Maps between isolated points on modular curves

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An isolated point is one which does not belong to such an infinite family.

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Let $x = [(E, \alpha)]_H \in X_H$ be a closed point, where $E/\mathbb{Q}(j(E))$ and $j(E) \notin \{0, 1728\}$. Let $y = f(x) = [(E, \alpha)]_{H'} \in X_{H'}$.

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Theorem

1. *Suppose that $H' = (G \cap H')H$. If x is isolated, then so is y.*
2. *Suppose that $G \cap H' = G \cap H$. If y is isolated, then so is x.*

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Theorem

Let $x \in X_1(2p)$ be a non-CM isolated point with $j(x) \in \mathbb{Q}$. Then $p = 37$ and $j(x) \in \{-9317, -162677523113838677\}$.