FORMULAS FOR DIOPHANTINE QUINTUPLES CONTAINING TWO PAIRS OF CONJUGATES IN SOME QUADRATIC FIELDS

ANDREJ DUJELLA, ZRINKA FRANUŠIĆ, VINKO PETRIČEVIĆ

Abstract. Let D be a positive integer which is not a perfect square. We consider Diophantine quintuples in the ring $\mathbb{Z}[\sqrt{D}]$ of the form

$$\{e, a \pm b\sqrt{D}, c \pm d\sqrt{D}\}$$

where a, b, c, d, e are integers. In this paper, we show that there exists a Diophantine quintuple of that form for certain values of D, including $D = 1 + n^2(n+1)^2$ and some other polynomials of degree 4, and we represent its elements also as polynomials in n.

1. Introduction

Let $\mathcal{R}$ be a commutative ring with the unity. *A Diophantine m-tuple in $\mathcal{R}$* is a set of m elements in $\mathcal{R}\backslash\{0\}$ with the property that the product of any two of its distinct elements increased by the unity is a square in $\mathcal{R}$. Diophantine m-tuples have been most studied for $\mathcal{R} = \mathbb{Z}$ and $\mathcal{R} = \mathbb{Q}$ where the major focus has been on finding an upper bound on m, i.e. on the size of such a set. Let us mention two important historical examples of such sets, $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ - the first Diophantine quadruple in $\mathbb{Q}$ (found by Diophantus himself) and $\{1, 3, 8, 120\}$ - the first Diophantine quadruple in $\mathbb{Z}$ (found by Fermat). There does not exist an integer Diophantine quintuple (see [11]) and, on the other hand, there are infinitely many rational Diophantine sextuples (see [7]). A brief overview of the results on Diophantine m-tuples, including various generalizations, can be found in [5, 6].

Any Diophantine triple $\{a_1, a_2, a_3\}$ can be extended to a Diophantine quadruple by adding one of the following two elements (if they are not equal to 0):

$$d_{\pm} = a_1 + a_2 + a_3 + 2a_1a_2a_3 \pm 2rst, \tag{1}$$

2020 *Mathematics Subject Classification.* 11D09, 11R11.
Key words and phrases. Diophantine quintuples, regular Diophantine quadruples, quadratic fields.

where $a_1a_2+1 = r^2$, $a_1a_3+1 = s^2$, $a_2a_3+1 = t^2$. Sets $\{a_1, a_2, a_3, d_-\}$ and $\{a_1, a_2, a_3, d_+\}$ (if $d_\pm \neq 0$) are called *regular* Diophantine quadruples. It is not difficult to show that the relation

$$(a_1 + a_2 - a_3 - a_4)^2 = 4(a_1a_2 + 1)(a_3a_4 + 1) \tag{2}$$

characterizes the property of being regular, i.e. $\{a_1, a_2, a_3, a_4\}$ is a regular Diophantine quadruple if and only if (2) holds. There is a conjecture saying that all Diophantine quadruples in $\mathbb{Z}$ are regular. If $d_-d_+ + 1 = \square$ and $d_\pm \neq 0$, then $\{a_1, a_2, a_3, d_+, d_-\}$ represents a Diophantine quintuple and such a set will be called a *biregular* Diophantine quintuple. Simply said, a biregular Diophantine quintuple includes two regular quadruples. Biregular quadruples in $\mathbb{Q}$ were studied in [4] and [8], and applied to construction of high-rank elliptic curves and rational Diophantine sextuples.

In this paper, we deal with biregular Diophantine quintuples containing two pairs of conjugates in the ring $\mathbb{Z}[\sqrt{D}]$, where D is a positive integer and not a perfect square, i.e. with quintuples of the form

$$\{e, a + b\sqrt{D}, a - b\sqrt{D}, c + d\sqrt{D}, c - d\sqrt{D}\}, \tag{3}$$

such that $a, b, c, d, e \in \mathbb{Z}$ and $c \pm d\sqrt{D}$ correspond to the regular extensions $d_\pm$ generated by the triple $\{e, a + b\sqrt{D}, a - b\sqrt{D}\}$.

Our work has been motivated by examples found by Gibbs in [10] and some of them are listed below. (Occasionally we denote an element $a + b\sqrt{D} \in \mathbb{Q}(\sqrt{D})$ by (a, b).)

D	Diophantine quintuple in $\mathbb{Z}[\sqrt{D}]$
2	$(3,0), (7,4), (7,-4), (119,84), (119,-84)$
5	$(4,0), (7,3), (7,-3), (50,22), (50,-22)$
13	$(6,0), (8,2), (8,-2), (166,46), (166,-46)$
17	$(12,0), (21,5), (21,-5), (438,106), (438,-106)$
29	$(4,0), (41,7), (41,-7), (2166,402), (2166,-402)$
34	$(5,0), (81,12), (81,-12), (16817,2884), (16817,-2884)$
37	$(4,0), (43,5), (43,-5), (7482,1230), (7482,-1230)$

Table 1

Gibbs conducted a search for Diophantine quintuples in $\mathbb{Z}[\sqrt{D}]$ for square free D with $|D| < 50$ and found 160 examples. All examples are found for positive D and all of them are biregular, i.e. include two regular quadruples. No example was found for $D \in \{23, 35, 42, 43, 47\}$.

We managed to find a Diophantine quintuple in $\mathbb{Z}[\sqrt{43}]$:

$$\{(-7512908, 1145708), (-195, 30), (0, 848), (195, 30), (7512908, 1145708)\}.$$

For other exceptions, we found many examples of "almost quintuples", meaning that only one condition out of ten is missing. Also, no Diophantine quintuples were found for negative D. In [1] Adžaga showed that there is no Diophantine m-tuple in imaginary quadratic number ring (i.e. with $D < 0$) with $m > 43$. It is also known that for $D = -1$ some particular Diophantine quadruples cannot be extended to Diophantine quintuples (see [2, 3, 9]).

We are interested in construction Diophantine quintuples in $\mathbb{Z}[\sqrt{D}]$ for infinite families of positive integers D. It is easy to obtain certain results of that type. Namely, if $\{a_1, a_2, a_3\}$ is a Diophantine triple in $\mathbb{Z}$ and $d_{\pm} \neq 0$, then $\{a_1, a_2, a_3, d_+, d_-\}$ is a Diophantine triple in $\mathbb{Z}[\sqrt{D}]$ for $D = d_+d_- + 1$. By taking $a_1 = n-1$, $a_2 = n+1$, $a_3 = 16n^3 - 4n$, we obtain $d_- = 4n$, $d_+ = 64n^5 - 48n^3 + 8n$ and $D = 256n^6 - 192n^4 + 32n^2 + 1$. Therefore, we are especially interested in families of D's which are asymptotically larger then this simply obtained family, i.e. in parametric families of D's where involved polynomials have degree smaller than 6.

One of our results of that shape in the following theorem.

Theorem 1. *Let n be a positive integer and $D = 1 + n^2(n+1)^2$. There exists a biregular Diophantine quintuple of the form* (3) *in $\mathbb{Z}[\sqrt{D}]$.*

2. Equations

If $\{z_1, z_2, z_3, z_4, z_5\}$ is a Diophantine quintuple in $\mathcal{R}$, then the following ten equations should be satisfied:

$$z_i z_j + 1 = \xi_{ij}^2, \ 1 \leq i < j \leq 5,$$

where $\xi_{ij} \in \mathcal{R}$. If a Diophantine quintuple in $\mathbb{Z}[\sqrt{D}]$ is of the form (3), then it suffices to fulfill only these equations:

$$e(a + b\sqrt{D}) + 1 = (u + v\sqrt{D})^2, \tag{4}$$

$$a^2 - Db^2 + 1 = x^2, \tag{5}$$

or

$$a^2 - Db^2 + 1 = x^2 D, \tag{6}$$

and

$$c^2 - Dd^2 + 1 = y^2, \tag{7}$$

or

$$c^2 - Dd^2 + 1 = y^2 D, \tag{8}$$

for $u, v, x, y \in \mathbb{Z}$. Since we assume that $c \pm d\sqrt{D}$ are regular extensions of the triple $\{e, a \pm b\sqrt{D}\}$, the conditions $e(c \pm d\sqrt{D}) + 1 = \square$, $(a \pm b\sqrt{D})(c \pm d\sqrt{D}) + 1 = \square$ are "automatically" fulfilled. Also, the condition $e(a - b\sqrt{D}) + 1 = \square$ can be omitted because (4) implies $e(a - b\sqrt{D}) + 1 = (u - v\sqrt{D})^2$.

According to (1) and putting $r = u + v\sqrt{D}$, $s = u - v\sqrt{D}$, $t = x\sqrt{D}$ we have

$$c \pm d\sqrt{D} = e + 2a + 2e(a^2 - Db^2) \pm 2(u^2 - Dv^2)x\sqrt{D}. \tag{9}$$

Also, since we have assumed that our quintuple should contain two pairs of conjugates in $\mathbb{Z}[\sqrt{D}]$, the possibility (5) is rejected (because it would yield a rational integer value of $c \pm d\sqrt{D}$). We further assume that the equation (7) should hold and we get

$$(2a + 2eDx^2 - e)^2 - 4D(u^2 - Dv^2)^2 x^2 + 1 = y^2, \tag{10}$$

where we substituted $a^2 - b^2 D = Dx^2 - 1$. Equation (4) splits into

$$ea + 1 = u^2 + Dv^2, \ eb = 2uv \tag{11}$$

and imply $(u^2 - Dv^2)^2 = (ea+1)^2 - (eb)^2 D = (ea+1)^2 - e^2(a^2 - Dx^2 + 1) = 1 + 2ae - e^2 + de^2 x^2$. Therefore (10) transforms into

$$(2a - e)^2 - 4Dx^2 + 1 = y^2. \tag{12}$$

So, if equations (11), (6), (12) (or equivalently (4), (6), (7)) are solvable in $e, a, b, u, v, x, y \in \mathbb{Z}$, then (3) represents a biregular Diophantine quintuple.

3. Solving the equations

From (11) we get

$$a = \frac{u^2 + Dv^2 - 1}{e}, \ b = \frac{2uv}{e}$$

and substituting into (6) yields

$$(u^2 + Dv^2 - 1)^2 - D(2uv)^2 + e^2 = Dx^2 e^2, \tag{13}$$

$$1 + e^2 - 2u^2 + u^4 - 2Dv^2 - 2Du^2v^2 + D^2v^4 = Dx^2e^2.$$

Obviously, $D \mid 1 + e^2 - 2u^2 + u^4$. Hence, assume that

$$1 + e^2 - 2u^2 + u^4 = kD, \ k \in \mathbb{Z}.$$

First, dividing (13) by D and then putting $D = (1 + e^2 - 2u^2 + u^4)/k$, we get

$$\frac{1}{k}(k^2 - 2kv^2 - 2ku^2v^2 + v^4 + e^2v^4 - 2u^2v^4 + u^4v^4) = x^2e^2.$$

The expression on the left side of the previous equality can be viewed as a quartic polynomial in u, $p(u) = \frac{1}{k}(k^2 - 2kv^2 - 2ku^2v^2 + v^4 + e^2v^4 - 2u^2v^4 + u^4v^4)$. It can be shown that if

$$k = \frac{e^2v^2}{4},$$

the discriminant of p equals zero and $p(u) = \frac{v^2\left(e^2-4u^2+4\right)^2}{4e^2} = \square$. So far, we have

$$D = \frac{4(1 + e^2 - 2u^2 + u^4)}{e^2v^2},$$

$$a = \frac{e^2\left(u^2 + 3\right) + 4\left(u^2 - 1\right)^2}{e^3}, \quad b = \frac{2uv}{e}.$$

An analogous procedure have to be carried out to fulfill (12). Taking into account the above, we get

$$\frac{-16e^2\left(u^4 - 6u^2 + 1\right) + e^4\left(e^2 - 4u^2 - 15\right) + 64u^2\left(u^2 - 1\right)^2}{e^4} = q(u) = y^2.$$

For $e = 4$ the discriminant of the polynomial q equals zero and

$$q(u) = \frac{1}{4}u^2(-3 + u^2)^2 = \square.$$

The only thing left is to find the parameters of u and v such that a, b, D given by

$$D = \frac{17 - 2u^2 + u^4}{4v^2},\ a = \frac{13 + 2u^2 + u^4}{16},\ b = \frac{uv}{2}$$

are integers. Obviously, u should be odd, $u = 2n + 1$, and with $v = 2$ we obtain

$$D = 1 + n^2(1 + n)^2,\ a = 1 + n + 2n^2 + 2n^3 + n^4,\ b = 1 + 2n.$$

So, the set

$$\begin{aligned} &\{4, 1 + n + 2n^2 + 2n^3 + n^4 \pm (1 + 2n)\sqrt{D}, \\ &6 - 14n + 4n^2 + 20n^3 - 22n^4 - 16n^5 + 32n^6 + 32n^7 + 8n^8 \\ &\pm (-6 + 14n - 2n^2 - 24n^3 + 8n^4 + 24n^5 + 8n^6)\sqrt{D}\} \end{aligned} \tag{14}$$

represents a biregular Diophantine quintuple in $\mathbb{Z}[\sqrt{D}]$, where $D = 1 + n^2(1+n)^2$, because the equations (4), (6), (7) are solvable in $\mathbb{Z}$. Indeed,

$$e(a + b\sqrt{D}) + 1 = (1 + 2n + 2\sqrt{D})^2,$$

$$(a + b\sqrt{D})(a - b\sqrt{D}) + 1 = ((-1 + n + n^2)\sqrt{D})^2,$$

$$(c + d\sqrt{D})(c - d\sqrt{D}) + 1 = (-1 + 6n^2 + 4n^3)^2.$$

This finishes the proof of Theorem 1.

For $D < 1000$ we list the examples of a Diophantine quintuples (14):

d	e	(a, b)	(c, d)
5	4	(7, 3)	(50, 22)
37	4	(43, 5)	(7482, 1230)
145	4	(157, 7)	(140670, 11682)
401	4	(421, 9)	(1158926, 57874)
901	4	(931, 11)	(6063786, 202014)

Table 2

Note that the first two rows in Table 2 correspond to examples from Table 1.

4. More examples

Here we try to find more solutions assuming that D is a polynomial of degree 4, as we obtained in the previous sections. Thus, let us take

$$D = D(n) = d_4n^4 + d_3n^3 + d_2n^2 + d_1n + d_0,$$

where $d_0, d_1, d_2, d_3, d_4 \in \mathbb{Z}$. Also, we assume that

$$u = u(n) = u_1n + u_0.$$

So, (11) gives

$$a = \frac{1}{e}(-1+u_0^2+d_0v^2+(2u_0u_1+d_1v^2)n+(u_1^2+d_2v^2)n^2+d_3v^2n^3+d_4v^2n^4),$$

$$b = \frac{2v}{e}(u_0 + u_1n).$$

Equation (6) implies that $D \mid a^2 + 1$ and therefore the polynomial remainder of these two polynomials, $D(n)$ and $(a^2+1)(n)$, equals zero.

By equating its coefficients to zero, we get

$$d_0 = \frac{1-2u_0^2+u_0^4+e^2}{u_1^4}d_4,\ d_1 = \frac{4(-1+u_0^2)u_0}{u_1^3}d_4,$$
$$d_2 = \frac{2(-1+3u_0^2)}{u_1^2}d_4,\ d_3 = \frac{4u_0}{u_1}d_4.$$

Also, the equation (6) yields that $(a^2+1-Db^2)/D = \square$. Hence, if $(a^2+1-Db^2)/D$ is considered as a polynomial in n, then for

$$d_4 = \frac{4u_1^4}{e^2v^2}$$

its discriminant equals zero and

$$\frac{1}{D}(a^2+1-Db^2) = \frac{(-4-e^2+4u_0^2+8nu_0u_1+4n^2u_1^2)^2v^2}{4e^4} = \square.$$

As argued in the previous sections, for $e=4$ we have $c^2-Dd^2+1=\square$. By all obtained, we have

$$\begin{aligned}
D &= \frac{1}{4v^2}(17-2u_0^2+u_0^4+(-4u_0u_1+4u_0^3u_1)n \\
&\qquad + (-2u_1^2+6u_0^2u_1^2)n^2+4u_0u_1^3n^3+u_1^4n^4),\\
a &= \frac{1}{16}(13+2u_0^2+u_0^4+(4u_0u_1+4u_0^3u_1)n \\
&\qquad + (2u_1^2+6u_0^2u_1^2)n^2+4u_0u_1^3n^3+u_1^4n^4),\\
b &= \frac{(u_0+u_1n)v}{2}.
\end{aligned}$$

We still have to choose u_0, u_1, v which would give integer values of D, a, b. For $v=2$, odd $u_0 = 2k+1$ and even $u_1 = 2l$, we obtain that $D, a, b \in \mathbb{Z}$. However, by taking $n_0 = k+ln$, we get $D = 1+n_0^2(n_0+1)^2$ and the quintuple (14). Nevertheless, for another choice of the parameter v, for instance $v=10$ and $u_0 = 23+50k$, $u_1 = 50l$ we get a new solution. If we put again $n_0 = k+ln$, we obtain

$$\begin{aligned}
D &= 697+6072n_0+19825n_0^2+28750n_0^3+15625n_0^4,\\
a &= 17557+152375n_0+496250n_0^2+718750n_0^3+390625n_0^4,\\
b &= 115+250n_0,\\
c &= 2392278510+41841233150n_0+319909592500n_0^2+1396567187500n_0^3\\
&\quad +3807366406250n_0^4+6637656250000n_0^5+7226562500000n_0^6\\
&\quad +4492187500000n_0^7+1220703125000n_0^8\\
d &= 90614010+1190152250n_0+6504293750n_0^2+18931875000n_0^3\\
&\quad +30953125000n_0^4+26953125000n_0^5+9765625000n_0^6.
\end{aligned}$$

We conclude with a table of examples obtained by extending the range of search from [10] (we omit examples from Table 1).

d	e	(a, b)	(c, d)
2	6	(31,15)	(6200,4384)
2	10	(13,9)	(176,124)
2	3	(39,20)	(4407,3116)
2	21	(17,12)	(97,68)
2	6	(403,279)	(81536,57652)
2	21	(97,68)	(6977,4932)
2	3	(7655,3828)	(175766455,124285652)
2	182	(107,75)	(72832,51500)
2	3	(44615,22308)	(5971316215,4222358188)
2	1974	(1379,975)	(1548400,1094884)
2	4074	(1297,831)	(2453263544,1734719288)
2	3	(8833479,4416740)	(234091018396407,165527346522964)
2	7665	(639,320)	(3119985873,2206163168)
5	28	(148,30)	(974948,436010)
5	416	(718,287)	(86262780,38577888)
5	104	(2467,493)	(1013140590,453090246)
5	3344	(3097,1379)	(556477890,248864478)
13	6	(268,22)	(786926,218254)
13	6	(86,20)	(26530,7358)
13	10	(148,34)	(137826,38226)
13	234	(44,10)	(297970,82642)
13	114	(122,32)	(358774,99506)
13	696	(278,77)	(289396,80264)
13	7794	(10652,2618)	(379787495194,105334099054)
17	12	(1211,285)	(2059138,499414)
17	12	(29635,6973)	(1239583250,300643098)
17	3192	(2240,527)	(1890993160,458633208)
17	12	(1955293,460069)	(5397399308486,1309061614854)
29	112	(17,1)	(58386,10842)
29	20	(17,3)	(1174,218)
29	44	(331,23)	(8292066,1539798)
34	5	(125745,18492)	(41853919985,7177888060)
37	390	(1708,238)	(640723886,105334358)
37	1146	(5026,700)	(16343520590,2686858234)
41	4032	(2082,325)	(33062532,5163500)
53	4	(33307,675)	(8681731610,1192527550)
58	90	(17,1)	(41704,5476)

d	e	(a,b)	(c,d)
61	1482	(782,100)	(4520182,578750)
73	4	(27,3)	(634,74)
73	8	(27,3)	(1214,142)
73	8	(162452,17803)	(52056888864,6092797992)
82	306	(173,19)	(200776,22172)
85	14	(132,6)	(402470,43654)
85	4	(3277,113)	(77233470,8377146)
97	3792	(1239,115)	(1913419134,194278278)
109	20	(33,3)	(4406,422)
113	1680	(1228,113)	(218696456,20573232)
130	6	(203,11)	(306160,26852)
145	4	(157,7)	(140670,11682)
229	1992	(15007,719)	(425594736326,28124091802)
401	4	(421,9)	(1158926,57874)
401	232	(782,25)	(167458932,8362500)
409	20	(143,7)	(16626,822)
493	15924	(11037,497)	(1271792334,57278646)
586	590	(3671,71)	(12416221632,512909388)
697	4	(17557,115)	(2392278510,90614010)
769	1400	(5321,187)	(3981276042,143568486)
901	4	(931,11)	(6063786,202014)
901	3540	(1832,61)	(25516444,850076)
1093	1056	(563,17)	(2308486,69826)
1765	4	(1807,13)	(23739330,565062)
1961	2	(1030,10)	(3461262,78162)

Table 3

Here is a brief description of our algorithm. For all square free D, $1 < D < 1000$, $1 \leq u, v \leq 10000$ and for all positive integers e such that $e \mid \gcd(u^2 + Dv^2 - 1, 2uv)$, we put $t = ((u + v\sqrt{D})^2 - 1)/e$ and test if $(N(t) + 1)/D$ equals a perfect square. If "yes", then $\{e, t, \bar{t}\}$ is a Diophantine triple. If $(2a - e)^2 - 4Dx^2 + 1 = \square$ or $D \cdot \square$, where $a = (t + \bar{t})/2$, then the Diophantine triple $\{e, t, \bar{t}\}$ can be exteded to a biregular Diophantine quintuple containing two pairs of conjugates $\{e, t, \bar{t}, t', \bar{t'}\}$ (where $t', \bar{t'}$ are given by (9) and $t' \neq 0$).

Note that entries for $d = 145$, 401, 697, 901, 1765 are special cases of our polynomials formulas for Diophantine quintuples.

Acknowledgements. The authors acknowledge support from the QuantiXLie Center of Excellence, a project co-financed by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (Grant KK.01.1.1.01.0004). A. D. and V. P. were supported by the Croatian Science Foundation under the project no. IP-2018-01-1313.

References

[1] N. Adžaga, *On the size of Diophantine m-tuples in imaginary quadratic number rings*, Bull. Math. Sci. **9(3)** (2019) 1950020 (10 pages).

[2] N. Adžaga, A. Filipin and Z. Franušić, *On the extensions of the Diophantine triples in Gaussian integers*, preprint.

[3] A. Bayad, A. Filipin and A. Togbé, *Extension of a parametric family of Diophantine triples in Gaussian integers*, Acta Math. Hungar. **148** (2016), 312–327.

[4] A. Dujella, *Diophantine triples and construction of high-rank elliptic curves over* $\mathbb{Q}$ *with three non-trivial* 2*-torsion points*, Rocky Mountain J. Math. **30** (2000), 157–164.

[5] A. Dujella, *Diophantine m-tuples*, `https://web.math.pmf.unizg.hr/~duje/dtuples.html`

[6] A. Dujella, *What is...a Diophantine m-tuple?*, Notices Amer. Math. Soc. **63** (2016), 772–774.

[7] A. Dujella, M. Kazalicki, M. Mikić and M. Szikszai, *There are infinitely many rational Diophantine sextuples*, Int. Math. Res. Not. IMRN **2017 (2)** (2017), 490–508.

[8] A. Dujella, M. Kazalicki and V. Petričević, *Rational Diophantine sextuples containing two regular quadruples and one regular quintuple*, Acta Mathematica Spalatensia, to appear.

[9] Z. Franušić, *On the extensibility of Diophantine triples* $\{k-1, k+1, 4k\}$ *for Gaussian integers*, Glas. Mat. Ser. III **43** (2008), 265–291.

[10] P. E. Gibbs, *Diophantine Quintuples over Quadratic Rings*, `https://www.researchgate.net/publication/323176085_Diophantine_Quintuples_over_Quadratic_Rings` (2018)

[11] B. He, A. Togbé and V. Ziegler, *There is no Diophantine quintuple*, Trans. Amer. Math. Soc. **371** (2019), 6665–6709.

Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia

Email address: duje@math.hr

Email address: fran@math.hr

Email address: vpetrice@math.hr