DIOPHANTINE QUADRUPLES IN THE RING OF INTEGERS OF THE PURE CUBIC FIELD $\mathbb{Q}(\sqrt[3]{2})$

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Abstract. We show that in the ring of integers of the pure cubic field $\mathbb{Q}(\sqrt[3]{2})$ there exits a $D(w)$-quadruple if and only if w can be represented as a difference of two squares of integers in $\mathbb{Q}(\sqrt[3]{2})$.

1. Introduction

Let R be a commutative ring with a unit 1 and let $w \in R$. A set of m distinct non-zero elements $\{w_1, \dots, w_m\} \subset R$ such that $w_i \cdot w_j + w$ is a perfect square in R for all $1 \leq i < j \leq m$ is called a *Diophantine m-tuple with the property $D(w)$* or a *$D(w)$-m-tuple* in R. If $w = 1$ then these sets are often called *Diophantine m-tuples.* They are named after the 3rd-century Greek mathematician Diophantus of Alexandria who first studied these sets and constructed the set $\{1, 33, 68, 105\}$ with the property $D(256)$. Fermat found the set $\{1, 3, 8, 120\}$ - the first $D(1)$-quadruple in $\mathbb{Z}$. Baker and Davenport in [3] showed that Fermat's quadruple cannot be extended to a $D(1)$-quintuple. The folklore conjecture says that there is no $D(1)$-quintuples in $\mathbb{Z}$. Dujella proved that are only finitely many $D(1)$-quintuples (see [8]).

The problem of existence of $D(w)$-quadruples was mainly considered in rings of integers of numbers fields. It all started with the fact that there does not exist a $D(n)$-quadruple in the ring of integers $\mathbb{Z}$ if $n \equiv 2 \pmod 4$. This simple statement was observed independently by several authors (see [4, 14, 16]). On the other hand, Dujella [5] showed that if $n \not\equiv 2 \pmod 4$ and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$ then there exists a $D(n)$-quadruple. It is interesting that the condition $n \not\equiv 2 \pmod 4$ is equivalent to the condition that n is representable as a difference of the squares of two integers. An analogous result for Gaussian integers was also found by Dujella [7]. He proved that there does not exist a $D(a + bi)$-quadruple in $\mathbb{Z}[i]$ if b is odd or $a \equiv b \equiv 2 \pmod 4$, i.e. if $a + bi$ is not representable as a difference of the squares of two

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elements in $\mathbb{Z}[i]$, and in contrary if $a+bi$ is not of such form and $a+bi \notin \{2,-2,1+2i,-1-2i,4i,-4i\}$ then a $D(a+bi)$-quadruple exists. Therefore, it is natural to state the following conjecture: *There exists a $D(w)$-quadruple if and only if w can be represented as a difference of two squares, up to finitely many exceptions.*

So far, the conjecture was shown to be true in rings of integers of some real quadratic fields. More precisely, the author proved that there exist infinitely many $D(w)$-quadruples if and only if w can be represented as a difference of two squares, in the ring of integers of $\mathbb{Q}(\sqrt{d})$ for $d=2$ and for all positive integers d such that one of Pellian equations $x^2-dy^2=\pm 2$ is solvable or such that $x^2-dy^2=4$ is solvable in odd numbers (see [11–13]). The assumption of solvability of these Pellian equations allows to have an effective characterization of integers that are representable as a difference of two squares. The similar result for complex quadratic fields is harder to obtain. Several authors contributed to the characterization of elements z of $\mathbb{Z}[\sqrt{-2}]$ for which a Diophantine quadruple with the property $D(z)$ exists (see [1, 10, 17]). One important difference between real and complex quadratic fields is that in the real case there exist infinitely many units and the methods for the construction of Diophantine quadruples usually use elements with a small norm.

In this paper we prove the above conjecture for the ring of integers of the pure cubic field $\mathbb{Q}(\sqrt[3]{2})$:

Theorem 1. *If w can be represented as a difference of two squares of integers in $\mathbb{Q}(\sqrt[3]{2})$ then there exits infinitely many Diophantine quadruples with the property $D(w)$ in the ring of integers of $\mathbb{Q}(\sqrt[3]{2})$. If w is not a difference of two squares of integers then a $D(w)$-quadruple does not exist in the ring of integers of $\mathbb{Q}(\sqrt[3]{2})$.*

The proof of the existence of $D(w)$-quadruples is based on the description of a difference of two squares of integers in $\mathbb{Q}(\sqrt[3]{2})$ and on applying polynomial formulas for Diophantine quadruples. The first step in proving the non-existence of certain $D(w)$-quadruples was made by Jukić Matić [15]. In the rings of integers of the cubic fields of the form $\mathbb{Q}(\sqrt[3]{d})$, where d is even, she described some elements w that cannot be written as a difference of two squares of integers and showed that $D(w)$-quadruples do not exist. These results are complemented in Section 5 and lead to a proof of the second statement of Theorem 1. In Section 4, the first statement of Theorem 1 is proved by effective constructions of our objects - Diophantine quadruples. This was accomplished thanks to a nice characterization of differences of two squares in the ring of integers of $\mathbb{Q}(\sqrt[3]{2})$ (Section 3).

We continue by exposing some well known facts relevant to the proof of Theorem 1 in the following section.

2. Preliminaries

Let us start with recalling some basic facts related to the structure of the ring of integers $\mathcal{O}_K$ of the pure cubic field $K = \mathbb{Q}(\sqrt[3]{2})$. Since $\sqrt[3]{2}$ is a root of the irreducible polynomial $x^3 - 2$ we have that $[K : \mathbb{Q}] = 3$. Also, $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$ is an integral basis for K, so $\mathcal{O}_K = \mathbb{Z}[1, \sqrt[3]{2}, \sqrt[3]{4}]$ (according to Theorem 7.3.2. in [2]).

As we said before, the methods for constructing $D(w)$-quadruples use elements with a small norm. It is a well-known fact that $\mathbb{Q}(\sqrt[3]{2})$ possesses an unique fundamental unit $\eta > 1$ and it is $\eta = 1 + \sqrt[3]{2} + \sqrt[3]{4}$ (Theorem 13.6.2. in [2]). All fundamental units in $\mathbb{Q}(\sqrt[3]{2})$ are $\pm\eta$ and $\pm\eta^{-1} = \pm(1 - \sqrt[3]{2})$. Also, the important roles have all (algebraic) units of $\mathcal{O}_K$, i.e. $\pm\eta^n$, $n \in \mathbb{Z}$.

Following lemmas are valid in any commutative ring R with an unit.

Lemma 1 (Theorem 1, [6])**.** *Let $\mu, \kappa \in R$. The set*

$$\{\mu, \mu(3\kappa + 1)^2 + 2\kappa, \mu(3\kappa + 2)^2 + 2\kappa + 2, 9\mu(2\kappa + 1)^2 + 8\kappa + 4\} \tag{2.1}$$

has the property $D(2\mu(2\kappa + 1) + 1)$.

We emphasize that the set (2.1) is considered as a $D(2\mu(2\kappa + 1) + 1)$-quadruple if it contains no equal elements or zero elements.

Lemma 2. *If $\{w_1, w_2, w_3, w_4\}$ is a $D(w)$-quadruple in R, then $\{w_1u, w_2u, w_3u, w_4u\}$ is a $D(wu^2)$-quadruple in R.*

3. On difference of two squares

Here, we describe the set of all integers that are representable as a difference of squares of two integers in $\mathbb{Q}(\sqrt[3]{2})$.

Theorem 2. *Let $\alpha, \beta, \gamma \in \mathbb{Z}$. Then $w = \alpha + \beta\sqrt[3]{2} + \gamma\sqrt[3]{4}$ can be represented as a difference of squares of two integers in $\mathbb{Q}(\sqrt[3]{2})$ if and only if*

$$\alpha \equiv 0, 1, 3 \pmod 4, \ \beta \equiv 0 \pmod 2 \ \textit{and} \ (\alpha, \beta, \gamma) \not\equiv (0, 0, 2) \pmod 4.$$

Proof. Suppose that

$$\alpha + \beta\sqrt[3]{2} + \gamma\sqrt[3]{4} = (a_1 + b_1\sqrt[3]{2} + c_1\sqrt[3]{4})^2 - (a_2 + b_2\sqrt[3]{2} + c_2\sqrt[3]{4})^2,$$

for some $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{Z}$, then

$$\begin{aligned} \alpha &= a_1^2 - a_2^2 + 4(b_1c_1 - b_2c_2), \\ \beta &= 2(c_1^2 - c_2^2 + b_1a_1 - b_2a_2), \\ \gamma &= b_1^2 - b_2^2 + 2(a_1c_1 - a_2c_2). \end{aligned}$$

It is easy to see that $\alpha \equiv 0,1,3 \pmod 4$ and $\beta \equiv 0 \pmod 2$. Also, if we assume that $\alpha \equiv \beta \equiv 0 \pmod 4$ and γ is even, then $a_1 \equiv a_2 \pmod 2$, $b_1 \equiv b_2 \pmod 2$, $c_1 \equiv c_2 \pmod 2$. Hence, $\gamma = b_1^2 - b_2^2 + 2(a_1c_1 - a_2c_2) \equiv 0 \pmod 4$, i.e. $\gamma \not\equiv 2 \pmod 4$.

The converse of the statement is proved by a series of equalities:

$$1 + 2\alpha + 2\beta\sqrt[3]{2} + (1 + 2\gamma)\sqrt[3]{4} =$$
$$(2\gamma-\alpha+1+(\alpha-\beta+1)\sqrt[3]{2}+(\beta-\gamma)\sqrt[3]{4})^2-(2\gamma-\alpha+(\alpha-\beta)\sqrt[3]{2}+(\beta-1-\gamma)\sqrt[3]{4})^2,$$
$$1 + 2\alpha + 2\beta\sqrt[3]{2} + 2\gamma\sqrt[3]{4} = (\alpha + 1 + \beta\sqrt[3]{2} + \gamma\sqrt[3]{4})^2 - (\alpha + \beta\sqrt[3]{2} + \gamma\sqrt[3]{4})^2,$$
$$4\alpha + 4\beta\sqrt[3]{2} + 4\gamma\sqrt[3]{4} = (\alpha + 1 + \beta\sqrt[3]{2} + \gamma\sqrt[3]{4})^2 - (\alpha - 1 + \beta\sqrt[3]{2} + \gamma\sqrt[3]{4})^2,$$
$$4\alpha + (4\beta + 2)\sqrt[3]{2} + 2\gamma\sqrt[3]{4} = (\alpha + \beta\sqrt[3]{2} + (\gamma + 1)\sqrt[3]{4})^2 - (\alpha + \beta\sqrt[3]{2} + \gamma\sqrt[3]{4})^2,$$
$$4\alpha+2\beta\sqrt[3]{2}+(2\gamma+1)\sqrt[3]{4} = (\alpha+(\beta+1)\sqrt[3]{2}+\gamma\sqrt[3]{4})^2-(\alpha+\beta\sqrt[3]{2}+\gamma\sqrt[3]{4})^2. \quad \square$$

4. The existence of $D(w)$-quadruples

In this section we construct a $D(w)$-quadruple for all $w \in \mathcal{O}_K$ of the form given in Theorem 2. The following lemma serves as a prototype or template for constructing $D(w)$-quadruples for the majority of cases.

Lemma 3. *Let $a, b, c \in \mathbb{Z}$ and $(a, b, c) \neq (0, -1, 0), (-1, 0, 0)$. The set*

$$\begin{aligned} D = \{&1 - \sqrt[3]{2},\\ &-21 - 32a - 9a^2 - 40b - 36ab - 18b^2 - 40c - 36ac - 36bc - 36c^2 +\\ &(-15 - 20a - 9a^2 - 32b - 18ab - 18b^2 - 40c - 36ac - 36bc - 18c^2)\sqrt[3]{2}+\\ &(-11 - 20a - 9a^2 - 20b - 18ab - 9b^2 - 32c - 18ac - 36bc - 18c^2)\sqrt[3]{4},\\ &-16 - 26a - 9a^2 - 40b - 36ab - 18b^2 - 40c - 36ac - 36bc - 36c^2 +\\ &(-12 - 20a - 9a^2 - 26b - 18ab - 18b^2 - 40c - 36ac - 36bc - 18c^2)\sqrt[3]{2}+\\ &(-11 - 20a - 9a^2 - 20b - 18ab - 9b^2 - 26c - 18ac - 36bc - 18c^2)\sqrt[3]{4},\\ &-75 - 116a - 36a^2 - 160b - 144ab - 72b^2 - 160c - 144ac - 144bc - 144c^2 +\\ &(-53 - 80a - 36a^2 - 116b - 72ab - 72b^2 - 160c - 144ac - 144bc - 72c^2)\sqrt[3]{2}+\\ &(-44 - 80a - 36a^2 - 80b - 72ab - 36b^2 - 116c - 72ac - 144bc - 72c^2)\sqrt[3]{4}\}\end{aligned}$$

is a $D(3 + 4a + (2 + 4b)\sqrt[3]{2} + 4c\sqrt[3]{4})$-quadruple in $\mathbb{Z}[1, \sqrt[3]{2}, \sqrt[3]{4}]$. The set

$$\{-19 + 5\sqrt[3]{2} + 8\sqrt[3]{4}, -1749109 - 1388265\sqrt[3]{2} - 1101867\sqrt[3]{4},$$
$$-1749060 - 1388224\sqrt[3]{2} - 1101831\sqrt[3]{4}, -6996319 - 5552983\sqrt[3]{2} - 4407404\sqrt[3]{4}\}$$

is a $D(3 - 2\sqrt[3]{2})$-quadruple and

$$\{-19 + 5\sqrt[3]{2} + 8\sqrt[3]{4}, -17524872 - 13909496\sqrt[3]{2} - 11039975\sqrt[3]{4},$$
$$-17524709 - 13909365\sqrt[3]{2} - 11039867\sqrt[3]{4}, -70099143 - 55637727\sqrt[3]{2} - 44159692\sqrt[3]{4}\}$$

is a $D(-1 + 2\sqrt[3]{2})$-quadruple.

Proof. Let $\mu = 1 - \sqrt[3]{2}$ and $\kappa = \alpha + \beta\sqrt[3]{2} + \gamma\sqrt[3]{4} \in \mathcal{O}_K$. Then

$$w = 2\mu(2\kappa + 1) + 1 = 4\alpha - 8\gamma + 3 + (-4\alpha + 4\beta - 2)\sqrt[3]{2} + (-4\beta + 4\gamma)\sqrt[3]{4}$$

is of the form $3+4a+(2+4b)\sqrt[3]{2}+4c\sqrt[3]{4}$. Now, let us make sure that for given $a, b, c \in \mathbb{Z}$ there exist $\alpha, \beta, \gamma \in \mathbb{Z}$ such that $w = 3 + 4a + (2 + 4b)\sqrt[3]{2} + 4c\sqrt[3]{4}$. Indeed, the linear system

$$4\alpha - 8\gamma + 3 = 3 + 4a,\ -4\alpha + 4\beta - 2 = 2 + 4b,\ -4\beta + 4\gamma = 4c,$$

i.e. the system

$$\alpha - 2\gamma = a,\ -\alpha + \beta = 1 + b,\ -\beta + \gamma = c$$

has the integer solution

$$\alpha = -a - 2b - 2c - 2,\ \beta = -a - b - 2c - 1,\ \gamma = -a - b - c - 1.$$

By putting this solution into (2.1) we obtain the set D. This is a $D(w)$-quadruple if it consists of non-zero distinct elements. Hence, we have to check whether there are equal elements or elements equal to zero in D and that requires solving 9 systems of three (nonlinear) equations. The results of the investigation are two integer solutions $(a, b, c) = (0, -1, 0), (-1, 0, 0)$ and corresponding sets $\{1 - \sqrt[3]{2}, 1 - \sqrt[3]{2}, 6 - 4\sqrt[3]{2}, 13 - 9\sqrt[3]{2}\}$ and $\{1 - \sqrt[3]{2}, 2 - 4\sqrt[3]{2}, 1 - \sqrt[3]{2}, 5 - 9\sqrt[3]{2}\}$ which do not represent $D(3 - 2\sqrt[3]{2})$ and $D(-1 + 2\sqrt[3]{2})$-quadruple respectively (because they have two equal elements). This situation can be resolved by putting $w = 3 - 2\sqrt[3]{2} = u^8(2\mu(2\kappa + 1) + 1)$, where $u = \mu = 1 - \sqrt[3]{2}$. Indeed, this holds for $\kappa = -7390 - 5865\sqrt[3]{2} - 4655\sqrt[3]{4}$ and according to Lemmas 1 and 2 we obtain the set $\{-19+5\sqrt[3]{2}+8\sqrt[3]{4}, -1749109-1388265\sqrt[3]{2}-1101867\sqrt[3]{4}, \ldots\}$. Similarly, we construct $D(-1+2\sqrt[3]{2})$-quadruple since $-1 + 2\sqrt[3]{2} = u^8(2\mu(2\kappa + 1) + 1)$ for $u = \mu = 1 - \sqrt[3]{2}$ and $\kappa = -23391 - 18565\sqrt[3]{2} - 14735\sqrt[3]{4}$. □

In what follows, the majority of cases are treated similarly by assuming that $w = u^2(2\mu(2\kappa + 1) + 1)$ for some $u, \mu \in \mathbb{Z}[1, \sqrt[3]{2}, \sqrt[3]{4}]$ and $\kappa \in \mathbb{Q}(\sqrt[3]{2})$. By multiplying the elements of the set (2.1) by u we obtain a $D(w)$-quadruple up to finitely many exceptions. These exceptions are resolved separately as in Lemma 3. The results for the existence of $D(w)$-quadruples for all w of the form $2a + 1 + 2b\sqrt[3]{2} + c\sqrt[3]{4}$, $a, b, c \in \mathbb{Z}$ are exposed in several tables in Section 6. To be more simple, we occasionally represent an integer $w = a+b\sqrt[3]{2}+c\sqrt[3]{4}$ as a triple $w = (a, b, c)$.

A Diophantine quadruple for each exception can be found as in Lemma 3. Precisely, each exception can be represented in a form $v^8u^2(2\mu(2\kappa+1)+1)$ or $v^{16}u^2(2\mu(2\kappa + 1) + 1)$, where μ and κ are given in tables for a corresponding exception and $v = 1 - \sqrt[3]{2}$. (Note that $w \cdot v^8 \equiv w \pmod 4$ and $w \cdot v^{16} \equiv w$

(mod 8)).

Since, we have shown the existence of $D(w)$-quadruples for $w = 2a+1+2b\sqrt[3]{2}+c\sqrt[3]{4}$, by multiplying w with u^2 where $u=\sqrt[3]{2}$, $u=\sqrt[3]{4}$ and $u=2$ we prove that there exist $D(4a+2b\sqrt[3]{2}+(1+2c)\sqrt[3]{4})$, $D(4a+(2+4b)\sqrt[3]{2}+4c\sqrt[3]{4})$ and $D(8a+4+8b\sqrt[3]{2}+4c\sqrt[3]{4})$-quadruples, respectively. The remaining cases are given in the tables in Section 7.

The exceptions are not a problem and we treat them the same way as before. Hence, we have proved that if w is a difference of two squares of integers then there exists a $D(w)$-quadruple and even more, there exist infinitely many $D(w)$-quadruples. Indeed, if v is an algebraic unit so is v^{-1} and an another $D(w)$-quadruple is obtained by multiplying the elements of $D(wv^2)$-quadruple by v^{-1}. So, the existence of infinitely many algebraic units in $\mathbb{Q}(\sqrt[3]{2})$ imply that there are infinitely many $D(w)$-quadruples. One may ask whether distinct quadruples by multiplying by units appear. The answer is positive due to the fact that the first element of a Diophantine quadruple created as described above is of the form μu, where μ and u are taken from a finite set of values. For instance, see Lemma 3.

5. The nonexistence of $D(w)$-quadruples

Lemma 4 (Lemma 3.1, Theorem 3.2, [15])**.** *If $w \in \mathcal{O}_K$ is of one of the following forms*

$$a+(2b+1)\sqrt[3]{2}+c\sqrt[3]{4},\ \ 4a+2+b\sqrt[3]{2}+c\sqrt[3]{4},$$

then there exist no $D(w)$-quadruples in $\mathcal{O}_K$.

Lemma 5. *If $w \in \mathcal{O}_K$ is of the form $4a+4b\sqrt[3]{2}+(4c+2)\sqrt[3]{4}$ then there exist no $D(w)$-quadruples in $\mathcal{O}_K$.*

Proof. Let $a,b,c \in \mathbb{Z}$ and $w = 4a+4b\sqrt[3]{2}+(4c+2)\sqrt[3]{4}$. Assume that $\{w_1,w_2,w_3,w_4\}$ is a $D(w)$-quadruple in $\mathcal{O}_K$. If $w_i = a_i+b_i\sqrt[3]{2}+c_i\sqrt[3]{4}$, i.e. $w_i=(a_i,b_i,c_i)$, $1\le i\le 4$, then

$$\begin{aligned} w_i\cdot w_j \bmod 4 \in S = \{&(0,0,2),(0,0,3),(0,2,2),(0,2,3),\\ &(1,0,1),(1,0,2),(1,2,0),(1,2,3)\}, \end{aligned}$$

for $1\le i<j\le 4$. We obtain that there do not exist four integers w_1,w_2,w_3,w_4 in $\mathcal{O}_K$ such that the above condition is fulfilled by testing all possibilities directly. To be more specific, we characterize congruence types modulo 4 of Diophantine pairs and triples. Let us recall, that the $D(w)$-pair $\{w_1,w_2\}$ has a congruence type $[(\alpha_1,\beta_1,\gamma_1),(\alpha_2,\beta_2,\gamma_2)]$ if $a_i\equiv\alpha_i \pmod 4$, $b_i\equiv\beta_i \pmod 4$, $c_i\equiv\gamma_i \pmod 4$, $i=1,2$. Without loss of generality, we assume that $\{w_1,w_2\}$ is ordered in an ascending order according to a congruence type

i.e. $\overline{\alpha_1\beta_1\gamma_1}_4 \leq \overline{\alpha_2\beta_2\gamma_2}_4$, where $\overline{\alpha\beta\gamma}_4 = \alpha \cdot 4^2 + \beta \cdot 4 + \gamma \cdot 1$ (- a number in the quaternary numeral system with digits α, β and γ). For instance, if $w_1 \mod 4 = (0,0,1)$ and if $\{w_1, w_2\}$ is a $D(w)$-pair, then

$$w_2 \mod 4 \in S_1 = \{(2,0,0),(2,0,1),(2,0,2),(2,0,3),(2,2,0),(2,2,1),(2,2,2),(2,2,3),(3,0,0),(3,0,1),(3,0,2),(3,0,3),(3,2,0),(3,2,1),(3,2,2),(3,2,3)\}.$$

Now, if $w_2 \mod 4 = (2,0,0)$ and if $\{w_2, w_3\}$ is a $D(w)$-pair, then

$$w_3 \mod 4 \in S_2 = \{(2,0,1),(2,0,3),(2,1,1),(2,1,3),(2,2,1),(2,2,3),(2,3,1),(2,3,3)\}.$$

So, if $\{w_1, w_2, w_3\}$ is a $D(w)$-triple and if $\{w_1, w_2\}$ has a congruence type $[(0,0,1),(2,0,0)]$ then

$$w_3 \mod 4 \in S_1 \cap S_2 = \{(2,0,1),(2,0,3),(2,2,1),(2,2,3)\}.$$

Finally, if $w_3 \mod 4 = (2,0,1)$ and if $\{w_3, w_4\}$ is a $D(w)$-pair, then

$$w_4 \mod 4 \in S_3 = \{(2,0,2),(2,2,0),(2,2,2),(3,1,0),(3,1,2),(3,3,0),(3,3,2)\}.$$

Under assumptions that $\{w_1, w_2, w_3, w_4\}$ is a $D(w)$-quadruple and $\{w_1, w_2, w_3\}$ has a congruence type $[(0,0,1),(2,0,0),(2,0,1)]$, it holds that

$$w_4 \in S_1 \cap S_2 \cap S_3 = \emptyset.$$

□

According to Theorem 2 and Lemmas 4, 5, we conclude the if $w \in \mathcal{O}_K$ is not representable a difference of two squares of integers, then a $D(w)$-quadruple does not exist. That proves the second statement of the Theorem 1.

6. List of tables for $D(2a+1+2b\sqrt[3]{2}+c\sqrt[3]{4})$-quadruples

$D(4a+3,2b,c)$-quadruples

w	μ	u	$k=(\alpha,\beta,\gamma)$	exceptions
$(3+4a,4b,4c)$	1	1	(a,b,c)	$(3,0,0)$, $(-1,0,0)$
(3+4a,4b,1+4c)	$(1-\sqrt[3]{2})^3$	$1-\sqrt[3]{2}$	$(-318-281a-354b-446c, -252-223a-281b-354c, -200-177a-223b-281c)$	$(-37,8,17)$, $(39,-12,-15)$
$(3+4a,4b,2+4c)$	$(1-\sqrt[3]{2})^2$	1	$(6+5a+6b+8c, 5+4a+5b+6c, 4+3a+4b+5c)$	$(3,-4,2)$, $(-1,4,-2)$
$(3+4a,4b,3+4c)$	$1-\sqrt[3]{2}$	$1-\sqrt[3]{2}$	$(-37-19a-24b-30c, -29-15a-19b-24c, -23-12a-15b-19c)$	$(-1,-8,7)$, $(3,4,-5)$
$(3+4a,2+4b,4c)$	$1-\sqrt[3]{2}$	1	$(-2-a-2b-2c, -1-a-b-2c, -1-a-b-c)$	$(3,-2,0)$, $(-1,2,0)$
(3+4a,2+4b,1+4c)	$(1-\sqrt[3]{2})^2$	$1-\sqrt[3]{2}$	$(128+73a+92b+116c, 102+58a+73b+92c, 81+46a+58b+73c)$	$(-13,-6,13)$, $(15,2,-11)$
$(3+4a,2+4b,2+4c)$	$(1-\sqrt[3]{2})^3$	1	$(-37-19a-24b-30c, -29-15a-19b-24c, -23-12a-15b-19c)$	$(-1,-6,6)$, $(3,6,-6)$
$(3+4a,2+4b,3+4c)$	1	$1-\sqrt[3]{2}$	$(12+5a+6b+8c, 10+4a+5b+6c, 8+3a+4b+5c)$	$(3,-6,3)$,$(-1,2,-1)$

$D(4a+1,4b,4c)$-quadruples

w	μ	u	$k=(\alpha,\beta,\gamma)$	exceptions
(5+8a,8b,4c)	2	1	$(a,b,\frac{c}{2})$	$(5,0,0)$, $(-3,0,0)$
(5+8a,4+8b,4c)	2	$(1-\sqrt[3]{2})^4$	$(20080+16001a+20160b+6350c, 15938+12700a+16001b+5040c, 25300+20160a+25400b+\frac{16001}{2}c)$	$(5,500,-400)$, $(-3,-300,240)$
(1+8a,4+8b,4c)	$2\sqrt[3]{2}$	1	$(\frac{b}{2},\frac{c}{2},a)$	$(1,4,0)$, $(1,-4,0)$
(1+8a,8b,4c)	$2\sqrt[3]{2}$	$(1-\sqrt[3]{2})^4$	$(1587+12700a+\frac{16001b}{2}+5040c, 2520+20160a+12700b+\frac{16001}{2}c, 2000+16001a+10080b+6350c$	$(-639,104,320)$, $(641,96,-480)$

$D(4a+1,4b,4c+1)$-quadruples

w	μ	u	$k=(\alpha,\beta,\gamma)$	exceptions
$(5+8a,4b,1+4c)$	$\sqrt[3]{2}$	$1-\sqrt[3]{2}$	$(6+8a+5b+6c, 5+6a+4b+5c, 4+5a+3b+4c)$	$(5,0,-3)$, $(-3,-4,5)$
$(1+8a,4b,1+4c)$	$\sqrt[3]{2}$	$(1-\sqrt[3]{2})^5$	$(309581+375968a+236845b+298406c, 245715+298406a+187984b+236845c, 195024+236845a+149203b+187984c)$	$(-595,980,-403)$ $(1637,-1104,-155)$

$D(4a+1,4b,4c+2)$-quadruples

w	μ	u	$\kappa=(\alpha,\beta,\gamma)$	exceptions
$(1+8a,4b,2+4c)$	$\sqrt[3]{4}$	1	$(\frac{c}{2}a,b)$	$(1,0,2)$, $(1,0,-2)$
$(5+8a,4b,2+4c)$	$2-\sqrt[3]{4}$	1	$(1+2a+b+\frac{c}{2}, 1+a+b+\frac{c}{2}, 2+2a+b+c)$	$(5,0,-2)$,$(-3,0,2)$

$D(4a+1,4b,4c+3)$-quadruples

w	μ	u	$\kappa=(\alpha,\beta,\gamma)$	exceptions
$(1+8a,4b,3+4c)$	$2+\sqrt[3]{2}+\sqrt[3]{4}$	$1-\sqrt[3]{2}$	$(1+2a+b+2c,1+2a+b+c,1+a+b+c)$	$(1,-4,3)$, $(1,0,-1)$
$(5+8a,4+8b,3+8c)$	4	$1+\sqrt[3]{2}+\sqrt[3]{4}$	$(-1+\frac{a}{2}+b-2c,-a+\frac{b}{2}+c,$ $\frac{a}{2}-b+\frac{c}{2})$	$(5,4,3)$, $(45,36,27)$ $(-35,-28,-21)$
$(5+8a,8b,3+8c)$	4	$(1+\sqrt[3]{2}+\sqrt[3]{4})^5$	$(278+\frac{521}{2}a-279b-62c,-248-31a+\frac{521}{2}b-279c,$ $21-\frac{279}{2}a-31b+\frac{521}{2}c)$	(5,205949008, -1044421)
$(5+8a,4+8b,7+8c)$	$2\sqrt[3]{4}$	$1+\sqrt[3]{2}+\sqrt[3]{4}$	$(a-b+\frac{c}{2},-2+a+b-2c,1-2a+\frac{b}{2}+c)$	$(37,28,23)$, $(-27,-20,-17)$
$(5+8a,8b,7+8c)$	$2\sqrt[3]{4}$	$(1+\sqrt[3]{2}+\sqrt[3]{4})^5$	$(281-279a-31b+\frac{521}{2}c,217+521a-279b-62c,$ $-527-62a+\frac{521}{2}-279c$	(125880325,0, 1096583)

$D(4a+1,4b+2,4c)$-quadruples

w	μ	u	$\kappa=(\alpha,\beta,\gamma)$	exceptions
$(1+8a,2+4b,4c)$	$\sqrt[3]{2}$	1	(b,c,a)	$(1,2,0)$, $(1,-2,0)$
$(5+8a,2+4b,4c)$	$2-\sqrt[3]{4}$	$(1+\sqrt[3]{2}+\sqrt[3]{4})^2$	$-3-6a+3b+\frac{c}{2},-1+a-3b+\frac{3}{2}c,4+6a+b-3c$	(133,106,84), $(13,10,8)$

$D(4a+1,4b+2,4c+1)$-quadruples

w	μ	u	$\kappa=(\alpha,\beta,\gamma)$	exceptions
(5+8a, 6+8b,1+4c)	2	$1-\sqrt[3]{2}$	$(8+5a+6b+2c,7+4a+5b+\frac{3}{2}c,11+6a+8b+\frac{5}{2}c)$	$(5,-10,5)$, $(-3,6,-3)$
(5+8a,2+8b ,1+4c)	2	$(1-\sqrt[3]{2})^5$	$(269625+236845a+298406b+93992c,214002+187984a+$ $236845b+\frac{149203}{2}c,339707+298406a+375968b+\frac{236845}{2}c)$	$(2605,-310,-1395)$, $(-1563,186,837)$
(1+8a, 6+8b,1+4c)	$2\sqrt[3]{2}$	$(1+\sqrt[3]{2}+\sqrt[3]{4})^3$	$(-23+24a-\frac{35}{2}b+\frac{3}{2}c,28+6a+24b-\frac{35}{2}c,$ $6-35a+3b+12c)$	$(6529,5182,4113)$, $(-4367,-3466,-2751)$
(1+8a, 2+8b,1+4c)	$2\sqrt[3]{2}$	$1-\sqrt[3]{2}$	$(2+4a+\frac{5}{2}b+\frac{3}{2}c,4+6a+4b+\frac{5}{2}c,3+5a+3b+2c)$	$(9,2,-7)$, $(-7,-6,9)$

$D(4a+1,4b+2,4c+2)$-quadruples

w	μ	u	$\kappa=(\alpha,\beta,\gamma)$	exceptions
(1+8a,2+4b,2+4c)	$\sqrt[3]{4}-\sqrt[3]{2}$	1	$(1+2a+b+2c,1+2a+b+c,1+a+b+c)$	$(1,2,-2)$, $(1,-2,2)$
(5+8a,2+4b,2+4c)	$2+\sqrt[3]{2}+\sqrt[3]{4}$	1	$(2a-b,b-c,-a+c)$	$(5,2,2)$, $(-3,-2,-2)$

$D(4a+1,4b+2,4c+3)$-quadruples

(a,b,c) (mod 8)	μ	u	$\kappa=(\alpha,\beta,\gamma)$	exceptions
(5+8a,2+4b,3+4c)	$2-\sqrt[3]{4}$	$1-\sqrt[3]{2}$	$36+24a+15b+\frac{19}{2}c,29+19a+12b+\frac{15}{2}c,46+30a+19b+12c)$	$(13,-14,3)$, $(-11,10,-1)$
(1+8a,2+4b,3+4c)	$\sqrt[3]{2}$	$1+\sqrt[3]{2}+\sqrt[3]{4}$	$(1-4a+b+2c,2a-2b+c,-1+a+b-2c)$	$(17,14,11)$, $(-7,-6,-5)$

7. List of tables for some $D(4a+2b\sqrt[3]{2}+2c\sqrt[3]{4})$-quadruples

$D(8a,4b,4c)$-quadruples

w	μ	u	$\kappa=(\alpha,\beta,\gamma)$	exceptions
$(8a,8b,8c)$	$1/2$	2	$(-1+a,b,c)$	$(0,0,0)$ (!0), $(8,0,0)$
$(8a,4+8b,8c)$	$(1-\sqrt[3]{2})/2$	2	$(-1-a-2b-2c,-a-b-2c,-a-b-c)$	$(8,-4,0)$, $(0,4,0)$
$(8a,8b,4+8c)$	$(1-\sqrt[3]{2})^2/2$	2	$(1+5a+6b+8c,1+4a+5b+6c,1+3a+4b+5c)$	$(8,-8,4)$, $(0,8,-4)$
$(8a,4+8b,4+8c)$	$(1-\sqrt[3]{2})^3/2$	2	$(-18-19a-24b-30c,-14-15a-19b-24c,-11-12a-15b-19c)$	$(0,-12,12)$, $(8,12,-12)$

$D(4a,4b+2,4c+2)$-quadruples

w	μ	u	$\kappa=(\alpha,\beta,\gamma)$	exceptions
$(8a,2+8b,2+4c)$	$1/\sqrt[3]{4}$	$\sqrt[3]{4}$	(c,a,b)	$(0,2,2)$, $(0,2,-2)$
$(4+8a,2+8b,2+4c)$	$(1-\sqrt[3]{2})/\sqrt[3]{4}$	$\sqrt[3]{4}$	$(-2-2a-2b-c,-1-a-2b-c,$ $-1-a-b-c)$	$(-4,2,2)$ $(4,2,-2)$
$(8a,6+8b,2+4c)$	$(1-\sqrt[3]{2})^2/\sqrt[3]{4}$	$\sqrt[3]{4}$	$6+6a+8b+5c,5+5a+6b+4c,4+4a+5b+3c)$	$(-8,6,2)$, $(8,-2,-2)$
$(4+8a,6+8b,2+4c)$	$(1-\sqrt[3]{2})^3/\sqrt[3]{4}$	$\sqrt[3]{4}$	$(-37-24a-30b-19c,-29-19a-24b-15c,$ $-23-15a-19b-12c)$	$(-12,14,-2)$, $(12,-10,2)$

$D(8a+4,8b+4,4c)$-quadruples

w	μ	u	$\kappa=(\alpha,\beta,\gamma)$	exceptions
$(4+16a,4+8b,8c)$	$1/\sqrt[3]{4}$	2	(b,c,a)	$(4,4,0)$, $(4,-4,0)$
$(4+16a,4+8b,4+8c)$	$(1-\sqrt[3]{2})^1/\sqrt[3]{4}$	2	$(-2-2a-b-2c,-1-2a-b-c,$ $-1-a-b-c)$	$(4,4,-4)$ $(4,-4,4)$
$(12+16a,4+8b,8c)$	$(1-\sqrt[3]{2})^2/\sqrt[3]{4}$	2	$(6+8a+5b+6c,5+6a+4b+5c,$ $4+5a+3b+4c)$	$(12,4,-8)$ $(-4,-4,8)$
$(12+16a,4+8b,4+8c)$	$(1-\sqrt[3]{2})^3/\sqrt[3]{4}$	2	$-37-30a-19b-24c,-29-24a-15b-19c,$ $-23-19a-12b-15c)$	$(28,-4,-12)$ $(-20,4,12)$

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