NONEXTENSIBILITY OF THE PAIR $\{1,3\}$ TO A DIOPHANTINE QUINTUPLE IN $\mathbb{Z}[\sqrt{-2}]$

ZRINKA FRANUŠIĆ, DIJANA KRESO

Abstract. We show that the Diophantine pair $\{1,3\}$ can not be extended to a Diophantine quintuple in the ring $\mathbb{Z}[\sqrt{-2}]$. This result completes the work of the first author and establishes nonextensibility of the Diophantine pair $\{1,3\}$ in $\mathbb{Z}[\sqrt{-d}]$ for all $d \in \mathbb{N}$.

1. Introduction and results

Let R be a commutative ring with unity 1. The set $\{a_1, a_2, \ldots, a_m\}$ in R such that $a_i \neq 0$, $i = 1, \ldots, m$, $a_i \neq a_j$, $a_ia_j + 1$ is a square in R for all $1 \leq i < j \leq m$ is called a *Diophantine m-tuple* in the ring R. The problem of constructing such sets was first studied by Diophantus of Alexandria who found a set of four rationals $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ with given property. Fermat found the first Diophantine quadruple in $\mathbb{Z}$ - the set $\{1, 3, 8, 120\}$. A Diophantine pair $\{a, b\}$ in the ring R, satisfying $ab + 1 = r^2$, can be extended to a Diophantine quadruple in R by adding elements $a + b + 2r$ and $4r(r + a)(r + b)$ provided that all elements are non-zero and different. Hence, in most of the rings Diophantine quadruples exist, but can we obtain Diophantine m-tuples of size greater that 4? The answer depends on the ring.

In the ring $\mathbb{Z}$ the folklore conjecture is that there are no Diophantine quintuples. In 1969, Baker and Davenport ([1]) showed that the set $\{1, 3, 8\}$ can not be extended to a Diophantine quintuple, which was the first result supporting the conjecture. This theorem was generalized first by Dujella ([4]) who showed that the set $\{k - 1, k + 1, 4k\}$ for $k \geq 2$ can not be extended to a Diophantine quintuple in $\mathbb{Z}$, and later by Dujella and Pethő in [8] who showed that not even the Diophantine pair $\{1, 3\}$ can be extended to a Diophantine quintuple in $\mathbb{Z}$. Greatest step towards proving the conjecture did Dujella ([6]) in 2004 who showed that there are no Diophantine sextuples and that there are only finitely many Diophantine quintuples. Also, in [7] it was proved that there are no Diophantine quintuples in the ring of polynomials with integers coefficients under assumption that not all elements are constant polynomials.

The size of Diophantine m-tuples can be greater than 4 in some rings. For instance, the set $\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\}$ is a Diophantine sextuple in $\mathbb{Q}$, found by Gibbs ([11]). Furthermore, we can construct Diophantine quintuples in the ring $\mathbb{Z}[\sqrt{d}]$ for some values of d; for instance $\{1, 3, 8, 120, 1678\}$ is a Diophantine quintuple in $\mathbb{Z}[\sqrt{201361}]$. It is natural to start investigating the upper bound for the size of Diophantine m-tuples in

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$\mathbb{Z}[\sqrt{d}]$ by focusing on a problem of extensibility of Diophantine triples $\{k-1, k+1, 4k\}$ with $k \notin \{0, \pm 1\}$ and Diophantine pair $\{1, 3\}$ to a Diophantine quintuple in $\mathbb{Z}[\sqrt{d}]$, since the problem in the ring $\mathbb{Z}$ was approached similarly, ([8]) and ([4]).

In [9] Franušić proved that the Diophantine pair $\{1, 3\}$ can not be extended to a Diophantine quintuple in $\mathbb{Z}[\sqrt{-d}]$ if d is a positive integer and $d \neq 2$. The case $d = 2$ was also considered and it was shown that if $\{1, 3, c\}$ is a Diophantine triple in $\mathbb{Z}[\sqrt{-2}]$ then $c \in \{c_k, d_l\}$, where the sequences (c_k) and (d_l) are given by:

$$c_k = \frac{1}{6}\big((2+\sqrt{3})(7+4\sqrt{3})^k + (2-\sqrt{3})(7-4\sqrt{3})^k - 4\big),\ k \geq 1, \tag{1}$$

$$d_l = -\frac{1}{6}\big((7+4\sqrt{3})^l + (7-4\sqrt{3})^l + 4\big),\ l \geq 0. \tag{2}$$

Sequences (c_k) and (d_l) can be given recurrently in the following way:

$$c_0 = 8, \quad c_1 = 120, \qquad c_{k+2} = 14c_{k+1} - c_k + 6,\ k \geq 1.$$

$$d_0 = -1, \quad d_1 = -3, \qquad d_{l+2} = 14d_{l+1} - d_l + 8,\ l \geq 0.$$

It is known that $\{1, 3, c_k, c_{k+1}\}$, $k \geq 1$, is a Diophantine quadruple in $\mathbb{Z}$ ([8]) and hence also in $\mathbb{Z}[\sqrt{-2}]$. The set $\{1, 3, d_l, d_{l+1}\}$ is a Diophantine quadruple in $\mathbb{Z}[\sqrt{-2}]$ since

$$d_l d_{l+1} + 1 = (c_l + 2)^2 \tag{3}$$

for every $l \geq 0$. The equation (3) easily follows from explicit formulas (1) and (2). The set $\{1, 3, c_k, d_l\}$ is not a Diophantine quadruple for $k \geq 1$ and $l \geq 0$ since $1 + c_k d_l$ is a negative odd number and hence can not be a square in $\mathbb{Z}[\sqrt{-2}]$. Therefore if there is an extension of the Diophantine pair $\{1, 3\}$ to a Diophantine quadruple in $\mathbb{Z}[\sqrt{-2}]$, then it is of the form $\{1, 3, c_k, c_l\}$, $l > k \geq 1$ or $\{1, 3, d_k, d_l\}$, $l > k \geq 0$. In the former case, the set can not be extended to a Diophantine quintuple in $\mathbb{Z}$ by [8], and consequently it can not be extended to a Diophantine quintuple in $\mathbb{Z}[\sqrt{-2}]$. It remains to examine the latter case. We formulate the following theorem.

Theorem 1.1. *Let k be a non-negative integer and d an integer. If $\{1, 3, d_k, d\}$ is a Diophantine quadruple in $\mathbb{Z}[\sqrt{-2}]$, where d_k is given by* (2)*, then $d = d_{k-1}$ or $d = d_{k+1}$.*

From Theorem 1.1 we immediately obtain the following corollary.

Corollary 1.2. *The Diophantine pair $\{1, 3\}$ can not be extended to a Diophantine quintuple in $\mathbb{Z}[\sqrt{-2}]$.*

The organization of the paper is as follows. In Section 2, assuming k to be minimal integer for which Theorem 1.1 does not hold, we translate the assumption in Theorem 1.1 into system of Pellian equations from which recurrent sequnces $\nu_m^{(i)}$ and $\omega_n^{(j)}$ are deduced, intersections of which induce solutions to the system. In Section 3 we use a congruence method introduced by Dujella and Pethő ([8]) to determine the fundamental solutions of Pellian equations. In Section 4 we give a lower bound for m and n for which the sequences $\nu_m^{(i)}$ and $\omega_n^{(j)}$ intersect. In Section 5 we use a theorem of Bennett ([3]) to establish an upper bound for k. Remainings cases are checked separately in Section 6 using linear forms in logarithms, Baker-Wüstholz theorem ([2]) and the Baker-Davenport method of reduction ([1]).

2. The system of Pellian equations

Let $\{1,3,d_k,d\}$ be a Diophantine quadruple in $\mathbb{Z}[\sqrt{-2}]$ where k is the minimal integer, for which the statement of Theorem 1.1 does not hold and assume $k \geq 6$. Clearly $d = d_l$ for some $l \geq 0$. There exist $x, y, z \in \mathbb{Z}$ such that

$$d+1 = -2x^2, \qquad 3d+1 = -2y^2, \qquad d_k d + 1 = z^2, \tag{4}$$

since $d+1$ and $3d+1$ are negative integers and $d_k d + 1$ is a positive integer.

The system (4) is equivalent to the following system of Pellian equations

$$z^2 + 2d_k x^2 = 1 - d_k, \tag{5}$$

$$3z^2 + 2d_k y^2 = 3 - d_k, \tag{6}$$

where

$$d_k + 1 = -2s_k^2, \qquad 3d_k + 1 = -2t_k^2, \tag{7}$$

for some $s_k, t_k \in \mathbb{Z}$. We may assume $s_k, t_k \in \mathbb{N}$. Conditions (7) follow from the fact that $\{1,3,d_k\}$ is a Diophantine triple in $\mathbb{Z}[\sqrt{-2}]$ and the fact that d_k+1 and $3d_k+1$ are negative integers.

The following propositions describe the set of positive integer solutions of the equations (5) and (6).

Proposition 2.1. *There exists $i_0 \in \mathbb{N}$ and $z_0^{(i)}, x_0^{(i)} \in \mathbb{Z}$, $i = 1,2,\ldots,i_0$ such that $(z_0^{(i)}, x_0^{(i)})$, $i = 1,2,\ldots,i_0$ are solutions of the equation (5) satisfying*

$$1 \leq z_0^{(i)} \leq \sqrt{-d_k(1-d_k)}, \qquad 1 \leq |x_0^{(i)}| \leq \sqrt{\frac{1-d_k^2}{2d_k}}.$$

For every solution $(z,x) \in \mathbb{N} \times \mathbb{N}$ of the equation (5), there exists $i \in \{1,2,\ldots,i_0\}$ and integer $m \geq 0$ such that

$$z + x\sqrt{-2d_k} = \big(z_0^{(i)} + x_0^{(i)}\sqrt{-2d_k}\big)\big(-2d_k - 1 + 2s_k\sqrt{-2d_k}\big)^m.$$

Proof. The fundamental solution of the related Pell's equation $z^2 + 2d_k x^2 = 1$ is $-2d_k - 1 + 2s_k\sqrt{-2d_k}$ since $(-2d_k-1)^2 + 2d_k \cdot (2s_k)^2 = 4d_k^2 + 4d_k + 1 - 4d_k(1+d_k) = 1$ and $-2d_k - 1 > 2s_k^2 - 1 = -d_k - 2$ is satisfied ([12, Theorem 105]). Further following arguments by Nagell ([12, Theorem108]) we obtain that there are finitely many integer solutions $(z_0^{(i)}, x_0^{(i)})$, $i = 1,2,\ldots,i_0$ of the equation (5) for which the following inequalities hold

$$1 \leq |z_0^{(i)}| \leq \sqrt{-d_k(1-d_k)}, \qquad 0 \leq |x_0^{(i)}| \leq \sqrt{\frac{1-d_k^2}{2d_k}},$$

and if $z + x\sqrt{-2d_k}$ is a solution in integers z, x of the equation (5), then

$$z + x\sqrt{-2d_k} = \big(z_0^{(i)} + x_0^{(i)}\sqrt{-2d_k}\big)\big(-2d_k - 1 + 2s_k\sqrt{-2d_k}\big)^m$$

for some $m \in \mathbb{Z}$ and $i \in \{1,2,\ldots,i_0\}$. Hence

$$z_0^{(i)} + x_0^{(i)}\sqrt{-2d_k} = \big(z + x\sqrt{-2d_k}\big)\big(-2d_k - 1 + 2s_k\sqrt{-2d_k}\big)^{-m},$$

wherefrom it can be easily deduced that if $z + x\sqrt{-2d_k}$ is a solution in positive integers z, x of the equation (5), then $z_0^{(i)} > 0$. Hence $1 \leq z_0^{(i)} \leq \sqrt{-d_k(1-d_k)}$ for all $i \in \{1, 2, \ldots, i_0\}$. Also, if $x_0^{(i)} = 0$ then $(z_0^{(i)})^2 = 1 - d_k$, which contradicts $z_0^{(i)} \leq \sqrt{-d_k(1-d_k)}$ for $k > 0$. Since $k \geq 6$ is assumed, the inequalities in Proposition 2.1 hold. To complete the proof it remains to show that $m \geq 0$. Assume to the contrary that $m < 0$. Then

$$\big(-2d_k - 1 + 2s_k\sqrt{-2d_k}\big)^m = \alpha - \beta\sqrt{-2d_k}$$

with $\alpha, \beta \in \mathbb{N}$ and $\alpha^2 + 2d_k\beta^2 = 1$. Since

$$z + x\sqrt{-2d_k} = \big(z_0^{(i)} + x_0^{(i)}\sqrt{-2d_k}\big)\big(\alpha - \beta\sqrt{-2d_k}\big),$$

we have $x = -z_0^{(i)}\beta + x_0^{(i)}\alpha$. By squaring $x_0^{(i)}\alpha = x + z_0^{(i)}\beta$ and substituting $\alpha^2 = 1 - 2d_k\beta^2$ we get

$$(x_0^{(i)})^2 = \beta^2(1-d_k) + x^2 + 2xz_0^{(i)}\beta > \beta^2(1-d_k) \geq 1 - d_k > \frac{1-d_k^2}{2d_k},$$

since x, $z_0^{(i)}$, β and k are positive integers, but this contradicts the upper bound for $x_0^{(i)}$. □

Using similar arguments we prove a similar proposition for the equation (6).

Proposition 2.2. *There exists $j_0 \in \mathbb{N}$ and $z_1^{(j)}, y_1^{(j)} \in \mathbb{Z}$, $j = 1, 2, \ldots, j_0$ such that $(z_1^{(j)}, y_1^{(j)})$, $j = 1, 2, \ldots, j_0$ are solutions of the equation (6) satisfying*

$$1 \leq z_1^{(j)} \leq \sqrt{-d_k(3-d_k)} \qquad 1 \leq |y_1^{(j)}| \leq \sqrt{\frac{(3-d_k)(1+3d_k)}{2d_k}}.$$

For every solution $(z, y) \in \mathbb{N} \times \mathbb{N}$ of the equation (3), there exists $j \in \{1, 2, \ldots, j_0\}$ and integer $n \geq 0$ such that

$$z\sqrt{3} + y\sqrt{-2d_k} = \Big(z_1^{(j)}\sqrt{3} + y_1^{(j)}\sqrt{-2d_k}\Big)\Big(-6d_k - 1 + 2t_k\sqrt{-6d_k}\Big)^n.$$

Finitely many solutions that satisfy the bounds given in Proposition 2.1 and Proposition 2.2 will be called *fundamental* solutions.

From Proposition 2.1 and Proposition 2.2 it follows that if (z, x) is a solution in positive integers of the equation (5), then $z = \nu_m^{(i)}$ for some $m \geq 0$ and $i \in \{1, 2, \ldots, i_0\}$ where

$$(8)\ \ \nu_0^{(i)} = z_0^{(i)},\ \nu_1^{(i)} = (-2d_k - 1)z_0^{(i)} - 4s_kd_kx_0^{(i)},\ \nu_{m+2}^{(i)} = (-4d_k - 2)\nu_{m+1}^{(i)} - \nu_m^{(i)},\ m \geq 0,$$

and if (z, y) is a solution in positive integers of the equation (6), then $z = \omega_n^{(j)}$ for some $n \geq 0$ and $j \in \{1, 2, \ldots, j_0\}$ where

$$(9)\ \ \omega_0^{(j)} = z_1^{(j)},\ \omega_1^{(j)} = (-6d_k - 1)z_1^{(j)} - 4t_kd_ky_1^{(j)},\ \omega_{n+2}^{(j)} = (-12d_k - 2)\omega_{n+1}^{(j)} - \omega_n^{(j)},\ n \geq 0.$$

Therefore we are looking for the intersection of sequences $\nu_m^{(i)}$ and $\omega_n^{(j)}$.

3. Congruences

Using the congruence method introduced by Dujella and Pethő ([8]) we determine the fundamental solutions of the equations (5) and (6).

Lemma 3.1.

$$\nu_{2m}^{(i)} \equiv z_0^{(i)} \ (mod \ \ -2d_k), \qquad \nu_{2m+1}^{(i)} \equiv -z_0^{(i)} \ (mod \ \ -2d_k),$$
$$\omega_{2n}^{(j)} \equiv z_1^{(j)} \ (mod \ \ -2d_k), \qquad \omega_{2n+1}^{(j)} \equiv -z_1^{(j)} \ (mod \ \ -2d_k),$$

for all $m,n \geq 0$, $i \in \{1,2,\ldots,i_0\}$, $j \in \{1,2,\ldots,j_0\}$.

Proof. Easily follows by induction. □

Lemma 3.2. *If the equation* $\nu_m^{(i)} = \omega_n^{(j)}$ *has a solution for some* $m,n \geq 0$, $i \in \{1,2,\ldots,i_0\}$, $j \in \{1,2,\ldots,j_0\}$, *then*

$$z_0^{(i)} = z_1^{(j)} \quad or \quad z_0^{(i)} + z_1^{(j)} = -2d_k.$$

Proof. Lemma 3.1 implies $z_0^{(i)} \equiv z_1^{(j)} \pmod{-2d_k}$ or $z_0^{(i)} \equiv -z_1^{(j)} \pmod{-2d_k}$. If $z_0^{(i)} \equiv -z_1^{(j)} \pmod{-2d_k}$ then $z_0^{(i)} + z_1^{(j)} \equiv 0 \pmod{-2d_k}$. From propositions 2.1 and 2.2 we get $0 < z_0^{(i)} + z_1^{(j)} \leq \sqrt{-d_k(1-d_k)} + \sqrt{-d_k(3-d_k)} < -d_k + 1 - d_k + 2 = -2d_k + 3$, which implies $z_0^{(i)} + z_1^{(j)} = -2d_k$.

If $z_0^{(i)} \equiv z_1^{(j)} \pmod{-2d_k}$, then $z_0^{(i)} = z_1^{(j)}$. Indeed, if $z_0^{(i)} > z_1^{(j)}$ then $0 < z_0^{(i)} - z_1^{(j)} < z_0^{(i)} \leq \sqrt{-d_k(1-d_k)} < -2d_k$, which is in contradiction with $z_0^{(i)} - z_1^{(j)} \equiv 0 \pmod{-2d_k}$. On the other hand, if $z_1^{(j)} > z_0^{(i)}$ then $0 < z_1^{(j)} - z_0^{(i)} < z_1^{(j)} \leq \sqrt{-d_k(3-d_k)} < -2d_k$, hence a contradiction is analogously obtained. □

Lemma 3.3.

$$\nu_m^{(i)} \equiv (-1)^m (z_0^{(i)} + 2d_k m^2 z_0^{(i)} + 4d_k s_k m x_0^{(i)}) \pmod{8d_k^2}$$
$$\omega_n^{(j)} \equiv (-1)^n (z_1^{(j)} + 6d_k n^2 z_1^{(j)} + 4d_k t_k n y_1^{(j)}) \pmod{8d_k^2}$$

for all $m,n \geq 0$, $i \in \{1,2,\ldots,i_0\}$, $j \in \{1,2,\ldots,j_0\}$.

Proof. Easily follows by induction.

□

Lemma 3.4. *If* $\nu_m^{(i)} = \omega_n^{(j)}$ *for some* $m,n \geq 0$, $i \in \{1,2,\ldots,i_0\}$, $j \in \{1,2,\ldots,j_0\}$, *then* $m \equiv n \pmod 2$.

Proof. If $m \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$, then Lemma 3.1 and Lemma 3.2 imply $z_0^{(i)} \equiv -z_1^{(j)} \pmod{-2d_k}$ and $z_0^{(i)} + z_1^{(j)} = -2d_k$. On the other hand, Lemma 3.3 implies

$$z_0^{(i)} + 2d_k m^2 z_0^{(i)} + 4d_k s_k m x_0^{(i)} \equiv -z_1^{(j)} - 6d_k n^2 z_1^{(j)} - 4d_k t_k n y_1^{(j)} \pmod{8d_k^2},$$

wherefrom, by substituting $z_0^{(i)} + z_1^{(j)} = -2d_k$ and dividing by $2d_k$, we obtain

$$-1 + m^2 z_0^{(i)} + 2s_k m x_0^{(i)} \equiv -3n^2 z_1^{(j)} - 2t_k n y_1^{(j)} \pmod{-4d_k}.$$

Since d_k is always odd, from (5) and (6) we see that $z_0^{(i)}$ and $z_1^{(j)}$ are even, hence the last congruence can not hold. Indeed, on the left side of the congruence is an odd number and on the right side is even, a contradiction. If m is odd and n even, a contradiction is obtained analogously. □

Therefore, the equations $\nu_{2m}^{(i)} = \omega_{2n+1}^{(j)}$ and $\nu_{2m+1}^{(i)} = \omega_{2n}^{(j)}$ have no solutions in integers $m, n \geq 0$, $i \in \{1, 2, \dots, i_0\}$, $j \in \{1, 2, \dots, j_0\}$. It remains to examine the cases when m and n are both even or both odd. In each of those cases we have $z_0^{(i)} = z_1^{(j)}$. Since

$$(z_0^{(i)})^2 - 1 = d_k(-2(x_0^{(i)})^2 - 1),$$

it follows that

$$\delta := \frac{(z_0^{(i)})^2 - 1}{d_k}$$

is an integer. Furthermore,

$$\delta + 1 = -2(x_0^{(i)})^2,\ 3\delta + 1 = -2(y_1^{(j)})^2,\ \delta d_k + 1 = (z_0^{(i)})^2.$$

Thus δ satisfies system (4) and clearly $\delta = d_l$ for some $l \geq 0$. Moreover, $\{1, 3, d_k, d_l\}$ is a Diophantine quadruple in $\mathbb{Z}[\sqrt{-2}]$ since $d_l \neq d_k$. Indeed, if $d_l = d_k$ then $d_k^2 + 1 = (z_0^{(i)})^2$, and since $d_k^2 \equiv 1 \pmod 4$ it follows that $(z_0^{(i)})^2 \equiv 2 \pmod 4$, a contradiction. In the following we show that $l = k-1$. Let us remind that k is a minimal integer such that the Theorem 1.1 does not hold. Assume $\delta > d_{k-1}$, that is $l < k-1$. Then the triple $\{1, 3, d_l\}$ can be extended to a Diophantine quadruple in $\mathbb{Z}[\sqrt{-2}]$ by d_k which differs from d_{l-1} and d_{l+1} since $l-1 < l+1 < k$ by assumption, which contradicts minimality of k. Therefore $l \geq k-1$. On the other hand, as $\delta d_k + 1 = (z_0^{(i)})^2 \leq -d_k(-d_k + 1)$ by Proposition 2.1, it follows that $\delta = d_l > d_k - 1$ and hence $l \leq k$. Since $d_l \neq d_k$ we have $d_l = d_{k-1}$. Hence $(z_0^{(i)})^2 = d_k d_{k-1} + 1$ and using (3) we obtain $z_0^{(i)} = z_0 = c_{k-1} + 2$. Furthermore, from (5), (6) and (7) we have $|x_0^{(i)}| = s_{k-1}$ and $|y_1^{(j)}| = t_{k-1}$. Moreover, since

$$s_k = \frac{1}{2\sqrt{3}}((2+\sqrt{3})^k - (2-\sqrt{3})^k),\ t_k = \frac{1}{2}((2+\sqrt{3})^k + (2-\sqrt{3})^k),$$

we have

$$2s_k s_{k-1} = c_{k-1},\ 2t_k t_{k-1} = 3c_{k-1} + 4, \tag{10}$$

which brings us to an important conclusion. If the system of Pellian equations (5) and (6), with k being the smallest integer for which Theorem 1.1 does not hold and under assumption $k \geq 6$, has a solution in positive integers, then the fundamental solutions of Pellian equations (5) and (6) are $(z_0, x_0^{\pm})$ and $(z_1, y_1^{\pm})$ respectively, where

$$z_0 = z_1 = 2(s_k s_{k-1} + 1), \tag{11}$$

$$x_0^{\pm} = \pm s_{k-1}, \quad y_1^{\pm} = \pm t_{k-1}. \tag{12}$$

4. The lower bound for m and n

By setting (11) and (12) into (8) and (9) and expanding we have

$$\begin{aligned}\nu_m^{\pm} &= \frac{1}{2}(2(s_k s_{k-1}+1) \pm s_{k-1}\sqrt{-2d_k}) \cdot (-2d_k - 1 + 2s_k\sqrt{-2d_k})^m + \\ &+ \frac{1}{2}(2(s_k s_{k-1}+1) \mp s_{k-1}\sqrt{-2d_k}) \cdot (-2d_k - 1 - 2s_k\sqrt{-2d_k})^m,\ m \geq 0,\end{aligned}$$

and

$$\begin{aligned}\omega_n^{\pm} &= \frac{1}{2\sqrt{3}}(2(s_k s_{k-1}+1)\sqrt{3} \pm t_{k-1}\sqrt{-2d_k}) \cdot (-6d_k - 1 + 2t_k\sqrt{-6d_k})^n + \\ &+ \frac{1}{2\sqrt{3}}(2(s_k s_{k-1}+1)\sqrt{3} \mp t_{k-1}\sqrt{-2d_k}) \cdot (-6d_k - 1 - 2t_k\sqrt{-6d_k})^n,\ n \geq 0.\end{aligned}$$

One intersection of these sequences is clearly $\nu_0^{\pm} = \omega_0^{\pm} = 2(s_k s_{k-1}+1)$, and hence further on we may assume $m, n \geq 1$. This intersection is related to the solution $d = d_{k-1}$ of (4) and implies that the triple $\{1, 3, d_k\}$ can be extended to a Diophantine quadruple in $\mathbb{Z}[\sqrt{-2}]$ by d_{k-1}. Another intersection is $\nu_1^- = \omega_1^-$. Indeed, (10) implies

$$s_k s_{k-1} + 1 = \frac{1}{3}(t_k t_{k-1} + 1) \tag{13}$$

and hence

$$\omega_1^- = -2 - 12d_k - 2s_k s_{k-1} - 12 d_k s_k s_{k-1} + 4 d_k t_k t_{k-1} = -2 - 4d_k - 2s_k s_{k-1} = \nu_1^-.$$

This intersection implies that the triple $\{1, 3, d_k\}$ can be extended to a Diophantine quadruple in $\mathbb{Z}[\sqrt{-2}]$ by d_{k+1}. Using (13) we write $\omega_n^{\pm}$ in the form

$$\begin{aligned}\omega_n^{\pm} &= \frac{1}{6}(2(t_k t_{k-1}+1) \pm t_{k-1}\sqrt{-6d_k}) \cdot (-6d_k - 1 + 2t_k\sqrt{-6d_k})^n + \\ &+ \frac{1}{6}(2(t_k t_{k-1}+1) \mp t_{k-1}\sqrt{-6d_k}) \cdot (-6d_k - 1 - 2t_k\sqrt{-6d_k})^n.\end{aligned}$$

Since

$$\begin{aligned}2(s_k s_{k-1}+1) - s_{k-1}\sqrt{-2d_k} &= 2 - \frac{\sqrt{-2d_{k-1}-2}}{\sqrt{-2d_k - 2} + \sqrt{-2d_k}} \\ &> 2 - \frac{\sqrt{-2d_k - 2}}{\sqrt{-2d_k - 2} + \sqrt{-2d_k}} > 1,\end{aligned}$$

it follows that

$$\nu_m^+ \geq \nu_m^- > \frac{1}{2}(-2d_k - 1 + 2s_k\sqrt{-2d_k})^m.$$

Furthermore,

$$\omega_n^- \leq \omega_n^+ < \frac{1}{2}(-6d_k - 1 + 2t_k\sqrt{-6d_k})^{n+1},$$

since

$$(2(t_k t_{k-1}+1) - t_{k-1}\sqrt{-6d_k}) \cdot (-6d_k - 1 - 2t_k\sqrt{-6d_k})^n < (-6d_k - 1 + 2t_k\sqrt{-6d_k})^n,$$

and

$$\frac{1}{3}(2(t_k t_{k-1}+1) + t_{k-1}\sqrt{-6d_k} + 1) < -6d_k - 1 + 2t_k\sqrt{-6d_k},$$

which can be verified by simple transformations and calculations using (7). Therefore, if one of the equations $\nu_m^{\pm} = \omega_n^{\pm}$ has solutions, then

$$\frac{1}{2}(-2d_k - 1 + 2s_k\sqrt{-2d_k})^m < \frac{1}{2}(-6d_k - 1 + 2t_k\sqrt{-6d_k})^{n+1},$$

wherefrom

$$\frac{m}{n+1} < \frac{\log(-6d_k - 1 + 2t_k\sqrt{-6d_k})}{\log(-2d_k - 1 + 2s_k\sqrt{-2d_k})}.$$

The expression on the right side of the inequality decreases when k increases. Since $k \geq 6$, it follows that $\frac{m}{n+1} < 1.072$. We may assume $n \geq 2$. Indeed for $n = 1$ it follows that $m \leq 2$ and as m and n are both even or both odd it follows that the only possibility is $m = 1$. We have already established the intersection $\nu_1^- = \omega_1^-$ and it can be easily verified that $\nu_1^+ \neq \omega_1^{\pm}$ and $\nu_1^- \neq \omega_1^+$. Now, it can be easily deduced that $m < n\sqrt{3}$. Hence, if the sequences $(\nu_m^{\pm})$ and $(\omega_n^{\pm})$ have any intersections besides two already established, then $n \geq 2$, m and n are of the same parity and $m < n\sqrt{3}$.

Proposition 4.1. *Let $n \geq 2$. If one of the equations $\nu_m^{\pm} = \omega_n^{\pm}$ has solutions then*

$$m \geq n \geq \frac{2}{3} \cdot \sqrt[4]{-d_k}.$$

Proof. If $m < n$, then $m \leq n - 2$ since m and n are of the same parity. From (8) and (9) using (10) one easily finds $\nu_0^+ < \omega_2^-$. Moreover, it can be easily shown by induction that $\nu_m^+ < \omega_{m+2}^-$ for $m \geq 0$. Indeed, sequences $(\nu_m^{\pm})$ and $(\omega_n^{\pm})$ are strictly increasing positive sequences, which can be easily checked by induction setting (11) and (12) into (8) and (9). Hence $\nu_{m+1}^+ < (-4d_k - 2)\nu_m^+$ and $\omega_{m+3}^- > (-12d_k - 3)\omega_{m+2}^-$, wherefrom it is clear that $\nu_m^+ < \omega_{m+2}^-$ implies $\nu_{m+1}^+ < \omega_{m+3}^-$, which completes the proof by induction. Since $\nu_m^- \leq \nu_m^+ < \omega_{m+2}^- \leq \omega_{m+2}^+$ it follows that if one of the equations $\nu_m^{\pm} = \omega_n^{\pm}$ has solutions, then $m + 2 > n$, a contradiction. Hence $m \geq n$.

For the other part of the statement assume to the contrary that $n < \frac{2}{3}\sqrt[4]{-d_k}$. Let us show how a contradiction can be obtained in the case $\nu_m^+ = \omega_n^+$. Other three case can be similarly resolved. Since m and n are of the same parity Lemma 3.3 implies that if $\nu_m^+ = \omega_n^+$, then

$$(c_{k-1} + 2)(m^2 - 3n^2 + m - 3n) \equiv 2(m - n) \pmod{-4d_k}, \tag{14}$$

and since (3) implies $(c_{k-1} + 2)^2 \equiv 1 \pmod{-d_k}$, we obtain

$$(m^2 - 3n^2 + m - 3n)^2 \equiv 4(m - n)^2 \pmod{-d_k}.$$

Moreover

$$(m^2 - 3n^2 + m - 3n)^2 \equiv 4(m - n)^2 \pmod{-4d_k} \tag{15}$$

since $(4, d_k) = 1$ and both sides of the congruence relation are divisible by 4 due to same parity of m and n. Under assumption $n < \frac{2}{3}\sqrt[4]{-d_k}$ one easily sees that the expressions on both sides of the congruence relation (15) are strictly smaller than $-4d_k$. Indeed, $0 \leq 2(m-n) \leq 2n(\sqrt{3}-1) < 2(\sqrt{3}-1)\frac{2}{3}\sqrt[4]{-d_k} < \sqrt{-4d_k}$ and $0 < -m^2+3n^2-m+3n \leq$

$2n^2+2n \le 3n^2 < \frac{12}{9}\sqrt{-d_k} < \sqrt{-4d_k}$. Therefore $-m^2+3n^2-m+3n = 2(m-n)$ which implies that $m \neq n$, hence $m > n$. Combining it with (14) we obtain

$$-(c_{k-1}+2)2(m-n) \equiv 2(m-n) \pmod{-4d_k},$$

wherefrom

$$-2s_k s_{k-1}(m-n) \equiv 3(m-n) \pmod{-2d_k}.$$

Since (7) implies $-2s_k^2 \equiv 1 \pmod{-d_k}$, by multiplying both sides by s_k we obtain

$$s_{k-1}(m-n) \equiv 3s_k(m-n) \pmod{-d_k},$$

and as $m-n \equiv 0 \pmod 2$ and $(d_k, 2) = 1$, it follows that

$$(m-n)(3s_k - s_{k-1}) \equiv 0 \pmod{-2d_k}. \tag{16}$$

On the other hand,

$$0 < m-n < n(\sqrt{3}-1) < (\sqrt{3}-1)\frac{2}{3}\sqrt[4]{-d_k} < 0.49 \cdot \sqrt[4]{-d_k}$$

and

$$0 < 3s_k - s_{k-1} \le 3s_k = 3\sqrt{\frac{-d_k-1}{2}} < 3 \cdot \sqrt{\frac{-d_k}{2}}$$

imply that

$$0 < (m-n)(3s_k - s_{k-1}) < 1.04\sqrt[4]{-d_k^3} < -2d_k.$$

Therefore we have a contradiction with (16). Completely analogously a contradiction is obtained in other three cases ($\nu_m^+ = \omega_n^-$, $\nu_m^- = \omega_n^+$, $\nu_m^- = \omega_n^-$). □

5. Application of a result of Bennett

Lemma 5.1. *Let*

$$\theta_1 = \sqrt{1+\frac{3}{3d_k}}, \qquad \theta_2 = \sqrt{1+\frac{1}{3d_k}}$$

and let (x, y, z) be a solution in positive integers of the system of Pellian equations (5) and (6). Then

$$\max\{\Big|\theta_1 - \frac{6s_k x}{3z}\Big|, \Big|\theta_2 - \frac{2t_k y}{3z}\Big|\} < (1-d_k)z^{-2}.$$

Proof. Clearly $\theta_1 = \frac{2s_k}{\sqrt{-2d_k}}$ and $\theta_2 = \frac{2t_k}{\sqrt{-6d_k}}$. Hence,

$$\Big|\theta_1 - \frac{6s_k x}{3z}\Big| = \Big|\frac{2s_k}{\sqrt{-2d_k}} - \frac{2s_k x}{z}\Big| = 2s_k\Big|\frac{z - x\sqrt{-2d_k}}{z\sqrt{-2d_k}}\Big|$$

$$= \frac{2s_k}{z\sqrt{-2d_k}} \cdot \frac{1-d_k}{z + x\sqrt{-2d_k}} < \frac{2s_k(1-d_k)}{\sqrt{-2d_k}} \cdot z^{-2} < (1-d_k)\cdot z^{-2}.$$

and

$$\Big|\theta_2 - \frac{2t_k y}{3z}\Big| = \Big|\frac{2t_k}{\sqrt{-6d_k}} - \frac{2t_k y}{3z}\Big| = \frac{2t_k}{\sqrt{3}}\Big|\frac{z\sqrt{3} - y\sqrt{-2d_k}}{z\sqrt{-2d_k}\sqrt{3}}\Big|$$

$$= \frac{2t_k}{3z\sqrt{-2d_k}} \cdot \frac{3-d_k}{z\sqrt{3} + y\sqrt{-2d_k}} < \frac{2t_k(3-d_k)}{3\sqrt{-6d_k}} \cdot z^{-2} < \frac{3-d_k}{3} \cdot z^{-2} < (1-d_k)\cdot z^{-2}.$$

□

In order to establish the lower bound for the same expression we use the following result by Bennett ([3]) on simultaneous rational approximations of square roots of rationals which are very close to 1.

Lemma 5.2 ([3], Theorem 3.2). *If a_i, p_i, q and N are integers for $0 \leq i \leq 2$ with $a_0 < a_1 < a_2$, $a_j = 0$ for some $0 \leq j \leq 2$, q nonzero and $N > M^9$ where*

$$M = \max\{|a_i| : 0 \leq i \leq 2\},$$

then we have

$$\max_{0\leq i\leq 2}\{\left|\sqrt{1+\frac{a_i}{N}} - \frac{p_i}{q}\right|\} > (130N\gamma)^{-1}q^{-\lambda}$$

where

$$\lambda = 1 + \frac{\log(33N\gamma)}{\log(1.7N^2\prod_{0\leq i<j\leq 2}(a_i - a_j)^{-2})}$$

and

$$\gamma = \begin{cases} \frac{(a_2-a_0)^2(a_2-a_1)^2}{2a_2-a_0-a_1}, & a_2 - a_1 \geq a_1 - a_0 \\ \frac{(a_2-a_0)^2(a_1-a_0)^2}{a_1+a_2-2a_0}, & a_2 - a_1 < a_1 - a_0. \end{cases}$$

By setting

$$\begin{array}{cccc} N = -3d_k, & a_0 = -3, & a_1 = -1, & a_2 = 0 \\ M = 3, & q = 3z, & p_1 = 6s_k x, & p_2 = 2t_k y \end{array}$$

we can apply the theorem above as $N = -3d_k > 3^9$ for $k \geq 6$. Therefore,

$$\max\{|\theta_1 - \frac{6s_k x}{3z}|, |\theta_2 - \frac{2t_k y}{3z}|\} > (130 \cdot (-3d_k)\gamma)^{-1} \cdot (3z)^{-\lambda},$$

where

$$\gamma = \frac{36}{5}, \qquad \lambda = 1 + \frac{\log(-99d_k \cdot \frac{36}{5})}{\log(1.7 \cdot 9d_k^2 \cdot \frac{1}{36})}.$$

Combining this result with Lemma 5.1 we have

$$z^{-\lambda+2} < (1 - d_k)(130 \cdot (-3d_k) \cdot \frac{36}{5}) \cdot 3^{\lambda}.$$

Since $\lambda < 2$ and $-d_k(1 - d_k) < 1.000000821d_k^2$ for $k \geq 6$, it follows that

$$z^{-\lambda+2} < 25272.03d_k^2,$$

and hence

$$(-\lambda + 2)\log z < \log(25272.03d_k^2).$$

Since

$$\frac{1}{2-\lambda} = \frac{1}{1 - \frac{\log(-99d_k \cdot \frac{36}{5})}{\log(1.7 \cdot 9d_k^2 \cdot \frac{1}{36})}} \leq \frac{\log(0.425d_k^2)}{\log(-0.00059d_k)}$$

we have

$$\log z < \frac{\log(25272.03d_k^2)\log(0.425d_k^2)}{\log(-0.00059d_k)}. \tag{17}$$

Furthermore, as $z = \nu_m^{\pm}$ for some $m \geq 0$, it follows that

$$z > \frac{1}{2}(-2d_k - 1 + 2s_k\sqrt{-2d_k})^m.$$

Since $2s_k\sqrt{-2d_k} > -2d_k - 2$ for $k \geq 0$ it follows that

$$z > \frac{1}{2}(-4d_k - 3)^m,$$

and since $(-4d_k - 3)^{-1} < \frac{1}{2}$ for $k \geq 1$ we have

$$z > (-4d_k - 3)^{m-1}.$$

Therefore,

$$\log z > (m-1)\log(-4d_k - 3),$$

and since $m \geq n \geq \frac{2}{3} \cdot \sqrt[4]{-d_k}$, it follows that $m - 1 > 0.5 \cdot \sqrt[4]{-d_k}$ and hence

$$\log z > 0.5 \cdot \sqrt[4]{-d_k} \cdot \log(-4d_k - 3).$$

Combining this result with (17) it follows that

$$\sqrt[4]{-d_k} < \frac{\log(25272.03d_k^2)\log(0.425d_k^2)}{0.5 \cdot \log(-0.00059d_k)\log(-4d_k - 3)}.$$

The expression on the right side of the inequality decreases when k increases, and hence by substituting $k = 6$ it follows that

$$\sqrt[4]{-d_k} < 20.477$$

and

$$-d_k < 175\,817$$

wherefrom $k \leq 5$, which contradicts the assumption $k \geq 6$. Therefore, a minimal integer k for which Theorem 1.1 does not hold, if such exists, is less or equal 5.

6. Remaining cases

To complete the proof, it remains to show that Theorem 1.1 holds also for $0 \leq k \leq 5$. In each case we have to solve a system of Pellian equations where one of the equations is always the Pell's equation $y^2 - 3x^2 = 1$ and the second one is

- $z^2 - 2x^2 = 2$ if $k = 0$,
- $z^2 - 6x^2 = 4$ if $k = 1$,
- $z^2 - 22y^2 = 12$ if $k = 2$,
- $z^2 - 902x^2 = 452$ if $k = 3$,
- $z^2 - 4182y^2 = 2092$ if $k = 4$,
- $z^2 - 58242y^2 = 29122$ if $k = 5$.

All the solutions in positive integers of $y^2 - 3x^2 = 1$ are given by $(x, y) = (x'_m, y'_m)$, where

$$x'_m = \frac{1}{2\sqrt{3}}(2+\sqrt{3})^m - \frac{1}{2\sqrt{3}}(2-\sqrt{3})^m,\ y'_m = \frac{1}{2}(2+\sqrt{3})^m + \frac{1}{2}(2-\sqrt{3})^m, m \geq 0.$$

So, the systems above can be reduced to finding the intersections of (x'_m) or (y'_m) and the following sequences:

- $x_n = \frac{1+\sqrt{2}}{2}(3+2\sqrt{2})^n + \frac{1-\sqrt{2}}{2}(3-2\sqrt{2})^n$, $n \geq 0$, if $k=0$,
- $x_n = \frac{1}{\sqrt{6}}(5+2\sqrt{6})^n - \frac{1}{\sqrt{6}}(5-2\sqrt{6})^n$, $n \geq 0$, if $k=1$,
- $y_n^{\pm} = \pm\frac{5+\sqrt{22}}{\sqrt{22}}(197 \pm 42\sqrt{22})^n \mp \frac{5-\sqrt{22}}{\sqrt{22}}(197 \mp 42\sqrt{22})^n$, $n \geq 0$, if $k=2$,
- $x_n^{\pm} = \pm\frac{61+2\sqrt{902}}{\sqrt{902}}(901 \pm 30\sqrt{902})^n \mp \frac{61-2\sqrt{902}}{\sqrt{902}}(901 \mp 30\sqrt{902})^n$, $n \geq 0$, if $k=3$,
- $y_n^{\pm} = \pm\frac{841+13\sqrt{4182}}{\sqrt{4182}}(37637 \pm 582\sqrt{4182})^n \mp \frac{841-13\sqrt{4182}}{\sqrt{4182}}(37637 \mp 582\sqrt{4182})^n$, $n \geq 0$, if $k=4$,
- $y_n^{\pm} = \pm\frac{23419+97\sqrt{58242}}{2\sqrt{58242}}(524177 \pm 2172\sqrt{58242})^n \mp$
  $\mp\frac{23419-97\sqrt{58242}}{2\sqrt{58242}}(524177 \mp 2172\sqrt{58242})^n$, $n \geq 0$, if $k=5$.

In what follows, we will briefly resolve the case $k=1$ just to demonstrate a method based on Baker's theory of linear forms in logarithms. Obviously, $x'_0 = x_0 = 0$, $x'_2 = x_1 = 4$ and we have to show that there are no other intersections of the sequences (x'_m) and (x_n). Let us assume that $m, n \geq 3$ and $x'_m = x_n$. By putting

$$P = \frac{1}{2\sqrt{3}}(2+\sqrt{3})^m,\ Q = \frac{1}{\sqrt{6}}(5+2\sqrt{6})^n,$$

we have

$$P - \frac{1}{12}P^{-1} = Q - \frac{1}{6}Q^{-1}.$$

Since

$$Q - P = \frac{1}{6}Q^{-1} - \frac{1}{12}P^{-1} > \frac{1}{6}(Q^{-1} - P^{-1}) = \frac{1}{6}P^{-1}Q^{-1}(P-Q),$$

we have $Q > P$. Furthermore, from

$$\frac{Q-P}{Q} = \frac{1}{6}Q^{-1}P^{-1} - \frac{1}{12}P^{-2} < \frac{1}{6}Q^{-1}P^{-1} + \frac{1}{12}P^{-2} < 0.25P^{-2}$$

we obtain

$$\begin{aligned} 0 < \log\frac{Q}{P} &= -\log\left(1 - \frac{Q-P}{Q}\right) < \frac{Q-P}{Q} + \left(\frac{Q-P}{Q}\right)^2 < \frac{1}{4}P^{-2} + \frac{1}{16}P^{-4} \\ &< 0.32P^{-2} < e^{-m}. \end{aligned}$$

The expression $\log\frac{Q}{P}$ can be written as a linear form of three logarithms in algebraic integers $\alpha_1 = 2+\sqrt{3}$, $\alpha_2 = 5+2\sqrt{6}$ and $\alpha_3 = \sqrt{2}$, i.e.

$$\Lambda = -m\log\alpha_1 + n\log\alpha_2 + \log\alpha_3 < e^{-m}.$$

Now, we can apply the famous theorem of Baker and Wüstholz from [2]:

Lemma 6.1. *If $\Lambda = b_1\alpha_1 + \cdots + b_l\alpha_l \neq 0$, where $\alpha_1, \ldots, \alpha_l$ are algebraic integers and $b_1, \ldots, b_l$ are rational integers, then*

$$\log|\Lambda| \geq -18(l+1)!\,l^{l+1}(32d)^{l+2}h'(\alpha_1)\cdots h'(\alpha_l)\log(2ld)\log B,$$

where $B = \max\{|\alpha_1|, \ldots, |\alpha_l|\}$, d is the degree of the number field generated by $\alpha_1, \ldots, \alpha_l$ over the rationals $\mathbb{Q}$,

$$h'(\alpha) = \frac{1}{d}\max\{h(\alpha), |\log\alpha|, 1\},$$

and $h(\alpha)$ denotes the logarithmic Weil height of α .

In our case $l = 3$, $d = 4$, $B = m$, $h'(\alpha_1) \leq 0.33$, $h'(\alpha_2) \leq 0.58$, $h'(\alpha_3) \leq 0.25$ and

$$m \leq 2 \cdot 10^{14} \log m.$$

Since the previous inequality does not hold for $m \geq M = 10^{16}$, we conclude that if there is a solution of $x'_m = x_n$ then $n \leq m < M = 10^{16}$. The upper bound for the solutions can be reduced by using the following lemma (originally introduced in [1]) :

Lemma 6.2 ([5], Lemma 4a))**.** *Let θ, β,α, a be a positive real numbers and let M be a positive integer. Let p/q be a convergent of the continued fraction expansion of θ such that $q > 6M$. If $\varepsilon = \|\beta q\| - M \cdot \|\theta q\| > 0$, where $\|\cdot\|$ denotes the distance from the nearest integer, then the inequality*

$$|m\theta - n + \beta| < \alpha a^{-m},$$

has no integer solutions m and n such that $\log(\alpha q/\varepsilon)/\log a \leq m \leq M$.

After we apply Lemma 6.2 on $\theta = \log \alpha_1 / \log \alpha_2$, $\beta = \log \alpha_3 / \log \alpha_2$, $\alpha = 1/\log \alpha_2$, $M = 10^{16}$ and $a = e$, we obtain a new upper bound $M = 38$ and by another application we obtain $M = 7$. By examining all the possibilities, we prove that the only solutions are $x = 0$ and $x = 4$.

All the other cases can be treated similarly. We get these explicit results which can be interpreted in terms of Theorem 1.1:

$k = 0$**:** $x_0 = x'_1 = 1 \Rightarrow d = d_1 = -3$,
$k = 1$**:** $x_0 = x'_0 = 0$, $x_1 = x'_2 = 4 \Rightarrow d \in \{d_0 = -1, d_2 = -33\}$,
$k = 2$**:** $y_0^+ = y'_1 = 2$, $y_1^- = y'_3 = 26 \Rightarrow d \in \{d_1 = -3, d_3 = -451\}$,
$k = 3$**:** $x_0^+ = x'_2 = 4$, $x_1^- = x'_4 = 56 \Rightarrow d \in \{d_2 = -33, d_4 = -6273\}$,
$k = 4$**:** $y_0^+ = y'_3 = 26$, $y_1^- = y'_5 = 362 \Rightarrow d \in \{d_3 = -451, d_5 = -87363\}$,
$k = 5$**:** $y_0^+ = y'_4 = 97$, $y_1^- = y'_6 = 1351 \Rightarrow d \in \{d_4 = -6273, d_6 = -1216801\}$.

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