ON THE EXTENSIBILITY OF DIOPHANTINE TRIPLES $\{k-1, k+1, 4k\}$ FOR GAUSSIAN INTEGERS

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Abstract. In this paper, we prove that if $\{k-1, k+1, 4k, d\}$, for $k \in \mathbb{Z}[i] \backslash \{0, \pm 1\}$, is a Diophantine quadruple in the ring of Gaussian integers, then $d = 16k^3 - 4k$.

1. Introduction

The set of non-zero elements $\{a_1, a_2, \ldots, a_m\}$ in a commutative ring R with 1 is called *Diophantine m-tuple* if $a_i a_j + 1$ is a perfect square in R for all $1 \leq i < j \leq m$. Let us mention few most famous historical examples of such sets: the first rational quadruple $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ found by Diophantus of Alexandria in third century AD, the first integer quadruple $\{1, 3, 8, 120\}$ found by Fermat in the seventeenth century, the first rational sextuple $\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\}$ found by Gibbs ([11]). There exist families of such sets, for instance quadruples $\{F_{2k}, F_{2k+2}, F_{2k+4}, 4F_{2k+1}F_{2k+2}F_{2k+3}\}$ (where F_k is k-th Fibonacci number) and $\{k-1, k+1, 4k, 16k^3-4k\}$ (which , actually, represent a generalization of the Fermat's quadruple).

In this paper, we deal with the extensibility of a particular family of triples $\{k-1, k+1, 4k\}$ in the ring of Gaussian integers $\mathbb{Z}[i]$. Here are some important results in the ring of integers. In 1969, Baker and Davenport in [2] showed that the Diophantine triple $\{1, 3, 8\}$ extends uniquely to the quadruple $\{1, 3, 8, 120\}$. Obviously, this result implies that $\{1, 3, 8\}$ cannot be extended to a Diophantine quintuple. In 1998, Dujella and Pethő in [9] proved that the Diophantine pair $\{1, 3\}$ can be extended to infinitely many quadruples, but it cannot be extended to a quintuple. Arkin, Hoggatt and Strauss showed that each Diophantine quadruple can be extended to quadruple (see [1]). Moreover, the following conjecture is very plausible: *If $\{a, b, c\}$ is a Diophantine triple, then there exists unique positive integer d such that $d > \max\{a, b, c\}$ and $\{a, b, c, d\}$ is a Diophantine quadruple.* This conjecture has already been proved for a large class of Diophantine triples (see [7] and [8]). Furthermore, as a consequence it was obtained that there

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is no Diophantine sextuple in $\mathbb{Z}$ and that there is only finitely many Diophantine quintuples ([8]). Concerning the family of triples $\{k-1, k+1, 4k\}$ in $\mathbb{Z}$, Dujella showed in [5] that for $k \neq 0, \pm 1$ this triple can be extended uniquely to a quadruple $\{k-1, k+1, 4k, 16k^3-4k\}$. Here, the analogous statement in $\mathbb{Z}[i]$ will be showed, i.e. we prove the following theorem

Theorem 1. *Let $k \in \mathbb{Z}[i]\backslash\{0, \pm 1\}$ and let $\{k-1, k+1, 4k, d\}$ be a Diophantine quadruple in $\mathbb{Z}[i]$. Then $d = 16k^3 - 4k$.*

In Section 2 we show that the original problem of extending the triple $\{k-1, k+1, 4k\}$ is equivalent to the problem of solving the following system of two Diophantine equations:

$$(k+1)x^2 - (k-1)y^2 = 2, \ 4kx^2 - (k-1)z^2 = 3k+1. \tag{1}$$

Solutions of each equation in (1) form linear recurrence sequences. If (1) is solvable then these sequences have the same initial term ($x_0 = 1$ which is related to a trivial solution of (1)), for all parameters $k \in \mathbb{Z}[i]$, $|k| > 5$. This is showed in Section 3 using some congruence conditions modulo $2k-1$ and $4k(k-1)$. In Section 4 we apply an analog of Bennett's theorem on simultaneous rational approximations of square roots which are close to one by rationals in the case of imaginary quadratic fields ([12]) and obtain that all solutions of (1), for $|k| \geq 350$, are $x = \pm 1$ and $x = \pm(4k^2 - 2k - 1)$. In Section 5, we solve our problem for $5 < |k| < 350$ by transforming the exponential equations into inequalities for linear forms in three logarithms of algebraic numbers, then applying Baker's theory on linear forms ([3]) and, finally, we reduce the upper bound for the solution of (1) by using a version of Baker-Davenport's reduction method ([2]) in Section 6.

All other cases ($1 \leq |k| \leq 5$) are solved separately in the last two sections. In the case $k = i$, instead of (1) we solve the following system of Pellian equations

$$y^2 + ix^2 = i + 1, \ z^2 - (2-2i)x^2 = -1 + 2i. \tag{2}$$

The set of solutions of (2) is described using [10] and then the same procedure as in Section 5 is applied. For some parameters $1 < |k| \leq 5$, we obtain, perhaps surprisingly, some extra solutions.

All computations are performed in *Mathematica 5.2.*

2. Solving a system of Diophantine equations

Let $k \in \mathbb{Z}[i]\backslash\{0, \pm 1\}$. Our aim is to determine all Diophantine quadruples of the form $\{k-1, k+1, 4k, d\}$ in $\mathbb{Z}[i]$. Thus, we have to solve the system

$$(k-1)d + 1 = x^2, \ (k+1)d + 1 = y^2, \ 4kd + 1 = z^2, \tag{3}$$

in $d, x, y, z \in \mathbb{Z}[i]$. By eliminating d in (3), we obtain the following system of Diophantine equations

$$(k+1)x^2 - (k-1)y^2 = 2, \tag{4}$$

$$4kx^2 - (k-1)z^2 = 3k+1. \tag{5}$$

It can be seen that the system of equations (4) and (5) is equivalent to the system (3). Indeed, if $x, y, z \in \mathbb{Z}[i]$ are the solutions of (4) and (5), than it follows that

$$(k+1)(x^2-1) = (k-1)(y^2-1),\ 4k(x^2-1) = (k-1)(z^2-1).$$

So, d is well defined by

$$d = \frac{x^2-1}{k-1} = \frac{y^2-1}{k+1} = \frac{z^2-1}{4k}.$$

We have to show that $d \in \mathbb{Z}[i]$. According to (4), we obtain that $(k+1)x^2 \equiv 0 (\mathrm{mod}\ (k-1))$, i.e. $2x^2 \equiv 0(\mathrm{mod}\ (k-1))$. Thus, we have that $2d \in \mathbb{Z}[i]$. Besides that, $2d$ can be represented as a difference of two squares of Gaussian integers, i.e. $2d = y^2 - x^2$. Hence, $2d$ must be of the form $2m+2ni$ or of the form $2d = 2m+1+2ni$, where $m, n \in \mathbb{Z}$ (see ([14, p. 449])). Suppose that $2d = 2m+1+2ni$. We can obtain a contradiction by showing that at least one of the numbers $(k-1)d+1$, $(k+1)d+1$ and $4kd+1$ is not a perfect square in $\mathbb{Z}[i]$. Let us note that k is of the form $2l+1$, $l \in \mathbb{Z}[i]$, because $(k-1)d$ is a Gaussian integer. If we assume that $2d \equiv 1(\mathrm{mod}\ 4)$ and $l \equiv 0(\mathrm{mod}\ 4)$, i.e. $k \equiv 1(\mathrm{mod}\ 8)$, then

$$y^2 = (k+1)d+1 \equiv 2(\mathrm{mod}\ 4),$$

and this is contradiction since $y^2 \mod 4 \in \{0,1,3,2i\}$. Similarly, we verify all the others possibilities ($2d \mod 4 \in \{3, 1+2i, 3+2i\}$ and $l \mod 4 \in \{1,2,3\}$). Hence, we conclude that $2d$ must be of the form $2m+2ni$, i.e that d is a Gaussian integer.

Our further step is to solve the system of equations (4) and (5) in $\mathbb{Z}[i]$. The following lemma describes the set of all solutions of the equation (4) in $\mathbb{Z}[i]$.

Lemma 1. *Let $k \in \mathbb{Z}[i] \backslash \{0, \pm 1, \pm i\}$. Then there exist $i_0 \in \mathbb{N}$ and $x_0^{(i)}, y_0^{(i)} \in \mathbb{Z}[i]$, $i = 1, \dots, i_0$, such that*

(i): *$(x_0^{(i)}, y_0^{(i)})$ is a solution of (4) for all $i = 1, \dots, i_0$,*
(ii): *the estimates*

$$|x_0^{(i)}|^2 \leq \frac{2|k-1|}{|k|-1}, \tag{6}$$

$$|y_0^{(i)}|^2 \leq \frac{2}{|k-1|} + \frac{2|k+1|}{|k|-1}, \tag{7}$$

hold for all $i = 1, \dots, i_0$,
(iii): *for each solution (x, y) of (4) there exist $i \in \{1, \dots, i_0\}$ and $m \in \mathbb{Z}$ such that*

$$x\sqrt{k+1} + y\sqrt{k-1} = (x_0^{(i)}\sqrt{k+1} + y_0^{(i)}\sqrt{k-1})(k + \sqrt{k^2-1})^m. \tag{8}$$

Proof. If (x,y) is a solution of (4), than (x_m,y_m) obtained by

$$x_m\sqrt{k+1}+y_m\sqrt{k-1}=(x\sqrt{k+1}+y\sqrt{k-1})(k+\sqrt{k^2-1})^m \tag{9}$$

is also a solution of (4) for all $m\in\mathbb{Z}$.

Let (x^*,y^*) be an element of the sequence $(x_m,y_m)_{m\in\mathbb{Z}}$ (defined by (9)) such the absolute value $|x^*|$ is minimal. We put

$$\begin{aligned}
x'\sqrt{k+1}+y'\sqrt{k-1} &= (x^*\sqrt{k+1}+y^*\sqrt{k-1})(k+\sqrt{k^2-1}),\\
x''\sqrt{k+1}+y''\sqrt{k-1} &= (x^*\sqrt{k+1}+y^*\sqrt{k-1})(k+\sqrt{k^2-1})^{-1}\\
&= (x^*\sqrt{k+1}+y^*\sqrt{k-1})(k-\sqrt{k^2-1}).
\end{aligned}$$

Due to minimality of $|x^*|$, we have that

$$\begin{aligned}
|x^*| &\le |x'|=|x^*k+y^*(k-1)|,\\
|x^*| &\le |x''|=|x^*k-y^*(k-1)|.
\end{aligned}$$

At least one of the expressions $|x^*k+y^*(k-1)|$ and $|x^*k-y^*(k-1)|$ must be greater or equal to $|x^*||k|$, since $|x^*k+y^*(k-1)|+|x^*k-y^*(k-1)|\ge 2|x^*||k|$. Let us assume that $|x^*k+y^*(k-1)|\ge|x^*||k|$. Hence,

$$|(x^*k)^2-(y^*(k-1))^2|\ge|x^*|^2|k|,$$

and

$$|(x^*)^2+2(k-1)|\ge|x^*|^2|k|.$$

Immediately, we obtain the estimate for $|x^*|$,

$$|x^*|^2\le\frac{2|k-1|}{|k|-1}.$$

This implies the estimate for $|y^*|$,

$$|(k-1)(y^*)^2|=|(k+1)(x^*)^2-2|\le|k+1|\frac{2|k-1|}{|k|-1}+2.$$

It is obvious that there exists only finitely many x^* an y^* such that above estimates are fulfilled. Finally, according to the definition of x^*, there exist $m_0\in\mathbb{Z}$ such that

$$x^*\sqrt{k+1}+y^*\sqrt{k-1}=(x\sqrt{k+1}+y\sqrt{k-1})(k+\sqrt{k^2-1})^{m_0}.$$

Therefrom, we obtain that

$$x\sqrt{k+1}+y\sqrt{k-1}=(x^*\sqrt{k+1}+y^*\sqrt{k-1})(k+\sqrt{k^2-1})^{-m_0}.$$

□

The solutions $(x_0^{(i)},y_0^{(i)})$, $i=1,\ldots,i_0$, defined in Lemma 1, will be called *fundamental solutions* of the equation (4).

Analogously, all solutions of (5) are given by the following lemma.

Lemma 2. *Let $k\in\mathbb{Z}[i]\backslash\{0,1\}$. Then there exist $j_0\in\mathbb{N}$ and $x_1^{(j)},z_1^{(j)}\in\mathbb{Z}[i]$, $j=1,\ldots,j_0$, such that*

(i): *$(x_1^{(j)},z_1^{(j)})$ is a solution of (5) for all $j=1,\ldots,j_0$,*

(ii): *the estimates*

$$|x_1^{(j)}|^2 \leq \frac{|k-1||3k+1|}{|2k-1|-1}, \tag{10}$$

$$|z_1^{(j)}|^2 \leq \frac{4|k||3k+1|}{|2k-1|-1} + \frac{|3k+1|}{|k-1|}, \tag{11}$$

hold for all $j = 1, \ldots, j_0$,

(iii): *for each solution* (x, z) *of (5) there exist* $j \in \{1, \ldots, j_0\}$ *and* $n \in \mathbb{Z}$ *such that*

$$x\sqrt{4k} + z\sqrt{k-1} = (x_1^{(j)}\sqrt{4k} + z_1^{(j)}\sqrt{k-1})(2k-1+\sqrt{4k(k-1)})^n. \tag{12}$$

Now, we create the sequences

$$v_0^{(i)} = x_0^{(i)},\ v_1^{(i)} = kx_0^{(i)} + (k-1)y_0^{(i)},\ v_{m+2}^{(i)} = 2kv_{m+1}^{(i)} - v_m^{(i)}, \tag{13}$$

$$v'^{(i)}_0 = x_0^{(i)},\ v'^{(i)}_1 = kx_0^{(i)} - (k-1)y_0^{(i)},\ v'^{(i)}_{m+2} = 2kv'^{(i)}_{m+1} - v'^{(i)}_m, \tag{14}$$

for all $m \in \mathbb{N}_0$ and $i = 1, \ldots, i_0$. If x is a solution of (4), then there exist a nonnegative integer m and $i \in \{1, \ldots, i_0\}$ such that $x = v_m^{(i)}$ or $x = v'^{(i)}_m$. Similarly, if x is a solution of (5), then there exist $n \geq 0$ and $j \in \{1, \ldots, j_0\}$ such that $x = w_n^{(j)}$ or $x = w'^{(j)}_n$, where

$$w_0^{(j)} = x_1^{(j)},\ w_1^{(j)} = (2k-1)x_1^{(j)} + (k-1)z_1^{(j)},\ w_{n+2}^{(j)} = 2(2k-1)w_{n+1}^{(j)} - w_n^{(j)}, \tag{15}$$

$$w'^{(j)}_0 = x_1^{(j)},\ w'^{(j)}_1 = (2k-1)x_1^{(j)} - (k-1)z_1^{(j)},\ w'^{(j)}_{n+2} = 2(2k-1)w'^{(j)}_{n+1} - w'^{(j)}_n. \tag{16}$$

Lemma 3. *Let* $k \in \mathbb{Z}[i]$ *and* $|k| > 3$. *Then* $x_0 = \pm 1$ *and* $y_0 = \pm 1$ *are the only fundamental solutions of (4) and all solutions are represented by the sequences* (v_m) *and* $(-v_m)$ *defined by*

$$v_0 = 1,\ v_1 = 2k-1,\ v_{m+2} = 2kv_{m+1} - v_m,\ m \in \mathbb{N}_0. \tag{17}$$

Proof. Suppose that x_0 is a fundamental solution of (4). Then the estimate (6) implies that

$$|x_0|^2 \leq 2(1 + \frac{2}{|k|-1}) < 4.$$

Hence, $|x_0|^2 = 1$ or $|x_0|^2 = 2$. Obviously, $x_0 = \pm 1$, $y_0 = \pm 1$ are the solutions of (4), for all k. Also, the following cases may appear:

- $x_0 = 0$, $y_0 = \pm(1+i)$, $k = 1+i$,
- $x_0 = 0$, $y_0 = \pm(1-i)$, $k = 1-i$,
- $x_0 = \pm(1+i)$, $y_0 = 0$, $k = -1-i$,
- $x_0 = \pm(1-i)$, $y_0 = 0$, $k = -1+i$,
- $x_0 = 0$, $y_0 = \pm i$, $k = 3$,
- $x_0 = \pm i$, $y_0 = 0$, $k = -3$

Evidently, these cases are not satisfying the condition $|k| > 3$. The rest of the assertion follows immediately from (13) and (14). □

Before proceeding further, let us recapitulate our results: For $|k| > 3$, the problem of solving (4) and (5) is reduced to looking for the intersections of recursive sequences, i.e. to solving the equations

$$v_m = \pm w_n,\ v_m = \pm w'_n,\ \ m, n \geq 0 \tag{18}$$

where we omitted the upper index (j).

3. Congruence method

In this section, we will determine all fundamental solutions of the equation (5) under the assumption that one of the equations in (18) is solvable. We will apply the congruence method which was first introduced by Dujella and Pethő in [5].

Lemma 4. *If (x_1, z_1) is a fundamental solutions of (5), then*

$$x_1 \bmod (2k-1) \in \{0, 1, -1\} \text{ or } z_1(k-1) \bmod (2k-1) \in \{0, 1, -1\}.$$

Proof. We have

$$(v_m \bmod (2k-1))_{m\geq 0} = (1, 0, -1, -1, 0, 1, 1, 0, -1, -1, \ldots),$$
$$(w_n \bmod (2k-1))_{n\geq 0} = (x_1, z_1(k-1), -x_1, -z_1(k-1), x_1, z_1(k-1), \ldots),$$
$$(w'_n \bmod (2k-1))_{n\geq 0} = (x_1, -z_1(k-1), -x_1, z_1(k-1), x_1, -z_1(k-1), \ldots).$$

These congruence relations are obtained by induction from (17), (15) and (16), respectively. The rest follows immediately from (18). □

In what follows, we will discuss all the possibilities given in Lemma (4).

• $\boldsymbol{x_1 \equiv 0 (\mathrm{mod}\ (2k-1))}$

In this case, we have that $x_1 = u(2k-1)$, for some $u \in \mathbb{Z}[i]$. Hence, $|x_1| \geq |2k-1|$ or $x_1 = 0$. If $x_1 \neq 0$, then (10) implies that

$$|2k-1|^2 \leq |x_1|^2 \leq \frac{|k-1||3k+1|}{|2k-1|-1}. \tag{19}$$

Therefrom, we obtain that

$$2(2|k|-1)^2(|k|-1) \leq |2k-1|^2(|2k-1|-1) \leq |k-1||3k+1| \leq (|k|+1)(3|k|+1).$$

Obviously, $2(2|k|-1)^2(|k|-1) - (|k|+1)(3|k|+1) > 0$, for $|k| > 3$ and this is in contrary with (19). So, for $|k| > 3$ there is no non-zero fundamental solution x_1 such that $x_1 \equiv 0 (\mathrm{mod}\ (2k-1))$.

The equation (5) has the solution $x_1 = 0$ if and only if $k \in \{0, -1, 1 \pm i, 5\}$.

• $\boldsymbol{x_1 \equiv \pm 1 (\mathrm{mod}\ (2k-1))}$

Let us assume that $x_1 = u(2k-1) \pm 1$ for some $u \in \mathbb{Z}[i]$. If $x_1 \neq \pm 1$, than $|x_1| \geq |2k-1| - 1$. According to (10), we obtain

$$(|2k-1|-1)^2 \leq \frac{|k-1||3k+1|}{|2k-1|-1}. \tag{20}$$

Further, if $|k| > 3$, then $(|2k-1|-1)^3 - |k-1||3k+1| \geq 8(|k|-1)^3 - (|k|+1)(3|k|+1) > 0$, but this contradicts (20). Hence, under the assumptions $|k| > 3$ and $x_1 \equiv \pm 1 (\text{mod } (2k-1))$, all fundamental solutions of (5) are $x_1 = \pm 1$.

• $\boldsymbol{z_1(k-1) \equiv 0(\text{mod } (2k-1))}$

Here, $z_1 = 0$ is a solution of (5) if and only if $k = 1$. If we assume that $z_1 \neq 0$, then $z_1 = u(2k-1)$ for some $u \in \mathbb{Z}[i]\backslash\{0\}$ (because $k-1$ and $2k-1$ are relatively prime). So, $|z_1| \geq |2k-1|$. As in the previous cases, according to (11), we obtain that there is no non-zero fundamental solution of (5) such that $z_1(k-1) \equiv 0(\text{mod } (2k-1))$ and $|k| > 4$

• $\boldsymbol{z_1(k-1) \equiv \pm 1(\text{mod } (2k-1))}$

We have that $z_1 \equiv \mp 2(\text{mod } (2k-1))$. The solution of (5) is $z_1 = \pm 2$ if and only if $k = 1$. If $z_1 \neq \pm 2$, then $z_1 = u(2k-1) \pm 2$ for some $u \in \mathbb{Z}[i]\backslash\{0\}$. According to (11), we get that there is no fundamental solution of (5) such that $z_1(k-1) \equiv \pm 1(\text{mod } (2k-1))$ and $|k| > 5$.

The above results can be resumed in the following lemma.

Lemma 5. *Let $k \in \mathbb{Z}[i]$ and $|k| > 5$. If at least one of the equations in (18) is solvable, then all fundamental solutions of the equation (5) are $x_1 = \pm 1$, $z_1 = \pm 1$ and related sequences (w_n) and (w'_n) are given by*

$$w_0 = 1,\ w_1 = 3k-2,\ w_{n+2} = 2(2k-1)w_{n+1} - w_n, \tag{21}$$

$$w'_0 = 1,\ w'_1 = k,\ w'_{n+2} = 2(2k-1)w'_{n+1} - w'_n, \tag{22}$$

for $n \in \mathbb{N}_0$.

Lemma 6. *The sequences (v_m), (w_n) and (w'_n) defined by (17), (21) and (22), respectively, satisfy the following congruences*

$$(v_m \text{ mod } 4k(k-1))_{m\geq 0} = (1, 2k-1, 2k-1, 1, 1, 2k-1, 2k-1\ldots),$$
$$(w_n \text{ mod } 4k(k-1))_{n\geq 0} = (1, 3k-2, -2k+3, 5k-4, -4k+5, 7k-6, -6k+7, \ldots),$$
$$(w'_n \text{ mod } 4k(k-1))_{n\geq 0} = (1, k, 2k-1, 2-k, 4k-3, -3k+4, 6k-5, -5k+6, \ldots).$$

Proof. This can be verified by induction method. □

Lemma 7. *Let $k \in \mathbb{Z}[i]$, $|k| > 5$ and let $x \in \mathbb{Z}[i]\backslash\{\pm 1\}$ be a solution of the system of equations (4) and (5). Then there exist $m, n \in \mathbb{N}$, $n \equiv 0$ or $\pm 2(\text{mod } 4k)$, such that*

$$x = v_m = w_n \text{ or } x = -v_m = -w_n \text{ or } x = v_m = w'_n \text{ or } x = -v_m = -w'_n,$$

where (v_m), (w_n) and (w'_n) are given by (17), (21) and (22), respectively.

Proof. If $v_m = \pm w_{2n+1}$ or $v_m = \pm w'_{2n+1}$, then Lemma 4 implies that $z_1(k-1)(\text{mod } (2k-1)) \in \{0, 1, -1\}$. But, there is no solution z_1 of (5) which satisfies these conditions.

Let $v_m = w_{2n}$. Then, according to Lemma 6, two cases may arise: $-2nk + 2n + 1 \equiv 1(\text{mod } 4k(k-1))$ or $-2nk + 2n + 1 \equiv 2k-1(\text{mod } 4k(k-1))$. Let us analyze each of them.

- If $-2nk+2n+1 \equiv 1 \pmod{4k(k-1)}$, then $-2n(k-1) \equiv 0 \pmod{4k(k-1)}$, i.e. $2n \equiv 0 \pmod{4k}$.
- If $-2nk+2n+1 \equiv 2k-1 \pmod{4k(k-1)}$, then $-2n(k-1)-2(k-1) \equiv 0 \pmod{4k(k-1)}$. Hence, $2n \equiv -2 \pmod{4k}$.

If we assume that $v_m = -w_{2n}$, then the following possibilities occur:

- If $2nk-2n-1 \equiv 1 \pmod{4k(k-1)}$, i.e. if $2nk-2n-2 = 4k(k-1)z$ for some $z \in \mathbb{Z}[i]$, then $(k-1)(n-2kz) = 1$. But, this equation is not solvable in $\mathbb{Z}[i]$ for $|k| > 5$.
- If $2nk-2n-1 \equiv 2k-1 \pmod{4k(k-1)}$, then $2nk-2n = 2k+4k(k-1)z$ for some $z \in \mathbb{Z}[i]$. Therefrom, we obtain that $(k-1)(n-2kz-1) = 1$ and this equation has no solution in $\mathbb{Z}[i]$ for $|k| > 5$.

Similarly, we show that the assumption $v_m = w'_{2n}$ implies that $2n \equiv 0 \pmod{4k}$ or $2n \equiv 2 \pmod{4k}$. Also, the assumption $v_m = -w'_{2n}$ leads to a contradiction.

□

Now, observe that $v_0 = w_0 = w'_0 = 1$ and $v_2 = w'_2 = -1-2k+4k^2$. So, $x = \pm 1$ and $x = \pm(4k^2-2k-1)$ are solutions of the system of equations (4), (5). The solution $x = \pm 1$ is not interesting for us, because it corresponds to $d = 0$ which presents a trivial extension of the triple $\{k-1, k+1, 4k\}$. On the other hand, the solution $x = \pm(4k^2-2k-1)$ corresponds to $d = 16k^3-4k$. Since we intend to prove that this is the unique nontrivial extension of the triple $\{k-1, k+1, 4k\}$, we have to show that the system of equations (4), (5) has no other solutions, but those given above. Our next step is to determine an upper bound for all solutions of (4), (5) that are different from the previous ones.

Lemma 8. *Let $k \in \mathbb{Z}[i]$ and $|k| > 5$. If $x \in \mathbb{Z}[i] \setminus \{\pm 1, \pm(4k^2-2k-1)\}$ satisfies the system of Diophantine equations (4), (5), then*

$$|x| \geq (4|k|-3)^{4|k|-3}.$$

Proof. According to Lemma 7, there exists $n > 2$, $n \equiv 0 \pmod{4k}$ or $n \equiv \pm 2 \pmod{4k}$, such that $x = \pm w_n$ or $x = \pm w'_n$. The sequence $(|w_n|)$ is increasing. Let us show this by induction. Obviously, $|w_0| \leq |w_1|$. Now, assume that $|w_n| \leq |w_{n+1}|$. From (21), we have that

$$|w_{n+2}| \geq |2(2k-1)w_{n+1}| - |w_n| \geq (|2(2k-1)| - 1)|w_{n+1}| \geq |w_{n+1}|.$$

Analogously, we obtain that $(|w'_n|)$ is an increasing sequence.

Now, let us show that $|w_n| \geq (4|k|-3)^{n-1}$, for all $n \in \mathbb{N}$. It can be easily verified that it is true for $n = 1$. Let us assume that the above inequality is true for some $n \in \mathbb{N}$. According to (21), we obtain that

$$|w_{n+1}| \geq (4|k|-2)|w_n| - |w_{n-1}| = (4|k|-3)|w_n| + |w_n| - |w_{n-1}|.$$

So, using the fact that $(|w_n|)$ is an increasing sequence, we get

$$|w_{n+1}| \geq (4|k|-3)(4|k|-3)^{n-1} \geq (4|k|-3)^n.$$

Further, we have that $|n| \geq 4|k|-2$, because $n \equiv 0 (\bmod 4k)$ or $n \equiv -2 (\bmod 4k)$ and $n \neq 0, 2$. Hence, $|w_n| \geq (4|k|-3)^{4|k|-3}$.

The same can be proved for the sequence (w'_n).

□

4. An application of the theorem on simultaneous approximations

In this section, we prove that if the parameter $|k|$ is large enough, then $x = \pm 1$ and $x = \pm(4k^2 - 2k - 1)$ give all solutions of the system of equations (4), (5). For that reason, we apply the following generalization of Bennett's theorem [4] on simultaneous rational approximations of square roots which are close to one.

Theorem 2. *([12]) Let $\theta_i = \sqrt{1 + \frac{a_i}{T}}$, $i = 1, 2$, with a_1 and a_2 pairwise distinct quadratic integers in the imaginary quadratic field K and let T be an algebraic integer of K. Further, let $M = \max\{|a_1|, |a_2|\}$, $|T| > M$ and*

$$\begin{aligned} l &= \frac{27}{64} \frac{|T|}{|T|-M}, \\ L &= \frac{27}{16|a_1|^2|a_2|^2|a_1-a_2|^2}(|T|-M)^2 > 1, \\ p &= \sqrt{\frac{2|T|+3M}{2|T|-2M}}, \\ P &= 16 \frac{|a_1|^2|a_2|^2|a_1-a_2|^2}{\min\{|a_1|, |a_2|, |a_1-a_2|\}^3}(2|T|+3M). \end{aligned}$$

Then

$$\max\left(\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right| \right) > c|q|^{-\lambda},$$

for all algebraic integers $p_1, p_2, q \in K$, where

$$\begin{aligned} \lambda &= 1 + \frac{\log P}{\log L}, \\ c^{-1} &= 4pP(\max\{1, 2l\})^{\lambda-1}. \end{aligned}$$

First, let us show the following technical lemma.

Lemma 9. *Let $k \in \mathbb{Z}[i]$, $|k| > 5$ and let $(x, y, z) \in \mathbb{Z}[i]^3$ be a solution of the system of equations (4), (5). Furthermore, let*

$$\theta_1^{(1)} = \pm\sqrt{\frac{k+1}{k-1}}, \quad \theta_1^{(2)} = -\theta_1^{(1)},$$

$$\theta_2^{(1)} = \pm\sqrt{\frac{k}{k-1}}, \quad \theta_2^{(2)} = -\theta_2^{(1)},$$

where the signs are chosen such that

$$\left|\theta_1^{(1)} - \frac{y}{x}\right| \leq \left|\theta_1^{(2)} - \frac{y}{x}\right|, \quad \left|\theta_2^{(1)} - \frac{z}{2x}\right| \leq \left|\theta_2^{(2)} - \frac{z}{2x}\right|.$$

Then, we obtain

$$\begin{aligned} \left|\theta_1^{(1)} - \frac{y}{x}\right| &\leq \frac{2}{\sqrt{|k^2-1|}} \cdot \frac{1}{|x|^2}, \\ \left|\theta_2^{(1)} - \frac{z}{2x}\right| &\leq \frac{1}{4}\frac{|3k+1|}{\sqrt{|k^2-k|}} \cdot \frac{1}{|x|^2}. \end{aligned}$$

Proof. We have

$$\left|\theta_1^{(1)} - \frac{y}{x}\right| = \left|(\theta_1^{(1)})^2 - \frac{y^2}{x^2}\right| \cdot \left|\theta_1^{(1)} + \frac{y}{x}\right|^{-1} = \frac{2}{|k-1||x|^2}\left|\theta_1^{(2)} - \frac{y}{x}\right|^{-1}.$$

Because of the assumptions on $\theta_1^{(1)}$ and $\theta_1^{(2)}$, we get

$$\left|\theta_1^{(2)} - \frac{y}{x}\right| \geq \frac{1}{2}\left(\left|\theta_1^{(1)} - \frac{y}{x}\right| + \left|\theta_1^{(2)} - \frac{y}{x}\right|\right) \geq \frac{1}{2}\left|\theta_1^{(1)} - \theta_1^{(2)}\right| = \left|\sqrt{\frac{k+1}{k-1}}\right|.$$

Hence,

$$\left|\theta_1^{(1)} - \frac{y}{x}\right| \leq \frac{2}{|k-1||x|^2}\left|\sqrt{\frac{k-1}{k+1}}\right|.$$

Similarly, according to

$$\begin{aligned} \left|\theta_2^{(1)} - \frac{z}{2x}\right| &\leq \frac{1}{|k-1||2x|^2}|4kx^2 - (k-1)z^2|\left|\theta_2^{(2)} - \frac{z}{2x}\right|^{-1} \\ &= \frac{|3k+1|}{4|k-1||x|^2}\left|\theta_2^{(2)} - \frac{z}{2x}\right|^{-1}, \\ \left|\theta_2^{(2)} - \frac{z}{2x}\right| &\geq \left|\sqrt{\frac{k}{k-1}}\right|, \end{aligned}$$

the other estimate is obtained. □

Now, we will apply Theorem 3 on $\theta_1^{(1)}$ and $\theta_2^{(1)}$. In our case, we have $a_1 = 2$, $a_2 = 1$, $T = k-1$, $M = 2$ and

$$l = \frac{27}{64}\frac{|k-1|}{|k-1|-2},\ L = \frac{27}{64}(|k-1|-2)^2,\ p = \sqrt{\frac{|k-1|+3}{|k-1|-2}},\ P = 128(|k-1|+3).$$

The condition $L > 1$ of Theorem 3 is satisfied, because $L > 0.43(|k|-3)^2$ and $|k| > 5$. So, we conclude that

$$\max\left\{\left|\theta_1^{(1)} - \frac{2y}{2x}\right|, \left|\theta_2^{(1)} - \frac{z}{2x}\right|\right\} > c|2x|^{-\lambda}, \tag{23}$$

where

$$\lambda = 1 + \frac{\log P}{\log L},\ c^{-1} = 4pP(\max\{1, 2l\})^{\lambda-1}.$$

If we assume that $|k| > 14$, then $\max\{1, 2l\} = 1$ and $c^{-1} = 4pP$.

Further, according to Lemma 9, we have

$$\max\left\{\left|\theta_1^{(1)} - \frac{y}{x}\right|, \left|\theta_2^{(1)} - \frac{z}{2x}\right|\right\} \leq \frac{1}{4}\frac{|3k+1|}{\sqrt{|k^2-k|}} \cdot \frac{1}{|x|^2},$$

and (23) implies

$$\frac{1}{4}\sqrt{\frac{|k-1|-2}{|k-1|+3}} \cdot \frac{1}{128(|k-1|+3)}|2x|^{-\lambda} < \frac{|3k+1|}{\sqrt{|k^2-k|}} \cdot \frac{1}{|2x|^2}.$$

Hence,

$$|2x|^{2-\lambda} \leq 2^9\frac{|3k+1|}{\sqrt{|k^2-k|}}\sqrt{\frac{(|k-1|+3)^3}{|k-1|-2}}. \tag{24}$$

Now, we use the estimate for x, $|x| \geq (4|k|-3)^{4|k|-3}$ (from Lemma 8), and after taking a logarithm of (24), we obtain

$$(2-\lambda)(\log 2 + (4|k|-3)\log(4|k|-3)) \leq \log\left(2^9\frac{|3k+1|}{\sqrt{|k^2-k|}}\sqrt{\frac{(|k-1|+3)^3}{|k-1|-2}}\right), \tag{25}$$

where log denotes the natural logarithm. This gives us an inequality for k, since

$$\lambda = 1 + \frac{7\log 2 + \log(|k-1|+3)}{\log\frac{27}{64} + 2\log(|k-1|-2)}.$$

Finally, let us assume that $|k| \geq 350$. Then $2-\lambda > 0.01$. The right side of (25) satisfies the following inequality

$$\log\left(2^9\frac{|3k+1|}{\sqrt{|k^2-k|}}\sqrt{\frac{(|k-1|+3)^3}{|k-1|-2}}\right) \quad \leq \quad \log(3|k|) + 7.$$

On the other hand, we obtain that the left side of (25) satisfies

$$0.01(\log 2 + (4|k|-3)\log(4|k|-3)) > \log(3|k|) + 7,$$

and that is a contradiction.

We just proved the following statement.

Theorem 3. *Let $k \in \mathbb{Z}[i]$ and $|k| \geq 350$. Then all solutions of the system of equations (4), (5) are given by $x = \pm 1$, $y = \pm 1$, $z = \pm 1$ and $x = \pm(4k^2-2k-1)$, $y = \pm(4k^2+2k-1)$, $z = \pm(8k^2-1)$.*

5. Linear forms in three logarithms

In this section, we study the case where $k \in \mathbb{Z}[i]$ and $5 < |k| < 350$. We will apply a method similar to those used in [2].

Let $x = v_m = w_m$ for some $m, n \in \mathbb{N}$. By solving the recurrences (17) and (21) for (v_m) and (w_n), we obtain

$$x = \frac{\sqrt{k-1}+\sqrt{k+1}}{2\sqrt{k+1}}(k+\sqrt{k^2-1})^m - \frac{\sqrt{k-1}-\sqrt{k+1}}{2\sqrt{k+1}}(k-\sqrt{k^2-1})^m,$$

$$x = \frac{\sqrt{k-1}+2\sqrt{k}}{4\sqrt{k}}(2k-1+2\sqrt{k^2-k})^n - \frac{\sqrt{k-1}-2\sqrt{k}}{4\sqrt{k}}(2k-1-2\sqrt{k^2-k})^n.$$

From now on, let us assume that $\mathrm{Re}(k) > 0$. Besides that, we will discus the case where $\mathrm{Re}(k) = 0$ and $\mathrm{Im}(k) > 0$. The other two cases ($\mathrm{Re}(k) < 0$ and $\mathrm{Re}(k) = 0$, $\mathrm{Im}(k) < 0$) can be avoided by taking a quadruple $\{-k+1, -k-1, -4k, -d\}$ instead of a quadruple $\{k-1, k+1, 4k, d\}$.

Let

$$P = \frac{\sqrt{k-1}+\sqrt{k+1}}{\sqrt{k+1}}(k+\sqrt{k^2-1})^m, \tag{26}$$

$$Q = \frac{\sqrt{k-1}+2\sqrt{k}}{2\sqrt{k}}(2k-1+2\sqrt{k^2-k})^n. \tag{27}$$

The equation $v_m = w_n$ implies that

$$P + \frac{2}{k+1}P^{-1} = Q + \frac{3k+1}{4k}Q^{-1}. \tag{28}$$

Now, we estimate the values $|P|$ and $|Q|$ and obtain that $|P| > 5$ and $|Q| > 9$. Further, according to (28) we have

$$||P|-|Q|| = |P-Q| \le \left|\frac{3k+1}{4k}\right| |Q|^{-1} + \left|\frac{2}{k+1}\right| |P|^{-1} < 0.2.$$

Hence, $|P| \le |Q| + 0.2 \le 1.03|Q|$, i.e. $|Q|^{-1} \le 1.03|P|^{-1}$ and

$$\left|\frac{P-Q}{P}\right| \le \left|\frac{3k+1}{4k}\right| |Q|^{-1}|P|^{-1} + \left|\frac{2}{k+1}\right| |P|^{-2} < 1.33|P|^{-2} < 0.06.$$

Finally, we obtain that

$$\left|\log\frac{|P|}{|Q|}\right| = \left|\log\left(1 - \frac{|P|-|Q|}{|Q|}\right)\right| < 1.33|P|^{-2} + (1.33|P|^{-2})^2 < 1.5|P|^{-2} < 16^{-m}.$$

The above expression can be written as a linear form in three logarithms:

$$\Bigg| m\log|k+\sqrt{k^2-1}| - n\log|2k-1+2\sqrt{k^2-k}| + \log\left|\frac{2\sqrt{k}(\sqrt{k-1}+\sqrt{k+1})}{\sqrt{k+1}(\sqrt{k-1}+2\sqrt{k})}\right|\Bigg| < 16^{-m} \tag{29}$$

and it is valid for all $k \in \mathbb{Z}[i]$ such that $\mathrm{Re}(k) > 0$ and $|k| > 5$.

We use the following theorem of Baker and Wüstholz ([3], p.20) to obtain a upper bound for m.

Theorem 4. *Let Λ be a nonzero linear form in logarithms of l algebraic numbers $\alpha_1,\dots,\alpha_l$ with rational integer coefficients $b_1,\dots,b_l$. Then*

$$\log\Lambda \geq -18(l+1)!\,l^{l+1}(32d)^{l+2}h'(\alpha_1)\cdots h'(\alpha_l)\log(2ld)\log B,$$

where $B=\max(|b_1|,\dots,|b_l|)$ and where d is the degree of the number field generated by $\alpha_1,\dots,\alpha_l$ over the rationals.

Here
$$h'(\alpha)=\max(h(\alpha),\frac{1}{d}|\log\alpha|,\frac{1}{d}),$$
where $h(\alpha)$ denotes the standard logarithmic Weil height of α ([3], p.22).

In our case, we have

$$\log|m\log\alpha_1-n\log\alpha_2+\log\alpha_3| \geq -18\cdot 4!\,3^4(32d)^5h'(\alpha_1)\cdots h'(\alpha_l)\log(6d)\log B,$$

where

$$\begin{aligned}\alpha_1 &= |k+\sqrt{k^2-1}|,\\ \alpha_2 &= |2k-1+2\sqrt{k^2-k}|,\\ \alpha_3 &= \left|\frac{2\sqrt{k}(\sqrt{k-1}+\sqrt{k+1}}{\sqrt{k+1}(\sqrt{k-1}+2\sqrt{k})}\right|.\end{aligned}$$

First, let us verify that the condition $\Lambda\neq 0$ in Theorem 4 is satisfied. Equivalently, we will show that $|P|\neq|Q|$. This condition is not trivially satisfied and it will be proved in the following lemma.

Lemma 10. *If $v_m=w_n$, then $|P|\neq|Q|$ for all $k\in\mathbb{Z}[i]\backslash\{0,\pm1\}$.*

Proof. Assume that $|P|=|Q|$. If $P=Q$, then (28) imply that $3k^2-4k+1=0$. The only integer solution of this equation is $k=1$ (which is not of our interest), so we conclude that $P\neq Q$.

Let us denote
$$\alpha=\sqrt{\frac{k+1}{k-1}},\quad \beta=\sqrt{\frac{k-1}{k}}.$$
According to (26) and (27), we have
$$P=a+b\alpha,\quad Q=c+d\beta,$$
where $a,b,c,d\in\mathbb{Q}[i]$. Moreover, the assumption $v_n=w_m$ implies that $a=c$, because $v_m=(a+b\alpha+a-b\alpha)/2$, i.e. $v_m=a$ and, similarly, $w_n=b$. Further, we have

$$|P|^2 = p+u\alpha+\overline{u\alpha}+q|\alpha|^2, \tag{30}$$
$$|Q|^2 = r+v\beta+\overline{v\beta}+s|\beta|^2, \tag{31}$$

where $p,q,r,s\in\mathbb{Q}$ and $u,v\in\mathbb{Q}[i]$.

At the moment, let us point out several facts that are crucial for our proof:

A: The complex numbers α, β are algebraic numbers of degree 2, for $k \in \mathbb{Z} \setminus \{0, \pm 1\}$.

B: The basis for $\mathbb{Q}[i](\alpha, \overline{\alpha})$ (considered as a vector space over $\mathbb{Q}[i]$) is $B_\alpha = \{1, \alpha, \overline{\alpha}, |\alpha|^2\}$ and, analogously, the basis for $\mathbb{Q}[i](\beta, \overline{\beta})$ is $B_\beta = \{1, \beta, \overline{\beta}, |\beta|^2\}$.

C: The set $B = \{1, \alpha, \overline{\alpha}, |\alpha|^2, \beta, \overline{\beta}, |\beta|^2\}$ is linearly independent.

Obviously, $|P|^2$ is an element of the algebraic extension field $\mathbb{Q}[i](\alpha, \overline{\alpha})$ (because $|\alpha|^2 = \alpha\overline{\alpha}$) and is uniquely represented in (30). Analogously, $|Q|^2 \in \mathbb{Q}[i](\beta, \overline{\beta})$ is uniquely represented in (31). Finally, the assumption $|P|^2 = |Q|^2$ implies that $u = q = v = s = 0$, because B is a linearly independent set. Hence, we have that $P = a$ and $Q = c$. So, $P = Q$ (because $a = c$), but we have already shown that this is not possible.

In what follows, we will prove the statements A,B and C.

Proof of A: Let us assume conversely that $\alpha \in \mathbb{Q}[i]$. Then $(k+1)/(k-1)$ is a perfect squares in $\mathbb{Q}[i]$. So, there exist $\rho, A, B \in \mathbb{Z}[i]$ such that

$$k + 1 = \rho A^2, \quad k - 1 = \rho B^2. \tag{32}$$

Therefrom follows that $2 = \rho(A^2 - B^2)$. Using the fact that $\mathbb{Z}[i]$ is a ring with unique factorization, we that only finitely many cases may occur: $(\rho, A^2 - B^2) \in \{(\pm 2, \pm 1), (\pm 2i, \mp i), (\pm 1, \pm 2), (\pm i, \mp 2i), (\pm(1+i), \pm(1-i)), (\pm(1-i), \pm(1+i))\}$. This implies that $k \in \{\pm 1, 0\}$. In the same way, the assumption that $\beta \in \mathbb{Q}[i]$, i.e. that $(k-1)/k$ is a perfect square in $\mathbb{Q}[i]$ implies that $k \in \{0, 1\}$.

Proof of B: If $\gamma \in \mathbb{Q}[i](\alpha, \overline{\alpha})$, then $\gamma = \sum q_{ij}\alpha^i\overline{\alpha}^j$, where $q_{ij} \in QQ[i]$. But, $\alpha^2, \overline{\alpha}^2 \in \mathbb{Q}[i]$ and $\alpha\overline{\alpha} = |\alpha|^2$ imply that $\gamma = q_0 + q_1\alpha + q_2\overline{\alpha} + q_3|\alpha|^2$ for some $q_0, q_1, q_2, q_3 \in \mathbb{Q}[i]$. Hence, the set B_α spans $\mathbb{Q}[i](\alpha, \overline{\alpha})$. Next we have to show that B_α is linearly independent. Suppose that the set $\{1, \alpha, \overline{\alpha}\}$ is linearly dependent. Hence, there exist $A, B \in \mathbb{Q}[i]$ such that

$$\overline{\alpha} = A + B\alpha.$$

By squaring the previous equation, we obtain

$$\overline{\alpha}^2 - A^2 - B^2\alpha^2 = 2AB\alpha$$

and, therefrom, we get that $\alpha \in \mathbb{Q}[i]$, a contradiction. So, $\{1, \alpha, \overline{\alpha}\}$ is linearly independent.

Further, if we assume that the set $\{1, \alpha, \overline{\alpha}, |\alpha|^2\}$ is linearly dependent, then

$$|\alpha|^2 = A + B\alpha + C\overline{\alpha} \tag{33}$$

for some $A, B, C \in \mathbb{Q}[i]$. Multiplication by α gives us

$$C|\alpha|^2 = -B\alpha^2 - A\alpha + \alpha^2\overline{\alpha}. \tag{34}$$

We can see that $C \neq 0$. Suppose the contrary. Then $|\alpha|^2 = A + B\alpha$ and by squaring we get that $2AB\alpha \in \mathbb{Q}[i]$. According to (33) and (34), it follows that

$$A+B\alpha+C\overline{\alpha}=-\frac{B}{C}\alpha^2-\frac{A}{C}\alpha+\frac{1}{C}\alpha^2\overline{\alpha}.$$

and because $\{1,\alpha,\overline{\alpha}\}$ is linearly independent, we have

$$A=-\frac{B}{C}\alpha^2,\ B=\frac{A}{C},\ C=\frac{1}{C}\alpha^2.$$

Therefore, $C^2=\alpha^2$, but it is a contradiction, because α^2 is not a perfect square in $\mathbb{Q}[i]$.

Proof of C: It suffices to prove that β, $\overline{\beta}$ and $|\beta|^2$ are not elements of $L[\{1,\alpha,\overline{\alpha},|\alpha|^2\}]$, i.e. of $\mathbb{Q}[i](\alpha,\overline{\alpha})$. Suppose that β can be represented as

$$\beta=A+B\alpha+C\overline{\alpha}+D|\alpha|^2,$$

for some $A,B,C,D\in\mathbb{Q}[i]$. By squaring, we get

$$\begin{aligned}\beta^2&=A^2+B^2\alpha^2+C^2\overline{\alpha}^2+D^2|\alpha|^4\\&\quad+2AB\alpha+2AC\overline{\alpha}+2AD|\alpha|^2+2BC|\alpha|+2BD\alpha^2\overline{\alpha}+2CD\overline{\alpha}^2\alpha,\end{aligned}$$

But, $\beta^2\in\mathbb{Q}[i]$ implies that the coefficients of the algebraic numbers α, $\overline{\alpha}$ i $|\alpha|^2$ are equal zero, i.e.

$$AB+CD\overline{\alpha}^2=0, \tag{35}$$

$$AC+BD\alpha^2=0, \tag{36}$$

$$AD+BC=0. \tag{37}$$

Now, (35) and (37) imply that $A^2=C^2\overline{\alpha}^2$, then (36) and (37) imply that $A^2=B^2\alpha^2$ and (35) and (36) imply that $A^2=D^2|\alpha|^4$. Hence, $\beta^2=4A^2$, a contradiction. Similarly, we can obtain that $\overline{\beta}$ and $|\beta|^2$ are not in $\mathbb{Q}[i](\alpha,\overline{\alpha})$. □

Lemma 11. *Let $k\in\mathbb{Z}[i]$ such that $|k|>5$ and $Re(k)>0$. If $v_m=w_n$, then $n\le m$.*

Proof. We showed that $|Q|\le|P|+0.2$ and, therefrom, $|Q|<1.04|P|$, i.e.

$$\left|1+\frac{\sqrt{k-1}}{2\sqrt{k}}\right|\alpha_2^n<1.04\left|1+\frac{\sqrt{k-1}}{\sqrt{k+1}}\right|\alpha_1^m.$$

After logarithming the above inequality, we get

$$n<\log\left(\frac{1.04\left|1+\frac{\sqrt{k-1}}{\sqrt{k+1}}\right|}{\left|1+\frac{\sqrt{k-1}}{2\sqrt{k}}\right|}\right)\frac{1}{\log\alpha_2}+m\frac{\log\alpha_1}{\log\alpha_2}. \tag{38}$$

We use the following useful inequalities:

$$\left|1+\frac{\sqrt{k-1}}{2\sqrt{k}}\right|>1,\ 1.04\left|1+\frac{\sqrt{k-1}}{\sqrt{k+1}}\right|\le 2.4,\ \log\alpha_2>2.$$

We see that

$$\frac{\log \alpha_1}{\log \alpha_2} < 1.$$

Indeed,

$$\frac{\alpha_1}{|k|} = \left| 1 + \sqrt{1 - \frac{1}{k^2}} \right| < 2.02, \ \frac{\alpha_2}{|k|} \geq \left(2 \left| 1 + \sqrt{1 - \frac{1}{k}} \right| - \frac{1}{|k|} \right) > 3.5.$$

Applying the above inequalities to (38), we obtain

$$n < \frac{1}{2} + m.$$

□

Our next aim is to determine standard logarithmic Weil height of α_i, $i = 1, 2, 3$. For that purpose we need minimal polynomials of these algebraic numbers. The minimal polynomials of the algebraic numbers $\beta_1 = k^2 - \sqrt{k^2 - 1}$, $\beta_2 = 2k - 1 + 2\sqrt{k^2 - k}$ and $\beta_3 = \dfrac{2\sqrt{k}(\sqrt{k-1} + \sqrt{k+1})}{\sqrt{k+1}(\sqrt{k-1} + 2\sqrt{k})}$ were given in [5],

$$\begin{aligned} q_1(x) &= x^2 - 2kx + 1, \\ q_2(x) &= x^2 - 2(2k - 1)x + 1, \\ q_3(x) &= (9k^4 + 24k^3 + 22k^2 + 8k + 1)x^4 - 16k(3k^3 + 7k^2 + 5k + 1)x^3 \\ &\quad + 48k^2(k^2 + 4k + 3)x^2 - 128k^2(k + 1)x + 64k^2. \end{aligned}$$

According to the proof of Theorem 9.11 in [13], we can determine the minimal polynomials of the algebraic numbers $\beta_i\overline{\beta_i} = |\alpha_i|$, $i = 1, 2, 3$ (and clearly of α_i, too).

The minimal polynomials of α_1, α_2 are, respectively,

$$p_1(x) = x^8 - 4(\mu^2 + \nu^2)x^6 + (8\mu^2 - 8\nu^2 - 2)x^4 - 4(\mu^2 + \nu^2)x^2 + 1, \tag{39}$$

$$p_2(x) = x^8 - 4(4(\mu^2+\nu^2-\mu)+1)x^6 + (32(\mu^2-\nu^2-\mu-2)+6)x^4 - 4(4(\mu^2+\nu^2-\mu)+1)^2x^2 + 1, \tag{40}$$

where $k = \mu + i\nu$. The minimal polynomial of α_3 is of degree 32,

$$p_3(x) = \sum_{i=0}^{16} a_i x^{2i} \tag{41}$$

and it was derived by the help of the program package Mathematica. We list only few of its coefficients a_i (because its coefficients are huge rational functions in μ, ν).

$$
\begin{aligned}
a_0 &= -\frac{2^{48}(\mu^2+\nu^2)^8}{((1+\mu)^2+\nu^2)^8((1+3\mu)^2+3\nu^2)^8},\\
a_1 &= \frac{2^{50}(\mu^2+\nu^2)^8}{((1+\mu)^2+\nu^2)^7((1+3\mu)^2+3\nu^2)^8},\\
a_2 &= -\frac{3\cdot 2^{45}(\mu^2+\nu^2)^8}{((1+\mu)^2+\nu^2)^7((1+3\mu)^2+3\nu^2)^8}(21+46\mu+13(\mu^2+\nu^2)),\\
a_3 &= \frac{2^{44}(\mu^2+\nu^2)^7}{((1+\mu)^2+\nu^2)^6((1+3\mu)^2+3\nu^2)^8}(3-26\mu+247\mu^2+300\mu^3+36\mu^4+191\nu^2+300\mu\nu^2\\
&\quad +72\mu^2\nu^2+36\nu^4),\\
a_4 &= -\frac{2^{38}(\mu^2+\nu^2)^6}{((1+\mu)^2+\nu^2)^6((1+3\mu)^2+3\nu^2)^8}(-1+52\mu+14\mu^2-3196\mu^3+20299\mu^4+48864\mu^5\\
&\quad +40476\mu^6+9648\mu^7+324\mu^8+170\nu^2-2780\mu\nu^2+32854\mu^2\nu^2+86112\mu^3\nu^2+91452\mu^4\nu^2\\
&\quad +28944\mu^5\nu^2+1296\mu^6\nu^2+13867\nu^4+37248\mu\nu^4+61476\mu^2\nu^4+28944\mu^3\nu^4+1944\mu^4\nu^4\\
&\quad +10500\nu^6+9648\mu\nu^6+1296\mu^2\nu^6+324\nu^8),\\
&\vdots\\
&\vdots\\
a_{14} &= -\frac{1536(\mu^2+\nu^2)^2}{((1+\mu)^2+\nu^2)((1+3\mu)^2+3\nu^2)^2}(21+46\mu+13\mu^2+13\nu^2),\\
a_{15} &= \frac{256(\mu^2+\nu^2)}{((1+\mu)^2+\nu^2)},\\
a_{16} &= -1.
\end{aligned}
$$

Further, for the purpose of determining the heights $h(\alpha_i)$, we have to find all roots of the minimal polynomials p_i or, if it is not possible, we have to bound them. With some algebraic manipulation, we can get all roots of p_1 and p_2. The roots of p_1 are

$$
\begin{aligned}
x_1,x_2 &= \pm|k+\sqrt{k^2-1}| = \pm\alpha_1,\\
x_3,x_4 &= \pm|k-\sqrt{k^2-1}|,\\
x_5,x_6 &= \pm\sqrt{|k|^2-|k^2-1|-\sqrt{(|k|^2-|k^2-1|)^2-1}},\\
x_7,x_8 &= \pm\sqrt{|k|^2-|k^2-1|+\sqrt{(|k|^2-|k^2-1|)^2-1}}.
\end{aligned}
$$

It can be showed that $|x_3| = |x_4| \le 1$ and that $|x_i| = 1$ for $i = 5,6,7,8$. So,

(42)
$$h(\alpha_1) = \frac{1}{8}\log(|x_1|\cdot|x_2|) = \frac{1}{4}\log|k+\sqrt{k^2-1}| \le \frac{1}{4}\log(2|k|+1) \le 1.64\ .$$

The roots of p_2 are

$$\begin{aligned} x_1, x_2 &= \pm\alpha_2, \\ x_3, x_4 &= \pm|2k-1-2\sqrt{k^2-k}|, \\ x_5, x_6 &= \pm\sqrt{|2k-1|^2-4|k^2-k|+\sqrt{(|2k-1|^2-4|k^2-k|)^2-1}}, \\ x_7, x_8 &= \pm\sqrt{|2k-1|^2-4|k^2-k|-\sqrt{(|2k-1|^2-4|k^2-k|)^2-1}}. \end{aligned}$$

As in the previous case, we obtain that

$$h(\alpha_2) = \frac{1}{8}\log(|2k-1+2\sqrt{k^2-k}|^2) \le \frac{1}{4}\log(4|k|+3) \le 1.82. \tag{43}$$

The estimate for $h(\alpha_3)$ will be less accurate than those $h(\alpha_1)$ and $h(\alpha_2)$, because most roots of p_3 cannot be found analytically. By calculation, we obtain following 8 roots:

$$\begin{aligned} x_1, x_2 &= \pm\alpha_3, \\ x_3, x_4 &= \pm\left|\frac{2\sqrt{k}(\sqrt{k-1}+\sqrt{k+1})}{\sqrt{k+1}(\sqrt{k-1}-2\sqrt{k})}\right|, \\ x_5, x_6 &= \pm\left|\frac{2k(\sqrt{k+1}-\sqrt{k-1})+\sqrt{2k(k-1)(\sqrt{k^2-1}-k)}}{\sqrt{k+1}(3k+1)}\right|, \\ x_7, x_8 &= \pm\left|\frac{2k(\sqrt{k+1}-\sqrt{k-1})-\sqrt{2k(k-1)(\sqrt{k^2-1}-k)}}{\sqrt{k+1}(3k+1)}\right|. \end{aligned}$$

We estimate the remaining roots on the following way:

$$|x_i| \le 32 \cdot \max\{|a_j|, 0 \le j \le 16\},\ i = 9, 10, \dots, 32,$$

where a_i represents a coefficient of normed polynomial for α_3. We give as an example the estimate for $|a_4|$,

$$\begin{aligned} |a_4| &= \frac{2^{38}(\mu^2+\nu^2)^6}{((1+\mu)^2+\nu^2)^6(1+3\mu)^2+3\nu^2)^8}|p(\mu,\nu)| \\ &\le \frac{2^{38}(\mu^2+\nu^2)^6}{(\mu^2+\nu^2)^6(9(\mu^2+\nu^2))^8}\sum|b_{ij}|(\mu^2+\nu^2)^4 \le 5.5\cdot 10^6, \end{aligned}$$

where $p(\mu,\nu) = \sum_{0\le i+j\le 8} b_{ij}\mu^i\nu^j = -1+52\mu+14\mu^2+\cdots+1296\mu^2\nu^6+324\nu^8.$
All coefficients are bounded by:

$$\max\{|a_j|, 0 \le j \le 16\} \le |a_8| < 1.65\cdot 10^8.$$

It can be seen that $|x_i| < 1$ for $i = 5, 6, 7, 8$. So, we have that

$$h(\alpha_3) \le \frac{1}{32}\log(a_n'\alpha_3^2|x_3||x_4|(32\cdot 1.65\cdot 10^8)^{24}), \tag{44}$$

where $a'_n = ((1+\mu)^2+\nu^2)^8((1+3\mu)^2+(3\nu)^2)^8$ represents the leading coefficient of the minimal polynomial of α_3 with integer coefficients. By using the estimates

$$\alpha_3^2 x_3 x_4 = 16\cdot\left|\frac{k}{k+1}\right|^2\cdot\left|\frac{2k+2\sqrt{k^2-1}}{3k+1}\right|^2 \le 16\left(1+\frac{1}{|k|-1}\right)^2 4\left(\frac{2|k|+1}{3|k|-1}\right)^2 < 62,$$

$$a'_n \le (1+2|k|+|k|^2)^8(1+6|k|+9|k|^2)^8 < 6.5\cdot 10^{52},$$

we get

$$h(\alpha_3) < \frac{1}{32}\log(6.5\cdot 10^{52}\cdot 62\cdot(32\cdot 1.65\cdot 10^8)^{24}) < 20.72.$$

Finally, we have everything for the application of Baker-Wüstholz theorem:

$$\begin{aligned} -m\log 16 &\ge \log|\Lambda| \ge -18\cdot 4!\cdot 3^4(32\cdot 2048)^5\cdot 1.64\cdot 1.82\cdot 18.69\cdot\log(6\cdot 2048)\log m \\ &> -2.5\cdot 10^{31}\log m. \end{aligned}$$

(We used that the degree $d \le [\mathbb{Q}(\alpha_1,\alpha_2,\alpha_3) : \mathbb{Q}(\alpha_1,\alpha_2)][\mathbb{Q}(\alpha_1,\alpha_2) : \mathbb{Q}(\alpha_1)][\mathbb{Q}(\alpha_1):\mathbb{Q}] \le 32\cdot 8\cdot 8$.)

Therefore, we obtain

$$\frac{m}{\log m} \le 2.5\cdot 10^{31}. \tag{45}$$

The inequality (45) is not valid for $m \ge 2\cdot 10^{33}$, so, we have that

$$|m\theta - n + \beta| < \alpha\cdot 16^{-m},\ m < 2\cdot 10^{33}, \tag{46}$$

for $\theta = \log\alpha_1/\log\alpha_2$, $\beta = \log\alpha_3/\log\alpha_2$, $\alpha = 1/\log\alpha_2$.

In the case of $k = i\nu$, $5 < \nu < 350$, the same conclusion, i.e. (46), can be obtained. The only difference is that we take

$$Q = \frac{\sqrt{k-1}-2\sqrt{k}}{2\sqrt{k}}(2k-1-2\sqrt{k^2-k})^n$$

in (27).

6. The reduction method

Our next step is reducing the upper bound of the solution of (46). We will use the reduction method similar to one described in [6, Lemma 4a)] (and originally introduced in [2]).

Lemma 12. *([6]) Let M be a positive integer and let p/q be a convergent of the continued fraction expansion of θ such that $q > 6M$. Furthermore, let $\varepsilon = \|\beta q\| - M\cdot\|\theta q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then the inequality*

$$|m\theta - n + \beta| < \alpha a^{-m}, \tag{47}$$

has no integer solutions m *and* n *such that*

$$\log(\alpha q/\varepsilon)/\log a \leq m \leq M.$$

We apply Lemma 12 to (46) for each $k \in \mathbb{Z}[i]$, $5 < |k| < 350$ such that $\mathrm{Re}(k) > 0$ or $\mathrm{Re}(k) = 0$, $\mathrm{Im}(k) > 0$. The reduction give us a new bound $M_0 = 33$, in all cases. The another application of the reduction in all cases give us that $m \leq 6$. By checking all the possibilities $0 < n \leq m \leq 6$, we conclude that the equation $v_m = w_n$ has only trivial solution $v_0 = w_0 = 1$.

Finally, we have to carry out the procedure described in sections 4 and 5 for the case $x = v_m = w'_n$. By solving the recurrence (22) for (w'_m), we obtain

$$x = \frac{2\sqrt{k} - \sqrt{k-1}}{4\sqrt{k}}(2k-1+2\sqrt{k^2-k})^n + \frac{2\sqrt{k} + \sqrt{k-1}}{4\sqrt{k}}(2k-1-2\sqrt{k^2-k})^n.$$

If $\mathrm{Re}(k) > 0$, then instead of (27) we take

$$Q = \frac{2\sqrt{k} - \sqrt{k-1}}{2\sqrt{k}}(2k-1+2\sqrt{k^2-k})^n,$$

and related algebraic numbers are α_1, α_2, α'_3. If $\mathrm{Re}(k) = 0$ and $\mathrm{Im}(k) > 0$, then we put

$$Q = \frac{2\sqrt{k} + \sqrt{k-1}}{2\sqrt{k}}(2k-1-2\sqrt{k^2-k})^n,$$

and we deal with α_1, α'_2, α_3. All estimates remain valid and we obtain that $v_2 = w'_2 = 4k^2 - 2k - 1$ and $v_0 = w'_0 = 1$ are the only solutions of the equation $v_m = w'_n$.

7. The case $1 < |k| \leq 5$

This case is interesting, because there are some extra fundamental solutions of (4) and (5) for certain parameters k. Precisely, these fundamental solutions of (4) also appear (besides $x = \pm 1$):

- $x_0 = 0$, $y_0 = \pm(1+i)$ for $k = 1+i$,
- $x_0 = 0$, $y_0 = \pm(1-i)$ for $k = 1-i$,
- $x_0 = 0$, $y_0 = \pm i$ for $k = 3$,

and for (5), we obtain

- $x_0 = 0$, $y_0 = \pm(1+2i)$ for $k = 1+i$,
- $x_0 = 0$, $y_0 = \pm(1-2i)$ for $k = 1-i$,
- $x_0 = \pm(1+i)$, $y_0 = \pm(2+3i)$ for $k = 3$,
- $x_0 = 0$, $y_0 = \pm 2i$ for $k = 5$,
- $x_0 = \pm i$, $y_0 = \pm 3i$ for $k = 5$,
- $x_0 = \pm 2$, $y_0 = \pm 4$ for $k = 5$.

Each of the above cases will be treated separately.

• $\mathbf{k=1+i}$

In Section 2,we showed that all solutions of (4) are given by recurrence sequences (13) and (14). Precisely, according to (13) the fundamental solution $x=0,\ y=1+i$ generates this recurrence sequence

$$u_0=0,\ u_1=-1+i,\ u_{m+2}=2(1+i)u-m+1-u_m,\ m\in\mathbb{N}_0,$$

and according to (14) we obtain the sequence $(-u_m)$. The fundamental solution $x=1,\ y=1$ generates the sequence

$$v_0=1,\ v_1=1+2i,\ v_{m+2}=2(1+i)v_{m+1}-v_m,\ m\in\mathbb{N}_0.$$

So, all solutions of (4) are $x=\pm u_m$ and $x=\pm v_m$. Further, (15) and (16) imply that all solutions of (5) are

$$q_0=0,\ q_1=-2+i,\ q_{n+2}=2(1+2i)q_{n+1}-q_n,\ n\in\mathbb{N}_0,$$

which corresponds to the fundamental solution $x=0,\ y=1+2i$, and

$$w_0=1,\ w_1=1+3i,\ w_{n+2}=2(1+2i)w_{n+1}-w_n,$$

$$w'_0=1,\ w'_1=1+i,\ w'_{n+2}=2(1+2i)w'_{n+1}-w'_n,\ n\in\mathbb{N}_0$$

which correspond to the fundamental solution $x=1,\ y=1$ Hence, all solutions of (5) are given by sequences $(\pm q_n)$, $(\pm w_n)$ and $(\pm w'_n)$ and then one of the following cases occur:

a): $v_m=w_n$ or $v_m=w'_n$,
b): $v_m=\pm q_n$,
c): $u_m=\pm w_n$ or $u_m=\pm w'_n$,
d): $u_m=\pm q_n$.

Case a) can be solved similarly as in previous sections. In what follows, we solve cases b), c), d).

b) Suppose that $v_m=\pm q_n$ for $m,n\in\mathbb{N}_0$. We apply the congruence method from Section 3 on sequences $(v_m \mod \delta)$ and $(q_n \mod \delta)$, where $\delta\in\{-1+2i,-4-4i\}$, and get that $v_{3m+1}=\pm q_n$, $m,n\in\mathbb{N}_0$, because

$$(v_m \bmod (-1+2i))=(-1+i,0,2i,2i,0,-1+i,-1+i,0,\ldots),$$
$$(q_n \bmod (-1+2i))=(0,0,0,\ldots).$$

The following sequences

$$(v_{3m+1} \bmod (-4-4i))=(-3-2i,-7,-7,-3-2i,-3-2i,-7,\ldots),$$
$$(q_n \bmod (-4-4i))=(0,-2+i,-4-2i,-2-i,-4,-6+i,-4+2i,-6-i,0,\ldots),$$

imply that $v_m\neq\pm q_n$, for all $m,n\in\mathbb{N}_0$.

c) Similarly, as in the case b), by applying the congruence method we obtain that there is no solution in this case.

d) By applying the congruence method as in the case b), we obtain that

$$u_{6m} = \pm q_{4n},\ \ m,n \in \mathbb{N}_0.$$

By repeating the procedures described in Sections 6 and 7, we conclude that the above equation has only the trivial solution $u_0 = q_0 = 0$. The meaning of this unexpected solution is that the Diophantine triple $\{i, 1+i, 2+i\}$ is extended by the element $d = i$, but such extension is not considered as a proper extension since i is already an element of the starting triple.

• $\mathbf{k = 1 - i}$
By conjugating, this case becomes the same as the previous one.

• $\mathbf{k = 3}$
The fundamental solutions of (4) are $(x, y) = (0, i)$ and $x, y = (1, 1)$. They generate two recurrence sequences

$$u_0 = 0,\ u_1 = 2i,\ u_{m+2} = 6u_{m+1} - u_m,$$
$$v_0 = 1,\ v_1 = 5,\ v_{m+2} = 6v_{m+1} - v_m.$$

The fundamental solutions of (5), $(x, z) = (1, 1)$ and $(x, z) = (1 + i, 2 + 3i)$, generate following sequences

$$w_0 = 1,\ w_1 = 4,\ u_{n+2} = 10w_{n+1} - w_n,$$
$$w'_0 = 1,\ w_1 = 3,\ u_{n+2} = 10w_{n+1} - w_n,$$
$$q_0 = 1 + i,\ q_1 = 9 + 11i,\ q_{n+2} = 6q_{n+1} - q_n,$$
$$q'_0 = 1 + i,\ q'_1 = 1 - i,\ q'_{n+2} = 6q_{n+1} - q_n.$$

Note that $q'_n = \overline{q_{n-1}}$. So, following cases should be analyzed:

a): $v_m = w_n$ or $v_m = w'_n$,
b): $v_m = \pm q_n$ or $v_m = \pm\overline{q_n}$,
c): $u_m = \pm w_n$ or $u_m = \pm w'_n$,
d): $u_m = \pm q_n$ or $u_m = \pm\overline{q_n}$.

By congruence method we obtain that the cases b), c) and d) have no solution.

• $\mathbf{k = 5}$
As in the previous case, the solutions can be obtained from $v_m = w_n$ or $v_m = w'_n$.

Finally, we solve $v_m = w_n$ and $v_m = w'_n$ for all k, $1 < |k| \le 5$. We apply methods given in Sections 6 and 7, and obtain that the only solution of $v_m = w_n$ is $m = n = 0$ and solutions of $v_m = w'_n$ are $m = n = 0$ and $m = n = 2$.

8. The case $k = \pm i$

The main difference of this case ($k = i$) is that the original problem (3) is equivalent with the following system of Pellian equations

$$y^2 + ix^2 = i + 1, \tag{48}$$
$$z^2 - (2-2i)x^2 = -1 + 2i. \tag{49}$$

The advantage of the above equations is that the solutions can be given immediately by using [10]. So, all solutions of (48) are given by

$$y_m^{(j)} + x_m^{(j)}\sqrt{-i} = \rho_j(i + (1-i)\sqrt{-i})^m,\ m \in \mathbb{N}_0,\ j = 1,2,3,4, \tag{50}$$

where $\rho_1 = 1 + \sqrt{-i}$, $\rho_2 = -1 + \sqrt{-i}$, $\rho_3 = -\rho_1$, $\rho_4 = -\rho_2$, and all solutions of (49) are
(51)
$$z_n^{(k)} + \tilde{x}_n^{(k)}\sqrt{2-2i} = \sigma_j(-1 + 2i + (1-i)\sqrt{2-2i})^n,\ n \in \mathbb{N}_0,\ k = 1,2,3,4,$$

and $\sigma_1 = 1 + \sqrt{2-2i}$, $\sigma_2 = 1 - \sqrt{2-2i}$, $\sigma_3 = -\sigma_1$, $\sigma_4 = -\sigma_2$.

Hence, our problem of solving the system of equation is reduced to

$$x_m^{(j)} = \tilde{x}_n^{(k)},\ m, n \in \mathbb{N}_0,$$

where $j, k \in \{1,2,3,4\}$. For $m = n = 0$, a trivial solution $x = 1$ is obtained and it corresponds to unregular extension of the Diophantine triple $\{i-1, i, 4i\}$ by $d = 0$.

Further, it can be shown that $x_{m+1}^{(1)} = x_m^{(2)} = -x_{m+1}^{(3)} = -x_m^{(4)} = x_m$ and that $\tilde{x}_n^{(3)} = -\tilde{x}_n^{(1)}$ and $\tilde{x}_n^{(4)} = -\tilde{x}_n^{(2)}$. So, it remains us to observe these four cases

$$x_m = \pm\tilde{x}_n^{(j)},\ m, n \in \mathbb{N}\ (j \in \{1,2\}). \tag{52}$$

Solutions x_m and $\tilde{x}_n^{(k)}$ $(k = 1,2)$ satisfy following recursions

$$x_{m+1}^{(j)} = 2ix_m^{(j)} - x_{m-1}^{(j)},\ m \geq 1,$$

$$\tilde{x}_{n+1}^{(k)} = 2(-1+2i)\tilde{x}_n^{(k)} - \tilde{x}_{n-1}^{(k)},\ n \geq 1.$$

By solving these recursions we obtain these formulas

$$x_m = (\frac{1}{2} + \frac{1+i}{2\sqrt{2}})(i(1+\sqrt{2}))^m + (\frac{1}{2} - \frac{1+i}{2\sqrt{2}})(i(1-\sqrt{2}))^m,$$
$$\tilde{x}_n^{(1)} = \frac{2-\sqrt{1+i}}{4}(-1+2i+2i\sqrt{1+i})^n + \frac{2+\sqrt{1+i}}{4}(-1+2i-2i\sqrt{1+i})^n,$$
$$\tilde{x}_n^{(2)} = -\frac{2+\sqrt{1+i}}{4}(-1+2i+2i\sqrt{1+i})^n - \frac{2-\sqrt{1+i}}{4}(-1+2i-2i\sqrt{1+i})^n.$$

Each of equations in (52) should be treated separately. By applying the methods given in Section 6. and 7., we obtain the solution $x = x_2 = \tilde{x}_2^{(1)} = -5 - 2i$ (which corresponds to $d = -20i$, i.e. to $16k^3 - 4k$ for $k = i$).

The case $k = -i$ can be solved in the same manner.

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