$D(-1)$-quadruples extending certain pairs in imaginary quadratic rings

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DEFINITION 1.

Let $m \geq 2$ be an integer and let R be a commutative ring with unity. Let $n \in R$ be a non-zero element and $\{a_1, \ldots, a_m\}$ a set of m distinct non-zero elements from R such that $a_i \cdot a_j + n$ is a square of an element of R, for $1 \leq i < j \leq m$. The set $\{a_1, \ldots, a_m\}$ is called a Diophantine m-tuple with the property $D(n)$ or simply a $D(n)$-m-tuple in R.

- $R = \mathbb{Z}$ and $n = 1$: the sets are known as Diophantine m-tuples;
- The first Diophantine quadruple in integers: Fermat's set $\{1, 3, 8, 120\}$
- He, Togbé and Ziegler;
- $n = -1$;
- Dujella, Filipin and Fuchs;
- Bonciocat, Cipu and Mignotte

A. Dujella, *Diophantine m-tuples page*,
`http://web.math.pmf.unizg.hr/~duje/dtuples.html`.

- generalizations of the problem of Diophantus;
- the ring $\mathbb{Z}[\sqrt{-k}]$, for certain $k > 0$.
- $k = 1, n = 1$: Dujella, Franušić, Bayad, Filipin, Togbé
- $k = 1, n = -1$; $\quad k \geq 2, n = -1$:

Y. Fujita, I. Soldo, *$D(-1)$-tuples in the ring $\mathbb{Z}[\sqrt{-k}]$ with $k>0$*, Publ. Math. Debrecen **100/1-2** (2022), 49–67.

Theorem 1.
Let k be an integer with $k \geq 2$ and a, b positive integers with

$$a < b \leq 8a - 3.$$

There does not exist a $D(-1)$-quadruple $\{a, b, c, d\}$ in the ring $\mathbb{Z}[\sqrt{-k}]$ with c, d negative integers.

Corollary 1.
Let k be an integer with $k \geq 2$ and a, b positive integers with

$$a < b \leq 8a - 3.$$

There does not exist a $D(-1)$-quintuple $\{a, b, c, d, e\}$ in $\mathbb{Z}[\sqrt{-k}]$ with c, d, e integers.

By using Corollary 1. we were able to prove the result concerning the extendibility of $D(-1)$-pairs $\{p^i, q^j\}$ to quintuples in the ring $\mathbb{Z}[\sqrt{-k}], k \geq 2$ with different primes p, q and positive integers i, j.

THEOREM 2.
Let i, j be positive integers, k an integer with $k \geq 2$, and p, q primes such that

$$p^i < q^j \leq 8p^i - 3.$$

There does not exist a $D(-1)$-quintuple of the form $\{p^i, q^j, c, d, e\}$ in the ring $\mathbb{Z}[\sqrt{-k}]$.

Y. FUJITA, I. SOLDO, *The non-existence of* $D(-1)$*-quadruples extending certain pairs in imaginary quadratic rings*, Acta Math. Hungar., to appear.

THEOREM 3.
Let k be a non-square integer with $k \geq 2$ and a, b positive integers with

$$a < b < 16a^2 - a - 2 + 2\sqrt{k(8a^2 + 3a + 1)}. \qquad (1)$$

Then, there does not exist a $D(-1)$-quadruple $\{a, b, c, d\}$ in $\mathbb{Z}[\sqrt{-k}]$ with c, d integers satisfying $d < 0 < c$.

THEOREM 4 (generalization).
Let k be a non-square integer with $k \geq 2$ and a, b positive integers with

$$a < b \leq 8a - 3.$$

Then, there does not exist a $D(-1)$-quadruple $\{a, b, c, d\}$ in $\mathbb{Z}[\sqrt{-k}]$ with c, d integers.

THEOREM 5.
Let i be a non-negative integer, j a positive integer, k a non-square integer with $k \geq 2$ and p, q primes with

$$p^i < q^j \leq 8p^i - 3.$$

Then, there does not exist a $D(-1)$-quadruple of the form $\{p^i, q^j, c, d\}$ in $\mathbb{Z}[\sqrt{-k}]$.

Finally, we considered if we could find some explicit parametric families of $D(-1)$-pairs $\{a, b\}$ satisfying $0 < a < b \leq 8a - 3$. Indeed, such examples exist and one can easily see that some a and b can be of the following form:

1) $a = F_{2n+1}$, $b = \{F_{2n+3}, F_{2n+5}\}$, where F_k is the kth Fibonacci number and n is a positive integer; For positive integers m, n, we can take:
2) $a = n^2 + 1$, $b = n^2 + 2n + 2$;
3) $a = n^2 + 1$, $b = 4n^2 + 4n + 5$;
4) $a = n^2 + 2n + 2$, $b = 4n^2 + 4n + 5$;
5) $a = 4n^2 + 4n + 5$, $b = 9n^2 + 6n + 10$;
6) $a = 9n^2 + 6n + 10$, $b = 16n^2 + 8n + 17$;
7) $a = 8n^4 + 8n^3 + 8n^2 + 4n + 1$,
   $b = 8n^4 + 24n^3 + 24n^2 + 12n + 5$.

Thank you for your attention!