About the equation $x^4 + dy^2 = z^p$

Modular Curves and Galois Representations, Zagreb

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Introduction

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Determine asymptotic conditions on d for the non-existence of primitive non-trivial solutions of the equation $x^4 + dy^2 = z^p$.

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- $d = 5, 6, 7$ (Pacetti, Villagra, 2022)

General approach

(a, b, c) non-trivial primitive solution of $x^4 + dy^2 = z^p$

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$$E_{(a,b,c)} : y^2 = x^3 + 4ax^2 + 2(a^2 + \sqrt{-d}b)x.$$

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$$\rho := \rho_{E_{(a,b,c)},p} \otimes \varkappa : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\overline{\mathbb{Q}_p}),$$

extends to a representation $\tilde{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$.

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Thus,

$$\tilde{\rho} \simeq \rho_{f,\mathfrak{p}}.$$

for some $f \in S_2(N_{(a,b,c)}, \varepsilon)$.

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Main goal

Under the assumption $K_f = K_g$. Obtain an elliptic curve E_g, relate it to $E_{(a,b,c)}$ for $p > M_g$, and arrive at a contradiction using properties of E_g that $E_{(a,b,c)}$ does not satisfy.

Asymptotic result

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Theorem (Pacetti, Villagra, G.)

Let d be a natural number congruent to 3 modulo 8 and such that the class number of $\mathbb{Q}(\sqrt{-d})$ is coprime to 6. Then there are non-trivial primitive solutions to the equation

$$x^4 + dy^2 = z^p$$

for p greater enough

Thank you for your attention!