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ON THE PETROVIĆ INEQUALITY FOR CONVEX FUNCTIONS

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Abstract. In this paper we give some generalizations of well-known Petrović's inequality for convex functions.

M. Petrović [14] (see also [10, p. 22]) has proved the following theorem:

THEOREM A. *If f is a convex function on the segment $I=[0, a]$, if $x_i \in I$ $(i=1, \ldots, n)$ and $x_1+\ldots+x_n \in I$, then*

$$f(x_1)+\ldots+f(x_n) \leqq f(x_1+\ldots+x_n)+(n-1) f(0). \tag{1}$$

T. Popoviciu [15] has considered the more general inequality

$$\sum_{i=1}^{n} p_i f(x_i) \leqq f(\sum_{i=1}^{n} p_i x_i)+(\sum_{i=1}^{n} p_i-1) f(0), \tag{2}$$

and he obtained that (2) holds for $p_i>0$ and $x_i \geqq 0$ $(i=1, \ldots, n)$.

P. M. Vasić [17] (see also [18]) showed that the result of T. Popoviciu is not valid, and proved that (2) holds if $p_1 \geqq 1$ and $x_i \geqq 0$ $(1 \leqq i \leqq n)$.

F. Giaccardi [7] has generalized the inequalities (1) and (2). The conditions for validity of such an inequality were weakened by P. M. Vasić and Lj. Stanković [19] (see also [9]). Their result is the following theorem:

THEOREM B. *Let $p_i \geqq 0$ and x_i $(i=1, \ldots, n)$ be real numbers which satisfy*

$$(x_i-x_0)(\sum_{k=1}^{n} p_k x_k-x_i) \geqq 0 \quad (i=1, \ldots, n), \quad \sum_{k=1}^{n} p_k x_k \neq x_0. \tag{3}$$

Then, for every convex function f,

$$\sum_{i=1}^{n} p_i f(x_i) \leqq A f(\sum_{i=1}^{n} p_i x_i)+B(\sum_{i=1}^{n} p_i-1) f(x_0) \tag{4}$$

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where

$$A=(\sum_{i=1}^{n} p_i x_i - x_0 \sum_{i=1}^{n} p_i)/(\sum_{i=1}^{n} p_i x_i - x_0), \quad B=\sum_{i=1}^{n} p_i x_i/(\sum_{i=1}^{n} p_i x_i - x_0). \tag{5}$$

If $x_0 = 0$, then (4) reduces to (2).

In [19], the following result was proved:

THEOREM C. *Let x_i, p_i $(i=1, \dots, n)$ be real numbers such that*

$$x_n(\sum_{i=1}^{n} p_i x_i - x_n) \geqq 0, \; p_n \geqq 0, \; x_n \sum_{i=1}^{n-1} p_i x_i \geqq 0, \; \sum_{i=1}^{n} p_i x_i \neq 0. \tag{6}$$

Then, for every convex function f,

$$P_n(p, x; f) \geqq P_{n-1}(p, x; f) \tag{7}$$

where

$$P_n(p, x; f) = f(\sum_{i=1}^{n} p_i x_i) - \sum_{i=1}^{n} p_i f(x_i) + (\sum_{i=1}^{n} p_i - 1) f(0).$$

For some further generalizations of the inequalities (2) and (4) see [8], [11], [17] and [19].

I. Olkin [12] (see also [4], [5]) has proved the following result:

THEOREM D. *Let $1 \geqq h_1 \geqq \dots \geqq h_n \geqq 0$ and $a_1 \geqq \dots \geqq a_n \geqq \geqq 0$. Let f be a convex function on $[0, a_1]$. Then*

$$(1-\sum_{k=1}^{n}(-1)^{k-1} h_k) f(0) + \sum_{k=1}^{n}(-1)^{k-1} h_k f(a_k) \geqq$$
$$\geqq f(\sum_{k=1}^{n}(-1)^{k-1} h_k a_k). \tag{8}$$

Results, which prevised this inequality ([2], [3], [16], [20], [21]), are given in [10, pp. 112 and 113].

R. E. Barlow, A. W. Marshall and F. Proschan [1] have proved a generalization of Theorem D (see Remark 2).

In [13] the following generalization of Theorem D was given:

THEOREM E. *Let the real numbers $p_1, \dots, p_n$ and $x_1 \geqq \dots \geqq \geqq x_n \geqq x_0 \geqq 0$ satisfy the conditions*

$$0 \leqq P_k \leqq 1 \; (k=1, \dots, n), \quad \sum_{i=1}^{n} p_i x_i > x_0 \; (\text{if } x_0 > 0),$$

where $P_k = \sum_{i=1}^{k} p_i$. If f is a convex function, then the reverse inequality in (4) holds.

In this paper we shall prove, starting with the Fuchs generalization of the Majorization theorem (see [6]) some generalizations of the previous results. Fuchs' result can be written in the following form:

THEOREM F. *Let* $a_1 \geqq \ldots \geqq a_s$, $b_1 \geqq \ldots \geqq b_s$ *and* $q_1, \ldots, q_s$ *be real numbers such that*

$$\sum_{i=1}^{k} q_i a_i \leqq \sum_{i=1}^{k} q_i b_i \ (k = 1, \ldots, s-1), \ \sum_{i=1}^{s} q_i a_i = \sum_{i=1}^{s} q_i b_i,$$

hold. Then, for every convex function f, *the inequality*

$$\sum_{s=1}^{s} q_i f(a_i) \leqq \sum_{i=1}^{s} q_i f(b_i) \tag{9}$$

is valid.

Now, we shall prove the following theorem:

THEOREM 1. *Let* $x_1 \geqq \ldots \geqq x_m \geqq x_0 \geqq x_{m+1} \geqq \ldots \geqq x_n$ $(m \in \in (0, 1, \ldots, n))$ *and* $p_1, \ldots, p_n$ *be real numbers such that* $x_i \in I$ $(i = = 0, 1, \ldots, n$; I *is an interval in* $R)$, $\sum_{i=1}^{n} p_i x_i \in I$ *and*

$$\sum_{i=1}^{k} p_i \Big(\sum_{r=1}^{n} p_r x_r - x_i \Big) \geqq 0 \ (1 \leqq k \leqq m), \ \sum_{i=k}^{n} p_i \Big(\sum_{r=1}^{n} p_r x_r - x_i \Big) \leqq 0 \tag{10}$$

$$(m + 1 \leqq k \leqq n).$$

Then, for every convex function f *on* I, *the inequality* (4) *holds. If the reverse inequalities hold in* (10), *then the reverse inequality in* (4) *holds.*

Proof. By substitutions

$$s = n + 1; \ a_i = x_i, \ q_i = p_i (i = 1, \ldots, m); \ a_{m+1} = x_0, \ q_{m+1} = B(1 - P_n);$$

$$a_i = x_{i-1}, \ q_i = p_{i-1} \ (i = m + 2, \ldots, n + 1);$$

$$b_i = \sum_{i=1}^{n} p_i x_i \quad (i = 1, \ldots, n + 1);$$

and

$$s = n + 1; \ a_i = \sum_{i=1}^{n} p_i x_i \quad (i = 1, \ldots, n + 1); \ b_i = x_i, \ q_i = p_i (i = = 1, \ldots, m); \ b_{m+1} = x_0, \ q_{m+1} = B(1 - P_n); \ b_i = x_{i-1}, \ q_i = p_{i-1}$$

$$(i = m + 2, \ldots, n + 1);$$

from Theorem F, we get Theorem 1.

Remark 1. For $x_1 \geqq \ldots \geqq x_n \geqq x_0$ the conditions (10) become

$$\sum_{i=1}^{k} p_i \left(\sum_{r=1}^{n} p_r x_r - x_i \right) \geqq 0 \quad (k = 1, \ldots, n), \tag{11}$$

and for $x_0 \geqq x_1 \geqq \ldots \geqq x_n$

$$\sum_{i=k}^{n} p_i \left(\sum_{r=1}^{n} p_r x_r - x_i \right) \leqq 0 \quad (k = 1, \ldots, n). \tag{12}$$

If $p_i \geqq 0$ $(i = 1, \ldots, n)$, the conditions (11) and (12) can be replaced by the conditions $\sum_{i=1}^{n} p_i x_i \geqq x_1$ and $\sum_{i=1}^{n} p_i x_i \leqq x_n$, which are completely equivalent to the condition (3).

From Theorem 1, for $x_0 = 0$ we get:

THEOREM 2. *Let the real numbers* $x_1 \geqq \ldots \geqq x_m \geqq 0 \geqq \geqq x_{m+1} \geqq \ldots \geqq x_n$ $(m \in (0, 1, \ldots, n))$ *and* $p_1, \ldots, p_n$ *satisfy the conditions* $x_i \in I$ $(i = 1, \ldots, n; 0 \in I)$, $\sum_{i=1}^{n} p_i x_i \in I$ *and* (10). *Then, for every convex function* f *on* I, *the inequality* (2) *holds. If the reverse inequalities in* (10) *hold, then the reverse inequality in* (2) *is valid.*

Now, we shall give some special cases of Theorems 1 and 2.

THEOREM 3. *Let* $x_1 \geqq \ldots \geqq x_m \geqq x_0 \geqq x_{m+1} \geqq \ldots \geqq x_n$ $(m \in \in (0, 1, \ldots, n))$ *be real numbers such that* $x_i \in I$ $(i = 0, 1, \ldots, n)$, $x_m \geqq 0$ *and* $x_{m+1} \leqq 0$.

(*a*) *If* p *is real n-tple such that*

$$0 \leqq P_k \leqq 1 \ (k = 1, \ldots, m), \ 0 \leqq \overline{P}_k \leqq 1 \ (k = m = 1, \ldots, n)$$
$$(\overline{P}_k = P_n - P_{k-1}), \tag{13}$$

then for every convex function $f : I \to R$, *the reverse inequality in* (4) *holds.*

(*b*) *If* $\sum_{i=1}^{n} p_i x_i \in I$ *and if either*

$$P_k \geqq 1 \ (k = 1, \ldots, m), \ \overline{P}_k \leqq 0 \ (k = m + 1, \ldots, n); \tag{14}$$

or

$$P_k \leqq 0 \ (k = 1, \ldots, m), \ \overline{P}_k \geqq 1 \ (k = m + 1, \ldots, n); \tag{15}$$

then (4) *holds.*

Proof. (*a*) For $k = 1, \ldots, m$ we have

$$\sum_{i=1}^{k} p_i \left(\sum_{r=1}^{n} p_r x_r - x_i \right) = \left(\sum_{r=1}^{n} p_r x_r - x_k \right) P_k + \sum_{i=1}^{k-1} P_i (x_{i+1} - x_i) =$$

$$= P_k \left(x_m P_n - x_k + \sum_{i=1}^{m-1} P_i (x_i - x_{i+1}) + \sum_{i=m+1}^{n} \overline{P}_i (x_i - x_{i-1}) \right) +$$

$$+ \sum_{i=1}^{k-1} P_i (x_{i+1} - x_i) = P_k (x_m P_n - x_k) + (1 - P_k) \sum_{i=1}^{k-1} P_i (x_{i+1} - x_i) +$$

$$+ P_k \sum_{i=k}^{m-1} P_i (x_i - x_{i+1}) + P_k \sum_{i=m+1}^{n} \overline{P}_i (x_i - x_{i-1}) =$$

$$= P_k (P_n - 1) x_m + (1 - P_k) \sum_{i=1}^{k-1} P_i (x_{i+1} - x_i) +$$

$$+ P_k \sum_{i=k}^{m-1} (P_i - 1)(x_i - x_{i+1}) + P_k \sum_{i=m+1}^{n} \overline{P}_i (x_i - x_{i-1}) = P_k (\overline{P}_{m+1} x_{m+1} +$$

$$+ (P_m - 1) x_m) + (1 - P_k) \sum_{i=1}^{k-1} P_i (x_{i+1} - x_i) +$$

$$+ P_k \sum_{i=k}^{m-1} (P_i - 1)(x_i - x_{i+1}) + P_k \sum_{i=m+2}^{n} \overline{P}_i (x_i - x_{i-1}) \leqq 0.$$

Analogously, for $k = m + 1, \ldots, n$ we have

$$\sum_{i=k}^{n} p_i \left(\sum_{r=1}^{n} p_r x_r - x_i \right) = \overline{P}_k (x_{m+1} (\overline{P}_{m+1} - 1) + x_m P_m) +$$

$$+ \overline{P}_k \sum_{i=1}^{m-1} P_i (x_i - x_{i+1}) + \overline{P}_k \sum_{i=m+2}^{k} (\overline{P}_i - 1)(x_i - x_{i-1}) +$$

$$+ (1 - \overline{P}_k) \sum_{i=k+1}^{n} \overline{P}_i (x_{i-1} - x_i) \geqq 0.$$

Using (13) we also have $x_1 \geqq \sum_{i=1}^{m} p_i x_i \geqq 0$ and $0 \geqq \sum_{i=m+1}^{n} p_i x_i \geqq \geqq x_n$ wherefrom we have $x_1 \geqq \sum_{i=1}^{n} p_i x_i \geqq x_n$.

Using the above identities we can prove (*b*).

From Theorem 3, for $x_0 = 0$ we get:

THEOREM 4. *Let* $x_1 \geqq \ldots \geqq x_m \geqq 0 \geqq x_{m+1} \geqq \ldots \geqq x_n$ $(m \in \in (0, 1, \ldots, n))$, $x_i \in I$ $(i = 1, \ldots, n)$ *and* $p_1, \ldots, p_n$ *be real numbers.*

(*i*) *If* (13) *holds, then for every convex function* $f: I \to R$ *the reverse inequality in* (2) *is valid.*

(*ii*) *If the conditions of Theorem* 3 (*b*) *are fulfilled, then* (2) *holds.*

Remark 2. In [1], the following result was given:

Let $x_1 \leqq \ldots \leqq x_m \leqq 0 \leqq x_{m+1} \leqq \ldots \leqq x_n$. (*a*) Reverse inequality holds in (2), if and only if (13) holds. (*b*) (2) holds if and only if there exists $j \leqq m$ such that $P_i \leqq 0$, $i < j$; $P_i \geqq 1$, $j \leqq i \leqq m$, $\bar{P}_i \leqq 0$, $i \geqq m+1$, or there exists $j \geqq m$ such that $P_i \leqq 0$, $i \leqq m$; $\bar{P}_i \geqq 1$, $m+1 \leqq i \leqq j$; $\bar{P}_i \leqq 0, i > j$.

For $j = 1$ and $j = n$, from (*b*), we get (*ii*) from Theorem 4.

Using Theorem 2, we can extend the conditions under which the inequality (7) holds. Namely, from it we obtain that inequality

$$f(q_1 a_1 + q_2 a_2) - q_1 f(a_1) - q_2 f(a_2) + (q_1 + q_2 - 1) f(0) \geqq 0 \quad (16)$$

holds if:

$a_1 \geqq a_2 \geqq 0$ and

$$q_1 (q_1 a_1 + q_2 a_2 - a_1) \geqq 0, \; q_1 (q_1 a_1 + q_2 a_2 - a_1) + q_2 (q_1 a_1 + q_2 a_2 - a_2) \geqq 0; \quad (17)$$

or $0 \geqq a_1 \geqq a_2$ and

$$q_2 (q_1 a_1 + q_2 a_2 - a_2) \leqq 0, \; q_1 (q_1 a_1 + q_2 a_2 - a_1) + q_2 (q_1 a_1 + q_2 a_2 - a_2) \leqq 0; \quad (18)$$

or $a_1 \geqq 0 \geqq a_2$ and

$$q_1 (q_1 a_1 + q_2 a_2 - a_1) \geqq 0, \quad q_2 (q_1 a_1 + q_2 a_2 - a_2) \leqq 0. \quad (19)$$

The reverse inequality in (16) holds if in (17), (18) and (19) the reverse inequalities holds.

Using the substitutions: $a_1 = x_n$, $a_2 = \sum_{i=1}^{n-1} p_i x_i$, $q_1 = p_n$, $q_2 = 1$; and $a_1 = \sum_{i=1}^{n-1} p_i x_i$, $a_2 = x_n$, $q_1 = 1$, $q_2 = p_n$, we get that (7) holds if

$$\sum_{i=1}^{n} p_i x_i \geqq x_n \geqq 0, \; \sum_{i=1}^{n} p_i x_i \geqq \sum_{i=1}^{n-1} p_i x_i \geqq 0; \quad (20)$$

or

$$x_n \geqq 0 \geqq \sum_{i=1}^{n} p_i x_i, \; x_n \geqq \sum_{i=1}^{n-1} p_i x_i \geqq \sum_{i=1}^{n} p_i x_i; \quad (21)$$

or if either of the conditions (20) and (21) hold, with the reverse inequalities.

The reverse inequality in (7) holds if

$$\sum_{i=1}^{n-1} p_i x_i \geqq x_n \geqq 0, \sum_{i=1}^{n-1} p_i x_i = \sum_{i=1}^{n} p_i x_i \geqq 0; \tag{22}$$

or

$$x_n \geqq 0 \geqq \sum_{i=1}^{n-1} p_i x_i, \; x_n \geqq \sum_{i=1}^{n} p_i x_i \geqq \sum_{i=1}^{n-1} p_i x_i; \tag{23}$$

or if either of the conditions (22) and (23) hold, with the reverse inequalities.

By combining the condition (21) and the corresponding condition with the reverse inequalities, we obtain (6).

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O PETROVIĆEVOJ NEJEDNAKOSTI ZA KONVEKSNE FUNKCIJE

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Sadržaj

U članku je dokazana slijedeća generalizacija poznate Petrovićeve nejednakosti za konveksne funkcije

TEOREMA 1. *Neka su* $x_1 \geqq \ldots \geqq x_m \geqq x_0 \geqq x_{m+1} \geqq \ldots \geqq x_n$ $(m \in (0,1, \ldots, n))$ *i* $p_1, \ldots, p_n$ *takvi realni brojevi da* $x_i \in I$ $(i=0,\ 1, \ldots, n)$, $\sum_{i=1}^{n} p_i x_i \in I$ *i da važi* (10). *Tada za svaku funkciju* f *konveksnu na* I *važi* (4), *gdje su* A *i* B *definisani sa* (5). *Ako u* (10) *važe suprotne nejednakosti, tada suprotna nejednakost važi i u* (4).

U specijalnom slučaju dobija se:

TEOREMA 3. *Neka su* $x_1 \geqq \ldots \geqq x_m \geqq x_0 \geqq x_{m+1} \geqq \ldots \geqq x_n$ $(m \in (0,\ 1, \ldots, n))$ *takvi realni brojevi da* $x_i \in I\ (i=0,\ 1, \ldots, n)$, $x_m \geqq 0$, $x_{m+1} \leqq 0$.

(*a*) *Ako je* p *takva realna* n*-torka da važi* (13), *tada za svaku konveksnu funkciju* $f: I \to R$, *važi suprotna nejednakost u* (4)

(*b*) *Ako* $\sum_{i=1}^{n} p_i x_i \in I$ *i ako važi ili* (14) *ili* (15) *tada važi* (4).

Za $x_0 = 0$, iz Teorema 1 i 3 dobijaju se respektivno Teoreme 2 i 4, a takođe su dobijeni neki rezultati koji se odnose na rafiniranje Petrovićeve nejednakosti.