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# COMPUTABLE SUBCONTINUA OF CIRCULARLY CHAINABLE CONTINUA

# DAVID TARANDEK

ABSTRACT. This paper explores, in computable metric spaces, circularly chainable continua which are not chainable. Given such a continuum K, if we endow it with semicomputability, its computability follows. Conditions under which semicomputability implies computability, typically topological, are extensively studied in the literature. When these conditions are not satisfied, it is natural to explore approximate approaches. In this article we investigate specific computable subcontinua of K. The main result establishes that, given two points on a semicomputable, circularly chainable, but non-chainable continuum K, one can approximate them by computable points such that there exists a computable subcontinuum connecting these approximations. As a consequence, given disjoint computably enumerable open sets U and V intersected by K, the intersection of K with the complement of their union necessarily contains a computable point, provided that this intersection is totally disconnected.

### 1. INTRODUCTION

A compact set  $K \subseteq \mathbb{R}$  is said to be *computable* if it can be effectively approximated by finitely many rational points with any given precision. A compact set  $K \subseteq \mathbb{R}$  is said to be *semicomputable* if there is a computable function  $f : \mathbb{R} \to \mathbb{R}$  such that the set of its zeros equals K.

In  $\mathbb{R}$  the following implication holds:

K computable  $\implies K$  semicomputable.

This also holds in more general ambient spaces such as  $\mathbb{R}^n$ , computable metric spaces and computable topological spaces.

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<sup>1</sup> 

#### D. TARANDEK

The converse,

# K semicomputable $\implies K$ computable,

does not hold. A well known counterexample from Miller [Mil02] goes like this: Let  $\gamma \in [0, 1]$  be a left-computable, but not computable, real number. Then the segment  $[\gamma, 1]$  is semicomputable, but not computable.

But, for each  $\epsilon > 0$  there exists  $\gamma_{\epsilon} \in \langle \gamma, \gamma + \epsilon \rangle$  such that  $[\gamma_{\epsilon}, 1]$  is computable. In that way we inner approximate the semicomputable set  $[\gamma, 1]$  with the computable sets  $[\gamma_{\epsilon}, 1]$ .

This example illustrates two common approaches in computable analysis.

- 1. Which topological conditions render a semicomputable set K computable?
- 2. If this conditions aren't met, under which conditions can K be inner approximated by a desired class of computable sets?
  - EXAMPLES 1. A semicomputable set homeomorphic to the circle  $S^1 \subseteq \mathbb{R}^2$  is computable [Mil02].
  - Specker [Spe59] shows that there exists a computable function  $f : \mathbb{R} \to \mathbb{R}$  which has zeros, but none of them are computable. It follows that  $K = f^{-1}(\{0\})$  is a semicomputable set that contains no computable points. In particular, K has no nonempty computable subsets.
  - It is shown that K from the previous example is homeomorphic to the Cantor set. It is compact, but not connected. However,  $(K \times [0,1]) \cup ([0,1] \times K) \subseteq \mathbb{R}^2$  is a compact and connected set, i.e., a *continuum*, which is semicomputable and has no nonempty computable subsets.
  - There exists a contractible, locally contractible semicomputable curve in  $\mathbb{R}^2$  which cannot be inner approximated by computable continua [Kih12].

Topological properties have a significant impact on the behavior of sets with respect to approaches **1**. and **2**. Among other objects, arcs and topological circles are often studied in the literature, as well as their respective generalizations, *chainable* and *circularly chainable* continua. Here are some important results concerning approach **1**.

- [IIj09] : In a computable metric space, every semicomputable circularly chainable continuum that is not chainable is computable.
- [CIV19]: The same holds in a computable topological space. Also, every semicomputable continuum K that is chainable from a to b, where a and b are computable points, is computable.
- [IS18] : A semicomputable manifold with a semicomputable boundary is computable.

We now present several important results regarding approach 2.

[IP18] : If A is a semicomputable arc in a computable metric space with endpoints a and b, then for every  $\epsilon > 0$  there exist computable points a' and b' such that  $d(a, a') < \epsilon$ ,  $d(b, b') < \epsilon$ , and a computable arc A' whose endpoints are a' and b' such that  $A' \subseteq A$ .

- [CHI21] : The same holds in a computable topological space. The same conclusion holds for semicomputable chainable continua under the additional assumption of decomposability.
- [IJ24]: The same holds even without the decomposability assumption, if K is a semicomputable continuum chainable from a to b, where a is a computable point.

The first goal is to prove that, given a circularly chainable, but not chainable, continuum K in a computable metric space and two distinct points  $a, b \in K$  we can find computable points  $a', b' \in K$  arbitrarily close to a and b and a computable subcontinuum L of K chainable from a' to b'. Some similarities and differences are highlighted when contrasting this result with the approach **2**.

The second goal is to generalize a well known result stated in Pour-El, Richards [PR89]:

THEOREM 1. Computable Intermediate Value Theorem. A computable function  $f : [0, 1] \to \mathbb{R}$  such that f(0) < 0, f(1) > 0 has a computable zero.

Some generalizations have been studied in [IP18], with the most prominent result being:

THEOREM 2. Let  $(X, d, \alpha)$  be a computable metric space and let U and V be disjoint c.e. open sets in X. Let  $S = X \setminus (U \cup V)$ . Suppose K is a continuum in X chainable from a to b, where  $a \in U$  and  $b \in V$ . Suppose K is a computable set and  $K \cap S$  is totally disconnected. Then  $K \cap S$  contains a computable point.

We plan to show that the modification of Theorem 2 holds, where K is a circularly chainable, but not chainable, continuum which intersects both Uand V.

Our choice of setting is a computable metric space. The result [AH23, Theorem 3.4], shows that every semicomputable set S in a computable topological space can be effectively embedded into the Hilbert cube and therefore computable topological spaces do not lead to a more general result.

### 2. Preliminaries

Here we state some basic definitions and facts about computable metric spaces. See [II]09; IS18; IP18; Wei00; Wei93].

Let  $k \in \mathbb{N} \setminus \{0\}$ . A function  $f : \mathbb{N}^k \to \mathbb{Q}$  is said to be *computable* if there exist computable (i.e., recursive) functions  $a, b, c : \mathbb{N}^k \to \mathbb{N}$  such that

$$f(x) = (-1)^{c(x)} \cdot \frac{a(x)}{b(x) + 1}$$

### D. TARANDEK

for each  $x \in \mathbb{N}^k$ . A function  $f : \mathbb{N}^k \to \mathbb{R}$  is said to be *computable* if there exists a computable function  $F : \mathbb{N}^{k+1} \to \mathbb{Q}$  such that

$$|f(x) - F(x,i)| < 2^{-i}$$

for all  $x \in \mathbb{N}^k$ ,  $i \in \mathbb{N}$ .

Let (X, d) be a metric space, and let  $\alpha$  be a sequence in X such that  $\alpha(\mathbb{N})$  is a dense set in (X, d). We say that  $(X, d, \alpha)$  is a *computable metric space* if the function

$$(i,j) \mapsto d(\alpha_i, \alpha_j) : \mathbb{N}^2 \to \mathbb{R}$$

is computable.

Let  $(X, d, \alpha)$  be a computable metric space. A point  $x \in X$  is said to be *computable* in  $(X, d, \alpha)$  if there exists a computable function  $f : \mathbb{N} \to \mathbb{N}$  such that

$$d\left(x,\alpha_{f(k)}\right) < 2^{-k}$$

for all  $k \in \mathbb{N}$ .

We fix computable functions

$$(j,i) \mapsto (j)_i : \mathbb{N}^2 \to \mathbb{N} \text{ and } j \mapsto \overline{j} : \mathbb{N} \to \mathbb{N}$$

such that

$$\{((j)_0, (j)_1, \dots, (j)_{\overline{j}}) \mid j \in \mathbb{N}\}$$

is the set of all nonempty finite sequences in  $\mathbb{N}$ .

For  $j \in \mathbb{N}$ , let

$$[j] := \{(j)_0, \dots, (j)_{\overline{i}}\}$$

Then each nonempty finite subset of  $\mathbb{N}$  is equal to [j] for some  $j \in \mathbb{N}$ .

If  $(X, d, \alpha)$  is a computable metric space,  $i \in \mathbb{N}$ , and  $q \in \mathbb{Q}$  with q > 0, we say that  $B(\alpha_i, q)$  is a *rational open ball* in this space. Here, for  $x \in X$  and r > 0, we denote by  $B(x, r) = \{y \in X : d(x, y) < r\}$  the open ball of radius rcentered at x. Let  $\tau_1, \tau_2 : \mathbb{N} \to \mathbb{N}$  be fixed computable functions such that

$$\{(\tau_1(i), \tau_2(i)) \mid i \in \mathbb{N}\} = \mathbb{N}^2$$

and let  $q: \mathbb{N} \to \mathbb{Q}$  be a fixed computable function whose image is the set of all positive rational numbers. Let  $(\lambda_i)_{i \in \mathbb{N}}$  be the sequence of points in X defined by  $\lambda_i = \alpha_{\tau_1(i)}$ , and let  $(\rho_i)_{i \in \mathbb{N}}$  be the sequence of rational numbers defined by  $\rho_i = q_{\tau_2(i)}$ . For  $i \in \mathbb{N}$ , we define

$$I_i = B(\lambda_i, \rho_i), \quad \hat{I}_i = \overline{B}(\lambda_i, \rho_i)$$

Note that  $\{I_i \mid i \in \mathbb{N}\}$  is the set of all rational open balls in  $(X, d, \alpha)$ . Therefore, the sequence  $(I_i)_{i \in \mathbb{N}}$  represents an effective enumeration of all rational open balls.

Let  $(X, d, \alpha)$  be a computable metric space. Any finite union of rational open balls in this space is said to be a *rational open set*. For  $j \in \mathbb{N}$  we define

$$J_j = \bigcup_{i \in [j]} I_i, \quad \hat{J}_j = \bigcup_{i \in [j]} \hat{I}_i.$$

Then  $(J_j)$  is an effective enumeration of all rational open sets in  $(X, d, \alpha)$ .

Let  $(X, d, \alpha)$  be a computable metric space. For every  $j \in \mathbb{N}$ , we define

$$fdiam(j) := diam\{\lambda_u \mid u \in [j]\} + 2\max\{\rho_u \mid u \in [j]\},\$$

and call it the *formal diameter* of  $J_j$ . This is formally a  $\mathbb{N} \to \mathbb{R}$  function of j, not of  $J_j$ . We define a function fmesh :  $\mathbb{N} \to \mathbb{R}$  by

$$\operatorname{fmesh}(l) = \max_{0 \le p \le \overline{l}} \operatorname{fdiam}((l)_p).$$

It is straightforward to conclude that the functions fdiam and fmesh are computable [IIj09, Proposition 2, Proposition 13].

Let (X, d) be a metric space,  $A, B \subseteq X$ , and  $\varepsilon > 0$ . We say that A and B are  $\varepsilon$ -close, and write  $A \approx_{\varepsilon} B$ , if

$$(\forall a \in A)(\exists b \in B) (d(a, b) < \varepsilon)$$
 and  $(\forall b \in B)(\exists a \in A) (d(a, b) < \varepsilon)$ .

If A and B are nonempty compact sets in (X, d), the number

$$\inf\{\varepsilon > 0 \mid A \approx_{\varepsilon} B\}$$

is called the *Hausdorff distance* from A to B, and it is denoted by  $d_H(A, B)$ .

It is not hard to check that, for  $\varepsilon > 0$ , we have  $d_H(A, B) < \varepsilon$  if and only if  $A \approx_{\varepsilon} B$ .

DEFINITION 1. Let  $(X, d, \alpha)$  be a computable metric space. We say that a compact set  $S \subseteq X$  is *computable* in  $(X, d, \alpha)$  if either  $S = \emptyset$  or there exists a computable function  $f : \mathbb{N} \to \mathbb{N}$  such that

$$S \approx_{2^{-k}} \{ \alpha_i \mid i \in [f(k)] \}, \text{ for all } k \in \mathbb{N}.$$

DEFINITION 2. Let  $(X, d, \alpha)$  be a computable metric space.

(i) A closed set  $S \subseteq X$  is said to be *computably enumerable (c.e.)* in  $(X, d, \alpha)$  if the set

$$\{i \in \mathbb{N} \mid I_i \cap S \neq \emptyset\}$$

is computably enumerable.

(ii) A compact set  $S \subseteq X$  is said to be *semicomputable* in  $(X, d, \alpha)$  if the set

$$\{j \in \mathbb{N} \mid S \subseteq J_j\}$$

is computably enumerable.

(iii) An open set  $U \subseteq X$  is said to be *computably enumerable open* in  $(X, d, \alpha)$  if

$$U = \bigcup_{i \in A} I_i,$$

for some c.e. subset A of  $\mathbb{N}$ .

#### D. TARANDEK

It is not hard to see that these definitions do not depend on the particular choices of the functions  $q, \tau_1, \tau_2$ , and  $j \mapsto [j]$ .

For compact sets K in  $(X, d, \alpha)$  we have the following important equivalence from [II]13, Proposition 2.6]:

K is computable  $\iff K$  is semicomputable and c.e.

We state the following basic facts about computable functions of type  $\mathbb{N}^k \to \mathbb{N}^n$  and  $\mathbb{N}^k \to \mathbb{R}$ :

PROPOSITION 1. (i) (**Projection theorem**) Let  $T \subseteq \mathbb{N}^{k+n}$  be a computably enumerable set. Then the set

$$S = \{ x \in \mathbb{N}^k \mid \exists y \in \mathbb{N}^n : (x, y) \in T \}$$

is computably enumerable.

(ii) (Single-valuedness theorem) Suppose  $T \subseteq \mathbb{N}^{k+n}$ ,  $S_1 \subseteq \mathbb{N}^k$ , and  $S_2 \subseteq \mathbb{N}^n$  are computably enumerable sets such that for each  $x \in S_1$  there exists  $y \in S_2$  with  $(x, y) \in T$ . Then there exists a partial computable (partial recursive) function  $f: S_1 \to \mathbb{N}^n$  such that  $f(S_1) \subseteq S_2$  and

$$(x, f(x)) \in T$$
 for each  $x \in S_1$ 

(iii) If  $S \subseteq \mathbb{N}^n$  is a computably enumerable set and  $f : \mathbb{N}^k \to \mathbb{N}^n$  is a computable function, then the set  $f^{-1}(S)$  is computably enumerable.

PROPOSITION 2. (i) If  $f, g : \mathbb{N}^k \to \mathbb{R}$  are computable functions, then f + g and f - g are computable.

(ii) If  $f, g: \mathbb{N}^k \to \mathbb{R}$  are computable functions, then the set

$$\{x \in \mathbb{N}^k \mid f(x) > g(x)\}\$$

is computably enumerable.

For  $m \in \mathbb{N}$ , let  $\mathbb{N}_m = \{0, \ldots, m\}$ . For  $n \ge 1$ , let

$$\mathbb{N}_m^n = \{ (x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{N}_m \}.$$

We say that a function  $\Phi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$  is effectively finitely valued or *e.f.v.* if the function  $\overline{\Phi} : \mathbb{N}^{k+n} \to \mathbb{N}$  defined by

$$\overline{\Phi}(x,y) = \chi_{\Phi(x)}(y), \quad x \in \mathbb{N}^k, \, y \in \mathbb{N}^n,$$

is computable (where  $\chi_S : \mathbb{N}^n \to \mathbb{N}$  denotes the characteristic function of  $S \subseteq \mathbb{N}^n$ ) and if there exists a computable function  $\varphi : \mathbb{N}^k \to \mathbb{N}$  such that

$$\Phi(x) \subseteq \mathbb{N}^n_{\varphi(x)}, \quad \forall x \in \mathbb{N}^k.$$

In the following proposition, we state some elementary facts about e.f.v. functions.



(B) A circular chain in (X, d)

PROPOSITION 3. (i) Let  $\Phi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$  and  $\Psi : \mathbb{N}^n \to \mathcal{P}(\mathbb{N}^m)$  be e.f.v. functions. Let  $\Lambda : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^m)$  be defined by:

$$\Lambda(x) = \bigcup_{z \in \Phi(x)} \Psi(z), \quad \forall x \in \mathbb{N}^k$$

Then  $\Lambda$  is an e.f.v. function.

(ii) Let  $\Phi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$  be e.f.v. and let  $T \subseteq \mathbb{N}^n$  be c.e. Then the set

$$S = \{ x \in \mathbb{N}^k \mid \Phi(x) \subseteq T \}$$

is c.e.

(iii) If  $\Phi, \Psi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$  are e.f.v. functions, then the sets

$$\{x \in \mathbb{N}^k \mid \Phi(x) = \Psi(x)\} \text{ and } \{x \in \mathbb{N}^k \mid \Phi(x) \subseteq \Psi(x)\}$$

are computable.

3. Chains and circular chains

DEFINITION 3. Let X be a set. Let  $\mathcal{C} = (C_0, \ldots, C_m)$  be a finite sequence of subsets of X. Then C is said to be a *chain* in X if for all  $i, j \in \{0, ..., m\}$ 

$$|i-j| > 1 \iff C_i \cap C_j = \emptyset.$$

The finite sequence  $\mathcal{C}$  is said to be a *circular chain* in X if for all  $i, j \in$  $\{0,\ldots m\}$ 

$$1 < |i - j| < m \iff C_i \cap C_j = \emptyset$$

If for all  $i, j \in \{0, \ldots m\}$ 

$$|i-j| > 1 \implies C_i \cap C_j = \emptyset,$$

we call  $\mathcal{C}$  a *quasi-chain*.

For a (circular) chain  $\mathcal{C} = (C_0, \ldots, C_m)$  and a natural number  $i \in \{0, \ldots, m\}$ we call  $C_i$  a link of C. If C is a chain,  $C_0$  and  $C_m$  are called *end links*.

If  $\mathcal{A} = (A_0, \ldots, A_n)$  is a finite sequence of bounded nonempty subsets of (X, d), we define

$$\operatorname{mesh}(\mathcal{A}) = \max_{0 \le i \le n} \operatorname{diam}(A_i).$$

Let  $\mathcal{C}$  be a (circular) chain and  $\epsilon > 0$ . If mesh( $\mathcal{C}$ )  $< \epsilon$ , we say that  $\mathcal{C}$  is an  $\epsilon$ -(circular) chain.

A (circular) chain in some metric space (X, d) is said to be an *open (circular) chain* in (X, d) if each of its links is an open set in (X, d). Similarly, a (circular) chain (X, d) is said to be a *compact (circular) chain* in (X, d) if each of its links is a compact set in (X, d).

REMARK 1. Let X be a set. We note the following:

- Let  $C = (C_0, \ldots, C_m)$  be a circular chain. Each link of C is a nonempty set by definition.
- If  $C_0$  is a nonempty set in X, the sequence  $(C_0)$  is both a chain and a circular chain. The same holds for the sequence  $(C_0, C_0)$ .
- Let  $m \geq 2$  and  $\mathcal{C} = (C_0, \ldots, C_m)$  be a chain in X. We claim: if  $i \neq j$ , then  $C_i \neq C_j$ . Namely, if |i-j| > 1, then  $C_i$  and  $C_j$  are disjont. Since they are nonempty, we have  $C_i \neq C_j$ . If |i-j| = 1, we can without loss of generality assume i < j. Then  $i \in \{0, \ldots m-1\}, j = i+1$ . Then we suppose the opposite,  $C_i = C_{i+1}$ . If i < m-1, then the link  $C_{i+2}$  intersects the link  $C_{i+1} = C_i$ , so we have derived a contradiction. If i = m - 1, then the link  $C_{m-2}$  intersects  $C_{m-1} = C_m$ , again a contradiction.
- For nonempty intersecting sets  $C_0$  and  $C_1$  in X, the sequence  $(C_0, C_1)$  is both a chain and a circular chain. It easily follows that collections of two-link chains and two-link circular chains coincide.
- Let  $\mathcal{C} = (C_0, \ldots, C_m)$  be a sequence of sets in X with  $m \geq 2$ . Then  $\mathcal{C}$  can not be both a chain and a circular chain. If  $\mathcal{C}$  is a chain, then  $C_0$  and  $C_m$  are disjoint, so  $\mathcal{C}$  is not a circular chain.
- Let  $C = (C_0, \ldots, C_m)$  be a chain in X. If m < 2, then we say that C is *trivial*. If  $m \ge 2$ , we say that C is *non-trivial*. To summarize: non-trivial chains cannot be circular, and their links are mutually distinct sets.

DEFINITION 4. Suppose X is a set,  $S \subseteq X$  and  $\mathcal{C} = (C_0, \ldots, C_m)$  a finite sequence of subsets of X. We say that  $\mathcal{C}$  covers S if  $S \subseteq C_0 \cup \cdots \cup C_m$ . If  $a, b \in X$ , we say that  $\mathcal{C}$  covers S from a to b if  $\mathcal{C}$  covers S and  $a \in C_0, b \in C_m$ .

Let (K, d) be a continuum. We say that (K, d) is a *(circularly) chainable* continuum if for each  $\epsilon > 0$  there exists an open  $\epsilon$ -(circular) chain  $(C_0, \ldots, C_m)$  in (K, d) which covers K. If  $a, b \in K$ , we say that (K, d) is a continuum chainable from a to b if for each  $\epsilon > 0$  there exists an open  $\epsilon$ -chain  $(C_0, \ldots, C_m)$  in (K, d) which covers K from a to b.

We state a well-known fact from [IP18]:



FIGURE 2. Closed topological sine curve

PROPOSITION 4. [IP18, Proposition 3.3, Proposition 6.11] Let (X, d) be a continuum. Then (X, d) is (circularly) chainable if and only if for each  $\epsilon > 0$  there exists a compact (circular)  $\epsilon$ -chain in (X, d) which covers X.

REMARK 2. Let (X, d) be a continuum. If X is not chainable, then there exists  $\epsilon_0 > 0$  such that no compact  $\epsilon_0$ -chain covers X. Using the Lebesgue number lemma, it can easily be seen that then no open  $\epsilon_0$ -chain covers X. If this is the case, we will say that X is not  $\epsilon_0$ -chainable.

EXAMPLES 2. Here we list some well-known examples (see [Nad92; CV77; Ilj09; Bur59]):

- Closed interval [0, 1] is chainable from 0 to 1, but not circularly chainable.
- An arc is a topological space homeomorphic to [0,1]. If A is an arc and  $f:[0,1] \to A$  a homeomorphism, then a = f(0) and b = f(1) are called its *endpoints*. A is then chainable from a to b.
- The unit circle  $S^1$  in  $\mathbb{R}^2$  is circularly chainable, but not chainable.
- Topological circles (homeomorphic images of  $S^1$ ) are also circularly chainable, but not chainable.
- Closed topological sine curve. Let

$$K = \left\{ (x, \sin\left(\frac{1}{x}\right)) : 0 < x \le 1 \right\} \cup \{(0, y) : y \in [-1, 1] \}.$$

Let a = (0, -1), b = (0, 0),  $c = (1, \sin 1)$ . K is a continuum chainable from a to c. K is not chainable from b to c.

- The Warsaw circle. Let K be the curve from the previous example. We define  $W = K \cup (\{0\} \times [-2, -1]) \cup ([0, 1] \times \{-2\}) \cup (\{1\} \times [-2, \sin 1])$ . W is a circularly chainable continuum.
- Let us consider the *double topological sine curve*:



FIGURE 3. The Warsaw circle



FIGURE 4. Double topological sine curve

$$D = \left\{ (x, \sin\left(\left|\frac{1}{x}\right|\right)) \mid x \in [-1, 1] \setminus \{0\} \right\} \cup \{(0, y) : y \in [-1, 1]\},\$$

and now we add some line segments to construct a closed curve

$$\begin{split} D' &= D \cup (\{-1\} \times [-2, \sin 1]) \cup ([-1, 1] \times \{-2\}) \\ & \cup (\{1\} \times [-2, \sin 1]). \end{split}$$

Then D' is a circularly chainable continuum, but it is not chainable. Notice that if a = (0,0) and  $b = (1, \sin 1)$  there doesn't exist a

subcontinuum of D' chainable from a to b. Furthermore, if subcontinuum L of D' is chainable from a = (0,0) to some  $b \in D'$ , then  $b \in \{0\} \times [-1,1]$ .

• A continuum is called *decomposable* if  $K = K_1 \cup K_2$ , where  $K_1$  and  $K_2$  are proper subcontinua of K. A continuum is called *indecomposable* if it is not decomposable. A decomposable continuum K is said to be 2-*indecomposable* if there exist no subcontinua  $K_1, K_2$  and  $K_3$  of K such that  $K = K_1 \cup K_2 \cup K_3$  and  $K_1 \not\subseteq K_2 \cup K_3$ ,  $K_2 \not\subseteq K_1 \cup K_3$  and  $K_3 \not\subseteq K_1 \cup K_2$ .

A continuum can be both chainable and circularly chainable. Then it is either indecomposable or 2-indecomposable. In this article, we focus on circularly chainable, but not chainable, continua.

REMARK 3. Nadler [Nad92] also uses the following terminology for continua:

- chainable: arc-like,
- circularly chainable: *circle-like*,
- circularly chainable but not chainable: proper circle-like.

DEFINITION 5 (subchain). Let  $C = (C_0, \ldots, C_m)$  be a circular chain. For  $k, l \in \{0, \ldots, m\}$  we define a following sequence. If  $k \leq l$  let  $C_{k,l} = (C_k, \ldots, C_l)$ . If k > l let  $C_{k,l} = (C_k, \ldots, C_m, C_0, \ldots, C_l)$ . If  $l \neq k-1$  and  $(k, l) \neq (0, m)$  we say that  $C_{k,l}$  is a subchain of C, and we write  $C_{k,l} \leq C$ .

- REMARK 4. Note that for l = k 1, sequence  $C_{k,l}$  is a circular chain, and therefore not a chain in general. Same holds for (k, l) = (0, m).
- If  $l \neq k-1$  and  $(k, l) \neq (0, m)$ , then  $\mathcal{C}_{k, l}$  is a chain.
- Also note that for  $l \neq k$  and  $m \geq 3$  we have  $C_{k,l} \neq C_{l,k}$ .
- We can obtain a subchain of C by omitting one of its links. For example, if  $1 \le k \le m-1$ , then by omitting  $C_k$  we get  $(C_{k+1}, C_{k+2}, \ldots, C_{k-1}) = C_{k+1,k-1}$ .
- Next we note that for nonempty subset  $C_0$  of X, sequence  $(C_0, C_0, C_0)$  is a circular chain but not a chain.
- Let  $m \geq 3$  and let  $\mathcal{C} = (C_0, \ldots, C_m)$  be a circular chain in X. If  $i \neq j$  then  $C_i \neq C_j$ . For  $i \in \{0, \ldots, m\}$  we omit the link  $C_i$  to obtain a chain whose all links are pairwise distinct (by Remark 1). From this it follows that all links of  $\mathcal{C}$  are pairwise distinct.
- Let  $C = (C_0, \ldots, C_m)$  be a circular chain in X. If m < 3, we will call such C trivial, and if  $m \ge 3$  we will call it non-trivial.
- By omitting a link of a non-trivial circular chain, we obtain a non-trivial chain.

DEFINITION 6. Suppose X is a set, let  $\mathcal{C} = (C_0, \ldots, C_m)$  and  $\mathcal{D} = (D_0, \ldots, D_n)$  be sequences of sets in X. If for each  $i \in \{0, \ldots, n\}$  there exists



 $j \in \{0, \ldots, m\}$  such that  $D_i \subseteq C_j$  we say that  $\mathcal{D}$  refines  $\mathcal{C}$ . If  $\mathcal{D}$  refines  $\mathcal{C}$  and both  $D_0 \subseteq C_0, D_n \subseteq C_m$ , we say that  $\mathcal{D}$  strictly refines  $\mathcal{C}$ .

DEFINITION 7. Let  $(C_0, \ldots, C_m)$  be a sequence of sets in X. We define the *counter-sequence* of C:

$$-\mathcal{C} = (C_m, C_{m-1}, \dots, C_0)$$

We note:

- If C is a (circular) chain, then -C is also a (circular) chain.
- For a chain  $\mathcal{C}'$  and a circular chain  $\mathcal{C}$  we have: If  $\mathcal{C}' \leq \mathcal{C}$ , then  $-\mathcal{C}' \leq -\mathcal{C}$ .

• If C is sequence of sets, -(-C)) = C.

Let X be a set and let  $\mathcal{C} = (C_0, \ldots, C_m)$  be a sequence of sets in X. Let links $(\mathcal{C}) = \{C_0, \ldots, C_m\}$ , the set of all links of  $\mathcal{C}$ . Obviously, links $(\mathcal{C}) =$ links $(-\mathcal{C})$ .

LEMMA 1. Let C and D be non-trivial chains such that links(C) = links(D). Then C = D or C = -D.

PROOF. Let  $C = (C_0, \ldots, C_m)$ ,  $\mathcal{D} = (D_0, \ldots, D_n)$ . From Remark 1 we conclude that  $|\operatorname{links}(\mathcal{C})| = m + 1$  and  $|\operatorname{links}(\mathcal{D})| = n + 1$ . From this we conclude that m = n. We have  $C_0 = D_i$ , for  $0 \leq i \leq n$ . We observe that the only sets that  $C_0$  intersects are itself and  $C_1$ . Every link  $D_i$  such that 0 < i < n intersects itself,  $D_{i-1}$  and  $D_{i+1}$ . Since  $D_{i-1} \neq D_{i+1}$ , we conclude that i = 0 or i = n.

If i = 0, we observe that  $C_1$  intersects  $C_0 = D_0$  but is itself not equal to it. The only link of  $\mathcal{D}$  that intersects  $D_0$  other than itself is  $D_1$ . Therefore  $C_1 = D_1$ . Similarly we conclude  $C_i = D_i$  for all  $0 \le i \le n$ , so  $\mathcal{C} = \mathcal{D}$ .

If i = n, we observe that  $C_1$  intersects  $C_0 = D_n$  but is not equal to it. The only link of  $\mathcal{D}$  that intersects  $D_n$  other than itself is  $D_{n-1}$ . Therefore  $C_1 = D_{n-1}$ . Similarly we conclude  $C_i = D_{n-i}$  for all  $0 \le i \le n$ , so  $\mathcal{C} = -\mathcal{D}$ . DEFINITION 8. Let X be a set and  $C = (C_0, \ldots, C_m)$  a circular chain in X. For  $0 \le k \le m$  we define

$$C_{+k} = (C_k, C_{k+1}, \dots, C_m, C_0, \dots, C_{k-1}).$$

In addition, we define  $C_{+(m+1)} = C_{+0}$ . Informally, the operation  $C \mapsto C_{+k}$  will be referred to as a *rotation*. The operation  $C \mapsto -C$  will be referred to as a *reflection*. We observe:

- $\mathcal{C}_{+0} = \mathcal{C};$
- for each  $0 \le k \le m$ ,  $\mathcal{C}_{+k}$  is a circular chain and  $\operatorname{links}(\mathcal{C}) = \operatorname{links}(\mathcal{C}_{+k});$
- for a chain  $\mathcal{C}'$ , if  $\mathcal{C}' \leq \mathcal{C}$ , then  $\mathcal{C}' \leq \mathcal{C}_{+k}$ , for all  $0 \leq k \leq m$ ;
- rotating and then reflecting a circular chain can be obtain by first reflecting and then rotating it,  $-C_{+k} = (-C)_{+(m-k+1)}$ .

LEMMA 2. Let  $\mathcal{C}$  and  $\mathcal{D}$  be non-trivial circular chains such that links $(\mathcal{C}) =$  links $(\mathcal{D})$ . Then there exists  $k = 0, \ldots, m$  such that  $\mathcal{C} = \mathcal{D}_{+k}$  or  $\mathcal{C} = -\mathcal{D}_{+k}$ .

PROOF. Let  $C = (C_0, \ldots, C_m)$ ,  $\mathcal{D} = (D_0, \ldots, D_n)$ . Next we define  $C' = (C_0, \ldots, C_{m-1})$ . This sequence is obtained from C by omitting its last link  $C_m$ . From Remark 4 we know that such sequence is a non-trivial chain. Then  $C_m = D_i$ , for some  $i \in \{0, \ldots, n\}$ . We define  $\mathcal{D}'$  by omitting  $D_i$  from  $\mathcal{D}$ , i.e.  $\mathcal{D}' = (D_{i+1}, D_{i+2}, \ldots, D_n, D_0, \ldots, D_{i-1})$ . Since links $(C') = \text{links}(\mathcal{D}')$ , either  $C' = \mathcal{D}'$  or  $C' = -\mathcal{D}'$ .

If we look  $\mathcal{C}' = \mathcal{D}'$  componentwise, we get  $(C_0, \ldots, C_{m-1}) = (D_{i+1}, D_{i+2}, \ldots, D_{i-1})$ . By appending  $C_m = D_i$  to the end of the sequences, we get  $(C_0, \ldots, C_m) = (D_{i+1}, D_{i+2}, \ldots, D_i)$ , i. e.  $\mathcal{C} = \mathcal{D}_{+(i+1)}$ .

If  $\mathcal{C}' = -\mathcal{D}'$  then we have  $(C_0, \ldots, C_{m-1}) = (D_{i-1}, D_{i-2}, \ldots, D_0, D_n, \ldots, D_{i+1})$ . By appending  $C_m = D_i$  to the end of the sequences, we get  $(C_0, \ldots, C_m) = (D_{i-1}, D_{i-1}, \ldots, D_0, D_n, \ldots, D_i)$ , i. e.  $\mathcal{C} = -\mathcal{D}_{+i}$ .

REMARK 5. Let  $C = (C_0, \ldots, C_m)$  be a nontrivial circular chain. Now we can describe the family of all circular chains  $\mathcal{D}$  such that links $(\mathcal{D}) = \text{links}(\mathcal{C})$ . It consists of m + 1 circular chains of the form  $C_{+k}, k \in \{0, \ldots, m\}$  (which are said to have the *same orientation* as C) and m + 1 circular chains of the form  $(-C)_{+k}, k \in \{0, \ldots, m\}$  (which are said to have the *opposite orientation* with regard to C).

PROPOSITION 5. Let (X, d) be a metric space and  $S \subseteq X$  a circularly chainable continuum. Then for each open circular chain  $\mathcal{C}$  that covers S and each  $\epsilon > 0$  there exists a compact  $\epsilon$ -circular chain  $\mathcal{D}$  in S which covers S and refines  $\mathcal{C}$ .

PROOF. Let  $C = (C_0, \ldots, C_m)$  be a circular chain that covers S and let  $\epsilon > 0$ . Then the set  $\{C_0 \cap S, \ldots, C_m \cap S\}$  is an open cover of S. Since S is compact, there exists  $\lambda > 0$ , a Lebesgue number of this cover. This means that for each  $A \subseteq S$  such that diam $(A) < \lambda$  there exists a member of this



FIGURE 6. Refinement

cover which contains A. By Proposition 4, there exists a compact  $\min\{\epsilon, \lambda\}$ circular chain  $\mathcal{D} = (D_0, \ldots, D_n)$  in S which covers S. For each  $i \in \{0, \ldots, n\}$ , since diam  $D_i < \lambda$ , there exists  $j \in \{0, \ldots, m\}$  such that  $D_i \subseteq C_j \cap S \subseteq C_j$ , so  $\mathcal{D}$  refines  $\mathcal{C}$ .  $\Box$ 

Two main results of this section follow, Lemma 3 and Theorem 3.

LEMMA 3. Let X be a set,  $\mathcal{C} = (C_0, \ldots, C_m)$  a circular chain in X and  $k \in \{1, \ldots, m-1\}$ . Let  $\mathcal{D} = (D_0, \ldots, D_n)$  be a compact circular chain in X which refines  $\mathcal{C}$ . Suppose there is a subchain  $\mathcal{D}' \leq \mathcal{D}$  whose one end link,  $D_a$  is contained in  $C_0$ , and other end link is  $D_i$ ,  $\mathcal{D}'$  refines  $(C_0, \ldots, C_{k-1})$ . Furthermore suppose there is a subchain  $\mathcal{D}' \leq \mathcal{D}$  whose one end link,  $D_b$  is contained in  $C_k$ , and other end link is  $D_j$ ,  $\mathcal{D}''$  refines  $(C_1, \ldots, C_k)$ . Let  $D_i$  and  $D_j$  intersect. Then there exists a subchain of  $\mathcal{D}$  which strictly refines  $(C_0, \ldots, C_k)$ .

PROOF. We start by observing that either  $\mathcal{D}' = \mathcal{D}_{a,i}$  or  $\mathcal{D}' = \mathcal{D}_{i,a}$ . Next, we consider the simple **Case I.** where  $\mathcal{D}' = \mathcal{D}_{0,i}$ . Then we consider **Case II.**, with  $\mathcal{D}' = \mathcal{D}_{a,i}$ , and **Case III.**  $\mathcal{D}' = \mathcal{D}_{i,a}$ . We reduce Case II. and Case III. to Case I.

**Case I.** Let  $\mathcal{D}' = \mathcal{D}_{0,i}$ . Subchain  $\mathcal{D}''$  can then be obtained in two ways: a)  $\mathcal{D}'' = \mathcal{D}_{j,b}$  and b)  $\mathcal{D}'' = \mathcal{D}_{b,j}$ .

First, a)  $\mathcal{D}'' = \mathcal{D}_{j,b}$ . We will now examine subcases dependent on the position of index b: 1. j < b < n, 2.  $b \leq j < n$ , 3.  $j \leq b = n$  and 4. b < j = n.



FIGURE 7. Subchain addition

1. If j < b < n, we consider  $\mathcal{D}_{0,b} = D_0, \ldots, D_b$ . In general we can imagine this sequence as either

A) 
$$D_0, ..., D_j, D_i, ..., D_b,$$
  
B)  $D_0, ..., D_i = D_j, ..., D_b$   
or C)  $D_0, ..., D_i, D_j, ..., D_b,$ 

three possibilities depending on the placement of intersecting links  $D_i$  and  $D_j$ . Notice that in such cases each link in  $\mathcal{D}_{0,b}$  is contained in  $\mathcal{D}'$  or in  $\mathcal{D}''$ . Since  $\mathcal{D}'$  refines  $\mathcal{C}_{0,k-1}$  and  $\mathcal{D}''$  refines  $\mathcal{C}_{1,k}$ , it follows that  $\mathcal{D}_{0,b}$  refines  $\mathcal{C}_{0,k}$ . Strict refinement follows from the fact that  $D_0 \subseteq C_0$  and  $D_b \subseteq C_k$ . Since b < n,  $\mathcal{D}_{0,b}$  is a subchain of  $\mathcal{D}$ .

**2.** If  $b \leq j < n$ , then we consider if  $b \leq i$ . If that is the case, then  $\mathcal{D}_{0,b}$  is a subsequence of  $\mathcal{D}_{0,i} = \mathcal{D}'$ .

If b > i, then  $\mathcal{D}_{0,b} = (D_0, \ldots, D_i, D_b = D_j)$ . Each link of  $\mathcal{D}_{0,b}$  is in  $\mathcal{D}'$ except  $D_b = D_j$  which is in  $\mathcal{D}'' = \mathcal{D}_{j,b}$ . Again it holds that each link in  $\mathcal{D}_{0,b}$  is contained in  $\mathcal{D}'$  or in  $\mathcal{D}''$ . We follow the arguments from the above paragraph to prove that  $\mathcal{D}_{0,b}$  is a subchain of  $\mathcal{D}$  that strictly refines  $\mathcal{C}_{0,k}$ .

**3.** If b = n, we observe that  $\mathcal{D}_{0,b}$  is now not a subchain of  $\mathcal{D}$ . We have now  $D_{b=n} \subseteq C_k$  and  $D_0 \subseteq C_0$ . Since  $D_n$  and  $D_0$  intersect, so do  $C_k$  and  $C_0$ . It follows  $k \in \{0, 1, m\}$ , but by the condition of the lemma only k = 1is possible. Since  $\mathcal{D}'$  refines  $(C_0, \ldots, C_{k-1} = C_0)$  it follows that  $D_i \subseteq C_0$ . Similarly,  $\mathcal{D}''$  refines  $(C_1, \ldots, C_k = C_1)$ , so  $D_j \subseteq C_1$ . If  $D_i$  is contained in  $C_1$ , then one-link chain  $(D_i)$  refines both  $C_0$  and  $C_1$  and therefore strictly refines  $(C_0, \ldots, C_{k=1})$ . If  $D_i \notin C_1$ , then  $D_i$  is not a link of  $\mathcal{D}'' = D_j, \ldots, D_b$ , and then i < j holds. If j = n and i = 0, then  $(D_j, D_i)$  is a subchain of  $\mathcal{D}$  which strictly refines  $-(C_0, \ldots, C_k)$ . Otherwise,  $(D_i, D_j)$  is a subchain of  $\mathcal{D}$  which strictly refines  $(C_0, \ldots, C_k)$ . 4. If b < j = n, then  $D_i$  and  $D_{n=j}$  intersect. Then  $i \in \{n-1, n, 0\}$  follows. If i = n-1 then the circular chain  $\mathcal{D}$  is equal to  $(D_0, \ldots, D_b, \ldots, D_{i=n-1}, D_{j=n})$ . Subsequence  $\mathcal{D}_{0,b}$  is a subchain, and links $(\mathcal{D}_{0,b}) \subseteq \text{links}(\mathcal{D}')$ , so again it follows that  $\mathcal{D}_{0,b}$  is the solution. If i = n then  $\mathcal{D}' = \mathcal{D}_{0,i=n} = \mathcal{D}$ . But  $\mathcal{D}$  is a circular chain, which contradicts condition of lemma which states that  $\mathcal{D}'$  is a subchain of  $\mathcal{D}$ . If i = 0 then  $\mathcal{D}'' = (D_{j=n}, D_{i=0}, \ldots, D_b)$ . We can see that  $\mathcal{D}_{0,b}$  is a chain and that  $\text{links}(\mathcal{D}_{0,b}) \subseteq \text{links}(\mathcal{D}'')$ .

b) Now,  $\mathcal{D}'' = \mathcal{D}_{b,j}$ . We consider the following subcases: 1.  $b \leq j < n$ , 2. j < b < n, 3. b = n, 4. b < j = n.

**1.**  $b \leq j < n$ . If  $b \leq i$ , then  $\mathcal{D}_{0,b}$  is a subsequence of  $\mathcal{D}_{0,i} = \mathcal{D}'$ .

If b > i, then  $\mathcal{D}_{0,b} = (D_0, \dots, D_i, D_b = D_j)$ .

In both cases each link in  $\mathcal{D}_{0,b}$  is contained in  $\mathcal{D}'$  or in  $\mathcal{D}''$ . Following the arguments from previous cases, we conclude that  $\mathcal{D}_{0,b}$  is subchain of  $\mathcal{D}$  which strictly refines  $\mathcal{C}_{0,k}$ .

**2.** If j < b < n first we observe that  $b \neq 1$ . If b = 1, then j = 0 follows. But then the sequence  $\mathcal{D}'' = \mathcal{D}_{b,j} = \mathcal{D}_{1,0}$  is not a subchain of  $\mathcal{D}$ . Next, we note that  $\mathcal{D}_{b,0}$  is a subsequence of  $\mathcal{D}_{b,j} = \mathcal{D}''$ . The fact that  $b \neq 1$  implies that  $\mathcal{D}_{b,0}$  is a subchain of  $\mathcal{D}$ . In addition,  $\mathcal{D}_{b,0}$  strictly refines  $-(C_0, \ldots, C_k)$ .

**3.** If b = n, we again conclude that the two-link chain  $(D_n, D_0)$  strictly refines  $-\mathcal{C}_{0,k}$ .

**4.** If b < j = n, we consider the sequence  $\mathcal{D}_{b,0} = (D_b, \dots D_{j=n}, D_0)$ . The links  $D_b, \dots D_{j=n}$  are contained in  $\mathcal{D}''$ , and the link  $D_0$  is contained in  $\mathcal{D}'$ . So, if  $\mathcal{D}_{b,0}$  is a chain, then it is the desired subchain of  $\mathcal{D}$ . If  $\mathcal{D}_{b,0}$  is not a chain, then b = 1. Now we have the two-link chain  $(D_0, D_{b=1})$  which strictly refines  $\mathcal{C}_{0,k}$ .

**Case II.** If  $\mathcal{D}' = \mathcal{D}_{a,i}$  then we consider the circular chain  $\mathcal{D}_{+a} = (D_a, D_{a+1}, \dots, D_{a-1}).$ 

- Subchains of  $\mathcal{D}$  and  $\mathcal{D}_{+a}$  coincide. Therefore  $\mathcal{D}'$  and  $\mathcal{D}''$  are subchains of  $\mathcal{D}_{+a}$ .
- Furthermore,  $\mathcal{D}' = (\mathcal{D}_{+a})_{0,i'}$ . We have in fact changed the enumeration of circular chain  $\mathcal{D}$  so that starting link of subchain  $\mathcal{D}'$  has index 0.
- We apply the method from **Case I.** to obtain subchain  $\mathcal{E}$  of  $\mathcal{D}_{+a}$  which strictly refines  $\mathcal{C}_{0,k}$  or  $-\mathcal{C}_{0,k}$ .  $\mathcal{E}$  is the desired solution, since it is a subchain of  $\mathcal{D}$ .

**Case III.** If  $\mathcal{D}' = \mathcal{D}_{i,a}$  first we rotate:  $\mathcal{D}_{+(a+1)} = (D_{a+1}, D_{a+2}, \dots, D_n, D_0, \dots, D_a)$ . Next we reflect:  $-\mathcal{D}_{+(a+1)} = (D_a, D_{a-1}, \dots, D_0, D_n, \dots, D_{a+1})$ . We note:

• If  $\mathcal{E} \leq \mathcal{D}$ , then  $-\mathcal{E} \leq -\mathcal{D}_{+(a+1)}$ . This entails that both  $-\mathcal{D}'$  and  $-\mathcal{D}''$  are subchains of  $-\mathcal{D}_{+(a+1)}$ .



FIGURE 8. Strict refinement

- Moreover,  $-\mathcal{D}' = (-\mathcal{D}_{+(a+1)})_{0,i'}$ , i.e. we have rotated and reflected circular chain  $\mathcal{D}$ , so that  $\mathcal{D}'$  again starts with link 0.
- Therefore, by applying method from Case I., we obtain a subchain *E* of −*D*<sub>+(a+1)</sub> such that it strictly refines *C*<sub>0,k</sub> or −*C*<sub>0,k</sub>.
- If  $\mathcal{E}$  strictly refines  $\mathcal{C}_{0,k}$ , then  $-\mathcal{E}$  strictly refines  $-\mathcal{C}_{0,k}$ . Therefore  $-\mathcal{E}$  is a subchain of  $\mathcal{D}$  that strictly refines  $\mathcal{C}_{0,k}$  or  $-\mathcal{C}_{0,k}$ .

REMARK 6. Notice that the previous lemma holds even without the compactness of  $\mathcal{D}$ .

THEOREM 3. Let (X, d) be a metric space,  $S \subseteq X$  a circularly chainable, but not chainable continuum. Let  $\epsilon_0 > 0$  be such that no compact  $\epsilon_0$ -chain covers S. Let  $\mathcal{C} = (C_0, \ldots, C_m)$  be an open circular  $\epsilon_0$ -chain that covers S. For each compact circular chain  $\mathcal{D} = (D_0, \ldots, D_n)$  which covers S and  $\mathcal{D} \leq \mathcal{C}$ and for each  $k \in \{2, \ldots, m-1\}$ , there exists a subchain  $\mathcal{D}'$  of  $\mathcal{D}$  such that  $\mathcal{D}'$ strictly refines  $(C_0, \ldots, C_k)$  or  $-(C_0, \ldots, C_k)$ .

PROOF. First we fix arbitrary  $\mathcal{D}$  which covers S and refines  $\mathcal{C}$ , then we fix arbitrary  $k \in \{2, \ldots, m-1\}$ . Let us suppose the following:

(\*) no subchain of  $\mathcal{D}$  strictly refines  $(C_0, \ldots, C_k)$  or  $-(C_0, \ldots, C_k)$ ,

and derive a contradiction.

Let us consider the set A of all  $i \in \{0, \ldots, n\}$  that are connected to  $C_0$ , i.e. such that there exists a subchain  $\mathcal{D}'$  of  $\mathcal{D}$  in which one end link is  $D_i$ , and the other is contained in  $C_0$  and  $\mathcal{D}'$  refines  $(C_0, \ldots, C_{k-1})$ . Similarly, let B be a set of all  $i \in \{0, \ldots, n\}$  that are connected to  $C_k$ , i.e. such that there exists a subchain  $\mathcal{D}''$  of  $\mathcal{D}$  in which one end link is  $D_i$ , and the other is contained in  $C_k$  and  $\mathcal{D}''$  refines  $(C_1, \ldots, C_k)$ . Now let

$$C'_{1} = \bigcup_{\substack{i \in B \\ D_{i} \subseteq C_{1}}} D_{i}$$

$$\vdots$$

$$C'_{k-1} = \bigcup_{\substack{i \in B \\ D_{i} \subseteq C_{k-1}}} D_{i}$$

$$C'_{k} = \bigcup_{\substack{D_{i} \subseteq C_{k-1}}} D_{i}$$

$$\vdots$$

$$C'_{m} = \bigcup_{\substack{D_{i} \subseteq C_{m}}} D_{i}$$

$$C'_{0} = \bigcup_{\substack{D_{i} \subseteq C_{m}}} D_{i}$$

$$C''_{1} = \bigcup_{\substack{i \in A \\ D_{i} \subseteq C_{1}}} D_{i}$$

$$\vdots$$

$$C''_{k-1} = \bigcup_{\substack{D_{i} \subseteq C_{k-1}}} D_{i}$$

Let us consider the sequence of sets  $\mathcal{C}' = (C'_1, \ldots, C'_k, \ldots, C'_m, C'_0, C''_1, \ldots, C''_{k-1})$ . After noting that some of this sets might be empty, we claim that the sequence a) covers S, b) is a quasi-chain.

a) We will prove that  $\mathcal{D}$  refines  $\mathcal{C}'$ . Then, since  $\mathcal{D}$  covers S, so does  $\mathcal{C}'$ . Let us choose arbitrary  $D_i$ , for  $i \in \{0, \ldots, m\}$ . Since  $\mathcal{D}$  refines  $\mathcal{C}$ ,  $D_i \subseteq C_0 \cup \cdots \cup C_m$ . If  $D_i$  is contained in one of the sets  $C_k, \ldots, C_m, C_0$ , then by definition  $D_i$  is contained in one of the sets  $C'_k, \ldots, C'_m, C'_0$ .

If  $D_i$  isn't contained in one of the sets  $C_k, \ldots, C_m, C_0$ , then it must be contained in one of the sets  $C_1, \ldots, C_{k-1}$ . In this case it will follow that  $i \in A$  or  $i \in B$ . Note that there exists a link of  $\mathcal{D}$  which isn't contained in  $C_1 \cup \cdots \cup C_{k-1}$ . Otherwise  $\mathcal{D}$  refines  $(C_1, \ldots, C_{k-1})$ . Since  $\mathcal{D}$  covers S, so does  $(C_1, \ldots, C_{k-1})$ . But then S is covered by an open  $\epsilon_0$ -chain. Using the Lebesgue number lemma, we can easily construct a compact  $\epsilon_0$ -chain that covers S. This is in contradiction with assumption that S can not be covered by a compact  $\epsilon_0$ -chain. Notice that if  $D_j$  is contained in one of the sets  $C_1, \ldots, C_{k-1}$ , then  $D_{j+1}$  is either also contained in one of the



FIGURE 9. Links of  $\mathcal{C}'$ 

sets  $C_1, \ldots, C_{k-1}$ , or  $D_{j+1} \subseteq C_0$  or  $D_{j+1} \subseteq C_k$ . Therefore there exists  $l \in \{0, \ldots, m\}$  such that  $(D_i, \ldots, D_{l-1})$  refines  $(C_1, \ldots, C_{k-1})$  and  $D_l \subseteq C_0$  or  $D_l \subseteq C_k$ .

- If  $D_l \subseteq C_0$ , then  $(D_i, \ldots, D_{l-1}, D_l)$  refines  $(C_0, \ldots, C_{k-1})$ , one end link is  $D_i$ , and the other end link,  $D_l$ , is contained in  $C_0$ . Therefore  $i \in A$ . It follows that  $D_i$  is contained in one of the sets  $C''_1, \ldots, C''_{k-1}$ .
- If  $D_l \subseteq C_k$ , then  $(D_i, \ldots, D_{l-1}, D_l)$  refines  $(C_1, \ldots, C_k)$ , one end link is  $D_i$ , and the other end link,  $D_l$ , is contained in  $C_k$ . Therefore  $i \in B$ . It follows that  $D_i$  is contained in one of the sets  $C'_1, \ldots, C'_{k-1}$ .

b) Now let us see that  $(C'_1, \ldots, C'_k, \ldots, C'_m, C'_0, C''_1, \ldots, C''_{k-1})$  is a quasichain. Let  $\mathcal{B} = \{C'_1, \ldots, C'_{k-1}\}, \mathcal{E} = \{C'_k, \ldots, C'_m, C'_0\}, \mathcal{A} = \{C''_1, \ldots, C''_{k-1}\}.$ First we choose two non-neighbouring links from same family, for instance  $C'_i, C'_j \in \mathcal{B}$ . Since by their definition  $C'_i \subseteq C_i, C'_j \subseteq C_j$ , and  $C_i$  and  $C_j$  being disjoint,  $C'_i$  and  $C'_j$  must also be disjoint. Similarly, one concludes that two non-neighbouring links are disjoint if they are both in  $\mathcal{E}$  or both in  $\mathcal{A}$ .

#### D. TARANDEK

Now let us observe two non-neighbouring links where one is in  $\mathcal{B}$  and the other in  $\mathcal{E}$ . We conclude that the only nontrivial possibility are sets  $C'_1$  and  $C'_0$ . Now, let us suppose that they aren't disjoint i.e. there exists  $x \in C'_1 \cap C'_0$ . That means that there exist  $i \in B$  and  $j \in \{0, \ldots, m\}$  such that  $x \in D_i \subseteq C_1$  and  $x \in D_j \subseteq C_0$ . So, to summarize, one-link chain  $(D_j)$  refines  $\mathcal{C}_{0,k-1}$ , there exists a subchain of  $\mathcal{D}$  which refines  $\mathcal{C}_{1,k}$ , whose one end link is in  $C_k$ , and the other end link  $D_i$  intersects  $D_j$ . By applying Lemma 3, it follows that there is a subchain of  $\mathcal{D}$  which strictly refines  $(C_0, \ldots, C_k)$  or  $-(C_0, \ldots, C_k)$ , so we have derived a contradiction with (\*). So  $C'_1$  and  $C'_0$  are disjoint.

If we observe two non-neighbouring links where one is in  $\mathcal{E}$  and the other in  $\mathcal{A}$ , we can see that only nontrivial possibilities are  $C'_k$  and  $C''_{k-1}$ . We proceed as in the previous paragraph.

If we take two non-neighbouring links where one is in  $\mathcal{B}$  and the other in  $\mathcal{A}$ , we can see that only nontrivial possibilities are  $C'_u$ ,  $C''_v$  such that  $C_u$ and  $C_v$  intersect. Let us suppose that  $C'_u$  and  $C''_v$  intersect, so there exist  $i \in B, j \in A$  and  $x \in X$  such that  $x \in D_i \subseteq C_u$  and  $x \in D_j \subseteq C_v$ . So there is a subchain of  $\mathcal{D}$  which refines  $\mathcal{C}_{0,k-1}$ , whose one end link is in  $C_0$  and the other is  $D_j$ , there also exists a subchain of  $\mathcal{D}$  which refines  $\mathcal{C}_{1,k}$ , whose one end link is in  $C_k$ , and the other end link  $D_i$  intersects  $D_j$ . By applying Lemma 3, it follows again that there is a subchain of  $\mathcal{D}$  which strictly refines  $(C_0, \ldots, C_k)$  or  $-(C_0, \ldots, C_k)$ , so we have derived a contradiction. So  $C'_u$  and  $C''_v$  are disjoint.

So by now we have proven that  $\mathcal{C}'$  is a quasi-chain which covers S. By construction,  $\mathcal{C}'$  refines  $\mathcal{C}$ , so  $\mathcal{C}'$  is an  $\epsilon_0$ -quasi-chain. By removing empty links, we construct a compact  $\epsilon_0$ -chain which covers S. But this is in contradiction with our assumption that no compact  $\epsilon_0$ -chain covers S. Therefore we must reject (\*), so there exists subchain  $\mathcal{D}'$  of  $\mathcal{D}$  such that  $\mathcal{D}'$  strictly refines  $\mathcal{C}_{0,k}$  or  $-\mathcal{C}_{0,k}$ .

### 4. Formal circular chains

Let  $(X, d, \alpha)$  be a computable metric space and let  $a \in X$  be a computable point in this space. Using Proposition 2 it is easy to conclude that the set  $\{i \in \mathbb{N} \mid a \in I_i\}$  is c.e. From this it follows that the set  $\{j \in \mathbb{N} \mid a \in J_j\}$  is also c.e. Consequently we have the following proposition:

PROPOSITION 6. Let  $(X, d, \alpha)$  be a computable metric space and let  $a \in X$  be a computable point in this space. Then

$$\{l \in \mathbb{N} \mid a \in J_{(l)_0}\}$$

is a c.e. set.

DEFINITION 9. Let  $(X, d, \alpha)$  be a computable metric space and let  $v, w \in \mathbb{N}$ . We say that  $I_v$  is *formally contained* in  $I_w$  and we write  $I_v \subseteq_F I_w$  if

$$d(\lambda_v, \lambda_w) + \rho_v < \rho_w.$$

Let  $a, b \in \mathbb{N}$ . We say that  $J_a$  is formally contained in  $J_b$  and we write  $J_a \subseteq_F J_b$  if

for each  $u \in [a]$  there exists  $v \in [b]$  such that  $I_u \subseteq_F I_v$ .

REMARK 7. Note that  $I_u \subseteq_F I_v$  denotes a relation between the indeces u and v, not between the sets  $I_u$  and  $I_v$ . The same holds for  $J_a \subseteq_F J_b$ . If  $I_u \subseteq_F I_v$ , then  $\hat{I}_u \subseteq I_v$ . Therefore  $J_a \subseteq_F J_b$  implies  $\hat{J}_a \subseteq J_b$ .

**PROPOSITION 7.** Let  $(X, d, \alpha)$  be a computable metric space. Then

$$A = \{ (u, w) \in \mathbb{N}^2 \mid I_u \subseteq I_v \}$$

is a c.e. set.

Proof.

$$(u,v) \in A \iff d(\lambda_u,\lambda_v) + \rho_u < \rho_v$$

and now we apply Proposition 2.

DEFINITION 10. Let  $(X, d, \alpha)$  be a computable metric space and let  $i, j \in \mathbb{N}$ . We say that  $I_i$  and  $I_j$  are formally disjoint, and we write  $I_i \diamond I_j$ , if

$$d(\lambda_i, \lambda_j) > \rho_i + \rho_j.$$

Let  $a, b \in \mathbb{N}$ . We say that  $J_a$  and  $J_b$  are formally disjoint, and we write  $J_a \diamond J_b$ , if

 $I_i \diamond I_j$ , for every  $i \in [a]$  and  $j \in [b]$ .

REMARK 8. Formal disjointness is not a relation between the sets  $I_i$  and  $I_j$ , but between the indeces *i* and *j*. The same holds for  $J_a \diamond J_b$ .

If  $I_i \diamond I_j$ , then  $\hat{I}_i \cap \hat{I}_j = \emptyset$ . Consequently,  $J_a \diamond J_b$  implies  $\hat{J}_a \cap \hat{J}_b = \emptyset$ .

The following proposition easily follows from Remark 7, Remark 8 and the fact that in a computable metric space  $\operatorname{Cl}(J_j) \subseteq \hat{J}_j$  holds.

PROPOSITION 8. Let  $(X, d, \alpha)$  be a computable metric space and let  $a, b \in \mathbb{N}$ . Then the following implications hold:

$$J_a \subseteq_F J_b \implies \operatorname{Cl}(J_a) \subseteq J_b,$$
  
$$J_a \diamond J_b \implies \operatorname{Cl}(J_a) \cap \operatorname{Cl}(J_b) = \emptyset.$$

DEFINITION 11. Let  $(X, d, \alpha)$  be a computable metric space. Let  $l \in \mathbb{N}$ . We define

$$\mathcal{H}_l = (J_{(l)_0}, \ldots, J_{(l)_{\overline{l}}}).$$

We say that *l* represents a formal circular chain or, shortly, that  $\mathcal{H}_l$  is a formal circular chain if for all  $i, j \in \{0, \ldots, \overline{l}\}$ 

We define

$$\bigcup \mathcal{H}_l = \bigcup_{k \in [l]} J_k.$$

 $1 < |i - j| < \bar{l} \implies J_{(l)_i} \diamond J_{(l)_j}.$ 

Using propositions 1, 2 and 3, it is not hard to prove the following proposition. For details, see [IIj09, Proposition 8] and [IIj13, Proposition 5.4.].

PROPOSITION 9. Let  $(X, d, \alpha)$  be a computable metric space. The following sets are c.e.:

$$\{(i,j) \in \mathbb{N}^2 \mid I_i \diamond I_j\}, \\ \{(a,b) \in \mathbb{N}^2 \mid J_a \diamond J_b\}, \\ \{(a,b) \in \mathbb{N}^2 \mid J_a \subseteq_F J_b\}, \\ \{l \in \mathbb{N} \mid \mathcal{H}_l \text{ is a formal circular chain}\}.$$

By [Ilj13, Proposition 5.3.] we have:

PROPOSITION 10. Let K be a semicomputable set in a computable metric space  $(X, d, \alpha)$ . The set

$$\{l \in \mathbb{N} \mid K \subseteq \bigcup \mathcal{H}_l\}$$

is c.e.

REMARK 9. Let  $\epsilon_0 > 0$  and let K be a circularly chainable, but not  $\epsilon_0$ -chainable, continuum in a computable metric space  $(X, d, \alpha)$ . Let  $l \in \mathbb{N}$  be such that  $\mathcal{H}_l$  is a formal circular chain which covers K and such that fmesh $(l) < \epsilon_0$ . Then  $(J_{(l_y)_0} \cap K, \ldots, J_{(l_y)_{\overline{l_y}}} \cap K)$  is a circular chain in K. Namely, non-neighbouring links do not intersect because  $\mathcal{H}_l$  is a formal circular chain. Neighbouring links intersect because otherwise there would exist an  $\epsilon_0$  chain in K covering K, which contradicts the fact that K is not  $\epsilon_0$ -chainable.

DEFINITION 12. Let  $(X, d, \alpha)$  be a computable metric space. Let  $l, n \in \mathbb{N}$ . We say that  $\mathcal{H}_l$  formally refines  $\mathcal{H}_n$  and we write

$$\mathcal{H}_l \leq \mathcal{H}_n$$

if for all  $i \in [l]$  there exists  $j \in [n]$  such that  $J_i \subseteq_F J_j$ .

Similarly, we say that  $(J_{a_0}, \ldots, J_{a_k})$  formally refines  $(J_{b_0}, \ldots, J_{b_l})$  if for each  $i \in \{0, \ldots, k\}$  there exists  $j \in \{0, \ldots, l\}$  such that  $J_{a_i} \subseteq_F J_{b_j}$ .

DEFINITION 13. Let  $(X, d, \alpha)$  be a computable metric space. Let  $l, l', k, k' \in \mathbb{N}$ . We write

$$\mathcal{H}_{l'}^{k'} \leq \mathcal{H}_{l}^{k}$$

if the following holds:

- $\mathcal{H}_{l'} \leq \mathcal{H}_l$ ,
- $(J_{(l')_0}, \ldots, J_{(l')_{k'}})$  formally refines  $(J_{(l)_0}, \ldots, J_{(l)_k})$ ,
- $J_{(l')_0} \subseteq_F J_{(l)_0}, J_{(l')_{k'}} \subseteq_F J_{(l)_k}.$

DEFINITION 14. Let  $(X, d, \alpha)$  be a computable metric space and let  $A \subseteq X, j \in \mathbb{N}$  and r > 0. We are going to write

 $A \subseteq_r J_j$ 

if the following holds:

$$A \subseteq J_j,$$
  

$$I_i \cap A \neq \emptyset, \text{ for each } i \in [j],$$
  

$$\rho_i < r, \text{ for each } i \in [j].$$

DEFINITION 15. Let  $(X, d, \alpha)$  be a computable metric space, let  $A, B \subseteq X$ and r > 0. We say that the number r is an (A, B)- separator if for all  $i, j \in \mathbb{N}$ the following implication holds:

$$(A \subseteq_r J_i \text{ and } B \subseteq_r J_j) \implies J_i \diamond J_j$$

DEFINITION 16. Let  $(X, d, \alpha)$  be a computable metric space, let K be a compact set in (X, d), suppose r > 0 and let  $a \in \mathbb{N}$ . We say that the number r is an (K, a)- augmentator if for each  $j \in \mathbb{N}$  the following implication holds:

$$K \subseteq_r J_j \implies J_j \subseteq_F J_a.$$

REMARK 10. Note if A, B are compact sets in (X, d), r is an (A, B)separator and  $r' \in \langle 0, r \rangle$ , then r' is also an (A, B)separator. Similarly, if r is
a (K, a)-augmentator and  $r' \in \langle 0, r \rangle$ , then r' is also an (K, a)augmentator.

Here we state four useful results: Lemma 4, Proposition 11, Lemma 5 and Proposition 12. The proofs can be found in [IP18, Lemma 4.8. - Proposition 4.13.].

LEMMA 4. Let  $(X, d, \alpha)$  be a computable metric space, let K be a nonempty compact set in (X, d) and let r > 0. Then there exists  $l \in \mathbb{N}$  such that  $K \subseteq_r J_l$ .

PROPOSITION 11. Let  $(X, d, \alpha)$  be a computable metric space and let A, B be disjoint nonempty compact sets in (X, d). Then there exists r > 0 such that r is an (A, B)-separator.

LEMMA 5. Let  $(X, d, \alpha)$  be a computable metric space, let  $A \subseteq X, j \in \mathbb{N}$ and r > 0 such that  $A \subseteq_r J_j$ . Then

$$\operatorname{fdiam}(j) < \operatorname{diam}(A) + 4r.$$

PROPOSITION 12. Let  $(X, d, \alpha)$  be a computable metric space, let  $a \in \mathbb{N}$ and let K be a compact set in (X, d) such that  $K \subseteq J_a$ . Then there exists r > 0 such that r is an (K, a)-augmentator. LEMMA 6. Let  $(X, d, \alpha)$  be a computable metric space. Let  $\mathcal{K}$  be a nonempty finite collection of nonempty compact sets in (X, d). Let A be a finite subset of  $\mathbb{N}$  and  $\epsilon > 0$ . Then for every  $K \in \mathcal{K}$  there exists  $i_K \in \mathbb{N}$ such that for all  $K, L \in \mathcal{K}, a \in A$  the following holds:

1.  $K \subseteq J_{i_K}$ ; 2.  $K \cap L = \emptyset \implies J_{i_K} \diamond J_{i_L}$ ; 3.  $K \subseteq J_a \implies J_{i_K} \subseteq_F J_a$ ; 4.  $\operatorname{fdiam}(i_K) < \operatorname{diam} K + \epsilon$ .

PROOF. We define  $\Delta = \{(K, K') \mid K, K' \in \mathcal{K}, K \cap K' = \emptyset\}$ . By Proposition 11 for every  $(K, K') \in \Delta$  there exists  $\mu_{K,K'} > 0$  which is a (K, K') – separator. If  $\Delta = \emptyset$  we define  $\mu_S = 1$ , otherwise  $\mu_S = \min\{\mu_{K,K'} \mid (K, K') \in \Delta\}$  (existence of the minimum follows from the finiteness of  $\mathcal{K}$ ).

Next, we define  $\Gamma = \{(K, a) \in \mathcal{K} \times A \mid K \subseteq J_a\}$ . By Proposition 12 for each  $(K, a) \in \Gamma$  there exists  $\mu_{K,a}$  which is an (K, a)-augmentator. If  $\Gamma = \emptyset$  we define  $\mu_A = 1$ , otherwise  $\mu_A = \min\{\mu_{K,a} \mid (K, a) \in \Gamma\}$ . Finiteness of  $\mathcal{K}$  and A implies the finiteness of  $\Gamma$ , which ensures the existence of the minimum.

Now we set  $\mu = \min\{\mu_S, \mu_A, \epsilon/8\}.$ 

- 1. By Lemma 4 for each  $K \in \mathcal{K}$  there exists  $i_K \in \mathbb{N}$  such that  $K \subseteq_{\mu} J_{i_K}$ .
- 2. Let  $K, L \in \mathcal{K}$  be disjoint. Then  $(K, L) \in \Delta$ , so  $\mu$  is a (K, L)-separator. Since  $K \subseteq_{\mu} J_{i_K}$  and  $L \subseteq_{\mu} J_{i_L}$ , we have that  $J_{i_K} \diamond J_{i_L}$  follows.
- 3. Let  $K \in \mathcal{K}, a \in A$  be such that  $K \subseteq J_a$ . Then  $(K, a) \in \Gamma$  and  $\mu$  is an (K, a)-augmentator. Now  $K \subseteq_{\mu} J_{i_K}$  implies  $J_{i_K} \subseteq_F J_a$ .
- 4. We know from Lemma 5 and from  $K \subseteq_{\mu} J_{i_K}$  that  $fdiam(i_K) \leq diam K + 4\mu$ . Then  $fdiam(i_K) \leq diam K + 4 \cdot \epsilon/8 < diam K + \epsilon$ .

LEMMA 7. Let  $(X, d, \alpha)$  be a computable metric space. Suppose S is a circularly chainable continuum with  $a \in S$ . Let  $n \in \mathbb{N}$  such that  $S \subseteq \bigcup \mathcal{H}_n$ . Let  $\epsilon > 0$ . Then there exists  $l \in \mathbb{N}$  such that:

- 1.  $S \subseteq \mathcal{H}_l;$
- 2.  $\mathcal{H}_l$  is a formal circular chain;
- 3. fmesh(l) <  $\epsilon$ ;
- 4.  $\mathcal{H}_l \leq \mathcal{H}_n;$
- 5.  $a \in J_{(l)_0}$ .

PROOF. The set links $(\mathcal{H}_n) = \{J_{(n)_0}, \ldots, J_{(n)_{\overline{n}}}\}\$  is an open cover of S in (X, d). Therefore,  $\{J_{(n)_0} \cap S, \ldots, J_{(n)_{\overline{n}}} \cap S\}\$  is an open cover of S in  $(S, d|_{S \times S})$ . There exists  $\lambda > 0$  which is the Lebesgue number of this cover, i.e. for every A subset of S such that diam  $A < \lambda$  there exists  $i \in \{0, \ldots, \overline{n}\}\$  such that  $A \subseteq J_{(n)_i}$ .

Let  $\mathcal{K} = (K_0, \ldots, K_m)$  be a compact  $\min\{\epsilon/2, \lambda\}$ -circular chain in S which covers S. Such circular chain exists because of Proposition 4. If  $a \in K_0$ , then

we proceed. If  $a \in K_i$  for some i > 0, now we rotate  $\mathcal{K}$  and notice that now a is contained in the zeroth link of  $\mathcal{K}_{+i}$  Without loss of generality we proceed with denoting  $\mathcal{K}_{+i}$  by  $\mathcal{K}$ .

Since diam  $K_i < \lambda$ , for each  $i \in \{0, \ldots, m\}$  there exists  $w_i \in \{0, \ldots, \overline{n}\}$ such that  $K_i \subseteq J_{(n)_{w_i}}$ . Now we can apply Lemma 6 with  $\mathcal{K} = \{K_0, \ldots, K_m\}$ , A = [n] and the parameter  $\epsilon/2$ . For each  $K_j \in \mathcal{K}$  we denote the corresponding  $i_{K_j}$  by  $k_j$ . There exists  $l \in \mathbb{N}$  such that  $((l)_0, \ldots, (l)_{\overline{l}}) = (k_0, \ldots, k_m)$ . Now,  $\mathcal{H}_l = (J_{k_0}, \ldots, J_{k_m})$ .

- 1. For each  $i \in \{0, ..., m\}$  we have  $K_i \subseteq J_{k_i}$ . Since  $\mathcal{K}$  covers S, so does  $\mathcal{H}_l$ .
- 2. Let  $u, v \in \{0, \ldots, m\}$  such that 1 < |u v| < m. Since  $\mathcal{K}$  is a circular chain,  $K_u$  and  $K_v$  are disjoint. Then by Lemma 6  $J_{k_u} \diamond J_{k_v}$ , therefore  $\mathcal{H}_l$  is a formal circular chain.
- 3. By Lemma 6, for every  $i \in \{0, ..., m\}$ , fdiam $(k_i) < \text{diam} K_i + \epsilon/2 < \epsilon$ . Therefore fmesh $(l) < \epsilon$ .
- 4. Let us fix an arbitrary  $i \in \{0, \ldots, m\}$ . Since  $K_i \subseteq J_{(n)w_i}$ , then again by Lemma 6  $J_{k_i} \subseteq J_{(n)w_i}$ , i.e.  $\mathcal{H}_l \leq \mathcal{H}_n$ .
- 5. By rotation of  $\mathcal{K}, a \in K_0 \subseteq J_{(l)_0}$ .

		1

THEOREM 4. Let  $(X, d, \alpha)$  be a computable metric space. Let K be a circularly chainable continuum in (X, d) and let  $\epsilon_0 > 0$  be such that no compact  $\epsilon_0$ -chain covers K. Let  $a, b \in K$  such that  $a \neq b$ . Then there exists  $l \in \mathbb{N}$  such that  $\mathcal{H}_l$  is a formal circular chain which covers K, fmesh $(l) < \min\{1, \epsilon_0\}$  and  $a \in J_{(l)_0}, b \in J_{(l)_k}$ , where  $k \in \{2, \ldots, \overline{l} - 1\}$ .

PROOF. Let r = d(a, b). We fix  $j \in \mathbb{N}$  such that  $K \subseteq J_j$ . Then there exists  $n \in \mathbb{N}$  such that  $((n)_0, \ldots, (n)_{\overline{n}}) = (j)$ . Thus  $K \subseteq \bigcup \mathcal{H}_n$  follows.

We apply Lemma 7 with  $\epsilon = \min\{1, \epsilon_0, r/4\}$ : there exists  $l \in \mathbb{N}$  such that  $\mathcal{H}_l$  is a formal circular chain which covers K with  $a \in J_{(l)_0}$  and fmesh $(l) < \epsilon$ .

Since  $\mathcal{H}_l$  covers K, we have  $b \in J_{(l)_k}$  for some  $k \in \{0, \ldots, \bar{l}\}$ . But we claim that  $\mathcal{H}_l$  is a circular chain as well. Namely, if there exist neighbouring links which are disjoint, then there exists an  $\epsilon_0$ -chain which covers K, but K is not  $\epsilon_0$ -chainable. From this it follows  $|i - j| \leq 1$  or  $|i - j| = \bar{l} \implies J_{(l)_i} \cup J_{(l)_j} \neq \emptyset$ . For this reason  $J_{(l)_0}$  intersects the sets  $J_{(l)_0}, J_{(l)_1}$  and  $J_{(l)_{\bar{l}}}$ . Let us now assume  $k \in \{0, 1, \bar{l}\}$ . Now  $J_{(l)_0}$  and  $J_{(l)_k}$  intersect and therefore  $d(a, b) \leq \operatorname{diam}(J_{(l)_0} \cup J_{(l)_k}) \leq \operatorname{diam} J_{(l)_0} + \operatorname{diam} J_{(l)_k} < 2\epsilon \leq \frac{r}{2}$ , which is a contradiction with d(a, b) = r. Therefore  $k \in \{2, \ldots, \bar{l} - 1\}$ .

Let  $(X, d, \alpha)$  be a computable metric space. We fix K, a semicomputable circularly chainable, but not chainable, continuum in (X, d). Let  $\epsilon_0 > 0$  be

such that no compact  $\epsilon_0$ -chain covers K. Without loss of generality we can assume that  $\epsilon_0 \in \mathbb{Q}$ .

We define the sets  $\Gamma$  and  $\Omega$ :

$$\Gamma = \{(l, k, l', k') \in \mathbb{N}^4 \mid \mathcal{H}_l, \mathcal{H}_{l'} \text{ formal circular chains which cover } K, \\ \mathcal{H}_{l'}^{k'} \leq \mathcal{H}_l^k, \operatorname{fmesh}(l') < \frac{1}{2} \operatorname{fmesh}(l), k \leq \overline{l}, k' \leq \overline{l'}\}; \\ \Omega = \{(l, k) \in \mathbb{N}^2 \mid \mathcal{H}_l \text{ formal circular chain, fmesh}(l) < \epsilon_0, \end{cases}$$

$$K \subseteq \bigcup \mathcal{H}_l, 2 \le k \le \overline{l} - 1 \}.$$

PROPOSITION 13. The sets  $\Gamma$  and  $\Omega$  are c.e.

PROOF. From Proposition 10 and Proposition 1 we know that  $\{(l,k) \in \mathbb{N}^2 \mid K \subseteq \bigcup \mathcal{H}_l\}$  is c.e. Similarly, using Propositions 9 and 1 we know that  $\{(l,k) \in \mathbb{N}^2 \mid \mathcal{H}_l \text{ a formal circular chain}\}$  is c.e. The set  $\{(l,k) \in \mathbb{N}^2 \mid \text{fmesh}(l) < \epsilon_0\}$  is c.e. due to Proposition 2. The fact that  $\{(l,k) \in \mathbb{N}^2 \mid 2 \leq k \leq \overline{l} - 1\}$  is computable and therefore c.e. is obvious. The set  $\Omega$  is an intersection of c.e. sets and therefore c.e.

We similarly conclude for  $\Gamma$ . The only challenging part is to prove that  $\{(l,k,l',k') \in \mathbb{N}^4 \mid \mathcal{H}_{l'}^{k'} \leq \mathcal{H}_l^k\}$  is c.e. We first prove that  $\Gamma_1 = \{(l,l') \in \mathbb{N}^2 \mid \mathcal{H}_{l'} \leq \mathcal{H}_l\}$  is c.e. The set  $\{(l,i,l',j) \in \mathbb{N}^4 \mid J_i \subseteq_F J_j \text{ and } j \in [l]\}$  is c.e due to Proposition 9 and Proposition 3. But then, using Proposition 1, so is  $\Gamma_2 = \{(l,i,l') \in \mathbb{N}^3 \mid \exists j \in \mathbb{N} \text{ such that } J_i \subseteq_F J_j \text{ and } j \in [l]\}$ . We then conclude:

$$(l, l') \in \Gamma_1 \iff \forall i \in [l'] \exists j \in [l] \text{ such that } J_i \subseteq_F J_j$$
$$\iff \forall i \in [l'] \ (l, i, l') \in \Gamma_2$$
$$\iff \{(l, i, l') \in \mathbb{N}^3 \mid i \in [l']\} \subseteq \Gamma_2.$$

The function  $\Phi : \mathbb{N}^2 \to \mathcal{P}(\mathbb{N}^3)$ ,  $\Phi(l, l') = \{(l, i, l') \in \mathbb{N}^3 \mid i \in [l']\}$  is an e.f.v. function. Then  $(l, l') \in \Gamma_1 \iff \Phi(l, l') \subseteq \Gamma_2$ . It follows from Proposition 3 that  $\Gamma_1$  is c.e. We proceed similarly with  $\{(l, k, l', k') \in \mathbb{N}^4 \mid (J_{(l')_0}, \ldots, J_{(l')_{k'}})$  formally refines  $(J_{(l)_0}, \ldots, J_{(l)_k})\}$ , and conclude that  $\Gamma$  is c.e.

PROPOSITION 14. For every  $(l,k) \in \Omega$  there exists  $(l',k') \in \Omega$  such that  $(l,k,l',k') \in \Gamma$ .

PROOF. Let us fix an arbitrary  $(l,k) \in \Omega$ . Now,  $\mathcal{H}_l = (J_{(l)_0}, \ldots, J_{(l)_{\overline{l}}})$  is a formal circular chain which covers K, and  $2 \leq k \leq \overline{l} - 1$ , fmesh $(l) < \epsilon_0$ . Let  $m = \overline{l}$  and  $\mathcal{C} = (C_0, \ldots, C_m) = (J_{(l)_0}, \ldots, J_{(l)_{\overline{l}}})$ . Now  $\mathcal{C}$  is a circular chain which covers K and mesh  $\mathcal{C} < \epsilon_0$ . We choose some compact circular chain  $\mathcal{E} = (E_0, \ldots, E_n)$  in K which covers K such that  $\mathcal{E} \leq \mathcal{C}$  and mesh  $\mathcal{E} < \frac{1}{4}$  fmesh(l)(this is possible due to Proposition 5). By Theorem 3 there exists a subchain  $\mathcal{E}'$  of  $\mathcal{E}$  which strictly refines  $(C_0, \ldots, C_k)$  or  $-(C_0, \ldots, C_k)$ . We have  $\mathcal{E}' = \mathcal{E}_{u,v}$ . Our goal is to align 1) the orientation and 2) the starting link of  $\mathcal{E}$  with  $(C_0, \ldots, C_k)$ . More precisely, we seek a circular chain  $\mathcal{D} = (D_0, \ldots, D_n)$  such that links $(\mathcal{D}) = \text{links}(\mathcal{E})$  and its subchain  $\mathcal{D}' = \mathcal{D}_{0,k'}$  such that links $(\mathcal{D}') = \text{links}(\mathcal{E}')$  and  $\mathcal{D}'$  refines  $\mathcal{C}_{0,k}$  with  $D_0 \subseteq C_0, D_{k'} \subseteq C_k$ , for some  $k' \leq n-1$ . That way we keep the strict refinement of the subchains, but we gain more control over their orientations.

1) If  $E_u \subseteq C_0$  and  $E_v \subseteq C_k$ , we proceed with  $\mathcal{D} = \mathcal{E}$ . If  $E_v \subseteq C_0$  and  $E_u \subseteq C_k$ , then we note that  $-\mathcal{E}' \leq -\mathcal{E}$ , so without loss of generality we proceed with  $\mathcal{D} = -\mathcal{E}$ .

2) Without loss of generality we can put  $\mathcal{D}' = (D_0, \ldots, D_{k'})$  for some  $k' \leq n-1$ . Else, if  $\mathcal{D}' = \mathcal{D}_{u,v}$  for some  $u \neq 0$ , we consider the circular chain  $\mathcal{D}_{+u}$  instead of  $\mathcal{D}$  and note that  $\mathcal{D}' = (\mathcal{D}_{+u})_{0,k'}$  for some  $k' \leq n-1$ .

Now we apply Lemma 6 with  $K = \{D_0, \ldots, D_n\}$ ,  $A = \{(l)_0, \ldots, (l)_{\overline{l}}\}$ and  $\epsilon = \frac{1}{4}$  fmesh(l). For each  $i \in \{0, \ldots, n\}$  there exists  $j_i \in \mathbb{N}$  such that  $D_i \subseteq J_{j_i}$ . We can now find  $l' \in \mathbb{N}$  such that  $((l')_0, \ldots, (l')_{\overline{l'}}) = (j_0, \ldots, j_n)$ . Then  $\mathcal{H}_{l'} = (J_{(l')_0}, \ldots, J_{(l')_{\overline{l'}}}) = (J_{j_0}, \ldots, J_{j_n})$ . Since  $\mathcal{D}$  covers K, so does  $\mathcal{H}_{l'}$ . If we choose  $a, b \in \{0, \ldots, n\}$  such that 1 < |a - b| < n, then  $D_a \cap D_b = \emptyset$ , and by Lemma 6  $J_{j_a} \diamond J_{j_b}$ , i.e.  $\mathcal{H}_{l'}$  is a formal circular chain. For each  $i \in \{0, \ldots, n\}$  there exists  $w_i \in \{0, \ldots, \overline{l}\}$  such that  $D_i \subseteq J_{(l)_{w_i}}$  and therefore, by Lemma 6  $J_{(l')_i} \subseteq_F J_{(l)_{w_i}}$ , so we have  $\mathcal{H}_{l'} \leq \mathcal{H}_l$ . Next, we note that fdiam $(j_i) < \operatorname{diam} D_i + \epsilon < \frac{1}{4}$  fmesh $(l) + \frac{1}{4}$  fmesh $(l) = \frac{1}{2}$  fmesh(l), hence fmesh $(l') < \frac{1}{2}$  fmesh(l). This entails fmesh $(l') < \epsilon_0$ .

We know that  $(D_0, \ldots, D'_k)$  refines  $(C_0, \ldots, C_k)$  with  $D_0 \subseteq C_0, D_{k'} \subseteq C_k$ and we want to show that  $(J_{(l')_0}, \ldots, J_{(l')_{k'}})$  formally refines  $(J_{(l)_0}, \ldots, J_{(l)_k})$ with  $J_{(l')_0} \subseteq_F J_{(l)_0}, J_{(l')_{k'}} \subseteq_F J_{(l)_k}$ . By Lemma 6  $D_0 \subseteq J_{(l)_0}$  and  $D_{k'} \subseteq J_{(l)_k}$ imply  $J_{(l')_0} \subseteq_F J_{(l)_0}$  and  $J_{(l')_{k'}} \subseteq_F J_{(l)_k}$ . Let  $i \in \{0, \ldots, k'\}$ . By refinement, there exists  $w_i \in \{0, \ldots, k\}$  such that  $D_i \subseteq J_{(l)_{w_i}}$ . That implies  $J_{(l')_i} \subseteq_F$  $J_{(l)_{w_i}}$ . Therefore  $\mathcal{H}_{l'}^{k'} \leq \mathcal{H}_l^k$  and we have proven  $(l, k, l', k') \in \Gamma$ .

To prove  $(l', k') \in \Omega$  it remains to show that  $2 \leq k' \leq \overline{l'} - 1$ . We note that  $J_{(l)_0} \cap J_{(l)_k} = \emptyset$  because of  $2 \leq k \leq \overline{l} - 1$ . But now, as mentioned,  $J_{(l')_0} \subseteq_F J_{(l)_0}$  and  $J_{(l')_{k'}} \subseteq_F J_{(l)_k}$  holds. We conclude  $J_{(l')_0} \cap J_{(l')_{k'}} = \emptyset$  and therefore  $2 \leq k' \leq \overline{l'} - 1$  and  $(l', k') \in \Omega$ .

Here we state a well-known result from computable analysis:

PROPOSITION 15. [IP18, Proposition 5.2.] Let  $k \in \mathbb{N} \setminus \{0\}, T \subseteq \mathbb{N}^k$  and  $a \in T$ . Suppose  $\varphi : T \to \mathbb{N}^k$  is a partial recursive function such that  $\varphi(T) \subseteq T$ . Let  $f : \mathbb{N} \to \mathbb{N}^k$  be the function defined by

$$f(0) = a, \quad f(y+1) = \varphi(f(y)).$$



FIGURE 10. Recursive refinements

Then f is computable.

Here we state a useful auxiliary result. Its proof can be found in [IIj09].

LEMMA 8. [II]09, Lemma 41.] Let (X, d) be a metric space which has compact closed balls. Let  $\mathcal{C}^k = (C_0^k, \ldots, C_{m_k}^k), k \in \mathbb{N}$  be a sequence of open chains such that  $\operatorname{Cl}(C_0^{k+1}), \ldots, \operatorname{Cl}(C_{m_{k+1}}^{k+1}))$  strictly refines  $(C_0^k, \ldots, C_{m_k}^k)$ , and  $\operatorname{mesh}(\mathcal{C}^k) < 2^{-k}, \forall k \in \mathbb{N}$ . Let

$$S = \bigcap_{k \in \mathbb{N}} (\operatorname{Cl}(C_0^{k+1}) \cup \dots \cup \operatorname{Cl}(C_{m_{k+1}}^{k+1})).$$

Then S is a continuum chainable from a to b, where  $a \in \bigcap_{k \in \mathbb{N}} C_0^k, b \in \bigcap_{k \in \mathbb{N}} C_{m_k}^k$ .

Before the main theorem, we state another useful result:

LEMMA 9. [IP18, Lemma 6.6.] Let  $(X, d, \alpha)$  be a computable metric space and let S be a nonempty compact set in (X, d). Suppose there exists a computable function  $f : \mathbb{N} \to \mathbb{N}$  such that, for each  $k \in \mathbb{N}$ , fmesh $(f(k)) < 2^{-k}, S \subseteq \bigcup \mathcal{H}_{f(k)}$  and each of the sets in the finite sequence  $\mathcal{H}_{f(k)}$  intersects S. Then S is a computable set.

THEOREM 5. Let  $(X, d, \alpha)$  be a computable metric space. Suppose  $K \subseteq X$  is a circularly chainable, but not chainable, semicomputable continuum. Let  $a, b \in K$  such that  $a \neq b$ . Then for every  $\epsilon > 0$  there exist computable points  $a', b' \in K$  such that  $d(a, a') < \epsilon, d(b, b') < \epsilon$  and a computable subcontinuum L of K chainable from a' to b'.

PROOF. Let  $\epsilon > 0$ .

By Proposition 13 we have that  $\Gamma$  and  $\Omega$  are c.e. By Proposition 14 we have that for every  $(l, k) \in \Omega$  there exists  $(l', k') \in \Omega$  such that  $(l, k, l'k') \in \Gamma$ .

It follows from Proposition 1 that there exists a partial computable function  $\varphi: \Omega \to \mathbb{R}^2$  such that  $\varphi(\Omega) \subseteq \Omega$  and  $(l, k, \varphi(l, k)) \in \Gamma$  for each  $(l, k) \in \Omega$ .

We should note that by Theorem 4 there exists  $(l_0, k_0) \in \Omega$  such that fmesh $(l_0) < \min\{1, \epsilon\}, a \in J_{(l_0)_0}, b \in J_{(l_0)_{k_0}}$ . Therefore we can apply Proposition 15 and introduce a computable function  $f : \mathbb{N} \to \mathbb{N}^2$  such that

$$f(0) = (l_0, k_0), \quad f(y+1) = \varphi(f(y))$$

Next, we introduce the componentwise notation of the function  $f: f(y) = (l_y, k_y)$ . We note that  $f(y) \in \Omega$  holds for every  $y \in \mathbb{N}$ . Therefore  $(f(y), \varphi(f(y))) \in \Gamma$  for every  $y \in \mathbb{N}$ . By definition of f this implies  $(f(y), f(y+1)) \in \Gamma$ , and then  $(l_y, k_y, l_{y+1}, k_{y+1}) \in \Gamma$  for every  $y \in \mathbb{N}$ . By definition of  $\Gamma$  we then have  $\mathcal{H}_{l_{y+1}}^{k_{y+1}} \leq \mathcal{H}_{l_y}^{k_y}$ , for every  $y \in \mathbb{N}$ . We also claim that  $\operatorname{fmesh}(l_y) < 2^{-y}, \forall y \in \mathbb{N}$ . Namely, that follows from  $\operatorname{fmesh}(l_0) < 1$  and  $\operatorname{fmesh}(l_{y+1}) < \frac{1}{2} \operatorname{fmesh}(l_y)$ .

Next, we fix  $y \in \mathbb{N}$ . We define:

$$\mathcal{C}^{y} = (C_{0}^{y}, \dots, C_{m_{y}}^{y}) = (J_{(l_{y})_{0}} \cap K, \dots, J_{(l_{y})_{k_{y}}} \cap K).$$

We try to see if conditions of Lemma 8 are met.

We claim that  $\mathcal{C}^y$  is a chain in K. Namely, from Remark 9 we know that  $(J_{(l_y)_0} \cap K, \ldots, J_{(l_y)_{\overline{l_y}}} \cap K)$  is a circular chain in K. Hence  $\mathcal{C}^y$  is its subchain. Next, we claim that

$$\operatorname{Cl}(C_0^{y+1}), \dots \operatorname{Cl}(C_{m_{y+1}}^{y+1}) \text{ refines } \mathcal{C}^y,$$
  
with 
$$\operatorname{Cl}(C_0^{y+1}) \subseteq C_0^y, \operatorname{Cl}(C_{m_{y+1}}^{y+1}) \subseteq C_{m_y}^y.$$

To prove this, first we apply Proposition 8 to  $\mathcal{H}_{l_{y+1}}^{k_{y+1}} \leq \mathcal{H}_{l_y}^{k_y}$  to establish that

$$(\operatorname{Cl}(J_{(l_{y+1})_0}), \dots, \operatorname{Cl}(J_{(l_{y+1})_{k_{y+1}}}))$$
 refines  $(J_{(l_y)_0}, \dots, J_{(l_y)_{k_y}})$ 

Because of this now

 $(Cl(J_{(l_{y+1})_0}) \cap K, \dots, Cl(J_{(l_{y+1})_{k_{y+1}}}) \cap K)$  refines  $(J_{(l_y)_0} \cap K, \dots, J_{(l_y)_{k_y}} \cap K)$ .

Now we use the simple topological fact  $\mathrm{Cl}(A\cap K)\subseteq \mathrm{Cl}(A)\cap K$  to conclude

 $\left(\operatorname{Cl}(J_{(l_{y+1})_0} \cap K), \dots, \operatorname{Cl}(J_{(l_{y+1})_{k_{y+1}}} \cap K)\right) \text{ refines } (J_{(l_y)_0} \cap K, \dots, J_{(l_y)_{k_y}} \cap K),$ 

from which  $\operatorname{Cl}(C_0^{y+1}), \ldots \operatorname{Cl}(C_{m_{y+1}}^{y+1})$  refines  $\mathcal{C}^y$  is evident.

Similarly, we conclude that  $\operatorname{Cl}(J_{(l_{y+1})_0} \cap K) \subseteq J_{(l_y)_0} \cap K$  and  $\operatorname{Cl}(J_{(l_{y+1})_{k_{y+1}}} \cap K) \subseteq J_{(l_y)_{k_y}} \cap K$ . This, by definition of  $\mathcal{C}^y$ , entails  $\operatorname{Cl}(C_0^{y+1}) \subseteq C_0^y$  and  $\operatorname{Cl}(C_{m_{y+1}}^{y+1}) \subseteq C_m^{y_{m_y}}$ .

By Lemma 8 we have obtained a continuum

$$L = \bigcap_{k \in \mathbb{N}} (\operatorname{Cl}(C_0^{k+1}) \cup \dots \cup \operatorname{Cl}(C_{m_{k+1}}^{k+1})),$$

chainable from  $a' \in \bigcap_{k \in \mathbb{N}} C_0^k$  to  $b' \in \bigcap_{k \in \mathbb{N}} C_{m_k}^k$ . By construction,  $L \subseteq K$ , so L is a subcontinuum of K. Next we notice that since  $a, a' \in J_{(l_0)_0}$  and fmesh $(l_0) < \epsilon$ ,  $d(a, a') < \epsilon$  follows. Similar holds for  $d(b, b') < \epsilon$ .

To prove that a' and b' are computable points first we fix  $y \in \mathbb{N}$ . Then  $a' \in J_{(l_y)_0}$ . The set  $J_{(l_y)_0}$  is the union of rational balls  $I_k$ , where  $k \in [(l_y)_0]$ . We can explicitly write one such k, namely  $k = ((l_y)_0)_0$ . The center of this  $I_k$ ,  $\lambda_{((l_y)_0)_0}$  is also contained in  $J_{(l_y)_0}$ . Since fmesh $(l_y) < 2^{-y}$ , this entails

$$d(a', \lambda_{((l_u)_0)_0}) < 2^{-y},$$

or, by definition of function  $\lambda$ ,

$$d(a', \alpha_{\tau_1((l_y)_0)_0}) < 2^{-y}.$$

Hence a' is a computable point, and similar argument can be brought forth for b'.

Now it is sufficient to prove that L is computable. We know that  $L \subseteq J_{(l_y)_0} \cup \cdots \cup J_{(l_y)_{k_y}}, \forall y \in \mathbb{N}$ . For each  $y \in \mathbb{N}$  there exists  $w \in \mathbb{N}$  such that  $((w)_0, \ldots, (w)_{\overline{w}}) = ((l_y)_0, \ldots, (l_y)_{k_y})$ . The set

$$\{(y,w) \in \mathbb{N}^2 \mid ((w)_0, \dots, (w)_{\overline{w}}) = ((l_y)_0, \dots, (l_y)_{k_y})\}$$

is computable. Therefore there exists a computable function  $f : \mathbb{N} \to \mathbb{N}$ such that  $((f(y))_0, \ldots, (f(y))_{\overline{f(y)}}) = ((l_y)_0, \ldots, (l_y)_{k_y})$ , for all  $y \in \mathbb{N}$ . Then  $L \subseteq \bigcup \mathcal{H}_{f(y)}, \forall y \in \mathbb{N}$ . For any given  $y \in \mathbb{N}$  we know that  $J_{(f(y))_0}$  and  $J_{(f(y))_{\overline{f(y)}}}$ intersect L, since  $a', b' \in L$ . But, since L is connected, every link of  $\mathcal{H}_{f(y)}$ intersects L, otherwise there would exist a separation of L. Then by Lemma 9 L is computable.

REMARK 11. First we note the most important part of the result: Theorem 5 ensures the existence of computable subcontinua of a semicomputable circularly chainable, but not chainable, continuum K.

One might wonder whether we have succeeded to approximate some semicomputable continuum with these computable subcontinua. The answer is that this result guarantees no such thing, because there is no obvious candidate for the semicomputable continuum being approximated. One such candidate might be K itself, but here we restate the fact that the topological properties of K ensure that it is computable as well. Any possible approximations are then not useful. We should also note that while there is a subcontinuum L of K chainable from a' to b', there need not be a subcontinuum of K chainable from a to b (see Example 2).

Instead, what we can approximate is any pair of given points a and b on K with computable points on a' and b' on K. On top of that, there exists a computable subcontinuum L of K chainable from a' and b'.



FIGURE 11. Computable intermediate value theorem

5. Computable intersection points

Our final result is motivated by the computable version of the **Intermediate value theorem**. We state it again here:

THEOREM 6. [PR89] A computable function  $f : [0,1] \to \mathbb{R}$  such that f(0) < 0, f(1) > 0 has a computable zero.

REMARK 12. We now make the following observations and try to obtain a setting for possible generalizations of the theorem.

- Let S be the x-axis, and  $\Gamma(f)$  the graph of f. Then  $\Gamma(f) \cap S$  contains a computable point.
- Let U be the lower and V the upper half-plane. The graph  $\Gamma(f)$  intersects both U and V.
- $\Gamma(f)$  is a computable subset of  $\mathbb{R}^2$ . It is also a continuum considering the Euclidean metric.
- This leads us to the following attempt: Suppose K is a continuum in  $\mathbb{R}^2$  which intersects both U and V. Then K certainly intersects S. Does

(5.1) K computable  $\implies K$  intersects S in a computable point hold?

In general, no! Let  $f : [0,1] \to \mathbb{R}$  be a nonnegative computable function which has zeroes, but none of them is computable [Spe59]. Let  $K = \Gamma(f) \cup \Gamma(-f)$ . Then K is a computable continuum, therefore it intersects S, but none of the points in  $K \cap S$  are computable.

• Some conditions under which (5.1) holds have been examined in [IP18]. One such condition is that K is an arc.

We restate this important result for clarity:



FIGURE 12. No computable intersection points

THEOREM 7. [IP18, Theorem 5.3] Let  $(X, d, \alpha)$  be a computable metric space and let U and V be disjoint c.e. open sets in X. Let  $S = X \setminus (U \cup V)$ . Suppose K is a continuum in X chainable from a to b, where  $a \in U$  and  $b \in V$ . Suppose K is a computable set and  $K \cap S$  is totally disconnected. Then  $K \cap S$  contains a computable point.

REMARK 13. A topological space X is said to be *totally disconnected* if every connected component of X is a one-point set. Note that any nonempty subspace of X is then totally disconnected as well.

We conclude with the main result of this section:

THEOREM 8. Let  $(X, d, \alpha)$  be a computable metric space and let U and V be disjoint c.e. open sets in X. Let  $S = X \setminus (U \cup V)$ . Suppose K is a circularly chainable, but not chainable, continuum in X which intersects both U and V. Suppose K is a semicomputable set and  $K \cap S$  is totally disconnected. Then  $K \cap S$  contains a computable point.

PROOF. We choose arbitrary points  $a \in K \cap U, b \in K \cap V$ . Since U and V are open, there exist  $r_1, r_2 > 0$  such that  $B(a, r_1) \subseteq U$  and  $B(b, r_2) \subseteq V$ . Let  $\epsilon = \min\{r_1, r_2\}$ . By Theorem 5 it follows that there exist computable points  $a', b' \in K$  such that  $d(a, a') < \epsilon$ ,  $d(b, b') < \epsilon$  and a computable subcontinuum L of K chainable from a' to b'. Note that  $a' \in L \cap U, b' \in L \cap V$ , and  $L \cap S$ is totally disconnected. Therefore, following Theorem 2,  $L \cap S$  contains a computable point, thus  $K \cap S$  contains it as well.

#### References

[AH23] Djamel Eddine Amir and Mathieu Hoyrup. Strong computable type. 2023. arXiv: 2210.08309 [math.L0]. URL: https://arxiv.org/ abs/2210.08309.

- [Bur59] C. E. Burgess. "Chainable continua and indecomposability". In: Pacific Journal of Mathematics (1959), pp. 653–659. DOI: https: //doi.org/10.2140/pjm.1959.9.653.
- [ČHI21] Vedran Čačić, Marko Horvat, and Zvonko Iljazović. "Computable subcontinua of semicomputable chainable Hausdorff continua". In: *Theoretical Computer Science* 892 (2021), pp. 155–169. ISSN: 0304-3975. DOI: https://doi.org/10.1016/j.tcs.2021.09.018. URL: https://www.sciencedirect.com/science/article/pii/ S0304397521005478.
- [CV77] C.O. Christenson and W.L. Voxman. Aspects of Topology. Marcel Dekker, Inc., New York, 1977.
- [ČIV19] E. Čičković, Z. Iljazović, and L. Validžić. "Chainable and circularly chainable semicomputable sets in computable topological spaces". In: Archive for Mathematical Logic 58 (2019), pp. 885–897.
- [IIj09] Z. Iljazović. "Chainable and Circularly Chainable Co-c.e. Sets in Computable Metric Spaces". In: Journal of Universal Computer Science 15.6 (2009), pp. 1206–1235.
- [IIj13] Z. Iljazović. "Compact manifolds with computable boundaries". In: Logical Methods in Computer Science 9.4 (2013), pp. 1–22.
- [IP18] Z. Iljazović and B. Pažek. "Computable intersection points". In: Computability 7 (2018), pp. 57–99.
- [IS18] Z. Iljazović and I. Sušić. "Semicomputable manifolds in computable topological spaces". In: Journal of Complexity 45 (2018), pp. 83– 114.
- [IJ24] Zvonko Iljazović and Matea Jelić. "Computable approximations of a chainable continuum with a computable endpoint". In: Archive for Mathematical Logic (2024), pp. 181–201. DOI: https://doi. org/10.1007/s00153-023-00891-5.
- [Kih12] T. Kihara. "Incomputability of Simply Connected Planar Continua". In: Computability 1.2 (2012), pp. 131–152.
- [Mil02] J.S. Miller. "Effectiveness for Embedded Spheres and Balls". In: Electronic Notes in Theoretical Computer Science 66 (2002), pp. 127– 138.
- [Nad92] S.B. Nadler. Continuum theory. Marcel Dekker, Inc., New York, 1992.
- [PR89] M.B. Pour-El and J.I. Richards. Computability in Analysis and Physics. Springer, Berlin, 1989.
- [Spe59] E. Specker. "Der Satz vom Maximum in der rekursiven Analysis". In: Constructivity in Mathematics. Ed. by A. Heyting. North Holland Publ. Comp., Amsterdam, 1959, pp. 254–265.
- [Wei93] K. Weihrauch. "Computability on computable metric spaces". In: Theoretical Computer Science 113 (1993), pp. 191–210.
- [Wei00] K. Weihrauch. Computable Analysis. Springer, Berlin, 2000.

#### REFERENCES

# Izračunljivi potkontinuumi cirkularno lančastih kontinuuma

# David Tarandek

Sažetak. Ovaj rad istražuje cirkularno lančaste kontinuume, koji nisu lančasti, u izračunljivim metričkim prostorima. Za proizvoljan takav kontinuum K vrijedi da iz njegove poluizračunljivosti slijedi izračunljivost. U literaturi se često proučavaju uvjeti pod kojima poluizračunljivost skupa implicira njegovu izračunljivost, s naglaskom na topološka svojstva. Kada ti uvjeti nisu zadovoljeni, prirodno je istraživati aproksimativne pristupe. U ovom članku fokusiramo se na specifične izračunljive potkontinuume od K. Glavni rezultat jest da svake dvije točke na poluizračunljivom, cirkularno lančastom, no ne i lančastom, kontinuumuKmožemo aproksimirati izračunljivim točkama tako da postoji izračunljivi potkontinuum L od K koji ih povezuje. Posljedično, ako su U i V disjunktni i izračunljivo prebrojivi otvoreni skupovi u izračunljivom metričkom prostoru, pokazujemo da, ako K siječe i U i V, tada njegov presjek s komplementom njihove unije nužno sadrži izračunljivu točku kada je taj presjek totalno nepovezan.

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