarrow) \cap M = \emptyset$, it must be $\frac{t}{\lambda} \leqslant 1$. On the other hand, $\frac{t}{\lambda} \geqslant t \geqslant t_1$; consequently $\frac{t}{\lambda} x \in [x_1, x] \cap M$ and $x \in \operatorname{Fr} M$. This ends the proof.

From Lemma 3 and Theorems 5 and 6 it follows:

PROPOSITION 4. *If* $M \subset X$ *is an H-smooth set, then* $\operatorname{Fr} M = \operatorname{Fr} \operatorname{Bn} M = \operatorname{Fr} \operatorname{Conv} M$.

In [2] it has been proved the following lemma

LEMMA 4. *If* $M \subset X$ *is an absorbent set, then*

$$\operatorname{Fr} M = \{x \in X : p_M(x) = 1\}.$$

In view of the properties of the Minkowski functional one can therefore obtain

LEMMA 5. *If* $M \subset X$ *is an absorbent and centric set, then*

$$M \cup \operatorname{Fr} M = \{x \in X : p_M(x) \leqslant 1\}$$

and

$$M \setminus \operatorname{Fr} M = \{x \in X : p_M(x) < 1\}.$$

The authors of [2] have introduced the concept of a radially bounded set, i. e. a set $M \subset X$ with the property that for each $x \in \in X \setminus \{0\}$ there exists a $\alpha_0 > 0$ such that $x \notin \alpha M$ for $\alpha > \alpha_0$.

For an absorbent set $M \subset X$ the following conditions are equivaent (see [2] Lemma 2):

(*i*) M is radially bounded;

(*ii*) $p_M(x) = 0$ if and only if $x = 0$;

(*iii*) $\{0\} \cup \operatorname{Fr} M$ is an absorbent set.

Consequently, if a set $M \subset X$ is absorbent, H-convex and satisfies one of the two equivalent conditions (*i*) or (*iii*), then the Minkowski functional p_M is a Hilbertian norm. It is easy to check that the following lemma is true.

LEMMA 6. *If $p : X \to R$ is a Hilbertian norm, then the sets $M_1 := \{x \in X : p(x) = 1\}$ and $M_2 := M_1 \cup \{0\}$ are H-smooth whereas $M_3 := \{x \in X : p(x) \leqslant 1\}$ is H-convex.*

The next two theorems yield a completion of Theorem 2 from [2].

THEOREM 7. *Let $M \subset X$ be an absorbent, radially bounded and H-smooth set. Then*

(*a*) $\operatorname{Fr} M$ *and* $\{0\} \cup \operatorname{Fr} M$ *are H-smooth sets;*

(*b*) *if M is an H-convex set, then so are $M \cup \operatorname{Fr} M$ and $M \setminus \operatorname{Fr} M$;*

(*c*) $\operatorname{Bn} M \cup \operatorname{Fr} M$ *and* $\operatorname{Bn} M \setminus \operatorname{Fr} M$ *are H-convex sets;*

(*d*) $\operatorname{Conv} M \cup \operatorname{Fr} M$ *and* $\operatorname{Conv} M \setminus \operatorname{Fr} M$ *are H-convex sets.*

Proof. The Minkowski functional of the set M is a Hilbertian norm. Assertions (*a*) and (*b*) follow immediately from Lemmas 4, 5 and 6. To prove (*c*) and (*d*) let us notice that $\operatorname{Bn} M = \operatorname{Conv} M$ is an absorbent, radially bounded and H-convex set as well as $\operatorname{Fr} \operatorname{Conv} M = \operatorname{Fr} \operatorname{Bn} M = \operatorname{Fr} M$. It remains to use (*b*).

The result below has been obtained in [2] under the additional assumption that M is a symmetric set. We will show that this assumption may be omitted.

THEOREM 8. *If $M \subset X$ is an absorbent, radially bounded set and $\operatorname{Fr} M$ is an H-smooth set, then the Minkowski functional p_M of the set M is a Hilbertian norm.*

Proof. The equivalence of conditions (*i*) and (*ii*) implies that $\{0\} \cup \operatorname{Fr} M$ is an absorbent set and, on account of Proposition 2, this union is H-smooth. Thus, the Minkowski functional $p_{\{0\} \cup \operatorname{Fr} M}$ is a Hilbertian semi-norm. However, $p_{\{0\} \cup \operatorname{Fr} M} = p_M$ (see [2] Lemma 3) and since M is radially bounded, we deal with a norm.

4. *Definition* 4. An absorbent subset M of a space X is said to be strictly convex if and only if it is convex and for any $x, y \in M$, $x \neq y$ and any $t \in (0, 1)$ there is $p_M(tx + (1-t)y) < 1$, where p_M denotes the Minkowski functional of M.

Now we are going to give a necessary and sufficient condition for an H-convex set to be strictly convex. In [2] it has been pointed out that each absorbent, H-smooth and radially bounded set is strictly convex. There exist, however, H-convex and strictly convex sets which are not radially bounded. As an example one can take the set $M := \{(x_1, x_2) \in R^2 : |x_1| < 1\}$.

Definition 5. An absorbent set $M \subset X$ is called strictly H-convex if and only if it is balanced and for any $x, y \in M$, $x \neq y$ and each $\alpha > 0$, $\beta > 0$ there exist numbers $\alpha_0 \geqslant 0$, $\beta_0 \geqslant 0$ fulfilling the conditions:

$$\alpha_0^2 + \beta_0^2 \leqslant 2(\alpha^2 + \beta^2); \tag{1}$$

$$\alpha x + \beta y \in \alpha_0 M; \tag{11}$$

$$\alpha x - \beta y \in \beta_0 M; \tag{12}$$

$$\alpha_0 < \alpha + \beta. \tag{13}$$

Remark 2. Every strictly H-convex set is H-convex.

Proof. Let $M \subset X$ be a strictly H-convex set. It is sufficient to show that M is H-smooth. Take $\alpha > 0$, $\beta > 0$, $x \in \alpha M$, $y \in \beta M$. Then we have $x = \alpha u$, $y = \beta v$ for some $u, v \in M$. If $u \neq v$, there exist numbers $\alpha_0 \geqslant 0$, $\beta_0 \geqslant 0$ fulfilling condition (1) and such that

$$x + y = \alpha u + \beta v \in \alpha_0 M, \; x - y = \alpha u - \beta v \in \beta_0 M.$$

If $u = v$, we put $\alpha_0 := \alpha + \beta$, $\beta_0 := |\alpha - \beta|$ getting

$$\alpha_0^2 + \beta_0^2 = 2(\alpha^2 + \beta^2), \; x + y = (\alpha + \beta) u \in \alpha_0 M,$$

$x - y = (\alpha - \beta) u = \beta_0 \operatorname{sgn}(\alpha - \beta) u \in \beta_0 M$. Thus, our remark is proved.

THEOREM 9. *Let $M \subset X$ be an absorbent set. The following three conditions are equivalent*:

(*a*) *M is a strictly H-convex set*;

(*b*) *M is a strictly convex and H-convex set*;

(*c*) *M is a strictly convex and H-smooth set.*

Proof. (*a*) $\Rightarrow$ (*b*). On account of Remark 2 it suffices to prove that M is strictly convex. The set M, being H-convex, is convex. Take $x, y \in M$, $x \neq y$, $t \in (0, 1)$. From the strict H-convexity of the set M, setting $\alpha := t$, $\beta := 1 - t$, we obtain, in particular, $tx + (1 - t) y = \alpha x + \beta y \in \alpha_0 M$, for some $\alpha_0 \geqslant 0$ with the property $\alpha_0 < \alpha + \beta = 1$. Hence:

$$p_M(tx + (1 - t) y) < 1.$$

(*b*) $\Rightarrow$ (*a*). The set M is balanced. Let us fix $x, y \in M$, $x \neq y$, $\alpha > 0$, $\beta > 0$. Then $\alpha x \in \alpha M$, $\beta y \in \beta M$ and, according to Theorem 1, there exist numbers $\alpha_1 \geqslant 0$, $\beta_0 \geqslant 0$ such that $\alpha_1^2 + \beta_0^2 \leqslant 2(\alpha^2 + \beta^2)$, $\alpha x + \beta y \in \alpha_1 M$, $\alpha x - \beta y \in \beta_0 M$ and $\alpha_1 \leqslant \alpha + \beta$.

Since M is a strictly convex set the following inequality holds:

$$p_M\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) < 1,$$

whence $p_M(\alpha x + \beta y) < \alpha + \beta$. If $\alpha_1 < \alpha + \beta$, we put $\alpha_0 := \alpha_1$. On the other hand, if $\alpha_1 = \alpha + \beta$, we can choose $\alpha_0 > 0$ such that

$$p_M(\alpha x + \beta y) < \alpha_0 < \alpha_1 = \alpha + \beta.$$

In both cases, the numbers α_0 and β_0 fulfil conditions (1), (11), (12) and (13).

The equivalence of conditions (*b*) and (*c*) is a consequence of Theorem 4.

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O H-GLATKIM I H-KONVEKSNIM SKUPOVIMA U LINEARNIM PROSTORIMA

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Sadržaj

U radu se istražuju svojstva H-glatkih i H-konveksnih skupova. Dokazano je da je H-konveksan skup konveksan. Nadalje se proučavaju centrične, balansirane i konveksne ljuske H-glatkih skupova. Dan je nuždan i dovoljan uvjet da H-konveksan skup bude striktno konveksan.