Some ℓ-adic properties of modular forms with Nebentypus and ℓ-regular partitions

(joint with Mostafa Ghazy)

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- For instance $b_2(n)$ is precisely the number of partitions of n into *odd* parts (which famously equals the number of partitions into *distinct* parts since $\frac{(1-q^{2m})}{1-q^m} = (1+q^m)$).

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- Recall Dedekind's $\eta(z) = q^{\frac{1}{24}} \prod_{m=1}^{\infty}(1-q^m)$. Note that $p(n)$ is related to $\frac{1}{\eta(z)}$ (weight $\frac{-1}{2}$), whereas $b_k(n)$ is related to $\frac{\eta(kz)}{\eta(z)}$ (weight 0, also different level and character).

- k-regular partitions are a well-studied variant of $p(n)$. For instance Dandurand and Penniston (2009), using the theory of complex multiplication, determined exact criteria for the ℓ-divisibility of $b_\ell(n)$ for $\ell \in \{5, 7, 11\}$, whereas Xia (2015), using theta function identities of Ramanujan, obtained congruences of the form

$$b_\ell(A(k)n + B(k)) \equiv C(k)b_\ell(n) \pmod{\ell}$$

for $\ell \in \{13, 17, 19\}$ and certain functions $A(k), B(k), C(k)$ depending on ℓ and k.

Congruences for ℓ-regular partitions modulo ℓ for small primes

Theorem (E. and Ghazy (2023))

For $5 \leq \ell \leq 31$ prime, $m \geq 1$, there exists $b_\ell(m)$ s.t. for $b_1 \equiv b_2 \pmod{2}$, $b_2 > b_1 \geq b_\ell(m)$, there exists $\mathfrak{B}_\ell(b_1, b_2, m)$ s.t. for $n \geq 0$ (and a certain c to be defined below)

$$b_\ell\left(\ell^{b_1} n + \frac{\ell^{b_1} c - \ell + 1}{24}\right) \equiv$$

$$\mathfrak{B}_\ell(b_1, b_2, m) b_\ell\left(\ell^{b_2} n + \frac{\ell^{b_2} c - \ell + 1}{24}\right) \pmod{\ell^m} (b_1 \text{ odd}),$$

$$b_\ell\left(\ell^{b_1} n - \frac{\ell^{b_1} c + \ell - 1}{24}\right) \equiv$$

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We illustrate Theorem 1 with $\ell = 17$. For $m = 1$ it applies for every pair of positive integers $b_1 < b_2$ with the same parity. We let $b_1 := 1$ and $b_2 := 3$. It turns out that $\mathfrak{B}(1, 3, 1) = 11$, and so

$$b_{17}(17^3 n + 1637) \equiv 11\ b_{17}(17n + 5) \pmod{17}$$

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- In some sense, some of our arguments amount to "taking a square root" of their results and arguments. For instance, certain weights of modular forms we study are half their counterparts, and generally live in spaces with quadratic Nebentypus.
- The above Theorem is a statement on a certain $\mathbb{Z}/\ell^m$ module being of rank ≤ 1 for $5 \leq \ell \leq 31$. A more general version holds for all primes ≥ 5 that we describe next.

The general setting for all $\ell \geq 5$

- We attach an integer c to primes $\ell \geq 5$ as follows

$$c = c(\ell) := 24 \left\lceil \frac{\ell - 1}{24} \right\rceil - (\ell - 1)$$

(so that $c + \ell - 1 \equiv 0 \pmod{24}$ and $0 \leq c < 24$. $c \in \{0, 20, 18, 14, 12, 8, 6, 2\}$).

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- Also set (following Folsom-Kent-Ono for $p(n)$)

$$\Phi_\ell(z) := \frac{\eta(\ell^2 z)}{\eta(z)},$$

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$$R_\ell(b; z) = \begin{cases} R_\ell(b-1; z)\Phi_\ell^c(z) \mid U(\ell) & \text{if } b \text{ is odd,} \\ R_\ell(b-1; z) \mid U(\ell) & \text{if } b \text{ is even.} \end{cases}$$

The spaces

Consider the following two infinite families of descending $\mathbb{Z}/\ell^m\mathbb{Z}$ modules

$$\Lambda^{\text{odd}}_{\ell,\text{reg}}(2b+1, m) := \text{Span}_{\mathbb{Z}/\ell^m\mathbb{Z}}\{R_\ell(2\beta+1; z) \pmod{\ell^m} : \beta \geq b\}$$

$$\Lambda^{\text{even}}_{\ell,\text{reg}}(2b, m) := \text{Span}_{\mathbb{Z}/\ell^m\mathbb{Z}}\{R_\ell(2\beta; z) \pmod{\ell^m} : \beta \geq b\}$$

The General Theorem

Theorem (E. and Ghazy (2023))

Let $\ell \geq 5$ be prime. For every $m \geq 1$ there exists $b_\ell(m)$ s.t.

1. *The nested sequence of $\mathbb{Z}/\ell^m\mathbb{Z}$ modules*

$$\Lambda^{odd}_{\ell,reg}(1,m) \supseteq \cdots \supseteq \Lambda^{odd}_{\ell,reg}(2b+1,m) \supseteq \cdots$$

   stabilizes for all b with $2b+1 \geq b_\ell(m)$. Moreover, if we denote the stabilized $\mathbb{Z}/\ell^m\mathbb{Z}$ module by $\Omega^{odd}_{\ell,reg}(m)$ then its rank is bounded above by $1 + \left\lfloor \frac{\ell-1}{12} \right\rfloor - \left\lceil \frac{\ell-1}{24} \right\rceil$.

2. *Likewise for*

$$\Lambda^{even}_{\ell,reg}(0,m) \supseteq \Lambda^{even}_{\ell,reg}(2,m) \supseteq \cdots \supseteq \Lambda^{even}_{\ell,reg}(2b,m) \supseteq \cdots$$

   and $\Omega^{even}_{\ell,reg}(m) \cong \Omega^{odd}_{\ell,reg}(m)$.

Twisted example of Serre's ℓ-adic modular forms

- $R_\ell(b; z)$ is of weight $\frac{c}{2} \in \{0,10,9,7,6,4,3,1\}$, level ℓ and quadratic character. Yet it is congruent modulo any power of ℓ to forms of level 1 (and increasing weights.)
- For $\ell = 23$, $c = 2$ and $m = 1$, we have $k_\ell = 34$

$$R_{23}(1; z) \equiv \Delta^2(z) E_{10}(z) \pmod{23}$$

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- $m = 3$, weight 5820

$$R_{23}(1;z) \equiv \Delta^2(z)E_{5796}(z) + 3289\Delta^3(z)E_{5784}(z) + 7682\Delta^4(z)E_{5772}(z) + 529\Delta^5(z)E_{5760}(z) + 11638\Delta^6(z)E_{5748}(z) \pmod{23^3}$$

Final remark

- In the course of the proof, we require the Eisenstein series of level ℓ and quadratic character χ. They are well-known (essentially going back to Hecke (1927)) to be

$$E_{k,\chi}(z) := 1 - \frac{2k}{B_{k,\chi}} \sum_{n=1}^{\infty} \left(\sum_{d|n, d>0} \chi(d) d^{k-1} \right) q^n,$$

$$F_{k,\chi}(z) := \sum_{n=1}^{\infty} \left(\sum_{d|n, d>0} \chi\left(\frac{n}{d}\right) d^{k-1} \right) q^n$$

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- They have the following beautiful symmetry in their q-expansion

A curious phenomenon

$$E_{3,\chi_{-7}} = 1 - \frac{7}{8}(q + 5q^2 - 8q^3 + 21q^4 - 24q^5 - 40q^6 + q^7 + 85q^8 + 73q^9 - 120q^{10} + 122q^{11} - 168q^{12} - 168q^{13} + 5q^{14} + 192q^{15} + 341q^{16} - 288q^{17} + 365q^{18} - 360q^{19} - 504q^{20} - 8q^{21} + 610q^{22} + 530q^{23} - 680q^{24} + \dots)$$

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$$F_{3,\chi_{-7}} = q + 5q^2 + 8q^3 + 21q^4 + 24q^5 + 40q^6 + 49q^7 + 85q^8 + 73q^9 + 120q^{10} + 122q^{11} + 168q^{12} + 168q^{13} + 245q^{14} + 192q^{15} + 341q^{16} + 288q^{17} + 365q^{18} + 360q^{19} + 504q^{20} + 392q^{21} + 610q^{22} + 530q^{23} + 680q^{24} + \dots$$

Hvala!