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Manuscript accepted for publication

THE HERMITE-HADAMARD INEQUALITY FOR $M_\varphi M_\psi$-h-CONVEX FUNCTIONS AND RELATED INTERPOLATIONS

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Abstract. In this paper we consider the Hermite-Hadamard inequality for $M_\varphi M_\psi$-h-convex functions. An $M_\varphi M_\psi$-h-convexity covers several particular types of generalized convexity such as a harmonic-h-convexity, a log-h-convexity, (h,p)-convexity, M_pA-h-convexity, $M_\varphi M_\psi$-convexity etc. The Hermite-Hadamard type inequalities with two and with n nodes are given. Special attention is paid to a dyadic partition of an interval and related interpolations.

1. Introduction

In recent decades we have witnessed the emergence of various types of convexity. In addition to the classical convexity, we find the following variants of convexity in the literature: s-convexity, Godunova-Levin convexity, P-convexity, h-convexity, strong convexity, m-convexity, MN-convexity, MT convexity, etc. For each type of convexity, one of the first results to be studied is the Hermite-Hadamard inequality. For the classical convexity, the Hermite-Hadamard inequality has the following statement.

For an integrable convex function $f : [a,b] \to \mathbb{R}$, the following sequence of inequalities holds:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{1}{2}[f(a)+f(b)]. \tag{1.1}$$

The natural question which arises in connection with this inequality is a question of its refinement. In recent literature, we find several articles on this topic. Here we have to mention article [9] where we find the following refinement.

Theorem A *Assume that $f : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$. Then for all $\lambda \in [0,1]$, we have*

$$f\left(\frac{a+b}{2}\right) \le m(\lambda) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M(\lambda) \le \frac{1}{2}[f(a)+f(b)], \tag{1.2}$$

2020 *Mathematics Subject Classification.* 26A51, 26A15.

Key words and phrases. the Hermite-Hadamard inequality, $M_\varphi M_\psi$-h-convex function, quasi-arithmetic mean, dyadic partition.

where

$$m(\lambda) := \lambda f\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda) f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right)$$

and

$$M(\lambda) := \frac{1}{2}\Big(f(\lambda b + (1-\lambda)a) + \lambda f(a) + (1-\lambda) f(b)\Big).$$

If $\lambda = \dfrac{1}{2}$, then points in the left refinement are $\dfrac{3a+b}{4}$ and $\dfrac{a+3b}{4}$, i.e

$$m\left(\frac{1}{2}\right) = \frac{1}{2} f\left(\frac{3a+b}{4}\right) + \frac{1}{2} f\left(\frac{a+3b}{4}\right)$$

in which we recognize the refinement which occurs in [15, p.37] and in articles about other type of convexity such as [2, 17].

Results from [9] were generalized in [7] for a more general class of functions. Namely, in [7], author obtained corresponding results for h-convex functions. Let us recall the definition of an h-convex function, [23].

Definition 1.1. *Let $h : J \to \mathbb{R}$ be a non-negative function, $\langle 0, 1\rangle \subseteq J$. A function $f : I \to \mathbb{R}$ is called h-convex if for any x, y from the interval I and any $t \in \langle 0, 1\rangle$ the following holds*

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y).$$

This concept covers some classes such as a class of convex functions, a class of s-convex functions in the second sense ($h(t) = t^s$, $s \in \langle 0, 1]$), a class of Godunova-Levin functions ($h(t) = \frac{1}{t}$), a class of P-convex functions ($h(t) = 1$). The Hermite-Hadamard inequality for an h-convex function was first given in [4] and [21] and has the following form:

Theorem B *If h is an integrable function, $h(\frac{1}{2}) \neq 0$, then for an integrable h-convex function $f : [a, b] \to \mathbb{R}$, the following sequence of inequalities holds:*

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq [f(a) + f(b)]\int_0^1 h(x)\,dx. \tag{1.3}$$

If f is h-concave, then the reversed signs of inequalities hold in (1.3).

The following Hermite-Hadamard-type result for an h-convex function can be found in [7] as a consequence of Theorem 2 from [7] and the corresponding Remark in the same paper.

Theorem C *If f is a non-negative, integrable, h-convex function on $[a,b]$ with $h \in L[0,1]$, $h(\frac{1}{2}) \neq 0$, then*

$$\begin{aligned} \delta_1 &\leq \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\leq \delta_2 \leq \Big[[h(1-\lambda)+\lambda]f(a)+[h(\lambda)+1-\lambda]f(b)\Big]\int_0^1 h(t)\,dt, \end{aligned} \tag{1.4}$$

where

$$\delta_1 := \frac{1}{2h(\frac{1}{2})}\left\{(1-\lambda)f\left[\frac{(1-\lambda)a+(\lambda+1)b}{2}\right]+\lambda f\left[\frac{(2-\lambda)a+\lambda b}{2}\right]\right\}$$

$$\delta_2 := \Big[f\big((1-\lambda)a+\lambda b\big)+(1-\lambda)f(b)+\lambda f(a)\Big]\int_0^1 h(t)\,dt.$$

Furthermore, if $\lambda \in \langle 0,1\rangle$ such that $h(\lambda) \neq 0$, then

$$\frac{1}{2h(\frac{1}{2})}\min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}f\left(\frac{a+b}{2}\right) \leq \delta_1. \tag{1.5}$$

A closer look into the proof of Theorem C gives that (1.4) is valid regardless of non-negativity of f. Non-negativity of f in points $\frac{(1-\lambda)a+(\lambda+1)b}{2}$ and $\frac{(2-\lambda)a+\lambda b}{2}$ is neccessary only in (1.5).

If $h(t) = t$, i.e. if f is a convex function, then the result of Theorem C collapses to the refinement of Hermite-Hadamard inequality (1.2). It is a refinement which involves two nodes $\dfrac{(1-\lambda)a+(\lambda+1)b}{2}$ and $\dfrac{(2-\lambda)a+\lambda b}{2}$. In paper [8], a result including n nodes was given. Here we give a version of that result for a real function of a real variable.

Theorem D *Let f be an h-convex with $h \in L[0,1]$, $f \in L[a,b]$, $h(\frac{1}{2}) \neq 0$. Then for any partition*

$$0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n = 1, \qquad \textit{with } n \geq 1$$

we have

$$\begin{aligned} &\frac{1}{2h(\frac{1}{2})}\sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)f\left(\left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right)a+\frac{\lambda_j+\lambda_{j+1}}{2}b\right) \\ &\qquad\leq \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\qquad\leq \sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)\times \\ &\qquad\times\Big[f\big((1-\lambda_j)a+\lambda_j b\big)+f\big((1-\lambda_{j+1})a+\lambda_{j+1}b\big)\Big]\int_0^1 h(t)\,dx. \end{aligned}$$

In papers [7] and [8], a case of h-concavity was not considered, but from the proofs it is clear that if f is h-concave, then inequalities in Theorems C and D hold with the reversed signs.

The topic of this paper is a counterpart of the Hermite-Hadamard inequality for a wider class of functions which covers h-convex functions.

Let φ be a continuous, strictly monotone function defined on the interval I. By M_φ we denote a quasi-arithmetic mean:

$$M_\varphi(x,y;t) := \varphi^{-1}(t\varphi(x) + (1-t)\varphi(y)), \quad x, y \in I, t \in [0,1].$$

It is obvious that the power mean M_p corresponds to $\varphi(x) = x^p$ if $p \neq 0$ and to $\varphi(x) = \log x$ if $p = 0$.

Definition 1.2. *Let φ and ψ be two continuous, strictly monotone functions defined on intervals I and K respectively. Let $h : J \to \mathbb{R}$ be a non-negative function, $\langle 0,1\rangle \subseteq J$ and let $f : I \to K$ such that $h(t)\psi(f(x)) + h(1-t)\psi(f(y)) \in \psi(K)$ for all $x, y \in I, t \in \langle 0,1\rangle$. We say that a function f is $M_\varphi M_\psi$-h-convex if*

$$f(M_\varphi(x,y;t)) \leq \psi^{-1}\Big(h(t)\psi(f(x)) + h(1-t)\psi(f(y))\Big) \tag{1.6}$$

for all $x, y \in I$ and all $t \in \langle 0,1\rangle$. If the sign of inequality is reversed in (1.6)*, then f is called $M_\varphi M_\psi$-h-concave.*

Some particular cases of $M_\varphi M_\psi$-h-convex functions have been recently investigated in last ten years. If $h(t) = t$, then $M_\varphi M_\psi$-h-convexity collapses to $M_\varphi M_\psi$-convexity which was described in [15]. Paper [1] consists several results about properties and the Jensen inequality for $M_\varphi M_\psi$-h-convex functions where M_φ, M_ψ are an arithmetic mean (A), a geometric mean (G) or a harmonic mean (H). Furthermore, an HA-h-convexity or harmonic-h-convexity was described in [3] and [19]. An HG-h-convexity is investigated in [19] and an AG-h-convexity or log-h-convexity in [20]. An AM_p-h-convexity or (h,p)-convexity was described in [11] while some properties of M_pA-h-convex functions were given in [6]. Properties of $M_\varphi A$-h-convex functions were studied in [24].

In the second section, we prove the Hermite-Hadamard inequality for an $M_\varphi M_\psi$-h-convex function. The third section is devoted to different interpolation results related to the Hermite-Hadamard inequality. We end this paper with results related to a dyadic partition of interval $[a,b]$.

In this paper, if some inequality has a number (n) then its reverse version, i.e. an inequality with another sign is denoted by (Rn).

2. The Hermite-Hadamard inequality

The following result gives a connection between the theory of h-convexity and the theory of $M_\varphi M_\psi$-h-convexity. As we will see below, it is the powerful tool used in many proofs.

Proposition 2.1. *Let φ and ψ be strictly monotone continuous functions defined on intervals I and K respectively.*

a) Let ψ be an increasing function. A function $f : I \to \mathbb{R}$ is $M_\varphi M_\psi$-h-convex ($M_\varphi M_\psi$-h-concave) if and only if $\psi \circ f \circ \varphi^{-1}$ is h-convex (h-concave).

b) Let ψ be an decreasing function. A function f is $M_\varphi M_\psi$-h-convex ($M_\varphi M_\psi$-h-concave) if and only if $\psi \circ f \circ \varphi^{-1}$ is h-concave (h-convex).

Proof. Let us suppose that ψ is increasing. For any $u, v \in \mathrm{Im}(\varphi)$ there exist $x, y \in I$ such that $\varphi(x) = u, \varphi(y) = v$. If f is $M_\varphi M_\psi$-h-convex and ψ is increasing, then for any $t \in \langle 0, 1\rangle$

$$\psi(f(\varphi^{-1}(t\varphi(x) + (1-t)\varphi(y)))) \leq h(t)\psi(f(x)) + h(1-t)\psi(f(y))$$

i.e.

$$(\psi \circ f \circ \varphi^{-1})(tu + (1-t)v) \leq h(t)(\psi \circ f \circ \varphi^{-1})(u) + h(1-t)(\psi \circ f \circ \varphi^{-1})(v).$$

So, $\psi \circ f \circ \varphi^{-1}$ is h-convex. Other cases are proved in a similar way. □

Theorem 2.2 (The Hermite-Hadamard inequality for an $M_\varphi M_\psi$-h-convex function)**.** *Let h be a non-negative function defined on the interval $J, \langle 0, 1\rangle \subseteq J$, $h(\frac{1}{2}) \neq 0$. Let φ and ψ be strictly monotone continuous functions defined on intervals I and K respectively such that φ is differentiable on $[a, b] \subseteq I$.*

a) If ψ is increasing, then for an $M_\varphi M_\psi$-h-convex function $f : [a, b] \to \mathbb{R}$ the following holds

$$\frac{1}{2h(\frac{1}{2})}(\psi \circ f)\Big(M_\varphi\big(a, b; \frac{1}{2}\big)\Big) \leq \frac{1}{\varphi(b) - \varphi(a)} \int_a^b \psi(f(x))\, \varphi'(x)\, dx$$
$$\leq \Big[\psi(f(a)) + \psi(f(b))\Big] \int_0^1 h(t)\, dt, \tag{2.7}$$

provided that all integrals exist.

If f is $M_\varphi M_\psi$-h-concave, then (R2.7) *holds.*

b) If ψ is decreasing, then for an $M_\varphi M_\psi$-h-convex function f (R2.7) *holds. If f is $M_\varphi M_\psi$-h-concave, then* (2.7) *holds.*

Proof. Let us suppose that ψ is increasing and f is $M_\varphi M_\psi$-h-convex. Then, by Proposition 2.1, a function $\psi \circ f \circ \varphi^{-1}$ is h-convex on $\varphi([a, b])$. If φ is increasing, then $\varphi([a, b]) = [\varphi(a), \varphi(b)]$, while if φ is decreasing, then $\varphi([a, b]) = [\varphi(b), \varphi(a)]$.

If φ is increasing, then applying (1.3) for a function $\psi \circ f \circ \varphi^{-1}$, we get

$$\frac{1}{2h(\frac{1}{2})}(\psi \circ f \circ \varphi^{-1})\Big(\frac{\varphi(a)+\varphi(b)}{2}\Big) \le \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} (\psi \circ f \circ \varphi^{-1})(x)\,dx$$

$$\le \Big[(\psi \circ f \circ \varphi^{-1})(\varphi(a)) + (\psi \circ f \circ \varphi^{-1})(\varphi(b))\Big]\int_0^1 h(t)\,dt.$$

After substitution $\varphi^{-1}(x) = u$, the integral in the middle term becomes $\displaystyle\int_a^b \psi(f(x))\,\varphi'(x)\,dx$ and inequality (2.7) is proved.

If φ is decreasing, then the middle term is $\frac{1}{\varphi(a)-\varphi(b)}\int_{\varphi(b)}^{\varphi(a)}(\psi \circ f \circ \varphi^{-1})(x)\,dx$ and after the same substitution we get $\frac{1}{\varphi(b)-\varphi(a)}\int_a^b \psi(f(x))\,\varphi'(x)\,dx$ and inequality (2.7) holds in this case.

All other cases are proved similarly. ∎

Remark 2.3. Some particular cases of the above inequality are known. If $h(t) = t$, then the Hermite-Hadamard-type inequality for HG-convex, GG-convex, M_pA-convex, AM_p-convex, $M_\varphi A$-convex and $M_\varphi M_\psi$-convex functions can be found in [16], [13], [10], [5], [22] and [14] respectively.

The Hermite-Hadamard inequality for HA-h-convex, AG-h-convex, AM_r-h-convex functions are given in [19], [20], [11] respectively.

When h has the form $h(t) = h_1(t^s)$ for the fixed $s \in \langle 0, 1]$, then results related to the Hermite-Hadamard inequality for h-convex functions are given in [18].

Note that Theorem 2.2 covers all the cases already mentioned. In the above-mentioned articles, the authors proved the Hermite-Hadamard type inequalities directly, *ab ovo*. But Proposition 2.1 allows us to prove such results much more elegantly using known results for h-convex functions.

3. Hermite-Hadamard type results with several nodes

In this section we direct our attention to Hermite-Hadamard-type results including two or more nodes. The section is finished with several results involving a dyadic partition of an interval. The following theorem is a generalization of Theorem C given in $M_\varphi M_\psi$-h-convexity settings. In fact, this is a Hermite-Hadamard-type result which on the left-hand side includes values of a function in two points:

$$\varphi^{-1}\left(\frac{(1-\lambda)\varphi(a)+(1+\lambda)\varphi(b)}{2}\right) \text{ and } \varphi^{-1}\left(\frac{(2-\lambda)\varphi(a)+\lambda\varphi(b)}{2}\right)$$

and which, in particular case, leads to the refinement of the Hermite-Hadamard inequality for an $M_\varphi M_\psi$-convex function.

Theorem 3.1. *Let h be a non-negative function defined on the interval $J, \langle 0,1\rangle \subseteq J$, $h(\frac{1}{2}) \neq 0$. Let φ and ψ be strictly monotone continuous functions defined on intervals I and K respectively such that φ is differentiable on $[a,b] \subseteq I$. Let $f : I \to \mathbb{R}$.*

(i) If ψ is increasing, then for an $M_\varphi M_\psi$-h-convex function f the following holds

$$\Delta_1 \leq \frac{1}{\varphi(b)-\varphi(a)} \int_a^b \psi(f(x))\, \varphi'(x)\, dx$$

$$\leq \Delta_2 \leq \Big\{[h(1-\lambda)+\lambda]\psi(f(a)) + [h(\lambda)+1-\lambda]\psi(f(b))\Big\} \int_0^1 h(t)\, dt, \tag{3.8}$$

where

$$\Delta_1 := \frac{1}{2h(\frac{1}{2})} \left\{(1-\lambda)(\psi\circ f)\Big(M_\varphi\Big(a,b;\frac{1-\lambda}{2}\Big)\Big) + \lambda(\psi\circ f)\,\Big(M_\varphi\Big(a,b;\frac{2-\lambda}{2}\Big)\Big)\right\}$$

$$\Delta_2 := \Big[\psi(f(M_\varphi(a,b;1-\lambda))) + (1-\lambda)\psi(f(b)) + \lambda\psi(f(a))\Big] \int_0^1 h(t)\, dt,$$

provided that all integrals exist.

Furthermore, if $h(\lambda), h(1-\lambda) \neq 0$ and $(\psi\circ f)\left(M_\varphi\left(a,b;\frac{1-\lambda}{2}\right)\right), (\psi\circ f)\left(M_\varphi\left(a,b;\frac{2-\lambda}{2}\right)\right) \geq 0$ for some $\lambda \in \langle 0,1\rangle$, then

$$\frac{1}{2h(\frac{1}{2})} \min\left\{\frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)}\right\} (\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big) \leq \Delta_1. \tag{3.9}$$

If f is $M_\varphi M_\psi$-h-concave, then (R3.8) *and* (R3.9) *(with change* $\min \to \max$*) hold.*

(ii) If ψ is decreasing and f is $M_\varphi M_\psi$-h-convex, then (R3.8) *and* (R3.9) *(with change* $\min \to \max$*) hold. If ψ is decreasing and f is $M_\varphi M_\psi$-h-concave, then* (3.8) *and* (3.9) *are valid.*

Proof. Let us prove the case when ψ is increasing. Other cases are done in the similar manner. Denote $G := \psi \circ f$. Since f is $M_\varphi M_\psi$-h-convex on I, then $G \circ \varphi^{-1}$ is h-convex on $\mathrm{Im}(\varphi)$ and applying Theorem C on function $G \circ \varphi^{-1}$, we get

$$\delta_1 = \frac{1}{2h(\frac{1}{2})} \left\{(1-\lambda)(G\circ\varphi^{-1})\left(\frac{(1-\lambda)\varphi(a)+(1+\lambda)\varphi(b)}{2}\right)\right.$$
$$\left. +\lambda(G\circ\varphi^{-1})\left(\frac{(2-\lambda)\varphi(a)+\lambda\varphi(b)}{2}\right)\right\}$$

$$\delta_2 = \Big[(G\circ\varphi^{-1})((1-\lambda)\varphi(a)+\lambda\varphi(b)) + (1-\lambda)G(b) + \lambda G(a)\Big] \int_0^1 h(t)\, dt.$$

The second term in (1.4) becomes $\frac{1}{\varphi(b)-\varphi(a)}\int_a^b \psi(f(x))\,\varphi'(x)\,dx$ and the fourth term in (1.4) becomes

$$\Big[\psi(f(M_\varphi(a,b;1-\lambda))) + (1-\lambda)\psi(f(b)) + \lambda\psi(f(a))\Big]\int_0^1 h(t)\,dt.$$

Since

$$\begin{aligned}
&(G\circ\varphi^{-1})\left(\frac{(1-\lambda)\varphi(a)+(1+\lambda)\varphi(b)}{2}\right) = (\psi\circ f)\left(M_\varphi\Big(a,b;\frac{1-\lambda}{2}\Big)\right)\\
&(G\circ\varphi^{-1})\left(\frac{(2-\lambda)\varphi(a)+\lambda\varphi(b)}{2}\right) = (\psi\circ f)\left(M_\varphi\Big(a,b;\frac{2-\lambda}{2}\Big)\right)\\
&(G\circ\varphi^{-1})\left((1-\lambda)\varphi(a)+\lambda\varphi(b)\right) = (\psi\circ f)(M_\varphi(a,b;1-\lambda))
\end{aligned}$$

we get (3.8).

Let us prove inequality (3.9). Let us rewrite δ_1 on this way:

$$\begin{aligned}
2h\left(\frac{1}{2}\right)\delta_1 &= \frac{1-\lambda}{h(1-\lambda)}h(1-\lambda)(G\circ\varphi^{-1})\left(\frac{(1-\lambda)\varphi(a)+(1+\lambda)\varphi(b)}{2}\right)\\
&+\frac{\lambda}{h(\lambda)}h(\lambda)(G\circ\varphi^{-1})\left(\frac{(2-\lambda)\varphi(a)+\lambda\varphi(b)}{2}\right)\\
&\geq \min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}\times\\
&\times\left\{(h(1-\lambda)(G\circ\varphi^{-1})\left(\frac{(1-\lambda)\varphi(a)+(1+\lambda)\varphi(b)}{2}\right)\right.\\
&\left.+\,h(\lambda)(G\circ\varphi^{-1})\left(\frac{(2-\lambda)\varphi(a)+\lambda\varphi(b)}{2}\right)\right\}\\
&\geq \min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}\times\\
&\times(G\circ\varphi^{-1})\left[(1-\lambda)\frac{(1-\lambda)\varphi(a)+(\lambda+1)\varphi(b)}{2}+\lambda\frac{(2-\lambda)a+\lambda b}{2}\right]\\
&= \min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}(G\circ\varphi^{-1})\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\\
&= \min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big).
\end{aligned}$$

□

Corollary 3.2. *Let the assumptions of Theorem* 3.1 *hold.*

(i) If ψ is increasing, then for an $M_\varphi M_\psi$-h-convex function $f: I \to \mathbb{R}$ the following holds:

$$\begin{aligned}
&\frac{1}{4h^2(\frac{1}{2})}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big)\\
&\qquad \le \frac{1}{4h(\frac{1}{2})}\left\{(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{4}\big)\Big)+(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{3}{4}\big)\Big)\right\}\\
&\qquad \le \frac{1}{\varphi(b)-\varphi(a)}\int_a^b \psi(f(x))\,\varphi'(x)\,dx\\
&\qquad \le \left\{(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big)+\frac{\psi(f(a))+\psi(f(b))}{2}\right\}\int_0^1 h(t)\,dt\\
&\qquad \le \left[\frac{1}{2}+h\Big(\frac{1}{2}\Big)\right]\Big[\psi(f(a))+\psi(f(b))\Big]\int_0^1 h(t)\,dt, \qquad (3.10)
\end{aligned}$$

provided that all integrals exist.

If f is $M_\varphi M_\psi$-h-concave, then (R3.10) *holds.*

(ii) If ψ is decreasing and f is $M_\varphi M_\psi$-h-convex, then (R3.10) *holds. If ψ is decreasing and f is $M_\varphi M_\psi$-h-concave, then* (3.10) *is valid.*

Proof. Firstly we consider the case when ψ is increasing and f is $M_\varphi M_\psi$-h-convex. The second and the third inequalities in (3.10) are simple consequences of Theorem 3.1 for $\lambda = \frac{1}{2}$. Let us prove the first and the fourth inequalities.

For an h-convex function F the following inequality holds:

$$F(A)+F(B) \quad \ge \quad \frac{1}{h(\frac{1}{2})}F\left(\frac{A+B}{2}\right). \qquad (3.11)$$

Numbers $A := \frac{\varphi(a)+3\varphi(b)}{4}$ and $B := \frac{3\varphi(a)+\varphi(b)}{4}$ satisfy:

$$\frac{A+B}{2}=\frac{\varphi(a)+\varphi(b)}{2}$$

and applying (3.11) on function $F := \psi\circ f\circ\varphi^{-1}$, we get

$$(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{4}\big)\Big)+(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{3}{4}\big)\Big) \ge \frac{1}{h(\frac{1}{2})}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big)$$

and the first inequality in (3.10) holds.

Let us prove the fourth inequality. From (3.11) we get

$$(\psi\circ f)\Big(M_\varphi(a,b;\frac{1}{2})\Big) \le h\Big(\frac{1}{2}\Big)\Big[\psi(f(a))+\psi(f(b))\Big]$$

and hence

$$(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big)+\frac{\psi(f(a))+\psi(f(b))}{2} \le \left[\frac{1}{2}+h\Big(\frac{1}{2}\Big)\right]\Big[\psi(f(a))+\psi(f(b))\Big]$$

and the fourth inequality in (3.10) is valid. ∎

Corollary 3.3. *Let h satisfies the assumptions of Theorem* 3.1*. Let f be a positive GG-h-convex function on $[a,b] \subseteq [0,\infty\rangle$. Then*

$$\begin{aligned}\Big(f(\sqrt{ab})\Big)^{\frac{1}{4h^2(\frac{1}{2})}} &\leq \Big[f(\sqrt[4]{a^3b})f(\sqrt[4]{ab^3})\Big]^{\frac{1}{4h(\frac{1}{2})}} \\ &\leq \exp\left(\frac{1}{\log b/a}\int_a^b \log f(x)\,\frac{dx}{x}\right) \\ &\leq \Big(f(\sqrt{ab})\sqrt{f(a)f(b)}\Big)^{H} \leq \Big(\sqrt{f(a)f(b)}\Big)^{H\left[\frac{1}{2}+h(\frac{1}{2})\right]}, \end{aligned} \tag{3.12}$$

where $H = \int_0^1 h(t)\,dt$ and provided that all integrals exist.

Proof. It is a consequence of Corollary 3.2 for $\psi = \varphi = \log$. ∎

Remark 3.4. Inequality (3.12) for $h(t) = t$ i.e. for GG-convex or multiplicatively convex function can be found in [15, p.62]. It is worth to mention that every polynomial with non-negative coefficients is GG-convex, every real analytic function $f(x) = \sum a_n x^n$ with $a_n \geq 0$ is GG-convex on $[0,R\rangle$ where R is the radius of convergence. Also, the Gamma function is GG-convex.

Corollary 3.5. *Let h satisfies the assumptions of Theorem* 3.1*. Let f be a function on $[a,b] \subseteq [0,\infty\rangle$ and $\varphi(x) = x^p$, $p \neq 0$.*

If $p > 0$ and f is $M_\varphi A$-h-convex, then

$$\begin{aligned}&\frac{1}{4h^2(\frac{1}{2})} f\left(\left(\frac{a^p+b^p}{2}\right)^{1/p}\right) \\ &\quad\leq \frac{1}{4h(\frac{1}{2})}\left\{f\left(\left(\frac{a^p+3b^p}{4}\right)^{1/p}\right) + f\left(\left(\frac{3a^p+b^p}{4}\right)^{1/p}\right)\right\} \\ &\quad\leq \frac{p}{b^p-a^p}\int_a^b f(x)x^{p-1}\,dx \\ &\quad\leq \left\{f\left(\left(\frac{a^p+b^p}{2}\right)^{1/p}\right) + \frac{f(a)+f(b)}{2}\right\}\int_0^1 h(t)dt \\ &\quad\leq \left[\frac{1}{2} + h\Big(\frac{1}{2}\Big)\right][f(a)+f(b)]\int_0^1 h(t)dt, \end{aligned} \tag{3.13}$$

provided that all integrals exist.

If $p < 0$ and f is $M_\varphi A$-h-convex, then (R3.13) *holds.*

Proof. It is a consequence of Corollary 3.2 for $\psi(x) = x$, $\varphi(x) = x^p$. ∎

Remark 3.6. If $h(t) = t$ and $p = 1$, then $4h^2(\frac{1}{2}) = 1$, $\frac{1}{2} + h(\frac{1}{2}) = 1$ and inequality (3.13) becomes the refinement of the Hermite-Hadamard inequality (1.1).

The following Hermite-Hadamard-type result involves more than two nodes.

Theorem 3.7. *Let h be a non-negative function defined on the interval $J, \langle 0,1\rangle \subseteq J$, $h(\frac{1}{2}) \neq 0$. Let φ and ψ be strictly monotone continuous functions defined on intervals I and K respectively such that φ is differentiable on $[a,b] \subseteq I$.*

(i) If ψ is increasing, then for an $M_\varphi M_\psi$-h-convex function $f : I \to \mathbb{R}$ and for a partition

$$0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n = 1, \qquad \text{with } n \geq 1$$

we have

$$\begin{aligned}
&\frac{1}{2h(\frac{1}{2})} \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j)(\psi \circ f)\Big(M_\varphi\big(a,b; 1 - \frac{\lambda_j + \lambda_{j+1}}{2}\big)\Big) \\
&\quad \leq \frac{1}{\varphi(b) - \varphi(a)} \int_a^b \psi(f(x))\varphi'(x)\, dx \\
&\quad \leq \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j)\Big\{(\psi \circ f)\big(M_\varphi(a,b; 1 - \lambda_j)\big) \\
&\quad + (\psi \circ f)\big(M_\varphi(a,b; 1 - \lambda_{j+1})\big)\Big\} \int_0^1 h(t)\, dt,
\end{aligned} \tag{3.14}$$

provided that all integrals exist.

If f is $M_\varphi M_\psi$-h-concave, then (R3.14) *holds.*

(ii) If ψ is decreasing and f is $M_\varphi M_\psi$-h-convex, then (R3.14) *holds. If ψ is decreasing and f is $M_\varphi M_\psi$-h-concave, then* (3.14) *is valid.*

Proof. Let ψ be increasing and f be $M_\varphi M_\psi$-h-convex. Denote $G := \psi \circ f$. Then a function $\psi \circ f \circ \varphi^{-1}$ is h-convex on $\varphi([a,b])$ and applying Theorem D on function $G \circ \varphi^{-1}$, we get

$$\begin{aligned}
&\frac{1}{2h(\frac{1}{2})} \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) G\left(\varphi^{-1}\left(\left(1 - \frac{\lambda_j + \lambda_{j+1}}{2}\right)\varphi(a) + \frac{\lambda_j + \lambda_{j+1}}{2}\varphi(b)\right)\right) \\
&\quad \leq \frac{1}{\varphi(b) - \varphi(a)} \int_a^b G(x)\varphi'(x)\, dx \\
&\quad \leq \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j)\Big\{G\left(\varphi^{-1}\left((1 - \lambda_j)\varphi(a) + \lambda_j\varphi(b)\right)\right) \\
&\quad + G\left(\varphi^{-1}\left((1 - \lambda_{j+1})\varphi(a) + \lambda_{j+1}\varphi(b)\right)\right)\Big\} \int_0^1 h(t)\, dt.
\end{aligned} \tag{3.15}$$

Using the fact that $G\left(\varphi^{-1}\left(\left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right)\varphi(a)+\frac{\lambda_j+\lambda_{j+1}}{2}\varphi(b)\right)\right) = (\psi \circ f)\Big(M_\varphi\big(a,b;1-\frac{\lambda_j+\lambda_{j+1}}{2}\big)\Big)$ etc, we get (3.14). Other cases are done in a similar manner. ∎

If a partition is equidistant, then the series of inequalities in (3.14) can be extended. Namely, we have the following result.

Theorem 3.8. *Let h be a non-negative function defined on the interval $J, \langle 0,1\rangle \subseteq J$, $h(\frac{1}{2}) \neq 0$. Let φ and ψ be strictly monotone continuous functions defined on intervals I and K respectively such that φ is differentiable on $[a,b] \subseteq I$. Let $f: I \to \mathbb{R}$. Let $n \geq 2$.*

(i) If ψ is increasing, then for an $M_\varphi M_\psi$-h-convex function f the following inequalities hold

$$\begin{aligned}
&\frac{1}{4h^2(\frac{1}{2})}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big)\\
&\qquad\leq l(n) \leq \frac{1}{\varphi(b)-\varphi(a)}\int_a^b \psi(f(x))\varphi'(x)\,dx \leq L(n)\\
&\qquad\leq \frac{1}{n}\Big[\psi(f(a))+\psi(f(b))\Big]\left\{1+2\sum_{j=1}^{n-1} h\left(\frac{j}{n}\right)\right\}\int_0^1 h(t)\,dt,
\end{aligned}\tag{3.16}$$

provided that all integrals exist and where

$$\begin{aligned}
l(n) &= \frac{1}{2nh(\frac{1}{2})}\sum_{j=0}^{n-1}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{2n-2j-1}{2n}\big)\Big)\\
L(n) &= \frac{2}{n}\int_0^1 h(t)\,dt\left\{\sum_{j=1}^{n-1}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{j}{n}\big)\Big)+\frac{\psi(f(a))+\psi(f(b))}{2}\right\}.
\end{aligned}$$

If f is $M_\varphi M_\psi$-h-concave, then (R3.16) *holds.*

(ii) If ψ is decreasing and f is $M_\varphi M_\psi$-h-convex, then (R3.16) *holds. If ψ is decreasing and f is $M_\varphi M_\psi$-h-concave, then* (3.16) *is valid.*

Proof. Let us suppose that ψ is increasing and f is $M_\varphi M_\psi$-h-convex. The second and the third inequalities in (3.16) are simply consequences of Theorem 3.7 when we apply it on points: $\lambda_j = \frac{j}{n}$. Let us prove the first inequality. Putting in (3.11) $F = \psi \circ f \circ \varphi^{-1} = G \circ \varphi^{-1}$ and

$$A = \frac{2n-2j-1}{2n}\varphi(a)+\frac{2j+1}{2n}\varphi(b), \quad B = \frac{2j+1}{2n}\varphi(a)+\frac{2n-2j-1}{2n}\varphi(b)$$

and since $A+B=\varphi(a)+\varphi(b)$, we get

$$\begin{aligned} G\Big(\varphi^{-1}\big(\frac{2n-2j-1}{2n}\varphi(a)+\frac{2j+1}{2n}\varphi(b)\big)\Big) \\ &+G\Big(\varphi^{-1}\big(\frac{2j+1}{2n}\varphi(a)+\frac{2n-2j-1}{2n}\varphi(b)\big)\Big) \\ &\geq \frac{1}{h\big(\frac{1}{2}\big)}G\Big(\varphi^{-1}\big(\frac{\varphi(a)+\varphi(b)}{2}\big)\Big), \end{aligned}$$

i.e.

$$\begin{aligned} G\Big(M_\varphi\big(a,b;\frac{2n-2j-1}{2n}\big)\Big)+G\Big(M_\varphi\big(a,b;\frac{2j+1}{2n}\big)\Big) \\ \geq \frac{1}{h\big(\frac{1}{2}\big)}G\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big). \end{aligned}$$

Let us write the sum $\sum_{j=0}^{n-1}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{2n-2j-1}{2}\big)\Big)$ twice and add the addend indexed by j from the first sum with the addend indexed by $(n-j-1)$ from the second sum. Then we get

$$\begin{aligned} 2\sum_{j=0}^{n-1}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{2n-2j-1}{2n}\big)\Big) \\ &=\sum_{j=0}^{n-1}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{2n-2j-1}{2n}\big)\Big)+(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{2j+1}{2n}\big)\Big) \\ &\geq \sum_{j=0}^{n-1}\frac{1}{h\big(\frac{1}{2}\big)}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big) \\ &=\frac{n}{h\big(\frac{1}{2}\big)}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big) \end{aligned}$$

and the first inequality in (3.16) follows.

In the proof of the fourth inequality in (3.16) we apply a definition of $M_\varphi M_\psi$-h-convexity on each addend in the sum and transform it:

$$\begin{aligned} \psi(f(a))+\psi(f(b))+2\sum_{j=1}^{n-1}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{j}{n}\big)\Big) \\ &\leq \psi(f(a))+\psi(f(b))+2\sum_{j=1}^{n-1}\Big(h\Big(\frac{j}{n}\Big)\psi(f(a))+h\Big(\frac{n-j}{n}\Big)\psi(f(b))\Big) \\ &=\Big[\psi(f(a))+\psi(f(b))\Big]\left\{1+2\sum_{j=1}^{n-1}h\Big(\frac{j}{n}\Big)\right\} \end{aligned}$$

and from this estimate the fourth inequality in (3.16) follows. ∎

In the following theorem we consider a particular partition of interval $[0,1]$, so-called a dyadic partition. Let $m \geq 1$ be an integer and let

$$\lambda_j := \frac{j}{2^m}, \qquad j = 0, 1, 2, \dots, 2^m.$$

Note that Corollary 3.2 contains result of this type for $m = 1$. In literature, there are no similar results for h-convex functions. Therefore, we can not use Proposition 2.1 in the proof of the following theorem.

Theorem 3.9. *Let h be a non-negative function defined on the interval $J, \langle 0,1\rangle \subseteq J$, $h(\frac{1}{2}) \neq 0$. Let φ and ψ be strictly monotone continuous functions defined on intervals I and K respectively such that φ is differentiable on $[a,b] \subseteq I$. Let $f : I \to \mathbb{R}$.*

(i) If ψ is increasing, then for an $M_\varphi M_\psi$-h-convex function f and $m \in \mathbb{N}$ the following holds

$$l(2^{m+1}) \geq \frac{1}{2h(\frac{1}{2})} l(2^m) \tag{3.17}$$

$$L(2^{m+1}) \leq \left(\frac{1}{2} + h\Big(\frac{1}{2}\Big)\right) L(2^m) \tag{3.18}$$

$$\begin{aligned} L(2^m) \leq{} & 8h^2\Big(\frac{1}{2}\Big) \int_0^1 h(t)dt \cdot l(2^m) + \frac{1}{2^m} \int_0^1 h(t)dt \Big\{ \psi(f(a)) + \psi(f(b)) \\ & - 2h\Big(\frac{1}{2}\Big)\psi\left(f\Big(M_\varphi(a,b,\frac{2^{m+1}-1}{2^{m+1}})\Big)\right) \\ & - 2h\Big(\frac{1}{2}\Big)\psi\left(f\Big(M_\varphi(a,b,\frac{1}{2^{m+1}})\Big)\right)\Big\}, \end{aligned} \tag{3.19}$$

where $l(n)$ and $L(n)$ are defined as in Theorem 3.8.

If f is $M_\varphi M_\psi$-h-concave, then (R3.17), (R3.18) *and* (R3.19) *hold.*

(ii) If ψ is decreasing and f is $M_\varphi M_\psi$-h-convex, then (R3.17), (R3.18) *and* (R3.19) *hold. If ψ is decreasing and f is $M_\varphi M_\psi$-h-concave, then* (3.17), (3.18) *and* (3.19) *hold.*

Proof. We prove the case when ψ is increasing and f is $M_\varphi M_\psi$-h-convex. We use notation: $F := \psi \circ f \circ \varphi^{-1}$, $A := \varphi(a)$ and $B := \varphi(B)$.

From Theorem 3.8 we get:

$$l(2^{m+1}) = \frac{1}{2^{m+2}h(\frac{1}{2})} \sum_{j=0}^{2^{m+1}-1} F\left(\frac{(2^{m+2}-2j-1)A + (2j+1)B}{2^{m+2}}\right).$$

Since

$$\begin{aligned}&\{0,1,2,\dots 2^{m+1}-1\} = \{0,2,4,\dots,2^{m+1}-2\}\cup\{1,3,5,\dots,2^{m+1}-1\}\\&=\{2k\,:\,k=0,1,\dots,2^m-1\}\cup\{2k+1\,:\,k=0,1,\dots,2^m-1\},\end{aligned}$$

we obtain

$$\begin{aligned}l(2^{m+1}) = \frac{1}{2^{m+2}h(\frac{1}{2})}&\left\{\sum_{k=0}^{2^m-1}F\left(\frac{(2^{m+2}-4k-1)A+(4k+1)B}{2^{m+2}}\right)\right.\\&\left.+\sum_{k=0}^{2^m-1}F\left(\frac{(2^{m+2}-4k-3)A+(4k+3)B}{2^{m+2}}\right)\right\}.\end{aligned}$$

Since F is h-convex, then $F(x)+F(y)\geq \frac{1}{h(\frac{1}{2})}F(\frac{x+y}{2})$. Putting in this inequality $x=\frac{(2^{m+2}-4k-1)A+(4k+1)B}{2^{m+2}}$ and $y=\frac{(2^{m+2}-4k-3)A+(4k+3)B}{2^{m+2}}$, we get that $l(2^{m+1})$ is bounded from below as follows

$$\begin{aligned}l(2^{m+1}) &\geq \frac{1}{2^{m+2}h(\frac{1}{2})}\sum_{k=0}^{2^m-1}\frac{1}{h(\frac{1}{2})}F\left(\frac{(2^{m+1}-2k-1)A+(2k+1)B}{2^{m+1}}\right)\\&=\frac{1}{2h(\frac{1}{2})}l(2^m).\end{aligned}$$

Hence (3.17) is proved.

Let us prove (3.18). Again, we split the sum in $L(2^{m+1})$ into two sums: one with odd indices and the second sum with even indices.

$$\begin{aligned}L(2^{m+1}) &= \frac{1}{2^m}\int_0^1 h(t)dt\left\{\frac{F(A)+F(B)}{2}+\sum_{k=1}^{2^m-1}F\left(\frac{(2^{m+1}-2k)A+2kB}{2^{m+1}}\right)\right.\\&\quad\left.+\sum_{k=0}^{2^m-1}F\left(\frac{(2^{m+1}-2k-1)A+(2k+1)B}{2^{m+1}}\right)\right\}\\&=\frac{1}{2^m}\int_0^1 h(t)dt\left\{\sum_{k=0}^{2^m-1}F\left(\frac{(2^{m+1}-2k-1)A+(2k+1)B}{2^{m+1}}\right)\right.\\&\quad+\left[\frac{1}{2}\sum_{k=1}^{2^m-1}F\left(\frac{(2^{m+1}-2k)A+2kB}{2^{m+1}}\right)+\frac{F(A)}{2}\right]\\&\quad\left.+\left[\frac{1}{2}\sum_{k=1}^{2^m-1}F\left(\frac{(2^{m+1}-2k)A+2kB}{2^{m+1}}\right)+\frac{F(B)}{2}\right]\right\}\\&=\frac{1}{2^m}\int_0^1 h(t)dt\left\{\sum_{k=0}^{2^m-1}F\left(\frac{[(2^m-k)A+kB]+[(2^m-k-1)A+(k+1)B]}{2\cdot 2^m}\right)\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{k=0}^{2^m-1} F\left(\frac{(2^{m+1}-2k)A+2kB}{2^{m+1}}\right)\\
&+\frac{1}{2}\sum_{r=0}^{2^m-1} F\left(\frac{(2^m-r-1)A+(r+1)B}{2^m}\right)\Bigg\}\\
&\le \frac{1}{2^m}\int_0^1 h(t)dt\left\{\sum_{k=0}^{2^m-1} h\Big(\frac{1}{2}\Big)F\left(\frac{(2^m-k)A+kB}{2^m}\right)\right.\\
&+\sum_{k=0}^{2^m-1} h\Big(\frac{1}{2}\Big)F\left(\frac{(2^m-k-1)A+(k+1)B}{2^m}\right)\\
&\left.+\frac{1}{2}\sum_{k=0}^{2^m-1} F\left(\frac{(2^m-k)A+kB}{2^m}\right)+\frac{1}{2}\sum_{r=0}^{2^m-1} F\left(\frac{(2^m-r-1)A+(r+1)B}{2^m}\right)\right\}\\
&=\frac{1}{2^m}\int_0^1 h(t)dt\left(\frac{1}{2}+h\Big(\frac{1}{2}\Big)\right)\times\\
&\times\left\{\sum_{k=0}^{2^m-1}\left[F\left(\frac{(2^m-k)A+kB}{2^m}\right)+F\left(\frac{(2^m-k-1)A+(k+1)B}{2^m}\right)\right]\right\}\\
&=\left(\frac{1}{2}+h\Big(\frac{1}{2}\Big)\right)L(2^m).
\end{aligned}
$$

Let us prove (3.19). Note that for $k=1,2,\ldots,2^m-1$

$$
\begin{aligned}
&\frac{(2^m-k)A+kB}{2^m}\\
&=\frac{1}{2}\left(\frac{(2^{m+1}-2k+1)A+(2k-1)B}{2^{m+1}}+\frac{(2^{m+1}-2k-1)A+(2k+1)B}{2^{m+1}}\right)
\end{aligned}
$$

Since F is h-convex, we get

$$
\begin{aligned}
\sum_{k=1}^{2^m-1}F\Big(\frac{(2^m-k)A+kB}{2^m}\Big) &\le \sum_{k=1}^{2^m-1} h\Big(\frac{1}{2}\Big)\Big\{F\left(\frac{(2^{m+1}-2k+1)A+(2k-1)B}{2^{m+1}}\right)\\
&\quad+F\left(\frac{(2^{m+1}-2k-1)A+(2k+1)B}{2^{m+1}}\right)\Big\}\\
&=h\Big(\frac{1}{2}\Big)\left[2\sum_{j=0}^{2^m-1} F\left(\frac{(2^{m+1}-2j-1)A+(2j+1)B}{2^{m+1}}\right)\right.\\
&\quad\left.-F\left(\frac{(2^{m+1}-1)A+B}{2^{m+1}}\right)-F\left(\frac{A+(2^{m+1}-1)B}{2^{m+1}}\right)\right].
\end{aligned}
$$

Adding on the both sides $\frac{F(A)+F(B)}{2}$ and using notations for l and L, we get

$$\frac{2^{m-1}}{\int_0^1 h(t)dt} L(2^m) \le 2^{m+2} h^2\Big(\frac{1}{2}\Big) \cdot l(2^m) + \frac{F(A)+F(B)}{2}$$
$$- h^2\Big(\frac{1}{2}\Big) F\left(\frac{(2^{m+1}-1)A+B}{2^{m+1}}\right) - h^2\Big(\frac{1}{2}\Big) F\left(\frac{A+(2^{m+1}-1)B}{2^{m+1}}\right)$$

and (3.19) is proved. □

If $h(\frac{1}{2}) \le \frac{1}{2}$, then the previous Theorem gives a sequence of interpolations of the Hermite-Hadamard inequality.

Corollary 3.10. *Suppose that the assumptions of Theorem* 3.9 *hold. Let* $h(\frac{1}{2}) \le \frac{1}{2}$.

If ψ is increasing and f is an $M_\varphi M_\psi$-h-convex integrable function such that $\psi \circ f \circ \varphi^{-1}$ is non-negative, then the following holds

$$\frac{1}{4h^2(\frac{1}{2})} (\psi \circ f)\Big(M_\varphi\big(a, b; \frac{1}{2}\big)\Big) \le l(2) \le l(2^2) \le \ldots \le l(2^m) \le \ldots$$
$$\le \frac{1}{\varphi(b)-\varphi(a)} \int_a^b \psi(f(x))\varphi'(x)\, dx$$
$$\le \ldots \le L(2^m) \le \ldots \le L(2^2) \le L(2)$$
$$\le \left[\frac{1}{2} + h\Big(\frac{1}{2}\Big)\right] \Big[\psi(f(a)) + \psi(f(b))\Big] \int_0^1 h(t)\, dt. \tag{3.20}$$

Additionally, if $\int_0^1 h(t)\, dt \le \frac{1}{2}$ and if $\psi \circ f \circ \varphi^{-1}$ is bounded on $\varphi([a, b])$, then

$$\lim_{m\to\infty} (L(2^m) - l(2^m)) = 0 \tag{3.21}$$

and

$$\lim_{m\to\infty} l(2^m) = \frac{1}{\varphi(b)-\varphi(a)} \int_a^b \psi(f(x))\varphi'(x)\, dx = \lim_{m\to\infty} L(2^m). \tag{3.22}$$

Proof. If $h(\frac{1}{2}) \le \frac{1}{2}$, then $\frac{1}{2h(\frac{1}{2})} \ge 1$ and $\frac{1}{2} + h\Big(\frac{1}{2}\Big) \le 1$ and from (3.17) and (3.18) we have that for any $m \ge 1$

$$l(2^{m+1}) \ge l(2^m) \quad \text{and} \quad L(2^{m+1}) \le L(2^m).$$

Hence, applying Theorem 3.8, Corollary 3.2 and above inequalities, we get (3.20).

If $h(\frac{1}{2}) \le \frac{1}{2}$ and $\int_0^1 h(t)\, dt \le \frac{1}{2}$, then $8h^2\Big(\frac{1}{2}\Big) \int_0^1 h(t)dt \le 1$ and (3.21) follows from (3.19). The sequence $(l(2^m))_m$ is a non-decreasing sequence, bounded from above with $\dfrac{1}{\varphi(b)-\varphi(a)}\displaystyle\int_a^b \psi(f(x))\varphi'(x)\,dx$, so, it is convergent. Similarly, $(L(2^m))_m$ is convergent and from (3.21) and from inequality

$$l(2^m) \leq \frac{1}{\varphi(b)-\varphi(a)}\int_a^b \psi(f(x))\varphi'(x)\,dx \leq L(2^m)$$

we get (3.22). □

Under assumptions of Corollary 3.10 we conclude that the larger m makes $l(2^m)$ and $L(2^m)$ closer to the integral mean of $\psi \circ f \circ \varphi^{-1}$. The behavior of convex functions involving dyadic partition is studied in [12]. Here we extend those results to a more general function class.

Conclusion. In this paper, we study Hermite-Hadamard-type inequalities for $M_\varphi M_\psi$-h-convex functions. Until now we have found similar results only for particular subclasses of the class of $M_\varphi M_\psi$-h-convex functions. The connection between h-convex function and $M_\varphi M_\psi$-h-convex function which is described in Proposition 2.1 has a crucial role in the proofs and the use of it makes proofs more elegant. It would be interesting to see how this method impacts the study of other properties of $M_\varphi M_\psi$-h-convex functions.

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Hermite-Hadamardova nejednakost za $M_\varphi M_\psi$-h-konveksne funkcije i odgovarajuće interpolacije

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Sažetak. U članku se promatra Hermite-Hadamardova nejednakost za $M_\varphi M_\psi$-h-konveksne funkcije. Kao što je poznato, $M_{\varphi M_\psi}$-h-konveksnost generalizira nekoliko klasa funkcija kao što su harmonijski-h-konveksne funkcije, logaritamski h-konveksne, (h,p)-konveksne, M_pA-h-konveksne, $M_\varphi M_\psi$ konveksne funkcije i druge. Dokazane su nejednakosti Hermite-Hadamardovog tipa koje uključuju dva i više čvorova, a posebna je pažnja posvećena dijadskoj particiji intervala i profinjenju nejednakosti koja se javlja u tom slučaju.

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