Determining biellipticity for quotient modular curves

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Modular curves and Galois representations.

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Bielliptic curves

$X_{|K}$ smooth projective curve defined over number field K.

- g_X genus of X
- $X(K)$ the set of K-rational points
- $\overline{K}$ a fix algebraic closure of K, and all L/K finite $L \subseteq \overline{K}$.
- By Faltings' theorem:
$$|X(K)| = \infty \Rightarrow g_X \leq 1 \,.$$
- The converse is not true : If $g \leq 1$, may happen $X(K) = \emptyset$.
- If $|X(K)| > 0$ and $g_X = 0$, then $|X(K)| = \infty$.
- If $|X(K)| > 0$ and $g_X = 1$, may happen $|X(K)| < \infty$.

Quadratic Points

Quadratic Points

Assume $g_X > 1$, and denote

$$\Gamma_2(X,K) := \cup_{[L:K]\leq 2} X(L).$$

Assume X has an involution u defined over K such that $g_{X_u} \leq 1$ where $X_u := X/u$ (hyperelliptic or bielliptic curve over K, respectively).

If $|X_u(K)| = \infty \Rightarrow |\Gamma_2(X,K)| = \infty$.

The converse is true (Abramovich-Harris-Silverman).

A weak result is as follows

Theorem (Harris-Silverman)

Assume $g_X \geq 2$. Then $\exists L/K$ finite field extension with $|\Gamma_2(X,L)| = \infty$ if and only if C is hyperelliptic or bielliptic (i.e., have a degree 2 map over $\overline{K}$

$$\varphi : X \to \mathbb{P}^1; \textit{ or } \ E$$

to the projective line or an elliptic curve)

For general global field k we have:

Proposition

Assume $g_X \geq 2$. Then,

1. *X_k is hyperelliptic if and only if there exists a (hyperelliptic) involution $w \in \mathrm{Aut}(X_{\overline{k}})$, having exactly $2g_X + 2$ fixed points. In particular, if X_k is hyperelliptic, then w is unique, defined over a finite purely inseparable extension ℓ/k of k, and it is called the hyperelliptic involution of X_k.*
2. *X_k is bielliptic if and only if there exists a (bielliptic) involution $\tilde{w} \in \mathrm{Aut}(X_{\overline{k}})$, having $2g_X - 2$ fixed points. If X_k is bielliptic and $g_X \geq 6$, then there is an unique bielliptic involution, which belongs to the center of $\mathrm{Aut}(X_{\overline{k}})$ and is defined over a finite purely inseparable extension ℓ of k.*

Theorem (Grauert-Samuel)

Let X_k defined over a global field k of characteristic $p>0$, and assume that X_k is conservative. Then $C(k)$ is always a finite set except possible when $X_k \otimes_k k^{sep}$ is isomorphic to a smooth projective curve C' defined over a finite field (isotrivial curve).

Theorem (Schweizer)

X_k defined over global field k of $char = p > 0$, conservative over k. Assume $g_X \geq 3$ and $Jac(X_k \times_k \overline{k})$ has no non-zero homomorphic images defined over $\overline{\mathbb{F}_{q=p^n}}$, then, there exist L/k finite such that $|\Gamma_2(X, L)| = \infty$ iff X is bielliptic or hyperelliptic.

Classical modular curves

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- Are algebraic curves corresponding to certain moduli problem classifying elliptic curves with additional data.
- Over $\mathbb{C}$, $X_{\Gamma,\mathbb{C}}$ corresponds to a Riemann surface obtained by completing by the cusps the

$$\mathbb{H}/\Gamma$$

Γ is a modular subgroup of $SL_2(\mathbb{R})$ commensurable with $SL_2(\mathbb{Z})$.

One of the more known are $\Gamma(N) \leq \Gamma_1(N) \leq \Gamma_\Delta(N) \leq \Gamma_0(N) \leq SL_2(\mathbb{Z})$ corresponding to natural maps

$$X(N)_\mathbb{C} \to X_1(N)_\mathbb{C} \to X_\Delta(N)_\mathbb{C} \to X_0(N)_\mathbb{C}$$

and each of such modular curves are algebraic curves over $\mathbb{Q}$ with good reduction for $p \nmid N$.

If X_Γ is bielliptic or hyperelliptic, exist an involution in $Aut(X_\Gamma)$.
Always $Norm_{SL_2(\mathbb{R})}\Gamma/\Gamma \leq Aut(X_\Gamma)$
$X_0(N)$, for $N \neq 37, 63, 108$

$$Norm_{SL_2(\mathbb{R})}\Gamma_0(N)/\Gamma_0(N) = Aut(X_0(N)),$$

where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | c \equiv 0 (mod\ N) \right\}$$

and for each $d|N$ with $(N, N/d) = 1$ we have $w_d = \frac{1}{\sqrt{d}} \begin{pmatrix} da & b \\ Nc & dk \end{pmatrix} \in SL_2(\mathbb{R})$
Atkin-Lehner involutins in $Aut(X_0(N))$ and

$$B(N) := \langle \{w_d\}_{d||N} \rangle \leq Aut(X_0(N)).$$

The bielliptic modular situations

For classical modular curves, has been determined all bielliptic curves between

- the curves $X_0(N)$ (B., 1999).
- the curves $X_1(N)$ (D. Jeon-C.H. Kim 2004)
- the curves $X(N)$ (D.Jeon -C.H. Kim, B.-Kontogeorgis-Xarles 2013)
- the intermediate curves $X_\Delta(N)$ (D. Jeon -C.H. Kim- A.Schweizer 2017).
- The modular curves $X_0^+(p^n) = X_0(p^n)/w_{p^n}$ (D. Jeon 2018).

In different works, we study which bielliptic curves appear in the families $X_0^{W_N}(N) = X_0(N)/W_N$, with $W_N \leq B(N)$ a non-trivial subgroup with $W_N \neq \langle w_N \rangle$.

The hyperelliptic $X_0^{W_N}(N)$ where determined by different works of Hasegawa and Hashimoto ($\sim$ 1997), and Hasegawa ($\sim$ 1999).

After finding the bielliptic curves (with few extra work), we could obtain the N such that $|\Gamma_2(X_0^{W_N}, \mathbb{Q})| = \infty$.

Bielliptic and quadratic

Proposition (Accola-Landman,Harris-Silverman)

If X is bielliptic and $f : X \to X'$ finite map then X' is bielliptic or hyperelliptic or $g_{X'} \leq 1$.

Theorem (Harris-Abramovich, available proof in Momose volume by B.)

Assume $g_X \geq 2$. Then $|\Gamma_2(X,K)| = \infty$ if and only if C is hyperelliptic over K or bielliptic over K where the elliptic quotient has positive K-rank.

Notation

- $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
- $B(N)$ group inv. Atkin-Lehner, $n = \#$ primes $p \mid N$, $|B(N)| = 2^n$.
- $X_0^*(N) = X_0(N)/B(N)$, $X_0^{W_N}(N) = X_0(N)/W_N$ with $W_N \leq B(N)$.
- g_N, g_N^* and $g_N^{W_N}$, genus of $X_0(N)$, $X_0^*(N)$ and $X_0(N)^{W_N}$, respectively.
- $J_0(N) = \mathrm{Jac}(X_0(N))$, $J_0^*(N) = \mathrm{Jac}(X_0^*(N))$ and $J_0^{W_N}(N) = \mathrm{Jac}(X_0^{W_N}(N))$.
- New_N set of normalized new forms if $S_2(\Gamma_0(N))^{\mathrm{new}}$.
- $\mathrm{New}_N^* = \mathrm{New}_N^{B(N)}$ subset of New_N invariant by $B(N)$.
- For $f \in \mathrm{New}_N$,
    - A_f is the a.v. associated by Shimura to f,
    - $a_m(f)$ is the m-th Fourier coefficient of f,
    - K_f the totally real number field $\mathbb{Q}(\{a_m(f)\}_m)$.
- ψ the Dedekind function.
- $\sigma_0(M)$ the number of positive divisors of M.
- A and B a. v. over K, $A \stackrel{K}{\sim} B$ mean isogeny over K.

Some facts about $X_0(N)$

We known that

- the map $S_2(\Gamma_0(N)) \cap K[[q]] \to \Omega^1_{X_0(N)/K}$, $h \mapsto h(q)dq/q$ is an $\simeq$.
- $J_0(N) \overset{\mathbb{Q}}{\sim} \prod_{M|N} \prod_{f\in \mathrm{New}_M \backslash G_{\mathbb{Q}}} A_f^{\sigma_0(N/M)}$.
- the set $\cup_{M|N} \cup_{f\in \mathrm{New}_M} \{f(q^d) \colon d|N/M\}$ is a base of $S_2(\Gamma_0(N))$.
- for a prime $p \nmid N$ and $f \in \mathrm{New}_M$, the characteristic polynomial of Frob_p acting in the Tate module of A_f is (by Eichler-Shimura)

$$\prod_{\sigma\colon K_f \hookrightarrow \overline{\mathbb{Q}}} x^2 - a_p(f^\sigma)x + p\,.$$

- for $p \nmid N$, $|X_0(N)(\mathbb{F}_{p^n})| = p^n + 1 - \sum_{i=1}^{2g_N} \alpha_i^n$, where

$$\prod_{i=1}^{2g_N}(x-\alpha_i) = \prod_{M|N} \prod_{f\in \mathrm{New}_M} (x^2 - a_p(f)x + p)^{\sigma_0(N/M)}\,.$$

$X_0^*(N)$, with N square-free

If N square-free:

- $J_0^*(N) \stackrel{\mathbb{Q}}{\sim} \prod_{M|N} \prod_{f \in \mathrm{New}_M^* \backslash G_{\mathbb{Q}}} A_f$.
- $\mathrm{End}_{\mathbb{Q}}(J_0^*(N)) \otimes \mathbb{Q} = \mathrm{End}_{\overline{\mathbb{Q}}}(J_0^*(N)) \otimes \mathbb{Q} \simeq \prod_j K_j$, with K_j totally real number fields.
- $\mathrm{Aut}(X_0^*(N)) = \mathrm{Aut}_{\mathbb{Q}}(X_0^*(N)) \hookrightarrow \prod_j K_j \Rightarrow \mathrm{Aut}(X_0^*(N)) \simeq (\mathbb{Z}/2\mathbb{Z})^m$.

Let be $E/\mathbb{Q}$ e.c. and $f_E \in \mathrm{New}_M$ the associated form to E. We say that (N, E) is bielliptic if E is $\mathbb{Q}$-isogeny to a bielliptic quotient of $X_0^*(N)$. Then,

cond$(E) = M|N$, $f_E \in \mathrm{New}_M^*$ and $\mathbb{Q}\sum_{d|N/M} d\, f_E(q^d)dq/q =$pullback de $\Omega^1_{E/\mathbb{Q}}$.

Lemma

Assume (N, E) is bielliptic. For a prime $p \nmid N$, we have the following inequalities

$$\text{(a)}\ \frac{\psi(N)}{2^n} \leq 12\frac{2|E(\mathbb{F}_{p^2})|-1}{p-1},\ \text{(b)}\ g_N^* \leq 2\frac{|E(\mathbb{F}_{p^2})|}{p-1},\ \text{(c)}\ g_N \leq 2^{n+1}\frac{|E(\mathbb{F}_{p^2})|}{p-1}.$$

Observe: $|E(\mathbb{F}_{p^2})| = (p+1)^2 - a_p(f_E)^2 \leq (p+1)^2$.

Sieves, N square-free

With the lemma we obtain a finite set $\mathcal{C}$ of possible N (Ogg argument).

- Second sieve (Cremona tables): Replace the set $\mathcal{C}$ by the set $\mathcal{P}$ of pairs (N,E), where $N\in\mathcal{C}$ and $E/\mathbb{Q}$ is an e.c. such that $f_E\in\mathrm{New}^*_M$, for $M|N$. We apply the lemma with $|E(\mathbb{F}_{p^2})|$.
- Third sieve (Cremona tables): If $N=M$, erase the pairs in $\mathcal{P}$ such that the strong Weil parametrization of E does not divide 2^{n+1}.
- Fourth sieve: For $p\nmid N$, write

$$P_p(n):=\mathrm{mod}\,[(\sum_{d|n}\mu(n/d)|X_0^*(N)(\mathbb{F}_{p^n})|)/n,2]$$

  where $\mathrm{mod}\,[r,2]\in\{0,1\}$ (denotes the class modulo 2) and μ the Moebius function. If $X_0^*(N)$ has an involution $u/\mathbb{Q}$, then

$$\sum_{n=0}^{k}(2n+1)P_p(2n+1)\leq 2g_N^*+2\,,\ \forall k\geq 0\,.$$

  We erase the pairs (N,E) when fails the inequality.
- Fifth sieve: erase the pairs such that

$$|X_0^*(N)((\mathbb{F}_{p^n})|>2|E((\mathbb{F}_{p^n})|\,,\ \text{for some prime } p\nmid N.$$

Hyperelliptic case, N square-free

Hasegawa, Hashimoto

$X_0^*(N)$ hyperellitic and N square-free $\Leftrightarrow$ $g_N^* = 2$.

Proposition

Assume N square-free and $g_N^* = 2$. The curve $X_0^*(N)$ is bielliptic if, and only if, $J_0^*(N) \overset{\mathbb{Q}}{\sim} E_1 \times E_2$ where E_1 E_2 are bielliptic quotients. In such situation, if $\omega_i \in \Omega^1_{X_0^*(N)/\mathbb{Q}}$, $1 \leq i \leq 2$, is the pulback of the regular differential of E_i, the functions $x = \omega_1/\omega_2$ and $y = dx/\omega_2$ satisfy the relation $y^2 = P(x)$ with $P(t) \in \mathbb{Q}[t]$ of degree 6. The automorphism group corresponds to $(x, y) \mapsto (\pm x, \pm y)$.

Proposition

For N square-free and $g_N^* = 2$. $X_0^*(N)$ is bielliptic if, and only if, $N \in \{106, 122, 129, 158, 166, 215, 390\}$. In such situation, $\mathrm{Aut}(X_0^*(N))$ is the Klein group.

Non-hyperelliptic pairs, N square-free

After sieve, the remaining pairs (N, E), ordered by genus are, always $g_N^* > 2$:

N	g_N^*	E
178	3	$89a$
183	3	$61a$
185	3	$37a$
246	3	$82a, 123b$
249	3	$83a, 249b$
258	3	$43a, 129a$
282	3	$141d$
290	3	$145a$
303	3	$101a$
310	3	$155c$
318	3	$53a, 106b$
430	3	$43a, 215a$
455	3	$65a$
462	3	$77a, 154a$
510	3	$102a$

N	g_N^*	E
202	4	$101a$
262	4	$131a$
354	4	$118a$
366	4	$61a$, $122a$
370	4	$185c$, $370a$
399	4	$57a$
426	4	$142b$
546	4	$91a$
570	4	$57a$, $190b$, $285b$

N	g_N^*	E
237	5	$79a$
402	5	$201c$
438	5	$219a$
645	5	$129a$ $215a$
714	5	$238b$
798	5	$399a$
910	5	$91a$, $455a$
690	6	$138a$
858	6	$143a$, $286c$
870	7	$145a$, $290a$

Petri's theorem

Fix an inmersion of K in $\mathbb{C}$. Denote by $K_h[x_1,\cdots,x_g]$ the homogenous polynomials of $K[x_1,\cdots,x_g]$.

Theorem of Petri (i)

Let X/K be a non-hyperelliptic curve with $g_X > 2$ and $\omega_1,\cdots,\omega_g$ a basis of $\Omega^1_{X/K}$. The curve X is obtained by the common zeros of the polynomial in

$$\mathcal{L} = \{Q \in K_h[x_1,\cdots,x_g] \colon Q(\omega_1,\cdots,\omega_g) = 0\}.$$

Petri's theorem

For $i>1$, denote $\mathcal{L}_i=\{Q\in\mathcal{L}\colon \deg Q=i\}$, K-v.s.

Observe $\dim\mathcal{L}_i\leq\dim\mathcal{L}_{i+1}$, because $x_j\mathcal{L}_i\subseteq\mathcal{L}_{i+1}\,\forall j\leq g$.

Theorem of Petri

Let be X/K a non-hyperelliptic curve of $g_X>2$ and $\omega_1,\cdots,\omega_g$ a basis of $\Omega^1_{X/K}$. X corresponds to the common zeros of the polynomials in

$$\mathcal{L}=\{Q\in K_h[x_1,\cdots,x_g]\colon Q(\omega_1,\cdots,\omega_g)=0\}\,.$$

More precisely,

- If $g_X=3$, $\dim\mathcal{L}_2=\dim\mathcal{L}_3=0$, $\mathcal{L}_4=K\cdot Q(x_1,x_2,x_3)\neq\{0\}$ and, for $i\geq 4$, $\mathcal{L}_i$ are multiple of Q. The zeroes of $\mathcal{L}$ are the one of Q (smooth plane quartic).
- If $g_X>3$, $\dim\mathcal{L}_2=(g-3)(g-2)/2$ and the zeroes of $\mathcal{L}$ are the ones in $\mathcal{L}_2$ and $\mathcal{L}_3$. If X is not trigonal or is not a quintic smooth plane curve, the zeroes for all $\mathcal{L}$ are the ones of $\mathcal{L}_2$.

If $g_X=4$ and X not hyperelliptic, its gonality is 3. In this situation $\dim\mathcal{L}_2=1$ and $\dim\mathcal{L}_3=5$. Therefore, an quation for X is given by a polynomial $Q_2\in\mathcal{L}_2$ ($Q_2\neq 0$) and a polynomial of $Q_3\in\mathcal{L}_3$ which is not multiple of Q_2.

Non-hyperelliptic involutions

Let be $u \in Aut_K(X)$ (X not hyperelliptic with $g_X > 2$), then

$$Q(u^*(\omega_1), \cdots, u^*(\omega_g)) = 0\,, \quad \forall Q \in \mathcal{L}\,.$$

If u is an involution and $\{\omega_i\}$ is a basis of eigenvectors, i.e. $u^*(\omega_i) = \varepsilon_i \omega_i$ with $\varepsilon_i = \pm 1$, then

$$Q(\varepsilon_1 x_1, \cdots, \varepsilon_g x_g) \in \mathcal{L}\,, \quad \forall Q \in \mathcal{L}\,. \tag{1}$$

Conversely, if the condition (1) is true, then the map

$$u \colon \omega_i \mapsto \varepsilon_i \omega_i \quad \text{or} \quad v \colon \omega_i \mapsto -\varepsilon_i \omega_i,\ 1 \le i \le g, \text{ is an involution of } X\,.$$

For $X = X_0^*(N)$, $J_0^*(N) \overset{\mathbb{Q}}{\sim} \prod A_{f_i}$. Because u acts in each A_{f_i} by $\pm\,\mathrm{Id}$, a basis of $\Omega^1_{X_0^*(N)/\mathbb{Q}}$ as union of basis of $\Omega^1_{A_{f_i}/\mathbb{Q}}$ are eigenvectors for u.

Proposition

Assume $X_0^*(N)$ is not hyperelliptic. Take $\omega_1, \cdots, \omega_{g_N^*}$ a basis of $\Omega^1_{X_0^*(N)/\mathbb{Q}}$ as previosly, such that ω_1 is the differential associated to e.c. E. The pair (N, E) is bielliptic if, and only if,

$$Q(-x_1, x_2, \cdots, x_{g_N^*-1}, x_{g_N^*}) \in \mathcal{L}_i\ \forall Q \in \mathcal{L}_i\ \forall i \ge 2\,. \tag{2}$$

Bielliptic involutions

The relation (2) is characterized by a $X/\mathbb{Q}$ as follows

- If $g_X = 3$ and $\mathcal{L}_4 = \langle Q_4(x_1, x_2, x_3)\rangle$:

$$Q(-x_1, x_2, x_3) \in \mathcal{L}\,, \forall Q \in \mathcal{L} \Leftrightarrow Q_4(x_1, x_2, x_3) = Q_4(-x_1, x_2, x_3)$$

- If $g_X > 3$

$$Q(-x_1, \cdots, x_g) \in \mathcal{L}_2\,, \forall Q \in \mathcal{L}_2$$
$$\Updownarrow$$
$$Q(x_1, \cdots, x_g) = \mathbb{Q}(-x_1, \cdots, x_g)\,, \forall Q \in \mathcal{L}_2\,.$$

and

$$Q(-x_1, \cdots, x_g) \in \mathcal{L}_3\,, \forall Q \in \mathcal{L}_3$$
$$\Updownarrow$$
$$Q(x_1, \cdots, x_g) - Q(-x_1, \cdots, x_g) \in x_1 \cdot \mathcal{L}_2\,, \forall Q \in \mathcal{L}_3\,.$$

Bielliptic curves by Petri's thm

Let us generalize the above criteria to determine when a non-hyperelliptic smooth curve X/K is bielliptic over K or not.

Proposition

Assume $\mathrm{Jac}(X) \stackrel{K}{\sim} E^m \times A$, with E an elliptic curve and A a.v. such that does not have E as a quotient defined over K. Let be $I_{g-m} \in M_{g-m}(K)$ identity matrix and $\{\omega_i\}$ a basis of $\Omega^1_{X/K}$ s.t. $\omega_1, \cdots, \omega_m$ and $\omega_{m+1}, \cdots, \omega_g$ are basis of the pullbacks of $\Omega^1_{E^m/K}$ and $\Omega^1_{A/K}$ resp.Then, the pair (X, E) is bielliptic over K if, and only if, exist a matrix $\mathcal{A} \in \mathrm{GL}_m(K)$ satisfying

$$Q((-x_1, x_2, \cdots, x_g) \cdot \mathcal{B}) \in \mathcal{L}'_i \ \forall Q \in \mathcal{L}_i \text{ and } \forall i \geq 2\,, \tag{3}$$

where $\mathcal{B}$ is the matrix $\left(\begin{array}{c|c} \mathcal{A} & 0 \\ \hline 0 & I_{g-m} \end{array}\right) \in \mathrm{GL}_g(K)$ y $\mathcal{L}'_i = \{Q((x_1, x_2, \cdots, x_g) \cdot \mathcal{B}))\colon Q \in \mathcal{L}_i\}$.

Note: $(\omega'_1, \cdots, \omega'_m) = \mathcal{A}^{-1}(\omega_1, \cdots, \omega_m)$ is a basis by eigenvectors of u in $\Omega^1_{E^m/K}$, with $u(\omega'_1) = \omega'_1$ and $u(\omega'_j) = -\omega'_j$ for $j \neq 1$.

Bielliptic curves $X_0^*(N)$, and quadratic points, N square-free

Theorem

Let $N > 1$ square-free integer. The modular curve $X_0^*(N)$ is bielliptic ($g_N^* \geq 2$) if, and only if, N appears in the following table

g_N^*	N
2	106, 122, 129, 158, 166, 215, 390
3	178, 183, 246, 249, 258, 290, 303, 318, 430, 455, 510
4	370

For such values of N, $\mathrm{Aut}(X_0^*(N))$ has order 2 if $g_N^* > 2$ and is the Klein group when $g_N^* = 2$.

Moreover, $|\Gamma_2(X_0^*(N), \mathbb{Q})| = \infty$ if, and only if, N is the previous list or in

$$\{67, 73, 85, 93, 103, 106, 107, 115, 122, 129, 133, 134, 146, 154, 158, 161, 165, 166, 167, 170, 177, 178, 183, 186, 191, 205, 206, 209, 213, 215, 221, 230, 246, 249, 255, 258, 266, 285, 286, 287, 290, 299, 303, 318, 330, 357, 370, 390, 430, 455, 510\}.$$

N non square-free. Preliminary steps

Lemma [3]

Let p be a prime. If for an integer $k \geq 2$, $X_0^*(p^k \cdot M)$ is bielliptic, then $X_0^*(p^{k-2} \cdot M)$ is hyperelliptic, bielliptic or has genus ≤ 1.

Corollary [3]

Let $N > 1$ s.t. $g_N^* \geq 2$. Let be M the biggest square-free integer s.t. $M|N$ and $val_p(N)$ is odd for each prime $p|M$. If $X_0^*(N)$ is bielliptic, then $X_0^*(M)$ is bielliptic or $g_M^* \leq 2$.

Proposition [Daeyeol Jeon]

Let be $N = p^k$ with p prime, $k > 1$ and $g_N^* \geq 2$. Then, $X_0^*(N)$ is bielliptic iff $N = 121 = 11^2$, or $N = 128 = 2^7$ ($g_{121}^* = 2$ and $g_{128}^* = 3$).

Lemma [2]

Let be (N, E) bielliptic over $\mathbb{Q}$. For a prime $p \nmid N$, the following results are satisfied:

$$\text{(a)}\ \frac{\psi(N)}{2^n} \leq 12\frac{2|E(\mathbb{F}_{p^2})| - 1}{p-1}\,,\ \text{(b)}\ g_N^* \leq 2\frac{|E(\mathbb{F}_{p^2})|}{p-1}\,,\ \text{(c)}\ g_N \leq 2^{n+1}\frac{|E(\mathbb{F}_{p^2})|}{p-1}\,.$$

N non-square free, $J_0^*(N)/\mathbb{Q}$ and $\Omega^1(X_0^*(N))$

- One main difference with N square-free is on the decomposition of $J_0^*(N)$ over $\mathbb{Q}$.

For N general, $M|N$ and $f \in \mathrm{New}_M$, write $H_f = \langle f(q^d) \colon d|N/N \rangle$.

- N square-free and $H_f^{B(N)} \neq \{0\} \Leftrightarrow f \in \mathrm{New}_M^*$. In this situation, $\dim H_f^{B(N)} = 1$ and

$$H_f^{B(N)} = \langle \sum_{d|N/M} w_d(f(q)) \rangle = \langle \sum_{d|N/M} d f(q^d) \rangle .$$

- If N is not square-free and $H_f^{B(N)} \neq \{0\}$, may occur $n_f := \dim H_f^{B(N)} > 1$.

Thus in the decomposition of $J_0^*(N)$,

$$J_0^*(N) \stackrel{\mathbb{Q}}{\sim} \prod_{M|N} \prod_{f \in \mathrm{New}_M / G_{\mathbb{Q}}} A_f^{n_f} ,$$

may appear $n_f > 1$. We need to determine a basis of $H_f^{B(N)}$ ($\forall f \in \mathrm{New}_M$ and $\forall M|N$) to determine a basis of $\Omega^1(X_0^*(N))$ (to apply Petri's theorem), main source Atkin-Lehner paper "Hecke operators...".

N not square-free, $J_0^*(N)/\mathbb{Q}$ and $\Omega^1(X_0^*(N))$

For an integer $d > 0$, B_d denote the operator

$$B_d \colon S_2(\Gamma_0(M) \to S_2(\Gamma_0(M \cdot d)\,, \quad f \mapsto f(q^d)\,.$$

Proposition [3]

For a prime $p \nmid M$ and $i \geq 0$, let be $f \in S_2(\Gamma_0(p^i \cdot M))^{B(M)}$ s. t. $w_{p^i}(f) = \varepsilon \cdot f$ ($\varepsilon = 1$ if $i = 0$). For $k > i$, let be $\mathcal{S}_f$ the v.s. of $S_2(\Gamma_0(p^k \cdot M))$ generated by the $k - i + 1$ l. i. $\{f, B_p(f), \cdots, B_p^{k-i}(f)\}$. Then,

(i) The following modular forms are a basis of $\mathcal{S}_f$:

$$g_j = (1 + pB_p)^{k-i-j}(1 - pB_p)^j f\,, \quad 0 \leq j \leq k - i\,.$$

and eigenvector for w_{p^k}: $w_{p^k}(g_j) = (-1)^j \varepsilon g_j$.

(ii) The dimension s_f of the v.s. $\mathcal{S}_f^{B(p^k \cdot M)}$ is

$$s_f = \begin{cases} \dfrac{k-i+1}{2} & \text{if } k-i \text{ is odd,} \\ \dfrac{k-i+1+\varepsilon}{2} & \text{if } k-i \text{ is even.} \end{cases}$$

Bielliptic curves, may be not over $\mathbb{Q}$.

- The other main difference when N non-square free is $\mathrm{End}_{\mathbb{Q}}(J_0^*(N)) \neq \mathrm{End}_{\overline{\mathbb{Q}}}(J_0^*(N))$.

Lemma [Silverman-Harris]

Let be X_K with $g_X \geq 6$. If X is bielliptic, then there exist an unique bielliptic involution and defined over K.

Lemma [Baker,González-Jiménez,González,Poonen=BGGP]

Let A be an a.v. defined over $\mathbb{Q}$ s.t. $A \overset{\mathbb{Q}}{\sim} \prod_{i=1}^m A_{f_i}^{n_i}$ for some $f_i \in \mathrm{New}_{N_i}$, with $A_{f_i} \overset{\mathbb{Q}}{\not\sim} A_{f_j}$ for $i \neq j$. Then $\mathrm{End}(A) = \mathrm{End}_{\mathbb{Q}}(A)$ iff for all quadratic Dirichlet character χ, $f_i \otimes \chi \neq f_j^{\sigma}$ for all $\sigma \in G_{\mathbb{Q}}$ and for all i and j.

Lemma [Pyle]

Let be $f \in \mathrm{New}_M$ without CM and s.t. $\dim A_f > 1$. If exists a prime p s.t. $a_p(f)^2 \notin \mathbb{Z}$, then A_f does not have an elliptic quotient over $\overline{\mathbb{Q}}$.

Lemma [BGGP]

Let be $f \in \mathrm{New}_M$ and χ_D the quadratic character associated to $K = \mathbb{Q}(\sqrt{D})$. Exists an isogeny between A_f and $A_{f\otimes\chi_D}$ defined over K.

N non square-free. Sieves

• By a morphism $X_0^*(N=ML)\to X_0^*(M)$ with M square-free (and L with certain properties) we are reduced to certain values of N.

• When the pair (N,E) is studied over $\mathbb{Q}$ (the general case if $g_N^*\geq 6$), we apply the inequality lemma counting over finite fields and similar sieves than when N was square-free,

execpt than the pairs (N,E) are now $f_E\in \mathrm{New}_M$ and

$$n_{f_E}=\dim\,\langle f_E(q^d)\colon d|N/M\rangle^{B(N)}\geq 1\,.$$

We determined list of N with $2\leq g_N^*\leq 5$.

• For $g_N^*\leq 5$, only appears possible pairs (N,E) with $E/\mathbb{Q}$ but with involution defined over a quadratic field extension K' (associated to a Dirichlet character χ).

• When $8|N$ or $9||N$ there are involutions coming from the normalizer of $\Gamma_0^*(N)$.

g_N^*	N
2	$88, 104, 112, 116, 135, 153, 168, 180, 184, 198, 204, 276, 284, 380$
3	$136, 144, 152, 162, 164, 171, 189, 196, 207, 234, 236, 240, 245, 248, 252, 270, 294, 312, 315, 348, 420, 476.$
4	$148, 160, 172, 176, 200, 224, 225, 228, 242, 260, 264, 275, 280, 300, 306, 308, 342, 350.$
5	$192, 208, 212, 216, 316, 364, 376, 378, 396, 414, 440, 444, 495, 572, 630.$

Example $g_N^* \leq 5$: $X_0^*(160)$, $g_{160}^* = 4$

$X_0^*(160)$ is not-hyperelliptic. The decomposition of $J_0^*(160)$ is
$J_0^*(160) \overset{\mathbb{Q}}{\sim} A_{f_1}^2 \prod_{i=3}^4 A_{f_i}$ with $A_{f_1} \overset{\mathbb{Q}}{\sim} E20a\,,\ A_{f_3} \overset{\mathbb{Q}}{\sim} E80b\,,\ A_{f_4} \overset{\mathbb{Q}}{\sim} E160a\,,$

$f_1 \in \mathrm{New}_{20}^{w_5}$, $f_3 \in \mathrm{New}_{80}^{w_5}$, $f_4 \in \mathrm{New}_{160}^*$ and $f_3 = f_1 \otimes \chi_{-1}$.

Because
$(1-2B_2)(1+2B_2)^2 = 1 + 2B_2 - 4B_2^2 - 8B_2^3\,,\ (1-2B_2)^3 = 1 - 6B_2 + 12B_2^2 - 8B_2^3,$

A basis of $\Omega^1_{X_0^*(160)/\mathbb{Q}}$: $\omega_i = h_i(q)\,dq/q$, $1 \leq i \leq 4$ with

$$
\begin{aligned}
h_1(q) &= f_1(q) + 2f_1(q^2) - 4f_1(q^4) - 8f_1(q^8)\,,\\
h_2(q) &= f_1(q) - 6f_1(q^2) + 12f_1(q^4) - 8f_1(q^8)\,,\\
h_3(q) &= f_3 - 2f_3(q^2)\,,\\
h_4(q) &= f_4(q)\,.
\end{aligned}
$$

Recall $\dim \mathcal{L}_2 = 1$, $\dim \mathcal{L}_3 = 5$.

Computing $Q_i(x,y,z,t) \in \mathcal{L}_i$ with $Q_i(h_1,h_2,h_3,h_4) = 0$, $i = 2,3$:

Example $g_N^* \leq 5$: $X_0^*(160)$, $g_{160}^* = 4$

$X_0^*(160)$ is not-hyperelliptic. The decomposition of $J_0^*(160)$ is
$J_0^*(160) \overset{\mathbb{Q}}{\sim} A_{f_1}^2 \prod_{i=3}^4 A_{f_i}$ with $A_{f_1} \overset{\mathbb{Q}}{\sim} E20a$, $A_{f_3} \overset{\mathbb{Q}}{\sim} E80b$, $A_{f_4} \overset{\mathbb{Q}}{\sim} E160a$,

$f_1 \in \mathrm{New}_{20}^{w_5}$, $f_3 \in \mathrm{New}_{80}^{w_5}$, $f_4 \in \mathrm{New}_{160}^*$ and $f_3 = f_1 \otimes \chi_{-1}$.

Because
$(1-2B_2)(1+2B_2)^2 = 1 + 2B_2 - 4B_2^2 - 8B_2^3$, $(1-2B_2)^3 = 1 - 6B_2 + 12B_2^2 - 8B_2^3$,

A basis of $\Omega^1_{X_0^*(160)/\mathbb{Q}}$: $\omega_i = h_i(q)\,dq/q$, $1 \leq i \leq 4$ with

$$\begin{aligned}
h_1(q) &= f_1(q) + 2f_1(q^2) - 4f_1(q^4) - 8f_1(q^8)\,,\\
h_2(q) &= f_1(q) - 6f_1(q^2) + 12f_1(q^4) - 8f_1(q^8)\,,\\
h_3(q) &= f_3 - 2f_3(q^2)\,,\\
h_4(q) &= f_4(q)\,.
\end{aligned}$$

Recall $\dim \mathcal{L}_2 = 1$, $\dim \mathcal{L}_3 = 5$.

Computing $Q_i(x,y,z,t) \in \mathcal{L}_i$ with $Q_i(h_1,h_2,h_3,h_4) = 0$, $i = 2,3$:

$$Q_2 = -48t^2 + 8tx + 3x^2 - 8ty + 6xy - y^2 + 36xz + 12yz - 8z^2\,.$$

$Q_3 = 20t^2x - 12tx^2 - 3x^3 - 20t^2y - 4ty^2 + 3xy^2 - 9x^2z + 6xyz + 3y^2z + 16tz^2 + 6xz^2 - 6yz^2$

$X_0^*(160)$ is not bielliptic over $\mathbb{Q}$

The pairs $(160, E80b)$ and $(160, E160a)$ need to study if they are bielliptic or not over $\mathbb{Q}$.

They are not because Q_2 is not even in z, also not in t.

The pair $(160, E20a)$ should be bielliptic over $\mathbb{Q}$ iff exist $\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$ such that the polynomials

$$R_2 := Q_2(a_1x + a_2y, b_1x + b_2y, z, t)\,,\ \ R_3 := Q_3(a_1x + a_2y, b_1x + b_2y, z, t)$$

satisfy

$\boxed{R_2 \text{ is even with } x}$ and $\boxed{R_3(x, y, z, t) - R_3(-x, y, z, t) = \lambda x\, R_2, \text{ for } \lambda \in \mathbb{Q}\,.}$

We can consider the situations with $a_1 = 0$ and $a_1 = 1$, to conclude

Not exist matrix $\mathcal{A}$ making R_2 even with respect x.

$X_0^*(160)$ is not bielliptic over $\mathbb{Q}$.

$X_0^*(160)$ is bielliptic over $\overline{\mathbb{Q}}$

The pair $(160, E)$ may only become bielliptic over $K' = \mathbb{Q}(i)$, with $E \overset{K'}{\sim} E20a$.

This will happen iff exist $\mathcal{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \in \mathrm{GL}_3(K)$ s.t.

$$R_2 := Q_2(a_1x + a_2y + a_3z, b_1x + b_2y + b_3z, c_1x + c_2y + c_3z, t)\,,$$

$$R_3 := Q_3(a_1x + a_2y + a_3z, b_1x + b_2y + b_3z, c_1x + c_2y + c_3z, t)$$

R_2 is even in x and $R_2 | (R_3(x, y, z, t) - R_3(-x, y, z, t))$.

Take $\mathcal{A} = \begin{pmatrix} i & i & 1 \\ 1 & -3 & 0 \\ 0 & -4\,i & 1 \end{pmatrix}$, and we obtain

$$\begin{aligned} R_2 = {} & 6t^2 + (2-6i)x^2 - 4ty + 3y^2 - 4itz + 6iyz - (1-6i)z^2\,, \\ R_3 = {} & 4tx^2 + 10t^2y + (6-6i)x^2y - 6ty^2 + 3y^3 + 10it^2z + (6+6i)x^2z \\ & -12ityz + 9iy^2z + 10tz^2 - (3-6i)yz^2 - (6-3i)z^3\,, \end{aligned}$$

Now R_2 and R_3 are even in x, therefore $X_0^*(160)$ is bielliptic over $\mathbb{Q}(i)$.

Remaining pairs (N, E), with $g_N^* > 5$

g_N^*	(N, E)
6	$(244, 61a), (272, 34a), (332, 83a), (332, 166a), (336, 42a), (336, 112a), (564, 94a), (620, 62a), (780, 65a), (780, 130c)$
7	$(320, 32a), (324, 27a), (360, 20a), (360, 30a), (450, 15a), (450, 75b), (456, 57a), (456, 76a), (456, 152a), (492, 123b), (504, 21a), (504, 36a), (504, 42a), (550, 55a), (550, 275a), (550, 550a), (558, 558a), (636, 53a), (660, 110b), (924, 77a), (924, 462a)$
8	$(408, 102a), (468, 26b), (468, 234b), (468, 234c), (480, 20a), (480, 24a), (480, 80b), (480, 160a), (540, 45a), (540, 54b), (990, 66a), (990, 99a), (1020, 102a)$
9	$(560, 56a), (560, 70a), (560, 280a), (1140, 190b), (1140, 285b)$
10	$(840, 20a), (840, 140b), (840, 210d), (1050, 175b)$
11	$(672, 112c), (672, 224a)$
13	$(1260, 21a), (1260, 70a), (1260, 90b), (1260, 210d)$

Example $g_N^* > 5$: $X_0^*(558)$, $g_{558}^* = 7$

$X_0^*(558)$ is not-hyperelliptic, not trigonal and $\dim \mathcal{L}_2 = 10$. The decomposition of $J_0^*(558)/\mathbb{Q}$:

$\prod_{i=1}^3 A_{f_i} \times A_{f_5}\,,\ A_{f_1} \overset{\mathbb{Q}}{\sim} 186c\,,\ A_{f_2} \overset{\mathbb{Q}}{\sim} E558a\,,\ f_1 \in \mathrm{New}_{186}^{B(62)}\,, f_2 \in \mathrm{New}_{558}^*\,,$
$f_3 \in \mathrm{New}_{93}^*\,, \dim A_{f_3} = 2\,, f_5 \in \mathrm{New}_{93}^{B(31)}\,, \dim A_{f_5} = 3$,

$g_1 = f_1$, $g_2 = f_2$, $\{g_3, g_4\}$ y $\{g_5, g_6, g_7\}$ basis of $\langle f_3^\sigma : \sigma \in G_\mathbb{Q}\rangle \cap \mathbb{Z}[[q]]$ and $\langle f_5^\sigma : \sigma \in G_\mathbb{Q}\rangle \cap \mathbb{Z}[[q]]$ resp.

Take $(1 + 2B_2)(1 \pm 3B_3) = 1 + 2B_2 \pm B_3 \pm 6B_6$,

a basis of $\Omega^1_{X_0^*(558)/\mathbb{Q}}$: $\omega_i = h_i(q)\, dq/q$, $1 \le i \le 7$ with

$$
\begin{aligned}
h_1(q) &= f_1(q) - 3f_1(q^3)\,,\\
h_2(q) &= f_2(q)\,,\\
h_3(q) &= g_3(q) + 2g_3(q^2) + 3g_3(q^3) + 6g_3(q^6)\,,\\
h_4(q) &= g_4(q) + 2g_4(q^2) + 3g_4(q^3) + 6g_4(q^6)\,,\\
h_5(q) &= g_5(q) + 2g_5(q^2) - 3g_5(q^3) - 6g_5(q^6)\,,\\
h_6(q) &= g_6(q) + 2g_6(q^2) - 3g_6(q^3) - 6g_6(q^6)\,,\\
h_7(q) &= g_7(q) + 2g_7(q^2) - 3g_7(q^3) - 6g_7(q^6)\,..
\end{aligned}
$$

Example $g_N^* > 5$: $X_0^*(558)$, $g_{558}^* = 7$

Let be $Q \in \mathbb{Q}_h[x_1, \cdots, x_7]$ of degree 2 (28 coefficients):

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + a_5x_5^2 + a_6x_6^2 + a_7x_7^2 + a_8x_1x_2 + a_9x_1x_3 +$$
$$a_{10}x_1x_4 + a_{11}x_1x_5 + a_{12}x_1x_6 + a_{13}x_1x_7 + a_{14}x_2x_3 + a_{15}x_2x_4 +$$
$$a_{16}x_2x_5 + a_{17}x_2x_6 + a_{18}x_2x_7 + a_{19}x_3x_4 + a_{20}x_3x_5 + a_{21}x_3x_6 +$$
$$a_{22}x_3x_7 + a_{23}x_4x_5 + a_{24}x_4x_6 + a_{25}x_4x_7 + a_{26}x_5x_6 + a_{27}x_5x_7 + a_{28}x_6x_7$$

$Q(h_1, \cdots, h_7) = 0$, we obtain $a_1, \cdots, a_{28}$ as linear combination of (recall $\mathcal{L}_2 = 10$) $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_9, a_{10}, a_{11}$.

More concretely one obtains, $a_8 = a_{14} = a_{15} = a_{16} = a_{17} = a_{18} = 0$.

Therefore, $Q(x_1, \cdots, x_7)$ is even in the variable x_2 $\forall Q \in \mathcal{L}_2$.

$X_0^*(558)$ is bielliptic over $\mathbb{Q}$ and the class of $\mathbb{Q}$-isogeny of the bielliptic quotient is $E558a$.

Results, N non square-free

Theorem[B-González][3]

Let be $N > 1$ a non-square free integer with $g_N^* \geq 2$. Then,

- The curve $X_0^*(N)$ is bielliptic over $\mathbb{Q}$ $\Leftrightarrow$ N appears in the next table

g_N^*	N
2	$88, 112, 116, 121, 153, 180, 184, 198, 204, 276, 284, 380$
3	$128, 144, 152, 164, 189, 196, 207, 234, 236, 240, 245, 248, 252, 294, 312, 315, 348, 420, 476$
4	$148, 172, 200, 224, 225, 228, 242, 260, 264, 275, 280, 300, 306, 342$
5	$364, 444, 495$
7	558

The curve $X_0^*(N)$ is bielliptic over $\overline{\mathbb{Q}}$, but not over $\mathbb{Q}$ $\Leftrightarrow$ $N = 160$.

- $\Gamma_2(X_0^*(N), \mathbb{Q}) = \infty \Leftrightarrow N$ appears in the following list

$$88, 104, 112, 116, 117, 121, 125, 128, 135, 136, 147, 148, 152, 153, 164$$
$$168, 171, 172, 176, 180, 184, 198, 204, 207, 224, 225, 228, 234, 236, 240$$
$$248, 252, 260, 264, 276, 279, 280, 284, 312, 315, 342, 348, 364, 380, 420$$
$$444, 476, 495, 558.$$

Results for $X_0^{W_N}(N)$, with N square-free

Theorem [B.-González-Kamel]

Let $N>1$ be square-free integer. Suppose that the genus of $X_0(N)/W_N$ is ≥ 2 for a non-trivial subgroupl W_N of $B(N)$ not equal to $\langle w_N \rangle$. The quotient modular curve $X_0(N)/W_N$ is bielliptic if and only if, exists $v\in B(N)\setminus W_N$ satisfying that the genus of $X_0(N)/\langle W_N, v\rangle$ is 1, except for the following quotient bielliptic modular curves of genus 4: $X_0(154)/\langle w_2, w_{77}\rangle$, $X_0(285)/\langle w_3, w_{95}\rangle$ and $X_0(286)/\langle w_2, w_{143}\rangle$.

$$End_{\overline{\mathbb{Q}}}(J_0^{W_N}(N)) = End_{\mathbb{Q}}(J_0^{W_N}(N)).$$

$J_0(N)^{W_N} \sim A_i^{n_i} \times \ldots$ with $n_i \geq 2$.

Results, $X_0^{W_N}(N)$, N non square-free

Theorem [B-Kamel-Schweizer]

Let $N>1$ be a no square-free integer. Assume that the genus of $X_0(N)/W_N$ is ≥ 2 for a non-trivial subgroup W_N of $B(N)$ not equal to $\langle w_N \rangle$. The quotient curve $X_0(N)/W_N$, denoted as a pair (N, W_N) is bielliptic if and only if appears in the table bellow:

1. It is a pair (N, W_N) with $|W_N|=2$ and N in the set

   $$\{40,48,52,63,68,72,75,76,80,96,98,99,100,108,124,188\},$$

   or is a pair (N, W_N) with $|W_N|=4$ and N in the set

   $$\{84,90,120,126,132,140,150,156,220.\},$$

   All such quotient modular curves are bielliptic over $\mathbb{Q}$ with a elliptic quotient given by $X_0^*(N)$, which has genus 1,
2. or is one of the following 29 pairs, listed by its genus:

Genus	(N, W_N)
2	$(44, \langle w_4 \rangle), (60, \langle w_{20} \rangle), (60, \langle w_4, w_3 \rangle)$
3	$(56, \langle w_8 \rangle), (60, \langle w_4 \rangle)$
4	$(60, \langle w_3 \rangle), (60, \langle w_5 \rangle), (112, \langle w_7 \rangle), (168, \langle w_3, w_{56} \rangle)$
5	$(84, \langle w_4 \rangle), (88, \langle w_{11} \rangle), (90, \langle w_9 \rangle)$ $(117, \langle w_9 \rangle), (120, \langle w_{15} \rangle), (126, \langle w_{63} \rangle), (168, \langle w_8, w_7 \rangle),$ $(168, \langle w_7, w_{24} \rangle), (180, \langle w_4, w_9 \rangle), (184, \langle w_{23} \rangle), (252, \langle w_4, w_{63} \rangle)$
6	$(104, \langle w_8 \rangle), (168, \langle w_8, w_3 \rangle)$
7	$(120, \langle w_{24} \rangle), (136, \langle w_8 \rangle), (252, \langle w_9, w_7 \rangle)$
9	$(126, \langle w_9 \rangle), (171, \langle w_9 \rangle), (252, \langle w_4, w_9 \rangle)$
10	$(176, \langle w_{16} \rangle)$

Steps for the determination if $X_0^{W_N}(N)$ is bielliptic or not

Some "steps":

- Consider the morphism $X_0^{W_N} \to X_0^*(N)$, to reduce to the set N where $X_0^*(N)$ is bielliptic, hyperelliptic or has genus≤ 1.
- Break the set N by the number of prime numbers that divides N (only 2,3 or 4 primes).
- Ad-hoc modifications of the programme for $X_0^*(N)$ to obtain Jacobian decomposition, in order to apply for $X_0^{W_N}(N)$.
- Sieves using the number of fixed points by no Atkin-Lehner involutions, when 4 or 9 divides N.

Further sieves

Castellnuovo, Unramified Criterium

Let be $\phi : X \to Y$ a degree d morphism. If X has a bielliptic involution v, them

$$g(X) \leq dg(Y) + d + 1$$

or the morphism ϕ factorizes through X/v.
In particular: An hyperelliptic curve o genus $g \geq 4$ is not bielliptic. A trigonal curve of genus > 4 is not bielliptic. A curve of genus $g \geq 6$ has at most a bielliptic involution.
Let be w an involution of X with more than 8 fixed points. Then, or w is a bielliptic involution or X is not bielliptic.
Let be X a genus g curve with a bielliptic involution v and let be G a subgroup of $Aut(X)$ such that $Y = X/G$ has genus $h \geq 2$.

(a) If the map $\phi : X \to Y$ is ramified, i.e. if $g - 1 > |G|(h - 1)$, and $g \geq 6$, then Y is hyperelliptic and v induces the hyperelliptic involution in Y.

(b) (Unramified covering criteria) If Y is not hyperelliptic, then should be bielliptic and the map $\phi : X \to Y$ should be unramified, i.e. $g - 1 = |G|(h - 1)$.

Fixed number of points by involutions

Searching bielliptic involutions

Let be G a subgroup of $Aut(X_0(N))$ such that any non-trivial element is an involution. Then the fixed points of such involutions are disjoint and the genus of $X_0(N)/G$ follows by

$$|G|(2g(X_0(N)/G)-2)+\sum_{w\in G}\#(w,X_0(N))=2g(X_0(N))-2.$$

Take $N=2^{\alpha}M$ with $\alpha\geq 2$ and M odd.

(a) Then S_2 is an involution of $X_0(N)$, defined over $\mathbb{Q}$, and commutes with all the AL involutions w_r with r odd. Also, $V_2=S_2w_{2^{\alpha}}S_2$ is an involution of $X_0(N)$, defined over $\mathbb{Q}$, and commutes with all w_r with $r||M$.

(b) If $\alpha\geq 3$, then V_2 also commutes with $w_{2^{\alpha}}$. Therefore, $V_2w_{2^{\alpha}}$ is an involution, and $S_2w_{2^{\alpha}}$ has order 4. In fact, $\langle S_2,w_{2^{\alpha}}\rangle\cong D_4$.

(c) If $\alpha=2$, then $\langle S_2,w_4\rangle$ is non-abelian of order 6 with $V_2=S_2w_4S_2=w_4S_2w_4$ as the third involution and S_2w_4 and w_4S_2 are of order 3.

Involutions

If $N=2^\alpha M$ with $\alpha\geq 2$ and M odd, then

$$X_0(N)/w_{2^\alpha}S_2w_{2^\alpha}=X_0(N/2).$$

Let u and v be two involutions such they commute in the curve X. Then uv is an involution and

$$\#(uv,X)=2\#(u,X/v)-\#(u,X).$$

Let be $N=2^\alpha M$ with $\alpha\geq 2$ and M odd. And let be $r||M$.

(a) $\#(V_2,X_0(N))=\#(w_{2^\alpha},X_0(N))$ and $\#(V_2w_r,X_0(N))=\#(w_{2^\alpha}w_r,X_0(N))$.

(b) $\#(S_2,X_0(N))=\#(w_{2^\alpha}S_2w_{2^\alpha},X_0(N))=(2g(X_0(N))-2)-2(2g(X_0(N/2))-2)$.

(c) $\#(S_2w_r,X_0(N))=\#(w_{2^\alpha}S_2w_{2^\alpha}w_r,X_0(N))=2\#(w_r,X_0(N/2))-\#(w_r,X_0(N))$.

(d) If $\alpha\geq 3$, then

$$\#(V_2w_{2^\alpha},X_0(N))=2\#(S_2,X_0(N/2))-\#(S_2,X_0(N)) \textit{ and}$$

$$\#(V_2w_{2^\alpha}w_r,X_0(N))=2\#(S_2w_r,X_0(N/2))-\#(S_2w_r,X_0(N)).$$

Involutions

Take $9||N$ and $S_3 = \begin{pmatrix} 1 & 1/3 \\ 0 & 1 \end{pmatrix}$.

(a) S_3 normalizes $\Gamma_0(N)$ and induces an automorphism in $X_0(N)$ of order 3 defined over $\mathbb{Q}(\sqrt{-3})$. Its Galois conjugate is S_3^2. Moreover, S_3 commute with the Atkin-Lehner involutions w_r with $r \equiv 1 \mod 3$, and for the $r \equiv 2 \mod 3$ we have that $w_r S_3 = S_3^2 w_r$ and $w_9 S_3$ has order 3.

(b) $V_3 = S_3 w_9 S_3^2$ is an involution in $X_0(N)$. With respect AL-involutions we have $w_r V_3 = \begin{cases} V_3 w_r & if\ r \equiv 1 \mod 3\ or\ r = 9\ and \\ V_3 w_9 w_r & if\ r \equiv 2 \mod 3 \end{cases}$
Moreover, if $r \equiv 2 \mod 3$ then $\langle V_3, w_r \rangle \cong D_4$ and $V_3 w_r$ have order 4 with $(V_3 w_r)^2 = w_9$.

(c) V_3 as involution in $X_0(N)$ is defined over $\mathbb{Q}(\sqrt{-3})$. Its $Gal(\mathbb{Q}(\sqrt{-3})/\mathbb{Q})$-conjugate is $V_3 w_9$. In particular, V_3 and $V_3 w_9$ have the same number of fixed points in $X_0(N)$.

(d) More in general, we have
$$\#(V_3 w_9, X_0(N)) = \#(V_3, X_0(N)) = \#(w_9, X_0(N))$$
and for $r \equiv 1 \mod 3$ we also have
$$\#(V_3 w_9 w_r, X_0(N)) = \#(V_3 w_r, X_0(N)) = \#(w_9 w_r, X_0(N)).$$

(e) V_3 as involution in $X_0(N)/W$ is defined over $\mathbb{Q}$ if and only if $w_9 \in W$.

Involutions

Suppose $4||N$ and write $N = 4M$. Let be W' a subgroup of $B(N)$ generated by w_4, $w_{m_1}, \ldots, w_{m_s}$ with $m_i||M$. Then,

$$X_0(N)/W' \cong X_0(N)/\langle S_2 w_4 S_2, w_{m_1}, \ldots, w_{m_s}\rangle = X_0(N)/\langle w_4 S_2 w_4, w_{m_1}, \ldots, w_{m_s}\rangle$$

$$= X_0(2M)/\langle w_{m_1}, \ldots, w_{m_s}\rangle.$$

Therefore, if $A \in GL_2(\mathbb{R})$ is a bielliptic involution of $X_0(2M)/\langle w_{m_1}, \ldots, w_{m_s}\rangle$, then $S_2 A S_2$ normalizes in $\langle \Gamma_0(N), W'\rangle$ and induces a bielliptic involution in $X_0(N)/W'$.
Suppose $9||N$. And W' a subgroup of $B(N)$ generated by $w_{n_1}, \ldots, w_{n_t}$ $(n_i||N)$ and denote $W" = \langle \{w_{n_i} w_9^{e(n_i)}\}_{i\in\{1,\ldots,t\}}\rangle$ where $e(m) = 0$ if $m \equiv 1 \mod 3$ or if $9||m$ and $m/9 \equiv 1 \mod 3$, and $e(m) = 1$ otherwise. Then V_3 induces an isomorphism

$$X_0(N)/W' \cong X_0(N)/W".$$

Quotient modular curves of genus ≥ 6

g_{W_N}	(N, W_N)	(w, E)	$\mathbb{Q}-Jacobiandecomp.$
6	$(104, \langle w_8 \rangle)$	$(V_2 w_{104}, E26a)$	$(E26a)^2 \times E26b \times E52a \times A_{f,104}$
	$(156, \langle w_4, w_{13} \rangle)$	$(w_3, E26b = X_0^*(156))$	$(E26b)^2 \times A_{f,39}^2$
	$(168, \langle w_8, w_3 \rangle)$	$(V_2 w_{168}, E14a)$	$(E14a)^2 \times E42a \times E56b \times E84b \times E168b$
	$(220, \langle w_5, w_{44} \rangle)$	$(w_4, E110b = X_0^*(220))$	$E11a \times E20a \times A_f \times 110b \times 110c$
	$(220, \langle w_{11}, w_{20} \rangle)$	$(w_4, E110b)$	$E44a \times E55a \times E110b \times A_f \times E220a$
7	$(120, \langle w_{24} \rangle)$	$(V_2 w_{40}, E15a)$	$(E15a)^2 \times (E20a)^2 \times E30a \times E40a \times E120a$
	$(124, \langle w_4 \rangle)$	$(w_{31}, E62a = X_0^*(124)$	$(A_{f_1,31})^2 \times E62a \times A_{f_3,62}$
	$(136, \langle w_8 \rangle)$	$(V_2 w_{136}, E17a)$	$(E17a)^2 \times E34a \times A_{f_3,64} \times A_{f_4,136}$
	$(252, \langle w_9, w_7 \rangle)$	$(V_3 w_7, E36a)$	$(E21a)^3 \times E36a \times (E42a)^2 \times E84b$
8	$(220, \langle w_4, w_5 \rangle)$	$(w_{11}, E110b = X_0^*(220))$	$(E11a)^2 \times A_f^2 \times E110b \times E110c$
9	$(126, \langle w_9 \rangle)$	$(V_3 w_7, E14a)$	$(E14a)^2 \times (E21a)^2 \times E42a \times (A_{f,63})^2$
	$(171, \langle w_9 \rangle)$	$(V_3 w_{171}, E19a)$	$(E19a)^2 \times E57a \times E57b \times E57c \times A_{f,171}, dim(A_f) = 4$
	$(252, \langle w_9, w_4 \rangle)$	$(V_3 w_7, E14a)$	$(E14a)^2 \times (E21a)^2 \times E42a \times (A_{f,63})^2$
10	$(176, \langle w_{16} \rangle)$	$(V_3 w_{176}, E11a)$	$(E11a)^3 \times E44a \times E88a \times A_{f_1,88} \times E176a \times A_{f_2,176}$
11	$(188, \langle w_4 \rangle)$	$(w_{47}, X_0^*(188) = E94a)$	$A_{f_1}^2 \times E94a \times A_{f_3}, dim(A_{f_1}) = 4$

Table, $g_{W_N} \geq 6$ Bielliptic

Quotient modular curves not defined over $\mathbb{Q}$

No morphism of degree two to an elliptic curve over $\mathbb{Q}$

$(252, \langle w_4, w_{63}\rangle)$	$(V_3, E14a)$
	$(V_3 w_7, E14a)$
$(126, \langle w_{65}\rangle)$	$(V_3, E14a)$
	$(V_3 w_9, E14a)$

Bielliptic quotient curves, with elliptic quotient not defined over $\mathbb{Q}$

$(63\langle w_9\rangle)$, genus 3. $J_0^{W_N} \sim_{\mathbb{Q}} X_0^*(63) \times A_{f,63}$, with $\dim(A_{f,63}) = 2$ and

$$A_{f,63} \sim_{\mathbb{Q}(\sqrt{-3})} E^2$$

with

$$E : Y^2 = -(26 + 6\sqrt{-3})X^3 - 27X^2 + 6\sqrt{-3}X + 1$$

We have $(w_7, X_0^*(63) = E21a)$ is a bielliptic pair over $\mathbb{Q}$.

BUT, we have two more bielliptic involutions not defined over $\mathbb{Q}$ with bielliptic quotient E (one conjugation of the other).

Bibliography

0 Bars, F.; Magma codes for quotient modular curves. https://github.com/FrancescBars/Magma-functions-on-Quotient-Modular-Curves

1 Bars, F.; González Rovira, J.: Bielliptic modular curves $X_0^*(N)$ with square-free levels. Math. Comp. 88 (2019), no. 320, 2939-2957.

2 Bars, F.; González Rovira, J.: Bielliptic modular curves $X_0^*(N)$,Journal of Algebra 559, 726–759, (2020).

3 Badr,E.;Bars,F.: Bielliptic smooth plane curves and quadratic points. International Journal of Number Theory 17 (04), 1047–1066 (2021).

4 F Bars, J González, M Kamel : Bielliptic quotient modular curves with N square-free. Journal of Number Theory 216, 380–402, (2020).

5 F.Bars, M.Kamel, A. Schweizer: Bielliptic quotient modular curves. Mathematics of Computation AMS Vol. 92, (2023), Pages 895–929