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A HOMOTOPY THEORY FOR GRAPHS

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Abstract. Some concepts and theorems are transferred from algebraic topology to graph theory and put in a combinatorial setting. The first part of this paper deals with combinatorial analogues to continuity, homotopy, path homotopy, contraction, retraction, fundamental group etc. The last analogue (called string fundamental group of a graph) is investigated in greater detail. It is shown that this group is isomorphic to the usual fundamental group of the graph if the girth is at least 5. The second part of this paper deals with string connected graphs, i. e., connected graphs with trivial string fundamental group. These graphs are characterized by embedding them in a special way into pseudosurfaces. Furthermore, combinatorial analogues to deformation retraction, homotopy equivalence etc. are developed.

PART ONE

1. Introduction

The aim of this paper is to transfer some concepts and theorems from algebraic topology to graph theory and to give them a *combinatorial* nature. There are some papers with similar ideas (see e. g. [5]—[8], [11]—[14]). In particular homotopy theories for graphs can be developed in several manners. One possibility is to assign to a graph X a topological space $T(X)$ in the following way: Let to each edge of the graph X correspond a 1-dimensional simplex in some R^n. Identify the vertices of the simplex according to the graph X and provide the union of the simplexes with the so called »weak topology« (a subset being open if the intersection with each simplex is open: see [9, p. 246]). Then one can apply the usual topological homotopy theory to $T(X)$. Dörfler [3] developed a homotopy theory for hypergraphs. He assigned in an analogous manner to each hypergraph X a topological space $T(X)$. His homotopy concept is not the topological one (it is of a combinatorial nature too), but closely related to the topological concept. The homotopy concept proposed in the present paper seems to be quite natural for graphs, but is in some details not as closely related to the topological concept as Dörfler's one.

The first part of this paper deals with combinatorial analogues to the topological concepts of continuity, homotopy, path homotopy, contraction, retraction, fundamental group etc. The second part will deal with analogues to simply connected spaces, deformation retraction, homotopy equivalence etc.

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We consider only finite graphs without loops and without multiple edges. The vertex set of a graph X is denoted by $V(X)$, the edge set by $E(X)$. If two vertices x and y are adjacent, we write $x \sim y$. We denote the edge connecting the vertices x and y with xy or sometimes $[x, y]$. Quite often we use the sets $N = \{0, 1, 2, \ldots\}$ and $N_n = \{0, 1, 2, \ldots, n\}$. For further graph theoretical terminology see [1] or [4], for algebraic topological terminology see e. g. [2], [9] or [10].

Let us take the following *gangster problem* as the starting point for our considerations: Suppose the vertices of a graph to be towns and the edges roads. In each town there is a member of a gangster syndicate. The gangsters decide to meet in one of the towns. For safety reasons they decide: Each day they will move from one town to an adjacent one or rest in the same town and if two of the gangsters are in adjacent towns originally, then at all steps of the journey these gangsters must be in adjacent towns or be in the same town. The problem is: For which graphs is it possible for the gangsters to meet in one of the towns?

Examples. A meeting is possible in the graph X as indicated in Fig. 1a. Obviously it is not possible in the graph C_5 (Fig. 1b).

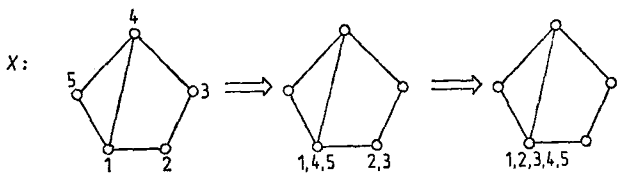

Fig. 1a

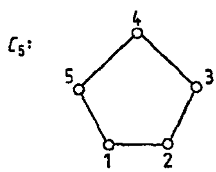

Fig. 1b

In the following the ideas of this problem and related ideas will be treated in a more formal way.

2. Net homotopy and net contraction

We begin by defining a graph theoretical version of continuous maps:

Let X, Y be graphs. A map $f : V(X) \to V(Y)$ is called a ***homomorphism*** from X to Y, if

$$x \sim y \Rightarrow f(x) \sim f(y) \quad \text{or} \quad f(x) = f(y)$$

for all $x, y \in V(X)$. Instead of $f: V(X) \to V(Y)$ we hereafter write $f: X \to Y$.

Note that the word »homomorphism« is used in the graph theoretic literature in different senses. (See e. g. [4] and [8]).

If $f: X \to Y$ and $f^{-1}: Y \to X$ are homomorphisms, then f is an isomorphism between the graphs X and Y. Thus isomorphism between graphs corresponds to homeomorphism between topological spaces.

Let f and g be homomorphisms from X to Y. A map $H\colon V(X) \times N_n \to V(Y)$ $(n \in N, n \geqslant 1)$ is called a *net homotopy from f to g*, if

(1) $x \sim y \Rightarrow H(x, t) \sim H(y, t)$ or $H(x, t) = H(y, t)$ for all $x, y \in V(X)$ and all $t \in N_n$,

(2) $H(x, t) \sim H(x, t+1)$ or $H(x, t) = H(x, t+1)$ for all $x \in V(X)$ and all $t \in N_{n-1}$,

(3) $H(x, 0) = f(x)$ and $H(x, n) = g(x)$ for all $x \in V(X)$.

If such a map exists, we call f *net homotopic to g* and write $f \simeq g$.

In this definition the conditions (1) and (2) again are graph theoretical versions of continuity. The set N_n can be regarded as a set of finitely many »points of time« and is used instead of the unit interval in algebraic topology.

The proof of the following proposition is left to the reader:

PROPOSITION 1. *The net homotopy relation $\simeq$ is an equivalence relation on the set of homomorphisms from a graph X to a graph Y.*

A graph X is said to be *net contractible to $x_0 \in V(X)$*, if the identity homomorphism $1_X: X \to X$ is net homotopic to the constant homomorphism $c_{x_0}: X \to X$ defined by $c_{x_0}(x) = x_0$ for all $x \in V(X)$. In other words: X is net contractible to x_0 if there exists a net homotopy $H\colon V(X) \times N_n \to V(X)$ such that $H(x, 0) = x$ and $H(x, n) = x_0$ for all $x \in V(X)$. The net homotopy H is called a *net contraction of X to x_0*. X is said to be *net contractible* if there exists a vertex $x_0 \in V(X)$ such that X is net contractible to x_0.

Examples. The graph X in Figure 1a is net contractible. A net contraction of X is indicated in the figure. Instead of writing $H(x, 0)$, $H(x, 1)$, $H(x, 2)$, ... in the figure each label x is moved. The graph C_5 in Figure 1b is not net contractible.

Our »gangster problem« now can be formulated: Characterize all net contractible graphs!

THEOREM 1. *Let X be net contractible to $x_0 \in V(X)$ and let y be any vertex of X. Then X is net contractible to y.*

Proof. Let $H: V(X) \times N_n \to V(X)$ be a net contraction of X to $x_0 \in V(X)$. Because X is net contractible, X is connected and therefore there is a path $x_0 x_1 x_2 \dots x_m (= y)$. Intuitively speaking we first move all labels to x_0 and then along this path to y. Formally we define a net contraction $K: V(X) \times N_{n+m} \to V(X)$ of X to y by

$$K(x, t) = \begin{cases} H(x, t) & \text{if } 0 \leqslant t \leqslant n, \\ x_{t-n} & \text{if } n \leqslant t \leqslant n+m. \end{cases}$$

Let Y be a subgraph of a graph X. A *net retraction* of X onto Y is a homomorphism $f: X \to Y$ such that $f(y) = y$ for all $y \in V(Y)$. When some net retraction of X onto Y exists, Y is called a *net retract* of X.

THEOREM 2. *A net retract of a net contractible graph is itself net contractible.*

Proof. Let $H: V(X) \times N_n \to V(X)$ be a net contraction of the graph X to a vertex $x_0 \in V(X)$ and let $f: X \to Y$ be a net retraction of X onto a subgraph Y. Define a map $K: V(Y) \times N_n \to V(Y)$ by

$$K(x, t) = f(H(x, t)).$$

We show that K is a net contraction of Y to $f(x_0) \in V(Y)$. In order to prove this we must show that K is a net homotopy from the identity homomorphism $1_Y: Y \to Y$ to the constant homomorphism $c_{f(x_0)}: Y \to Y$. It is left to the reader to check the conditions (1), (2), (3) in the definition of a net homotopy.

3. Strings and string homotopy

We define a graph P_m by $V(P_m) = N_m$ and $E(P_m) = \{[i, i+1] \mid i \in N_{m-1}\}$, if $m \geqslant 1$, and $E(P_m) = \emptyset$, if $m = 0$. Here $[i, i+1]$ denotes the edge joining the vertices i and $i+1$. P_m is a path in the ordinary graph theoretical sense. Different from this is the topological concept of a path, which is a continuous map from the unit interval into a topological space. We define a graph theoretical analogue to this concept:

A *string in a graph* X is a homomorphism $\sigma: P_m \to X$ $(m \in N)$. An *elementary extension* of σ is a string $\sigma^*: P_{m+g} \to X$ $(g \in N)$ defined by

$$\sigma^*(i) = \begin{cases} \sigma(i) & \text{if } 0 \leqslant i \leqslant k, \\ \sigma(k) & \text{if } k \leqslant i \leqslant k+g \\ \sigma(i-g) & \text{if } k+g \leqslant i \leqslant m+g \end{cases}$$

for a fixed $k \in N_m$. A string σ^e constructed from σ by a sequence of elementary extensions is called an *extension* of σ.

Examples. Figure 2a shows a string σ in a graph X, Figure 2b an elementary extension σ^* of σ and Figure 2c an extension σ^e of σ. Intuitively speaking one gets an elementary extension of σ by inserting g vertices in P_m between k and $k+1$, whereby these g vertices get the same image as k.

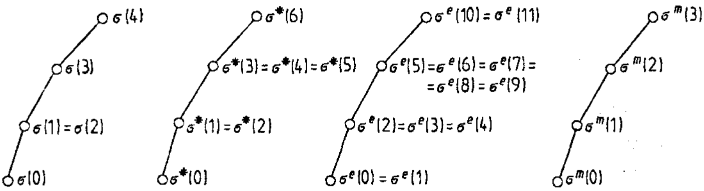

Fig. 2a Fig. 2b Fig. 2c Fig. 2d

If τ is an extension of σ then we call σ a *reduction* of τ. If σ^m is a reduction of σ such that there is no reduction ϱ of σ^m with $\varrho \neq \sigma^m$ then we call σ^m a *minimal reduction* of σ.

Examples. The string σ in Figure 2a is a reduction of σ^* in Figure 2b and a reduction of σ^e in Fig. 2c. A minimal reduction σ^m of σ is shown in Figure 2d.

Obviously to each string σ in X there is a unique minimal reduction σ^m. The relation »is an extension of« is a partial order on the set of all extensions of a fixed string σ in X. The set of extensions of σ forms a lattice with respect to this relation. σ^m is the least element of this lattice.

Now we define an analogue to the topological concept of path homotopy: Two strings $\sigma : P_k \to X$ and $\tau : P_r \to X$ are called *string homotopic*, denoted $\sigma \overset{s}{\simeq} \tau$, provided that $\sigma(0) = \tau(0)$, $\sigma(k) = \sigma(r)$ and that there is a net homotopy $H : V(P_m) \times N_n \to V(X)$ from an extension $\sigma^e : P_m \to X$ to an extension $\tau^e : P_m \to X$ such that $H(0, t) = \sigma(0) = \tau(0)$ and $H(m, t) = \sigma(k) = \tau(r)$ for all $t \in N_n$.

Examples. The strings $\sigma : P_4 \to X$ and $\tau : P_3 \to X$ shown in Figure 3a are string homotopic, because $\sigma(0) = \tau(0)$, $\sigma(4) = \tau(3)$ and there are extensions $\sigma^e : P_4 \to X$, $\tau^e : P_4 \to X$ (shown in Figure 3b) and a net homotopy $H : V(P_4) \times N_2 \to V(X)$ from σ^e to τ^e (shown in Figure 3c) such that $H(0, t) = \sigma(0) = \tau(0)$ and $H(4, t) = \sigma(4) = \tau(3)$ for all $t \in N_2$.

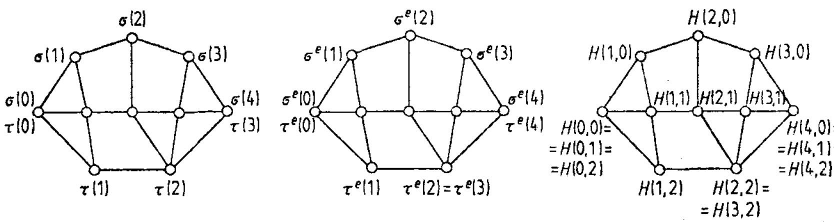

Fig. 3a Fig. 3b Fig. 3c

LEMMA 1. *Let* $\sigma : P_m \to X$ *and* $\tau : P_m \to X$ *be strings in a graph* X *with* $\sigma \simeq \tau$ *and* $H : V(P_m) \times N_n \to V(X)$ *a net homotopy from* σ *to* τ *with* $H(0, t) = \sigma(0) = \tau(0)$, $H(m, t) = \sigma(m) = \tau(m)$ *for all* $t \in N_n$. *If* $\sigma^e : P_r \to X$ *is an extension of* σ *then there exists a net homotopy* $H^e : V(P_r) \times N_n \to V(X)$ *from* σ^e *to some extension* τ^e *of* τ *such that* $H^e(0, t) = \sigma(0) = \tau(0)$ *and* $H^e(r, t) = \sigma(m) = \tau(m)$ *for all* $t \in N_n$.

Proof. Let σ^* be an elementary extension of $\sigma : P_m \to X$ as defined before. Intuitively speaking one gets a net homotopy H^* from σ^* to some elementary extension τ^* of τ by defining the labels of the inserted g vertices to be the label of the vertex k at all steps of the homotopy H. Formally we define $H^* : V(P_{m+g}) \times N_n \to V(X)$ by

$$H^*(i, t) = \begin{cases} H(i, t) & \text{if } 0 \leqslant i \leqslant k, \\ H(k, t) & \text{if } k + 1 \leqslant i \leqslant k + g, \\ H(i - g, t) & \text{if } k + g \leqslant i \leqslant m + g. \end{cases}$$

It is easy to see that H^* is a net homotopy from σ^* to some elementary extension τ^* of τ and that $H^*(0, t) = \sigma(0) = \tau(0)$ and $H^*(m + g, t) = \sigma(m) = \tau(m)$. By repeated application of this construction one gets a net homotopy $H^e : V(P_r) \times N_n \to V(X)$ from σ^e to some extension τ^e of τ such that $H^e(0, t) = \sigma(0) = \tau(0)$ and $H^e(r, t) = \sigma(m) = \tau(m)$.

PROPOSITION 2. *String homotopy* $\stackrel{s}{\simeq}$ *is an equivalence relation on the set of all strings in a graph* X.

Proof. Reflexivity and symmetry are clear. So we only have to show the transitivity. Let $\sigma : P_k \to X, \tau : P_m \to X$ and $\omega : P_n \to X$ be strings in X with $\sigma \stackrel{s}{\simeq} \tau$ anv $\tau \stackrel{s}{\simeq} \omega$. Then $\sigma(0) = \tau(0)$, $\sigma(k) = \tau(m)$ and $\tau(0) = \omega(0), \tau(m) = \omega(n)$. From this it follows that $\sigma(0) = \omega(0)$ and $\sigma(k) = \omega(n)$. Because $\tau \stackrel{s}{\simeq} \sigma$ there exists a net homotopy $H : V(P_r) \times N_p \to V(X)$ from an extension τ' of τ to an extension σ' of σ such that $H(0, t) = \tau(0) = \sigma(0)$ and $H(r, t) = \tau(m) = \sigma(k)$ for all $t \in N_p$. Because $\tau \stackrel{s}{\simeq} \omega$ there exists a net homotopy $K : V(P_s) \times N_q \to V(X)$ from an extension τ'' of τ to an extension ω'' of ω such that $K(0, t) = \tau(0) = \omega(0)$ and $K(s, t) = \tau(m) = \omega(n)$ for all $t \in N_q$. Let $\tau^e : P_u \to X$ be defined as sup $\{\tau', \tau''\}$ (in the lattice formed by all extensions of τ). By Lemma 1, there exists a net homotopy $H^e : V(P_u) \times N_p \to V(X)$ from τ^e to some extension σ^e of σ with $H^e(0, t) = \tau(0) = \sigma(0)$ and $H^e(u, t) = \tau(m) = \omega(n)$ for all $t \in N_p$ and there exists a net homotopy $K^e : V(P_u) \times N_q \to V(X)$ from τ^e to some extension ω^e of ω with $K^e(0, t) = \tau(0) = \omega(0)$ and $K^e(u, t) = \tau(m) = \omega(n)$ for all $t \in N_q$. We define a map $L^e : V(P_u) \times N_{p+q} \to V(X)$ by

$$L^e(i,t) = \begin{cases} H^e(i, p-t) & \text{if } 0 \leqslant t \leqslant p, \\ K^e(i, t-p) & \text{if } p \leqslant t \leqslant p+q. \end{cases}$$

It is easy to see that L^e is a net homotopy from σ^e to ω^e with $L^e(0,t) = \sigma(0) = \omega(0)$ and $L^e(u,t) = \sigma(k) = \omega(n)$ for all $t \in N_{p+q}$. Hence $\sigma \overset{s}{\simeq} \omega$.

4. The string fundamental group of a graph

If $\sigma : P_k \to X$ and $\tau : P_m \to X$ are strings in a graph X with $\sigma(k) = \tau(0)$, then their *string product* is a string $\sigma * \tau : P_{k+m} \to X$ defined by

$$\sigma * \tau(i) = \begin{cases} \sigma(i) & \text{if } 0 \leqslant i \leqslant k, \\ \tau(i-k) & \text{if } k \leqslant i \leqslant k+m. \end{cases}$$

It is easy to see that $(\sigma * \tau) * \omega = \sigma * (\tau * \omega)$, if both sides are defined. Therefore we are allowed to write $\sigma * \tau * \omega$.

A *string loop* in a graph X at $x_0 \in V(X)$ is a string $\alpha : P_m \to X$ with $\alpha(0) = \alpha(m) = x_0$. The vertex x_0 is referred to as the *base vertex* of α. Two string loops α and β in X having common base vertex are *string homotopic modulo* x_0, denoted $\alpha \simeq_{x_0} \beta$, provided that they are string homotopic as strings.

Since no ambiguities will occur we will write $\alpha \simeq_{x_0} \beta$ instead of $\alpha \overset{s}{\simeq}_{x_0} \beta$.

The proof of the following lemma is left to the reader:

LEMMA 2. *Let $\alpha, \alpha', \beta, \beta'$ be string loops in a graph X at $x_0 \in V(X)$. If $\alpha \simeq_{x_0} \alpha'$, and $\beta \simeq_{x_0} \beta'$, then $\alpha * \beta \simeq_{x_0} \alpha' * \beta'$.*

It is evident that string homotopy modulo x_0 is an equivalence relation on the set of all string loops in X with base vertex x_0. Therefore this set is partitioned into equivalence classes. The equivalence class of the string loop α at x_0 is denoted by $[\alpha]$ and is called the *string homotopy class* of α. The set of all such string homotopy classes is denoted by $S(X, x_0)$. If $[\alpha], [\beta] \in S(X, x_0)$, then the *product* $[\alpha] \circ [\beta]$ is defined as follows:

$$[\alpha] \circ [\beta] = [\alpha * \beta].$$

(Lemma 2 insures that $[\alpha] \circ [\beta]$ is well defined.) A *null loop* at x_0 is a constant string loop $\nu : P_m \to X$ defined by $\nu(i) = x_0$ for all $i \in N_m$.

Note that there are infinitely many null loops at x_0 (contrary to algebraic topology). But all null loops at x_0 are string homotopic modulo x_0 and hence yield the same string homotopy class.

If $\sigma : P_m \to X$ is a string in X, then the string $\overline{\sigma} : P_m \to X$ defined by $\overline{\sigma}(i) = \sigma(m - i)$ for $i \in N_m$ is called the *reverse string* of σ.

THEOREM 3. *The set $S(X, x_0)$ is a group under the operation $\circ$.*

Proof. The details are left to the reader. We only make some remarks concerning the neutral element and the inverse elements. If $\alpha \in S(X, x_0)$ and ν is an arbitrary null loop at x_0, then $\alpha * \nu$ is an extension of α and therefore $\alpha * \nu \simeq_{x_0} \alpha$. Thus $[\alpha] \circ [\nu] = [\alpha]$.
If $\alpha \in S(X, x_0)$, $\alpha : P_m \to X$ and ν is an arbitrary null loop at x_0, then there exists a net homotopy $H : V(P_{2m}) \times N_m \to V(X)$ from $\alpha * \overline{\alpha}$ to an extension $\nu^e : P_{2m} \to X$ of ν given by

$$H(i, t) = \begin{cases} \alpha(\max\{0, i - t\}) & \text{if } 0 \leqslant i \leqslant m, \\ \overline{\alpha}(\min\{m, i + t\}) & \text{if } m \leqslant i \leqslant 2m. \end{cases}$$

Obviously $H(0, t) = H(2m, t) = x_0$ for all $t \in N_m$. Thus $\alpha * \overline{\alpha} \simeq_{x_0} \nu$ and therefore $[\alpha] \circ [\overline{\alpha}] = [\nu]$.
The group $S(X, x_0)$ with the operation $\circ$ is called the *string fundamental group* of X at x_0.

THEOREM 4. *If a graph X is connected and x_0, x_1 are vertices of X, then the string fundamental groups $S(X, x_0)$ and $S(X, x_1)$ are isomorphic.*

Proof. Because X is connected, there exists a string $\sigma : P_m \to X$ with $\sigma(0) = x_0$ and $\sigma(m) = x_1$. If α is a string loop at x_0, then $\overline{\sigma} * \alpha * \sigma$ is a string loop at x_1. We define a function $\hat{\sigma} : S(X, x_0) \to S(X, x_1)$ by

$$\hat{\sigma}([\alpha]) = [\overline{\sigma} * \alpha * \sigma] \quad \text{for} \quad [\alpha] \in S(X, x_0).$$

It is easy to see that $\hat{\sigma}([\alpha])$ is independent of the choice of representative from $[\alpha]$ and therefore $\hat{\sigma}$ is well defined. It is left to the reader to show that $\hat{\sigma}$ is an isomorphism from $S(X, x_0)$ to $S(X, x_1)$. (Compare [2], p. 67, Theorem 4.3.)

Because of Theorem 4, we sometimes omit the base vertex in the notation for the string fundamental group of a connected graph X and speak of the string fundamental group $S(X)$. But as in algebraic topology, Theorem 4 does not guarantee that the isomorphism between $S(X, x_0)$ and $S(X, x_1)$ is unique. Different strings from x_0 to x_1 may lead to different isomorphisms.

In the following we assign to a graph X a topological space $T(X)$ as described in the introduction. The vertices of X and the corresponding points in $T(X)$ are denoted by the same symbol. Let $G(T(X))$ be the fundamental group of $T(X)$.

In which way is $S(X)$ related to $G(T(X))$? In order to give a (partial) answer to this question, we need some further definitions.

By I we denote the interval $[0, 1]$ in R^1 and by $\langle a, b\rangle$ the simplex in $R^n\,(n \geqslant 1)$ with vertices a, b. If $H : V(P_m) \times N_n \to V(X)$ is a net homotopy from a string $\sigma : P_m \to X$ to a string $\tau : P_m \to X$, then let $p_{i,t} : I \to T(X)$ be a path from $H(i, t)$ to $H(i + 1, t)$ $(0 \leqslant i \leqslant m - 1)$ which is chosen in such a way that

$$p_{i,t}(I) = \begin{cases} \langle H(i, t), H(i + 1, t)\rangle & \text{if } H(i, t) \neq H(i + 1, t), \\ \{H(i, t)\} & \text{if } H(i, t) = H(i + 1, t). \end{cases}$$

If $p : I \to T(X)$ and $q : I \to T(X)$ are paths in $T(X)$ with $p(1) = q(0)$, we denote their path product by $p * q$. (Compare [2], p. 63.)

LEMMA 3. *Let X be a graph with girth $\geqslant 5$ and $\sigma : P_m \to X$ and $\tau : P_m \to X$ string loops in X at $x_0 \in V(X)$. If $H : V(P_m) \times N_n \to V(X)$ is a net homotopy from σ to τ with $H(0, t) = H(m, t) = x_0$ for all $t \in N_n$, then the loop $p_{0,0} * p_{1,0} * \dots * p_{m-1,0}$ at x_0 is homotopic modulo x_0 to the loop $p_{0,n} * p_{1,n} * \dots * p_{m-1,n}$.*

Proof. We consider the step from t to $t + 1$. What can happen to $H(i, t)$ and $H(i + 1, t)$ at this step? All possible cases are shown in Figure 4. (Cases which differ only by changing the roles of $H(i, t)$ and $H(i + 1, t)$ are only stated once.) Because X has girth $\geqslant 5$, the cases in Figure 4g-j cannot occur. Thus we only have to investigate the cases in Figure 4a-f. We show that in all these cases the path $p_{i,t}$ is homotopic to the path $p_{i,t+1}$. We do this by writing down homotopies which are indicated in Figure 4a-f by certain names. Note that the simplexes corresponding to the edges of X are chosen in some $R^n\,(n \geqslant 1)$. In Figure 4f note that the union of the simplexes corresponding to the two edges is homeomorphic to a straight line segment in some $R^n\,(n \geqslant 1)$, therefore it suffices in this case to find a homotopy from $p_{i,t}$ to $p_{i,t+1}$ under the assumption that $H(i - 1, t)$, $H(i, t)$ and $H(i + 1, t)$ lie on a straight line in some $R^n\,(n \geqslant 1)$. Because we consider i and t fixed for the moment, we use the parameters $r, s \in I$ and define the following homotopies:

identity: $\mathrm{id}(s, r) = p_{i,t}(s)$,

contraction: $\mathrm{co}(s, r) = p_{i,t}(s) + r\,(H(i, t) - p_{i,t}(s))$,

dilatation: $\mathrm{di}(s, r) = H(i, t) + r\,(p_{i,t+1}(s) - H(i, t))$,

reflexion: $\mathrm{re}(s, r) = p_{i,t}(s) + r\,(p_{i,t}(1 - s) - p_{i,t}(s))$,

translation: $\mathrm{tr}(s, r) = p_{i,t}(s) + r\,(p_{i,t+1}(s) - p_{i,t}(s))$.

To write down the corresponding homotopies, with the roles of $H(i, t)$ and $H(i + 1, t)$ changed, is left to the reader. All these homotopies are »straight line homotopies« and this can be interpreted as if the point $H(i, t)$ moves with »constant velocity« along a straight line to the point $H(i, t + 1)$ and analogously the point $H(i + 1, t)$ to the point

$H(i+1, t+1)$. Thus, if f_0 is a homotopy from $p_{0,t}$ to $p_{0,t+1}$ and f_1 a homotopy from $p_{1,t}$ to $p_{1,t+1}$, both homotopies of the kinds mentioned above, it follows that $f_0(1, r) = f_1(0, r)$ for all $r \in I$. (This result also can be attained by direct computation for all possible pairs f_1, f_2.) Thus the map h_1 defined by

$$h_1(s, r) = \begin{cases} f_0(2s, r) & \text{for} \quad 0 \leqslant s \leqslant \frac{1}{2}, \\ f_1(2s-1, r) & \text{for} \quad \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

is a homotopy from $p_{0,t} * p_{1,t}$ to $p_{0,t+1} * p_{1,t+1}$.

Next we construct a homotopy from $p_{0,t} * p_{1,t} * p_{2,t}$ to $p_{0,t+1} * * p_{1,t+1} * p_{2,t+1}$. Let f_2 be a homotopy of $p_{2,t}$ to $p_{2,t+1}$ of one of the kinds mentioned above. Because $f_1(1, r) = f_2(0, r)$ for all $r \in I$ and $h_1(1, r) = f_1(1, r)$ it follows that $h_1(1, r) = f_2(0, r)$ for all $r \in I$. Thus the map h_2 defined by

$$h_2(s, r) = \begin{cases} h_1(2s, r) & \text{for} \quad 0 \leqslant s \leqslant \frac{1}{2}, \\ f_2(2s-1, r) & \text{for} \quad \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

is a homotopy from $p_{0,t} * p_{1,t} * p_{2,t}$ to $p_{0,t+1} * p_{1,t+1} * p_{2,t+1}$.

Proceeding in this way we can construct a homotopy h_{m-1} from $p_{0,t} * p_{1,t} * \dots * p_{m-1,t}$ to $p_{0,t+1} * p_{1,t+1} * \dots * p_{m-1,t+1}$. Because $H(0, t) = H(0, t+1) = x_0$ and $H(m, t) = H(m, t+1) = x_0$, it follows that $h_{m-1}(0, r) = h_{m-1}(1, r) = x_0$ for all $r \in I$. Thus $p_{0,t} * * p_{1,t} * \dots * p_{m-1,t}$ is homotopic modulo x_0 to $p_{0,t+1} * p_{1,t+1} * \dots * p_{m-1,t+1}$. Because this holds for all $t \in N_{n-1}$, it follows that $p_{0,0} * * p_{1,0} * \dots * p_{m-1,0}$ is homotopic modulo x_0 to $p_{0,n} * p_{1,n} * \dots * * p_{m-1,n}$.

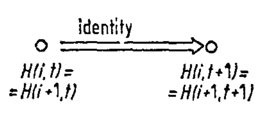

Fig. 4a

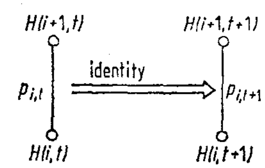

Fig. 4b

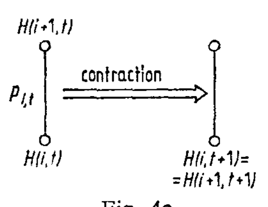

Fig. 4c

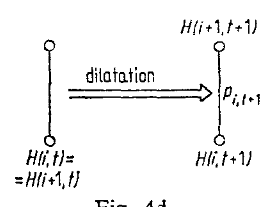

Fig. 4d

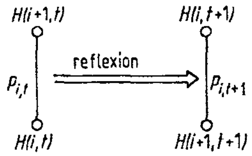

Fig. 4e

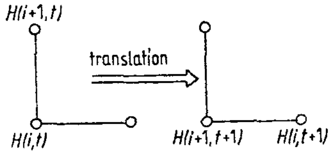

Fig. 4f

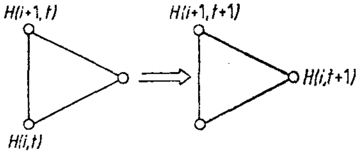

Fig. 4g

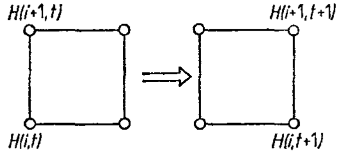

Fig. 4h

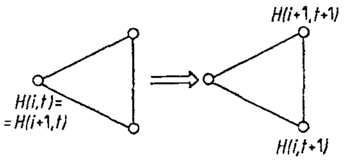

Fig. 4i

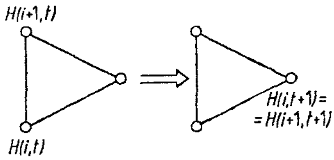

Fig. 4j

THEOREM 5. *If the connected graph X has girth $\geqslant 5$, then $S(X)$ and $G(T(X))$ are isomorphic.*

Proof. We choose a vertex $x_0 \in V(X)$ as base vertex for the string loops in X and the corresponding point in $T(X)$ as base point for the loops in $T(X)$. Let $\alpha : I \to T(X)$ be a loop in $T(X)$ at x_0. If one traverses the unit interval from 0 to 1, let $v_0, v_1, \ldots, v_m$ be the points corresponding to vertices occuring in this order. Then we define a string loop $\alpha_s : P_m \to X$ in X at x_0 by $\alpha_s(i) = v_i$ $(0 \leqslant i \leqslant m)$. In this way we can assign to each loop in $T(X)$ at x_0 a string loop α_s in X at x_0. Now we define a map $\Phi([\alpha]) = [\alpha_s]$.

Φ is a homomorphism:

$$\Phi([\alpha] \circ [\beta]) = \Phi([\alpha * \beta]) = [\alpha * \beta]_s = [\alpha_s * \beta_s] = [\alpha_s] \circ [\beta_s] = \\ = \Phi([\alpha]) \circ \Phi([\beta]).$$

Φ is one-to-one: Suppose $\Phi([\alpha]) = \Phi([\beta])$. Then $[\alpha_s] = [\beta_s]$ and therefore $\alpha_s \simeq_{x_0} \beta_s$. Hence there is a net homotopy $H : V(P_m) \times N_n \to V(X)$ from an extension $\alpha_s^e : P_m \to X$ of α_s to an extension $\beta_s^e : P_m \to X$ of β_s with $H(0, t) = H(m, t) = x_0$ for all $t \in N_n$. We put $\alpha_s^e = \sigma$ and $\beta_s^e = \tau$. Then, by Lemma 3, the loop

$p_{0,0} * p_{1,0} * \ldots * p_{m-1,0}$ is homotopic modulo x_0 to the loop $p_{0,n} * p_{1,n} * * \ldots * p_{m-1,n}$. But since $p_{0,0} * p_{1,0} * \ldots * p_{m-1,0}$ is homotopic modulo x_0 to α and $p_{0,n} * p_{1,n} * \ldots * p_{m-1,n}$ is homotopic modulo x_0 to β, it follows that α is homotopic modulo x_0 to β. Therefore $[\alpha] = [\beta]$.

Φ maps $G(T(X), x_0)$ onto $S(X, x_0)$: Let $[\alpha_s] \in S(X, x_0)$. Let $\alpha_s^m : P_k \to X$ be the minimal reduction of α_s. Let $U \subseteq T(X)$ be the union of all points $\alpha_s^m(i)$ $(i \in V(P_k))$ and simplexes $\langle \alpha_s^m(i), \alpha_s^m(i + + 1)\rangle$ $(i \in N_{k-1})$. Define a loop $\alpha : I \to T(X)$ in $T(X)$ in such a way that $\alpha(I) = U$ and the points $\alpha_s^m(0), \alpha_s^m(1), \ldots, \alpha_s^m(k)$ occur in this order if one traverses the unit interval I from 0 to 1. Then $\Phi([\alpha]) = [\alpha_s^m] = [\alpha_s]$. This completes the proof that Φ is an isomorphism from $G(T(X))$ to $S(X)$.

Remark. If X has girth $\leqslant 4$, the theorem is not true. E. g. for the complete graph K_3 the group $S(K_3)$ is trivial, but $G(T(K_3))$ is infinite cyclic.

Because $G(T(X))$ is a free group (see [9], p. 197), the following corollary of Theorem 5 is an immediate consequence.

COROLLARY. *If the connected graph X has girth $\geqslant 5$, then $S(X)$ is a free group.*

PART TWO

This part of the paper deals with analogues to simply connected spaces, deformation retraction, homotopy equivalence etc.

5. String connected graphs

We call a graph X *string connected* if it is connected and its string fundamental group $S(X)$ is trivial.

In order to characterize string connected graphs we need the concept of a pseudosurface (see [15, p. 48]) which is defined as follows. Let A denote a set of $\sum_{i=1}^{t} n_i m_i \geqslant 0$ distinct points of S_k (closed orientable 2-manifold of genus k), with $1 < m_1 < m_2 < \ldots < m_t$. Partition A into n_i sets of m_i points each, $i = 1, 2, \ldots, t$. For each set of the partition, identify all the points of that set. The resulting topological space is called a *pseudosurface*, and is designated by $S(k; n_1(m_1), n_2(m_2), \ldots, n_t(m_t))$. Each point resulting from an identification of m_i points of S_k is called a *singular point*. If a graph G is embedded in a pseudosurface, we assume that each singular point is occupied by a vertex of G; such a vertex is called a *singular vertex*.

Let C be a cycle in a graph Y. We call Y a *pseudoplanar (planar) net* of C if Y can be embedded in a pseudosurface $S(0; n_1(m_1), n_2(m_2), \ldots, n_t(m_t))$ (a sphere $S(0; 0(0), 0(0), \ldots, 0(0)) = S_0$) in such a way that one region is bounded by C and all other regions are triangles or quadrangles. We call such an embedding of Y a *proper embedding* of Y. If $C \subseteq Y \subseteq X$ and Y is a pseudoplanar (planar) net of C, we call Y a pseudoplanar (planar) net of C in the graph X. If there exists a pseudoplanar (planar) net of C in X, we say that C has a pseudoplanar (planar) net in X. An example of a planar net Y_1 of a cycle C is shown in Figure 5a, an example of a pseudoplanar net Y_2 (which is not a planar net of C) is shown in Figure 5b. One gets Y_2 from Y_1 by identifying the vertices a and b. The example in Figure 5b shows that a cycle C having a pseudoplanar net in a graph X must

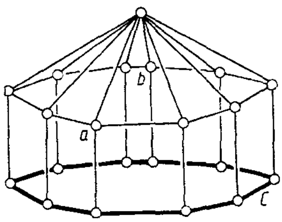

Fig. 5a

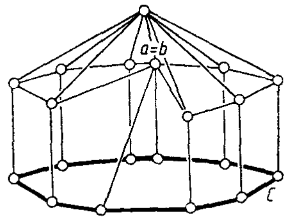

Fig. 5b

not have a planar net in X. If in the following we speak of a pseudosurface, we always mean a pseudosurface of the form $S(0; n_1(m_1), n_2(m_2), \ldots, n_t(m_t))$. If a pseudoplanar (planar) net Y of C is embedded properly in a pseudosurface (a sphere), we call each region, except the one bounded by C, a *region of* Y.

Let us distinguish in the following between a cycle and a circuit. In a cycle $x_0 x_1 \ldots x_m (= x_0)$ each x_i is adjacent to x_{i+1} $(0 \leqslant i \leqslant m - 1)$ and all vertices, except x_m and x_0, are pairwise distinct. In a circuit $x_0 x_1 \ldots x_m (= x_0)$ each x_i is adjacent to x_{i+1} $(0 \leqslant i \leqslant m - 1)$, but the vertices need not be pairwise distinct. Furthermore, by $A \triangle B$ we denote the symmetric difference of the sets A and B and, if R is a region, by $E(R)$ the set of edges in the boundary of R.

LEMMA 4. *Let* $C = x_0 x_1 \ldots x_m (= x_0)$, $m \geqslant 5$, *be a cycle and let* Y *be a planar net of* C. *If* Y *is embedded properly in a sphere and* Y *has* r *regions, then there exists a region* R *of* Y *such that the graph* C', *induced by* $E(C) \triangle E(R)$, *is a cycle containing* x_0 *that has a planar net* Y' *embedded properly in this sphere with* Y' *having* $r - 1$ *regions.*

Proof. We distinguish two cases.

Case 1. There is a path $x_i y_1 y_2 \ldots y_r x_j$ with $1 \leqslant i < j \leqslant m - 1$ and $y_k \notin V(C)$ for $1 \leqslant k \leqslant r$ (Figure 6a). Among all these paths we choose one for which the number of regions of Y inside the cycle $x_0 x_1 \ldots x_i y_1 y_2 \ldots y_r x_j x_{j+1} \ldots x_0$ is maximum. Then the cycle $x_i y_1 y_2 \ldots y_r x_j x_{j-1} \ldots x_{i+1} x_i$ must be the boundary of a region R which has the desired properties.

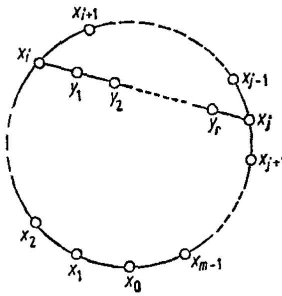

Fig. 6a

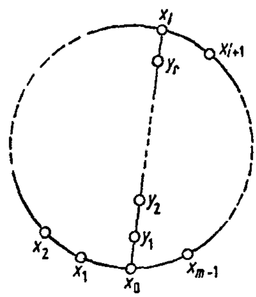

Fig. 6b

Case 2. There is no path as in Case 1. But since $m \geqslant 5$, there must exist a path $x_0 y_1 y_2 \dots y_r x_i$ with $1 \leqslant i \leqslant m-1$ and $y_k \notin V(C)$ for $1 \leqslant k \leqslant r$ (Figure 6b). Among all these paths we choose one for which the number of regions of Y inside the cycle $x_0 x_1 \dots x_i y_r y_{r-1} \dots \dots y_1 x_0$ is maximum. Then the cycle $x_0 y_1 y_2 \dots y_r x_i x_{i+1} \dots x_0$ must be the boundary of a region R, which has the desired properties.

LEMMA 5. *Let* $C = x_0 x_1 \dots x_m (= x_0)$, $m \geqslant 5$, *be a cycle and let Y be a pseudoplanar net of C. If Y is embedded properly in a pseudosurface* $S(0; n_1(m_1), n_2(m_2), \dots, n_t(m_t))$ *and Y has r regions, then there exists a region R of Y such that the graph C' induced by* $E(C) \triangle E(R)$ *is a cycle contaning x_0 that has a pseudoplanar net Y' embedded properly in this pseudosurface with Y' having $r-1$ regions.*

Proof. We form a new graph $\overline{Y}$ from Y by undoing the identifications of the m_i points of S_0 to one singular vertex of $S(0; n_1(m_1), n_2(m_2), \dots, n_t(m_t))$ for $i = 1, 2, \dots, t$. (This process is shown in Figure 7 for a singular point with $m_i = 2$, occupied by a singular vertex u of X.) Formally this can be done as follows. Each singular vertex

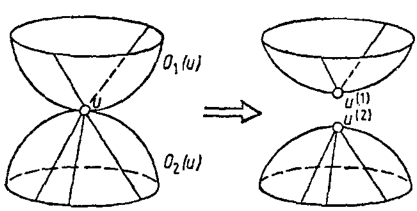

Fig. 7

$u \in V(Y)$ has an open neighbourhood in $S(0; n_1(m_1), n_2(m_2), \dots \dots, n_t(m_t))$ homeomorphic to the union of open discs $O_1(u), O_2(u), \dots \dots, O_{n_u}(u)$. We can choose $O_1(u), O_2(u), \dots, O_{n_u}(u)$ so small that each edge incident with u has a nonempty intersection with exactly one of $\dot{O}_1(u), \dot{O}_2(u), \dots, \dot{O}_{n_u}(u)$, where $\dot{O}_i(u) = O_i(u) \setminus \{u\}$, $i =$

$= 1, 2, \ldots, n_u$. For each singular vertex $u \in V(Y)$ we delete all edges incident with u and replace u by n_u vertices $u^{(1)}, u^{(2)}, \ldots, u^{(n_u)}$. We join each vertex $u^{(i)}$ $(1 \leqslant i \leqslant n_u)$ with each nonsingular vertex v of Y for which the edge uv in Y has a nonempty intersection with $\dot{O}_i(u)$. Furthermore, we join two vertices $u^{(i)}$ and $v^{(j)}$ $(1 \leqslant i \leqslant n_u, 1 \leqslant j \leqslant n_v)$ if the edge uv in Y has a nonempty intersection with $\dot{O}_i(u)$ and a nonempty intersection with $\dot{O}_j(v)$. In this way the cycle C turns into a cycle $\bar{C}$ and the pseudoplanar net Y of C into a planar net $\bar{Y}$ of $\bar{C}$ which can be embedded in a sphere in such a way that $\bar{Y}$ has r regions. Then, by Lemma 4, there exists a region $\bar{R}$ of $\bar{Y}$ such that the graph $\bar{C}'$ induced by $E(\bar{C}) \triangle E(\bar{R})$ is a cycle containing x_0 or $x_0^{(j)}$ and having a planar net $\bar{Y}'$ embedded in this sphere with $\bar{Y}'$ having $r - 1$ regions.

Now we go back from $\bar{Y}$ to Y by identifying all vertices $u^{(1)}, u^{(2)}, \ldots, u^{(n_u)}$ to u for each singular vertex $u \in V(Y)$. Since all vertices of $\bar{C}$ are nonsingular vertices and no two vertices in the boundary of $\bar{R}$ can be identified (because $\bar{R}$ is a triangle or a quadrangle), no two vertices of $\bar{C}'$ can be identified and thus $\bar{C}'$ turns into a cycle C' in Y. Furthermore, the region $\bar{R}$ turns into a region R of Y in the pseudosurface considered and C' is induced by $E(C) \triangle E(R)$. Since $\bar{C}'$ contains x_0 or $x_0^{(j)}$, the cycle C' contains x_0. Furthermore, the planar net $\bar{Y}'$ of $\bar{C}'$ turns into a pseudoplanar net Y' of C' embedded in the pseudosurface considered with Y' having $r - 1$ regions.

LEMMA 6. *If $a : P_m \to X$ is a string loop in X at x_0 and $C(a) = a(0)\, a(1) \ldots a(m)\ (= a(0))$ a cycle having a pseudoplanar net in X, then $a \simeq_{x_0} v$, where v is a constant string loop at x_0.*

Proof. We show the lemma by induction with respect to the number of regions of the pseudoplanar net of $C(a)$ generated by a proper embedding in a pseudosurface. The assertion obviously is true for all string loops a at x_0, where $C(a)$ has a pseudoplanar net Y in X properly embeddable in a pseudosurface with one region. We assume that the assertion is true for all string loops a at x_0, where $C(a)$ has a pseudoplanar net Y in X properly embeddable in a pseudosurface with fewer than r regions $(r \geqslant 2)$. Now let $a : P_m \to X$ be a string loop at x_0, where $C(a)$ has a pseudoplanar net Y in X properly embeddable in a pseudosurface with r regions. If $C(a)$ also has a pseudoplanar net in X properly embeddable in a pseudosurface with fewer than r regions, we are through. Therefore we can assume that $C(a)$ has no such pseudoplanar net in X. Thus $m \geqslant 5$. Let Y be embedded properly in a pseudosurface with r regions. By Lemma 5, there exists a region R of Y such that the graph $C'(a)$ induced by $E(C(a)) \triangle E(R)$ is a cycle in X containing x_0 and having a pseudoplanar net Y' embedded properly in this pseudosurface with $r - 1$ regions. We define a string loop $a' : P_k \to X$ in such a way that $C'(a) =$

$= \alpha'(0)\,\alpha'(1) \ldots \alpha'(k)\ (= \alpha'(0))$ and α and α' are both oriented clockwise or counterclockwise. (An example is shown in Figure 8.)

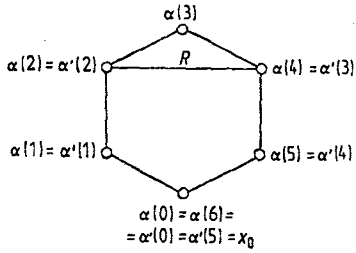

Fig. 8

Since R is a triangle or a quadrangle, it follows that $\alpha \simeq_{x_0} \alpha'$. (This is easy to show by inspecting all possible cases for this triangle or quadrangle.) By the induction hypothesis $\alpha' \simeq_{x_0} \nu$, where ν is a constant string loop at x_0. Thus $\alpha \simeq_{x_0} \nu$.

THEOREM 6. *A graph X is string connected iff it is connected and each cycle of X has a pseudoplanar net in X.*

Proof. (1) Let X be string connected. Then X is connected. Thus we only have to show that each cycle of X has a pseudoplanar net in X. If X is a tree, this is true. Therefore we can assume that X is not a tree. Let $C = x_0\, x_1 \ldots x_r\ (= x_0)$ be an arbitrary cycle of X. We define a string loop $\alpha : P_r \to X$ by $\alpha(i) = x_i$ for $0 \leqslant i \leqslant r$. Since X is string connected, $\alpha \simeq_{x_0} \nu$, where ν is a constant string loop at x_0. Thus there exists a net homotopy $H : V(P_m) \times N_n \to V(X)$ from an extension $\alpha^e : P_m \to X$ of α to an extension $\nu^e : P_m \to X$ of ν. To simplify the situation, we can assume that there is no index i, $1 \leqslant$ $\leqslant i \leqslant k - 1$, with $\alpha^e(i-1) = \alpha^e(i+1)$ and $\alpha^e(i) \neq \alpha^e(i-1)$ because each extension of α is net homotopic to such an extension. Now we change the graph X and the net homotopy H in the following way. If $u \in V(X)$ and there are s pairs (i, t) with $H(i, t) = u$, arranged in any order, then we delete all edges incident with u and replace u by a complete graph $K_s(u)$ with vertices $u^{(1)}, u^{(2)}, \ldots, u^{(s)}$. We join each $u^{(j)}$ with every vertex $v^{(k)}$ if $uv \in E(X)$. In this way the graph X turns into a graph $\bar{X}$. Furthermore, we replace the net homotopy $H :$ $: V(P_m) \times N_n \to V(X)$ by a net homotopy $\bar{H} : V(P_m) \times N_n \to$ $\to V(\bar{X})$, defined by $\bar{H}(i, t) = u^{(j)}$ for the j-th pair (i, t) with $H(i, t) =$ $= u$. Thereby the cycle C in X turns into a cycle $\bar{C} = \bar{H}(0, 0)\, \bar{H}(1, 0) \ldots$ $\ldots \bar{H}(m, 0)\ (= \bar{H}(0, 0))$ in $\bar{X}$, the string loop $\alpha^e : P_m \to X$ in X into a string loop $\bar{\alpha}^e : P_m \to \bar{X}$ in $\bar{X}$ with $\bar{\alpha}^e(i) = \bar{H}(i, 0)$ for $0 \leqslant i \leqslant m$ and the constant string loop $\nu^e : P_m \to X$ in X into a string loop $\bar{\nu}_e :$ $: P_m \to \bar{X}$ in $\bar{X}$ with $\bar{\nu}_e(i) = \bar{H}(i, n)$ for $0 \leqslant i \leqslant m$. The homotopy

$\bar{H}$ has the property that $\bar{H}(i, t) \neq \bar{H}(i', t')$ for $(i, t) \neq (i', t')$. (The situation is depicted in Figure 9.) Now we define a graph $\bar{Y}$ (shown in Figure 9) by

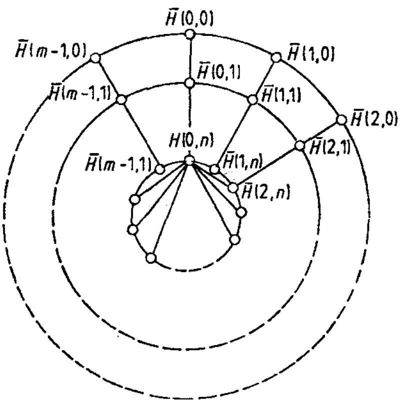

Fig. 9

$$V(\bar{Y}) = \{\bar{H}(i, t) \mid 0 \leqslant i \leqslant m-1,\ 0 \leqslant t \leqslant n\} \text{ and}$$
$$\begin{aligned} E(\bar{Y}) = {} & \{[\bar{H}(i, t), \bar{H}(i+1, t)] \mid 0 \leqslant i \leqslant m-1,\ 0 \leqslant t \leqslant n\} \cup \\ & \cup \{[\bar{H}(i, t), \bar{H}(i, t+1)] \mid 0 \leqslant i \leqslant m-1, 0 \leqslant t \leqslant n-1\} \cup \\ & \cup \{[\bar{H}(0, n), \bar{H}(i, n)] \mid 2 \leqslant i \leqslant m-2\}. \end{aligned}$$

The graph $\bar{Y}$ is a planar net of $\bar{C}$ in $\bar{X}$. If we now go back from $\bar{X}$ to X by identifying the vertices $u^{(1)}, u^{(2)}, \ldots, u^{(s)}$ of each $K_s(u)$, the cycle $\bar{C}$ turns back into the cycle C and the planar net $\bar{Y}$ of $\bar{C}$ in $\bar{X}$ into a pseudoplanar net Y of C in X. Thus C has a pseudoplanar net in X.

(2) Conversely let each cycle of X have a pseudoplanar net in X. To show that X is string connected it suffices to show that for all $x_0 \in V(X)$ and all string loops α at x_0 it follows that $\alpha \simeq_{x_0} \nu$, where ν is a constant string loop at x_0. If $\alpha : P_k \to X$ is a string loop in X we define the length of α by $l(\alpha) = k$. We now procede by induction with respect to $l(\alpha)$.

The assertion obviously is true for all $x_0 \in V(X)$ and all string loops α at x_0 with $l(\alpha) = 0$ (because in this case α is itself a constant string loop at x_0). We assume that the assertion is true for all $x_0 \in V(X)$ and all string loops α at x_0 with $l(\alpha) < m$. Let $x_0 \in V(X)$ and $\alpha : P_m \to X$ be a string loop at x_0 with $l(\alpha) = m$. Let α^m be the minimal reduction of α. We distinguish two cases:

Case 1. $\alpha^m \neq \alpha$. Then $l(\alpha^m) < l(\alpha)$. Thus by the induction hypothesis $\alpha^m \simeq_{x_0} \nu$, where ν is a constant string loop at x_0. Since $\alpha \simeq_{x_0} \alpha^m$ it follows that $\alpha \simeq_{x_0} \nu$.

Case 2. $a^m = a$. In this case $a(0)\, a(1) \dots a(m)\, (= a(0))$ is a circuit in X. If this circuit is a cycle in X, then by hypothesis this cycle has a pseudoplanar net in X and by Lemma 6 it follows that $a \simeq_{x_0} \nu$, where ν is a constant string loop at x_0. Thus we can assume that this circuit is not a cycle. Then there exists an index j, with $0 \leqslant j \leqslant m-2$, such that $a(j) = a(j+g)$, $2 \leqslant g \leqslant m-j$. We define a string loop $\beta : P_{g+1} \to X$ at $y_0 = a(j)$ by

$$\beta(i) = a(j+i) \text{ for } 0 \leqslant i \leqslant g$$

and a string loop $\gamma : P_m \to X$ at x_0 by

$$\gamma(i) = \begin{cases} a(i) & \text{for } i < j \text{ or } i > j+g, \\ y_0 & \text{for } j \leqslant i \leqslant j+g. \end{cases}$$

Since $l(\beta) < l(a)$, it follows by the induction hypothesis that $\beta \simeq_{x_0} \mu$, where μ is a constant string loop at y_0. From this it is easy to conclude that $a \simeq_{x_0} \gamma$. Let γ^m be the minimal reduction of γ. Since $\gamma(j) = \gamma(j+1) = \dots = \gamma(j+g)$, it follows that $l(\gamma^m) < l(\gamma) = l(a)$. Thus by the induction hypothesis $\gamma^m \simeq_{x_0} \nu$, where ν is a constant string loop at x_0. From $a \simeq_{x_0} \gamma \simeq_{x_0} \gamma^m \simeq_{x_0} \nu$, it follows that $a \simeq_{x_0} \nu$.

6. Net deformation retraction and net homotopy equivalence

Let Y be a subgraph of a graph X. A *net deformation retraction* of X onto Y is a net homotopy $H : V(X) \times N_n \to V(X)$ such that

(1) $H(x, 0) = x$ and $H(x, n) \in V(Y)$ for all $x \in V(X)$,

(2) $H(y, t) = y$ for all $y \in V(Y)$ and all $t \in N_n$.

When some net deformation retraction of X onto Y exists, Y is called a *net deformation retract* of X.

Example. The subgraph Y shown in Figure 10b of the graph X shown in Figure 10a is a net deformation retract of X. A net deformation retraction $H: V(X) \times N_1 \to V(X)$ of X onto Y is given by $H(x_i, 0) = x_i$ $(1 \leqslant i \leqslant 6)$, $H(y_i, 0) = y_i$ $(1 \leqslant i \leqslant 5)$, $H(x_i, 1) = y_i$ $(1 \leqslant i \leqslant 5)$, $H(x_6, 1) = y_1$ and $H(y_i, 1) = y_i$ $(1 \leqslant i \leqslant 5)$.

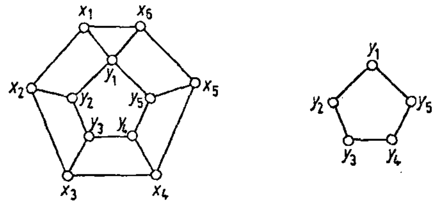

Fig. 10a Fig. 10b

A *net contraction* of a graph X to a vertex $x_0 \in V(X)$ is just a net deformation retraction of X onto the trivial graph Y with $V(Y) = \{x_0\}$ and $E(Y) = \emptyset$. Each net deformation retract of X is a net retract of X. One gets a net retraction f from a net deformation retraction H by putting $f(x) = H(x, n)$. Conversely, a net retract of X need not be a net deformation retract of X. E. g. the subgraph Y shown in Figure 11b of the graph X shown in Figure 11a is a net retract of X. A net retraction of X onto Y is given by $f(x_i) = y_i$ $(1 \leqslant i \leqslant 5)$, $f(x_6) = y_1$ and $f(y_i) = y_i$ $(1 \leqslant i \leqslant 5)$. But Y is not a net deformation retract of X.

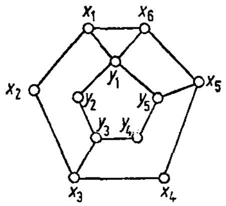

Fig. 11a

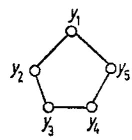

Fig. 11b

THEOREM 7. *If Y is a net deformation retract of the connected graph X, then $S(X)$ and $S(Y)$ are isomorphic.*

The proof is left to the reader (compare the analogon in algebraic topology, e. g. [2], p. 75).

In the next definition note that the composition of two homomorphisms is again a homomorphism.

Two graphs X and Y are said to be net *homotopy equivalent* provided there exist homomorphisms $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq 1_X$ and $fg \simeq 1_Y$, where 1_X and 1_Y are the identity homomorphisms on X, Y respectively. The homomorphism f is called a *net homotopy equivalence* from X to Y and g a *net homotopy inverse* for f.

PROPOSITION 3. *Net homotopy equivalence is an equivalence relation for graphs.*

The proof is left to the reader. (Compare the analogon in algebraic topology, e. g. [2], p. 118.)

THEOREM 8. *If Y is a net deformation retract of X, then Y and X are net homotopy equivalent.*

The proof is left to the reader. (Compare the analogon in algebraic topology, e. g. [2], p. 119.)

We consider a homomorphism $f : X \to Y$. If α is a string loop in X at $x_0 \in V(X)$, then $f\alpha$ is a string loop in Y at $y_0 = f(x_0)$. Let

$f : X \to Y$ be a homomorphism with $f(x_0) = y_0$. Then the homomorphism $f_* : S(X, x_0) \to S(Y, y_0)$ defined by

$$f_*([\alpha]) = [f\alpha] \quad \text{for} \quad [\alpha] \in S(X, x_0)$$

is called the *homomorphism induced by f*.

The homomorphism f_* is well defined, i. e., if $\alpha \simeq_{x_0} \beta$, then $f\alpha \simeq_{x_0} f\beta$; for if $H : V(P_m) \times N_n \to V(X)$ is a net homotopy from an extension α^e of α to an extension β^e of β with $H(0, t) = H(m, t) = x_0$ for all $t \in N_n$, then $K : V(P_m) \times N_n \to V(Y)$ defined by $K(i, t) = f(H(i, t))$ is a net homotopy from an extension $(f\alpha)^e$ of $f\alpha$ to an extension $(f\beta)^e$ of $f\beta$ with $K(0, t) = K(m, t) = y_0$ for all $t \in N_n$. The proof that f_* is actually a homomorphism is left to the reader. The proof of the following Lemma is also left to the reader.

LEMMA 7. *If $f : X \to Y$ and $g : Y \to Z$ are homomorphisms, then $(gf)_* = g_* f_*$.*

LEMMA 8. *Let X be a graph and $x_0 \in V(X)$. Let $H : V(X) \times N_n \to V(X)$ be a net homotopy with $H(x_0, 0) = y_0$ and $H(x_0, n) = y_n$. Let $\sigma : P_n \to X$ be a string defined by $\sigma(i) = H(x_0, i)$ for all $i \in N_n$. If $\alpha : P_m \to X$ is a string loop at y_0 and $\beta : P_m \to X$ the string at y_n defined by $\beta(i) = H(\alpha(i), n)$ for all $i \in N_m$, then $\alpha \simeq_{y_0} \sigma * \beta * \overline{\sigma}$.* (See Figure 12).

Proof. We define an extension $\alpha^e : P_{2n+m} \to X$ of α by

$$\alpha^e(i) = \begin{cases} y_0 & \text{if } 0 \leqslant i \leqslant n, \\ \alpha(i - n) & \text{if } n \leqslant i \leqslant m + n, \\ y_0 & \text{if } m + n \leqslant i \leqslant 2n + m. \end{cases}$$

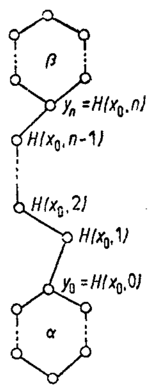

Fig. 12.

Then we define a map $K : V(P_{2n+m}) \times N_n \to V(X)$ by

$$K(i, t) = \begin{cases} \sigma(\max\{0, t + i - n\}) & \text{if } 0 \leqslant i \leqslant n, \\ H(\alpha^e(i), t) & \text{if } n \leqslant i \leqslant n + m, \\ \sigma(\max\{0, m + n + t - i\}) & \text{if } n + m \leqslant i \leqslant 2n + m. \end{cases}$$

The reader may check that K is a net homotopy from α^e to $\sigma * \beta * \overline{\sigma}$, especially that $K(i, 0) = \alpha^e(i)$ and $K(i, n) = \sigma * \beta * \overline{\sigma}(i)$. Furthermore $K(0, t) = K(2n + m, t) = \sigma(0) = y_0$ for all $t \in N_n$. Thus $\alpha \simeq_{y_0} \sigma * \beta * \overline{\sigma}$.

THEOREM 9. *If $f : X \to Y$ is a net homotopy equivalence with $f(x_0) = y_0$, then $S(X, x_0)$ and $S(Y, y_0)$ are isomorphic.*

Proof. Let $g : Y \to X$ be a net homotopy inverse to f and let $H : V(X) \times N_n \to V(X)$ be a net homotopy from gf to 1_X. Let $g(y_0) = x_1, f(x_1) = y_1$ and define a string $\sigma : P_n \to X$ in X by

$$\sigma(i) = H(x_0, i) \quad \text{for all} \quad i \in N_n.$$

Thus $\sigma(0) = H(x_0, 0) = (gf)(x_0) = g(y_0) = x_1$ and $\sigma(n) = H(x_0, n) = x_0$. If α is any string loop at x_0, then it follows by Lemma 8 that $gf\alpha \simeq_{x_1} \sigma * \alpha * \bar{\sigma}$. From this, it follows by Lemma 7 that

$$(g_* f_*)([\alpha]) = (gf)_*([\alpha]) = [gf\alpha] = [\sigma * \alpha * \bar{\sigma}] =: \hat{\hat{\sigma}}([\alpha]).$$

It is easy to show (as for $\hat{\sigma}$ in the proof of Theorem 4) that $\hat{\hat{\sigma}}$ is an isomorphism from $S(X, x_0)$ to $S(X, x_1)$. Therefore $g_* f_*$ is also an isomorphism from $S(X, x_0)$ to $S(X, x_1)$.

By completely analogous arguments one can show that $f_* g_*$ is an isomorphism from $S(Y, y_0)$ to $S(Y, y_1)$. From this it follows that f_* and g_* are themselves isomorphisms between $S(X, x_0)$ and $S(Y, y_0)$.

7. Net contractible graphs

THEOREM 10. *Every net contractible graph is string connected.*

Proof. Let X be a net contractible graph. Then X is connected. Furthermore there is a vertex $x_0 \in V(X)$ and a net homotopy $H : V(X) \times N_n \to V(X)$ such that

$$H(x, 0) = x \text{ and } H(x, n) = x_0$$

for all $x \in V(X)$.

Let $\alpha : P_m \to X$ be an arbitrary string loop at x_0. We must show that $\alpha \simeq_{x_0} \nu$, where ν is a constant string loop at x_0. We define a string loop σ at x_0 by

$$\sigma(i) = H(x_0, i) \text{ for all } i \in N_n.$$

By Lemma 8 (with $y_0 = y_n = x_0$ and $\beta = \nu$) it follows that $\alpha \simeq_{x_0} \sigma * \nu * * \bar{\sigma} \simeq_{x_0} \nu$. Hence $\alpha \simeq_{x_0} \nu$. Therefore $S(X, x_0)$ is trivial and X is string connected.

The following theorem can easily be shown in analogy to the corresponding theorem in algebraic topology (see e. g. [2], p. 119).

THEOREM 11. *A graph X is net contractible iff it is net homotopy equivalent to a trivial graph.*

Theorem 11 is an answer to the »gangster problem« formulated in the beginning of this paper. The answer consists of a characterization of net contractible graphs; but it is not a good answer since it is not easy to decide whether a given graph is net homotopy equivalent to a trivial graph or not.

Open Problem. Find a better characterization of net contractible graphs (e. g., similar to that of string connected graphs in Theorem 6)!

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TEORIJA HOMOTOPIJE ZA GRAFOVE

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Sadržaj

Rad je podijeljen u dva dijela. U prvom dijelu se uvode neki osnovni pojmovi kao što su mrežna homotopija, nitna homotopija, mrežna kontrakcija, mrežna retrakcija itd. i istražuje mrežna fundamentalna grupa grafa. U drugom dijelu se razvija kombinatorički analogon pojma jednostavno povezanog topološkog prostora. Takvi grafovi se nazivaju nitno povezani grafovi koji su ujedno karakterizirani. Nadalje su razvijeni kombinatorički analogoni pojmova deformacione retrakcije, homotopske ekvivalencije itd. i u vezi s njima dokazani neki teoremi.