A DIOPHANTINE PROBLEM IN $\mathbb{Z}[(1+\sqrt{d})/2]$

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Abstract. We characterize the existence of infinitely many Diophantine quadruples with the property $D(z)$ in the ring $\mathbb{Z}[(1+\sqrt{d})/2]$, where d is a positive integer such that the Pellian equation $x^2 - dy^2 = 4$ is solvable, in the terms of the representability of z as a difference of two squares.

1. Introduction

A *Diophantine quadruple* in a commutative ring R with the unit 1 is the set of four distinct non-zero elements with the property that the product of each two distinct elements increased by 1 is a perfect square in R. These sets owe its name to Diophantus of Alexandria, the first one (as we belive) who found such a set among rational numbers, $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$. Many centuries later, Fermat found another interesting set, $\{1, 3, 8, 120\}$, the first Diophantine quadruple consisting of integers.

In this paper, we deal with generalized Diophantine quadruples called *Diophantine quadruples with the property* $D(r)$, where $r \in R$, i.e. with the sets of four distinct non-zero elements in R such that the product of each two distinct elements increased by r is a perfect square in R. Often, we use the shorter term $D(r)$*-quadruples*. The natural problem that arises here is to describe the set of all $r \in R$ such that $D(r)$-quadruple exists. This problem has been solved in certain rings. In the ring of integers $\mathbb{Z}$, Brown in [3], Gupta, Singh in [10] and Mohanty, Ramasamy in [14] proved independently that if an integer n is congruent to 2 modulo 4, then the $D(n)$-quadruple in $\mathbb{Z}$ does not exist. On the contrary, Dujella showed in [4] that if n is not congruent to 2 modulo 4 and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\} = S$, then a $D(n)$-quadruple exists. It is interesting that we can state these results as follows: *a* $D(n)$*-quadruple exists in* $\mathbb{Z}$ *if and only if* n *can be represented as a difference of two squares of integers, up to finitely many possible exceptions.* Let us mention that it was conjectured that for $n \in S$, a $D(n)$-quadruple does not exist and some partial results were obtained in [3], [7], [8]. In the ring of Gaussian integers $\mathbb{Z}[i]$, the analogous statement has been shown by Dujella in [6]. In $\mathbb{Z}[\sqrt{d}]$ such that $d \equiv 3 \pmod 4$ and that one of the equations $x^2 - dy^2 = \pm 2$ is solvable, a stronger result is valid. Precisely, it was proved

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by the author in [12] that *there exist infinitely many $D(z)$-quadruples if and only if z can be represented as a difference of two squares in* $\mathbb{Z}[\sqrt{d}]$. This assertion was also proved in the ring $\mathbb{Z}[\sqrt{2}]$ ([11]) and only partially proved in [1].

Our intention here is to show *the equivalence between the existence of infinitely many $D(z)$-quadruples and the representability of z as a difference of two squares in some rings of the form* $\mathbb{Z}[(1+\sqrt{d})/2]$. We assume that the equation $x^2 - dy^2 = 4$ is solvable in odd numbers. This assumption allow us to have a nice characterization of elements in $\mathbb{Z}[(1+\sqrt{d})/2]$ that are representable as a difference of two squares of elements in $\mathbb{Z}[(1+\sqrt{d})/2]$. Specifically, according to [9], we have the following result: $z \in \mathbb{Z}[(1+\sqrt{d})/2]$ can be represented as a difference of two squares in $\mathbb{Z}[(1+\sqrt{d})/2]$ if and only if z is one of the following forms $2m+1+2n\sqrt{d}$, $2m+(2n+1)\sqrt{d}$, $4m+4n\sqrt{d}$, $4m+2+(4n+2)\sqrt{d}$, $\dfrac{2m+1}{2}+\dfrac{2n+1}{2}\sqrt{d}$, where $m,n \in \mathbb{Z}$. The proof of the existence of $D(z)$-quadruples is based on an effective construction (see Section 2). On the other hand, if z is not of the above form then we show that the assumption of the existence of a $D(z)$-quadruple leads to a contradiction (see Section 3).

Let us recall several facts concerning our assumption on solvability of the Pellian equation $x^2 - dy^2 = 4$, where $d \equiv 1 \pmod 4$ and $\gcd(x,y) = 1$. The problem of giving necessary and sufficient conditions for the solvability of this equation is called *Eisenstein's problem* in the literature and many authors have worked on this problem (see [13], [15], [16] for instance). For an impression, we list all $d \in \mathbb{N}$ less than 200 such that our assumption is fulfilled: $5^{\pm}$, $13^{\pm}$, 21^{+}, $29^{\pm}$, 45^{+}, $53^{\pm}$, $61^{\pm}$, 69^{+}, 77^{+}, $85^{\pm}$, 93^{+}, $109^{\pm}$, 117^{+}, $125^{\pm}$, 133^{+}, $149^{\pm}$, 157^{+}, 165^{+}, $173^{\pm}$, $181^{\pm}$, where the superscript $+$ denotes that the equation $x^2 - dy^2 = 4$ is solvable, while the superscript $\pm$ denotes that both of the equations $x^2 - dy^2 = \pm 4$ are solvable. (If $x^2 - dy^2 = -4$ is solvable in odd numbers, then $x^2 - dy^2 = 4$ is also solvable in odd numbers.)

2. The existence of Diophantine quadruples

The assumptions in this section are that $d \in \mathbb{N}$ is not a perfect square and that the Pellian equation $x^2 - dy^2 = 4$ is solvable in odd integers x and y. Consequently, we have that $d \equiv 5 \pmod 8$.

For each element $z \in \mathbb{Z}[(1+\sqrt{d})/2]$ such that $z = a^2 - b^2$, where $a, b \in \mathbb{Z}[(1+\sqrt{d})/2]$, Diophantine quadruple with the property $D(z)$ will be constructed. These elements are described precisely in the following lemma.

Lemma 1. [9, Theorem 1] *An element $z \in \mathbb{Z}[(1+\sqrt{d})/2]$ can be represented as a difference of two squares in $\mathbb{Z}[(1+\sqrt{d})/2]$ if and only if z has one of* *the following forms*

$$2m+1+2n\sqrt{d},\ 2m+(2n+1)\sqrt{d},\ 4m+4n\sqrt{d},$$
$$4m+2+(4n+2)\sqrt{d},\ \frac{2m+1}{2}+\frac{2n+1}{2}\sqrt{d},$$

where $m,n\in\mathbb{Z}$.

Constructions of some quadruples arise from polynomial formulas for Diophantine quadruples derived in [4] and [5]. The results we need are collected in the following lemmas.

Lemma 2. [5, Theorem 1] *The sets*

$$\{m,(3k+1)^2m+2k,(3k+2)^2m+2k+2,9(2k+1)^2m+8k+4\},$$
$$\{m,mk^2-2k-2,m(k+1)^2-2k,m(2k+1)^2-8k-4\}$$

have the property $D(2m(2k+1)+1)$.

Lemma 3. [4] *The set*

$$\{1,9k^2-8k,9k^2-2k+1,36k^2-20k+1\}$$

has the property $D(8k)$ *and the set*

$$\{4,9k^2-5k,9k^2+7k+2,36k^2+4k\},$$

has the property $D(8k+1)$.

Let us mention that by the expression a *set with the property* $D(z)$ we mean that the product of each two distinct elements of this set increased by z is a perfect square, but we allow that some of the elements could be equal or equal to zero (unlike in the case of Diophantine quadruples).

Often, we use the following simple property.

Lemma 4. *Let* $\{z_1,z_2,z_3,z_4\}$ *be the set with the property* $D(z)$. *Then the set* $\{z_1w,z_2w,z_3w,z_4w\}$ *has the property* $D(zw^2)$.

The following technical lemma allows us to apply Lemma 2.

Lemma 5. *For each* $M,N\in\mathbb{Z}$, *there exist* $k\in\mathbb{Z}[(1+\sqrt{d})/2]$ *and* $m=\alpha+\beta\sqrt{d}\in\mathbb{Z}[\sqrt{d}]$ *such that*

- **a:** $2M+1+2N\sqrt{d}=2m(k+1)+1$, $k\in\mathbb{Z}[\sqrt{d}]$ *and* $\alpha^2-d\beta^2=1$,
- **b:** $4M+3+(4N+2)\sqrt{d}=2m(2k+1)+1$, α,β *are odd and* $\alpha^2-d\beta^2=4$,
- **c:** $2M+(2N+1)\sqrt{d}=m(2k+1)+1$, α,β *are odd and* $\alpha^2-d\beta^2=4$,
- **d:** $2M+1+(2N+1)\sqrt{d}=m(2k+1)+1$, $\alpha^2-d\beta^2=\pm1$,
- **e:** $\frac{2M+1}{2}+\frac{2N+1}{2}\sqrt{d}=\frac{m}{2}(2k+1)+1$, α,β *are odd and* $\alpha^2-d\beta^2=4$.

Proof. The proofs of all statements are similar. Thus we provide the proof only for the statement **c**.

Let (α, β) be a solution of the equation $x^2 - dy^2 = 4$ in odd integers. The equation $2M+1+(2N+1)\sqrt{d} = m(2k+1)+1$ is equivalent to the following linear system

$$\begin{array}{rcrcl} 2\alpha\gamma & + & 2\beta d\delta & = & 2M-\alpha-1, \\ 2\beta\gamma & + & 2\alpha\delta & = & 2N+1-\beta, \end{array}$$

where $k = \gamma + \delta\sqrt{d}$. We solve this system in unknowns γ, δ and obtain that

$$\gamma = \pm((2M-\alpha-1)\alpha - (2N+1-\beta)\beta d)/8,$$
$$\delta = \pm((2N+1-\beta)\alpha - (2M-\alpha-1)\beta)/8.$$

We can choose α, β such that $2M-\alpha-1 \equiv 2N+1-\beta \equiv 0 (\text{mod } 4)$. Then the numerators in the above expressions for γ, δ are either both congruent to 0 or both congruent to 4 modulo 8 and we showed that $k \in \mathbb{Z}[(1+\sqrt{d})/2]$. $\square$

Lemma 6. *Let $M, N \in \mathbb{Z}$. There exist $p, q \in \mathbb{Z}$ of different parity and $w = \frac{\alpha}{2} + \frac{\beta}{2}\sqrt{d}$ such that α, β are solutions of the equation $x^2 - dy^2 = 4$ in odd numbers and that*

$$\frac{2M+1}{2} + \frac{2N+1}{2}\sqrt{d} = (p + q\sqrt{d})w^2. \tag{1}$$

Proof. The equation (1) can be understood as the system in unknowns p and q, which solutions are given by

$$p = ((2M+1)\frac{\alpha^2+d\beta^2}{2} - (2N+1)\alpha\beta d)/4, \tag{2}$$

$$q = ((2N+1)\frac{\alpha^2+d\beta^2}{2} - (2M+1)\alpha\beta)/4. \tag{3}$$

We choose α, β such that $\alpha\beta \equiv 3 \pmod 4$ if $M \equiv N \pmod 2$, and $\alpha\beta \equiv 1 \pmod 4$ if $M \not\equiv N \pmod 2$. Then it is clear that p, q are integers. Moreover, p and q are of different parity since the difference of the numerators in (2) and (3) is

$$2(M-N)(\frac{\alpha^2+d\beta^2}{2} + \alpha\beta) + (2N+1)\alpha\beta(1-d) \equiv 0 + 4 (\text{mod } 8). \tag{4}$$

$\square$

Theorem 1. *If $z \in \mathbb{Z}[(1+\sqrt{d})/2]$ can be represented as a difference of two squares in $\mathbb{Z}[(1+\sqrt{d})/2]$, then there exist infinitely many Diophantine quadruples with the property $D(z)$ in $\mathbb{Z}[(1+\sqrt{d})/2]$.*

Proof. The proof splits into five parts. Each part corresponds to one of the forms of z given in Lemma 1.

1) Let $z = 2M+1+2N\sqrt{d}$, $M, N \in \mathbb{Z}$. According to Lemmas 2 and 5**a**, there exist $k = \gamma+\delta\sqrt{d}$ and $m = \alpha+\beta\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ such that $z = 2m(k+1)+1$ and that the set

$$D = \{m, (\frac{k}{2})^2 m - k - 2, (\frac{k}{2}+1)^2 m - k, (k+1)^2 m - 4k - 4\} \tag{5}$$

has the property $D(z)$. The elements of the set D are not necessarily in $\mathbb{Z}[(1+\sqrt{d})/2]$. This condition will be fulfilled if and only if γ, δ are of the same parity, i.e. if and only if M, N are of the different parity. Hence, if z is of the form $4M+3+4N\sqrt{d}$ or $4M+1+(4N+2)\sqrt{d}$, then there exists a set in $\mathbb{Z}[(1+\sqrt{d})/2]$ with the property $D(z)$. Let us note that if z is of the form $4M+3+4N\sqrt{d}$, then $D \subset \mathbb{Z}[\sqrt{d}]$ (because related γ, δ are even).

If the set D consists of non-zero, distinct elements of $\mathbb{Z}[(1+\sqrt{d})/2]$, then D is the $D(z)$-quadruple. In what follows, we consider the case when at least two elements of D are equal or the case when an element of D is zero. Suppose that these cases occur for some z_0 of the form $4M+3+4N\sqrt{d}$ (or $4M+1+(4N+2)\sqrt{d}$). We know that $z_0 = 2(2k_0+1)m_0+1$, for some $k_0, m_0 \in \mathbb{Z}[\sqrt{d}]$ (from Lemma 5**a**). Let S be the set of all numbers $2(2k+1)m_0+1$, $k \in \mathbb{Z}[\sqrt{d}]$, of the form $4M+3+4N\sqrt{d}$ such that the related set (5) does not represent a Diophantine quadruple, precisely such that this set has at least two elements equal or at least one element equal to zero. Further, let $w = s+t\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ be a solution of the Pell's equation $x^2-dy^2=1$. Then z_0w^2 is of the same form as z_0, i.e. of the form $4M+3+4N\sqrt{d}$. Also, we can assume that $z_0w^2 \notin S$, because S is finite. Hence, related set given by (5) represents the $D(z_0w^2)$-quadruple. Multiplying each element of this set by $s-t\sqrt{d}$, we obtain the $D(z_0)$-quadruple, according to Lemma 4.

Since m from set (5) is an arbitrary solution of the Pell's equation $x^2-dy^2=1$, there exists infinitely many $D(z)$-quadruples for z of the above forms.

From now on, we will show only the existence of a set with the property $D(z)$ in $\mathbb{Z}[(1+\sqrt{d})/2]$ (for certain z), since then the same argument as above will give the existence of infinitely many $D(z)$-quadruples.

Lemmas 5**b** and 2 imply that the set

$$\{m, (3k+1)^2m+2k, (3k+2)^2m+2k+2, 9(2k+1)^2m+8k+4\} \tag{6}$$

has the property $D(4M+3+(4N+2)\sqrt{d})$.

It remain to prove the existence of a set with the property $D(4M+1+4N)\sqrt{d})$. According to Lemma 2, the set

$$\{2, 2(3k+1)^2+2k, 2(3k+2)^2+2k+2, 18(2k+1)^2+8k+4\}$$

has the property $D(8M+5+8N\sqrt{d})$ for $k = M+N\sqrt{d}$ and the property $D(8M+1+(8N+4)\sqrt{d})$ for $k = \dfrac{2M-1}{2} + \dfrac{2N+1}{2}\sqrt{d}$ and according to Lemma 3, the set

$$\{4, 9k^2-5k, 9k^2+7k+2, 36k^2+4k\}$$

has the property $D(8M+1+8N\sqrt{d})$ for $k=M+N\sqrt{d}$, and the property $D(8M+5+(8N+4)\sqrt{d})$ for $k=\frac{2M+1}{2}+\frac{2N+1}{2}\sqrt{d}$.

For all other cases we give only a sketch of the proof.

2) The existence of $D(2M+(2N+1)\sqrt{d})$-quadruple is a direct consequence of Lemmas 2 and 5**c**.

3) The existence of $D(4m+4n\sqrt{d})$-quadruples can be shown in four steps. Lemma 3 implies that the set

$$\{1, 9k^2-8k, 9k^2-2k+1, 36k^2-20k+1\}$$

has the property $D(8M+8N\sqrt{d})$, where $k=M+N\sqrt{d}$.

Multiplying the elements of a $D(2M+(2N+1)\sqrt{d})$-quadruple by 2, we obtain the $D(8M+(8N+4)\sqrt{d})$-quadruple.

Lemmas 2 and 5**d** give us the set

$$\{\frac{m}{2}, k^2\frac{m}{2}-k-2, (k+1)^2\frac{m}{2}-k, (2k+1)^2\frac{m}{2}-4k-4\}$$

with the property $D(2M+1+(2N+1)\sqrt{d})$. Multiplying the elements of this set by 2, we get a set with the property $D(8M+4+(8N+4)\sqrt{d})$ in $\mathbb{Z}[(1+\sqrt{d})/2]$.

The $D(8M+4+8N\sqrt{d})$-quadruple can be obtained by multiplying the elements of the $D(2M+1+2N\sqrt{d})$-quadruple by 2.

4) According to Lemmas 2 and 5**e**, we obtain that the set

$$\{\frac{m}{2}, k^2\frac{m}{2}-k-2, (k+1)^2\frac{m}{2}-k, (2k+1)^2\frac{m}{2}-4k-4\}$$

has the property $D(\frac{2M+1}{2}+\frac{2N+1}{2}\sqrt{d})$. It is easy to see that we get a set with the property $D(4M+2+(4N+2)\sqrt{d})$ by multiplying the elements of the above set by 2.

5) This case follows from Lemma 6, and already proved cases **1** and **2**. □

3. The nonexistence of a Diophantine quadruple

Suppose that z cannot be represented as a difference of two squares in $\mathbb{Z}[(1+\sqrt{d})/2]$. Then, according to Lemma 1, z must be of the form $4m+2+4n\sqrt{d}$, $4m+(4n+2)\sqrt{d}$ or $2m+1+(2n+1)\sqrt{d}$, $m,n\in\mathbb{Z}$. For the elements of these forms we will show that $D(z)$-quadruple does not exist. The only assumption required here is that $d\equiv 5 \pmod 8$

Theorem 2. *Let $z \in \mathbb{Z}[(1+\sqrt{d})/2]$ be of the form $4m+2+4n\sqrt{d}$, $4m+(4n+2)\sqrt{d}$ or $2m+1+(2n+1)\sqrt{d}$, where $m,n \in \mathbb{Z}$. Then a $D(z)$-quadruple in $z \in \mathbb{Z}[(1+\sqrt{d})/2]$ does not exist.*

Proof. Suppose that z is of the form $4m+2+4n\sqrt{d}$ and that the $D(4m+2+4n\sqrt{d})$-quadruple exists. According to its definition, there exist $u_i+v_i\sqrt{d} \in \mathbb{Z}[(1+\sqrt{d})/2]$, $i=1,2,3,4$, such that

$$(u_i+v_i\sqrt{d})(u_j+v_j\sqrt{d})+4m+2+4n\sqrt{d}=(\gamma_{ij}+\delta_{ij}\sqrt{d})^2,$$

for $1 \le i < j \le 4$ and for some $\gamma_{ij}+\delta_{ij}\sqrt{d} \in \mathbb{Z}[(1+\sqrt{d})/2]$. Thus, we have

$$\begin{aligned} u_iu_j+v_iv_jd+4m+2 &= \gamma_{ij}^2+d\delta_{ij}^2, \\ u_iv_j+u_jv_i+4n &= 2\gamma_{ij}\delta_{ij}. \end{aligned}$$

We change the variables as follows: $x_i=2u_i$ and $y_i=2v_i$ for $i=1,2,3,4$, $\xi_{ij}=2\gamma_{ij}$ and $\eta_{ij}=2\delta_{ij}$ for $1 \le i < j \le 4$. So, we obtain that

$$\begin{aligned} x_ix_j+y_iy_jd+16m+8 &= \xi_{ij}^2+d\eta_{ij}^2, \\ x_iy_j+x_jy_i+16n &= 2\xi_{ij}\eta_{ij}, \end{aligned} \tag{7}$$

where $x_i, y_i, \xi_{ij}, \eta_{ij} \in \mathbb{Z}$ and $x_i \equiv y_i \pmod 2$, $\xi_{ij} \equiv \eta_{ij} \pmod 2$. Now, let us assume that $d \equiv 5 \pmod{16}$. It can be shown that in this case

$$(\xi_{ij}^2+d\eta_{ij}^2, 2\xi_{ij}\eta_{ij}) \mod 16 \in S,$$

where $S=\{(0,0),(4,0),(6,2),(6,14),(8,8),(14,6),(14,10)\}$. Thus, if $(x_i+y_i\sqrt{d})/2$ and $(x_j+y_j\sqrt{d})/2$ are the elements of the quadruple, then it has to hold that

$$(x_ix_j+5y_iy_j+8, x_iy_j+x_jy_i) \mod 16 \in S. \tag{8}$$

Further, we show that there are no integers x_i, y_i, $i=1,2,3,4$, of the same parity such that condition (8) is fulfilled. For instance, let $x_1 \equiv 1 \pmod{16}$ and $y_1 \equiv 5 \pmod{16}$. Then, if we require that $(x_2+y_2\sqrt{d})/2$ is the next element of the quadruple, then (8) implies that

$$(x_2+25y_2+8, y_2+5x_2) \mod 16 \in S.$$

Therefrom, we get that

$$(x_2,y_2) \mod 16 \in T,$$

where $T = \{(0,6),(0,14),(2,6),(2,14),(3,1),(3,3),(3,11),(4,2),(4,10),(6,2),(6,10),(7,7),(7,13),(7,15),(8,6),(8,14),(10,6),(10,14),(11,3),(11,9),(11,11),(12,2),(12,10),(14,2),(14,10),(15,5),(15,7),(15,15),(16,6),(16,14)\}$. Next, let us assume that $x_2 \equiv 0 \pmod{16}$ and $y_2 \equiv 6 \pmod{16}$. If $(x_3+y_3\sqrt{d})/2$ is the third element of the quadruple, then we check the condition (8) for $i=1,2$ and $j=3$. Thereby, we obtain that $(x_3,y_3) \mod 16 \in T$ and that $(14y_3+8, 6x_3) \mod 16 \in S$. These two conditions imply that $(x_3,y_3) \mod 16 \in V$, where $V=\{(3,1),(7,13),(11,9),(15,5)\}$. Finally, let us assume that $x_3 \equiv 3 \pmod{16}$ and $y_3 \equiv 1 \pmod{16}$. Then $(x_4+y_4\sqrt{d})/2$, the forth element of the quadruple, must satisfy the following conditions: $(x_4, y_4) \mod 16 \in V$ and $(3x_4 + 5y_4 + 8, 3y_4 + x_4) \in S$. But these conditions cannot both be satisfied, because $(3x_4 + 5y_4 + 8, 3y_4 + x_4) \mod 16 \in \{(6,6),(14,14)\} \cap S = \emptyset$, for all $(x_4, y_4) \mod 16 \in V$.

So far, we showed that the set $\{(x_1 + y_1\sqrt{d})/2, (x_2 + y_2\sqrt{d})/2, (x_3 + y_3\sqrt{d})/2\}$ such that $x_1 \equiv 1 (\mathrm{mod}\ 16)$, $y_1 \equiv 5 (\mathrm{mod}\ 16)$, $x_2 \equiv 0 (\mathrm{mod}\ 16)$, $y_2 \equiv 6 (\mathrm{mod}\ 16)$ and $x_3 \equiv 3 (\mathrm{mod}\ 16)$, $y_3 \equiv 1 (\mathrm{mod}\ 16)$ cannot be extended to a $D(4m + 2 + 4n\sqrt{d})$-quadruple. All the other cases are checked similarly by the assistance of computer.

The analogous result is also true for $d \equiv 13 (\mathrm{mod}\ 16)$, and the only difference is that we take $S' = \{(0,0),(4,0),(6,6),(6,10),(8,8),(14,2),(14,14)\}$ instead of S in (8).

Now, if z is of the form $4m + (4n + 2)\sqrt{d}$, then a similar procedure as in the previous case can be performed. Instead of condition (8), the following condition is used

$$(x_i x_j + 5y_i y_j, x_i y_j + x_j y_i + 8) \mod 16 \in S,$$

under the assumption that $d \equiv 5 (\mathrm{mod}\ 16)$.

Finally, if z is of the form $2m + 1 + (2n + 1)\sqrt{d}$, then, in the same manner, we verify each of the four cases, $4m \pm 1 + (4n \pm 1)\sqrt{d}$. For each case, we use, instead of (8), the following condition

$$(x_i x_j + 5y_i y_j \pm 4, x_i y_j + x_j y_i \pm 4) \mod 16 \in S.$$

□

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