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THE MAXIMUM CARDINALITY OF ESSENTIAL FAMILIES IN REGULAR OR NORMAL SPACES

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Abstract. Let X be a regular or normal space (T_1 not required) with infinite weight and $\mathcal{C}$ be an essential family in X. We will show that $\operatorname{card} \mathcal{C} \leq \operatorname{wt} X$. This implies that every essential family in a separable metrizable space is countable.

1. Introduction

The Main Result of this paper is Theorem 4.2. It states that if X is a regular or normal space (T_1 not required) with infinite weight and $\mathcal{C}$ is an essential family in X, then $\operatorname{card} \mathcal{C} \leq \operatorname{wt} X$. We will give the definition of an essential family in Definition 3.2(1). It is different from the classical one (Definition 3.1(1)) in order to accommodate anomalies that occur in spaces that are not normal. However, it agrees with the classical one for normal spaces. Let us remark that essential families have provided a useful tool in the study of dimension, especially for separable metrizable or compact Hausdorff spaces (see [LR], [Mi], [RSW], [Ru1], [Ru2], [Ru3], [Ru4], [Sa], [Wa]). We shall explain in Section 3 how one may use our alternate definition of an essential family to define the dimension of a space.

The inspiration for this research arose in December 2023 when we asked J. van Mill if a separable metrizable space of (necessarily countable) infinite weight could have an uncountable essential family. The author then proved that this was not possible assuming the continuum hypothesis. Subsequently (in a private communication) van Mill presented a short elegant proof without employing the continuum hypothesis. Our proof is significantly different from either of those we just mentioned. Corollary 4.4 provides this result on separable metrizable spaces of infinite weight.

2. Disjoint Closed Pairs

In this paper map will mean continuous function.

Definition 2.1. *Let X be a space and (A, B) an ordered pair of subsets of X. One says that (A, B) is a disjoint closed (open) pair in X if each of A and B is closed (open) in X and $A \cap B = \emptyset$. A partition P in X of a disjoint closed pair (A, B) is a closed subset P of X having the property that $X \setminus P = U \cup V$ such that (U, V) is a disjoint open pair in X, $A \subset U$, and $B \subset V$.*

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In the following when a space X is given and (A, B) is a disjoint closed (open) pair in X, then we shall frequently omit saying "in X." Similarly we shall do the same for a partition P of a disjoint closed pair.

Definition 2.2. *For each disjoint closed pair $C = (A, B)$ in a space X, put $C^+ = A$, $C^- = B$, and $C^0 = X \setminus (A \cup B)$.*

Observe that in Definition 2.2, if P is a partition of C, then $P \subset C^0$, the latter being open in X. Lemma 2.3 provides us with a useful characterization of normal spaces.

Lemma 2.3. *Let X be a space. Then X is normal if and only if each disjoint closed pair has a partition.* □

Suppose that X is a space that is not normal, and let C be a disjoint closed pair that has no partition. Then $C^+ \neq \emptyset \neq C^-$, for otherwise $\emptyset$ would be a partition of C. It is also true that $C^* = (C^-, C^+)$ is a disjoint closed pair that has no partition, and $C \neq C^*$. Hence, $\{C\}$, $\{C^*\}$, and $\{C, C^*\}$ are "classical" essential families (see [LR], [Sa], and especially that the property 3.1(1) of Definition 3.1(1) below is fulfilled trivially) in X, and the latter has cardinality 2. What is more, if $C^+ \neq \emptyset \neq C^-$ and C has precisely one partition, then the same anomaly also occurs. Hence if one is attempting to use the classical notion of essential family to define dimension in such a space X, then a certain distortion arises. On the other hand, if C has nonempty "parallel" partitions, i.e., nonempty partitions P and S such that $P \cap S = \emptyset$, then the difficulty described above vanishes. This motivates our Definition 2.4.

Definition 2.4. *For each space X, let $\mathfrak{C}(X)$ denote the set of nonempty collections of disjoint closed pairs in X. Denote by $\mathfrak{P}(X)$ the subset of $\mathfrak{C}(X)$ consisting of those $\mathcal{C}$ having the property that for each $C \in \mathcal{C}$ either $\emptyset$ is a partition of C or C has nonempty partitions P and S with $P \cap S = \emptyset$.*

Definition 2.5. *Let X be a space and $\mathcal{C} \in \mathfrak{C}(X)$. Then by $\mathcal{C}^0$ we mean, $\bigcap\{C^0 \mid C \in \mathcal{C}\}$.*

Lemma 2.6. *Let X be a space and $\mathcal{C} \in \mathfrak{P}(X)$ ($\mathcal{C} \in \mathfrak{C}(X)$). Then for each nonempty subset $\mathcal{D} \subset \mathcal{C}$, $\mathcal{D} \in \mathfrak{P}(X)$ ($\mathcal{D} \in \mathfrak{C}(X)$).* □

Lemma 2.7. *Let X be a normal space and C be a disjoint closed pair such that $\emptyset$ is not a partition of C. Then*

(1) $C^+ \neq \emptyset \neq C^-$,
(2) *every map $f : X \to [0, 1]$ having the property that $f(C^+) \subset \{0\}$ and $f(C^-) \subset \{1\}$ is surjective,*
(3) *there exists a map $f : X \to [0, 1]$ enjoying the properties that $f(C^+) = \{0\}$ and $f(C^-) = \{1\}$, and*
(4) *there exist nonempty partitions P, S of C with $P \cap S = \emptyset$, such that for all $x \in C^0$, there are an open subset U_x of X such that $x \in U_x \subset \mathrm{cl}_X U_x \subset C^0$ and an element $R \in \{P, S\}$ with $\mathrm{cl}_X U_x \cap R = \emptyset$.*

Proof. If either $C^+ = \emptyset$ or $C^- = \emptyset$, then $\emptyset$ is a partition of C in violation of our hypothesis. So (1) holds. Let $f : X \to [0, 1]$ be a map as in (2). Applying (1), we get $f(C^+) = \{0\}$ and $f(C^-) = \{1\}$. If f is not surjective, then there is $0 < x < 1$ with $x \notin \mathrm{im}(f)$. Thus $\emptyset = f^{-1}(x)$ is a partition of C, which is impossible. This gives us (2). Of course (3) is true by Urysohn's

Lemma and (1). To prove (4), use (3) and (2) to get a surjective map $f : X \to [0,1]$ having the property that $f(C^+) = \{0\}$ and $f(C^-) = \{1\}$. Then put $P = f^{-1}(1/3)$ and $S = f^{-1}(2/3)$. From this it is easy to see that (4) is true. □

According to Definition 2.4 and Lemma 2.7(4) we get,

Lemma 2.8. *For every normal space X, $\mathfrak{C}(X) = \mathfrak{P}(X)$.* □

3. Essential Families

Let us first present the definition of an essential family in a space X in the "classical sense."

Definition 3.1. *Let X be a space and $\mathcal{C} \in \mathfrak{C}(X)$. We say that the family $\mathcal{C}$ is*

1. **essential** *in X in the* **classical sense** *if for every collection $\{P_C \mid C \in \mathcal{C}\}$ of respective partitions P_C of C, $\bigcap\{P_C \mid C \in \mathcal{C}\} \neq \emptyset$;*
2. **inessential** *in X in the* **classical sense** *if it is not essential in X in the classical sense, i.e., if there exists a collection $\{P_C \mid C \in \mathcal{C}\}$ of respective partitions P_C of C such that $\bigcap\{P_C \mid C \in \mathcal{C}\} = \emptyset$.*

Definition 3.2(1) provides us with a new notion of an essential family in a space. We replace $\mathfrak{C}(X)$ by $\mathfrak{P}(X)$. Lemma 2.8, however, shows that for the class of normal spaces, the concepts are one and the same.

Definition 3.2. *Let X be a space and $\mathcal{C} \in \mathfrak{P}(X)$. We say that the family $\mathcal{C}$ is*

1. **essential** *in X if for every collection $\{P_C \mid C \in \mathcal{C}\}$ of respective partitions P_C of C, $\bigcap\{P_C \mid C \in \mathcal{C}\} \neq \emptyset$;*
2. **inessential** *in X if it is not essential in X, i.e., if there exists a collection $\{P_C \mid C \in \mathcal{C}\}$ of respective partitions P_C of C such that $\bigcap\{P_C \mid C \in \mathcal{C}\} = \emptyset$.*

Lemma 3.3. *Let X be a space and $\mathcal{C} \in \mathfrak{P}(X)$. If $\mathcal{C}$ is essential in X and $C \in \mathcal{C}$, then $\emptyset$ is not a partition of C, and there exist nonempty partitions P and S of C with $P \cap S = \emptyset$.* □

Making use of Definition 3.2(1) and taking a cue from Theorem 2.4, Definition 2.5, and Definition 2.6 of [LR], we could make the following "new" definition of various notions of dimension.

Definition 3.4. *Let X be a space. Then,*

1. $\dim X = -1$ *if $X = \emptyset$;*
2. $\dim X = 0$ *if $X \neq \emptyset$ and there is no essential family in X;*
3. *if $n \in \mathbb{N}$, then $\dim X = n$ if there exists an essential family $\mathcal{C}$ in X with $\operatorname{card} \mathcal{C} = n$, and for every essential family $\mathcal{C}_0$ in X, $\operatorname{card} \mathcal{C}_0 \leq n$;*
4. *X is infinite-dimensional if for all $n \in \mathbb{N}$, there exists an essential family $\mathcal{C}$ in X with $\operatorname{card} \mathcal{C} = n$;*
5. *X is strongly infinite-dimensional if there exists a countably infinite essential family in X;*
6. *X is weakly infinite-dimensional if it is infinite-dimensional and not strongly infinite-dimensional.*

In relation to Definition 3.4(5), we do not know if there has ever been a term used to express the "dimension" of a space that has an essential family of a certain uncountable cardinality $\aleph$ but not one of a cardinality greater than $\aleph$. If a nonempty space X has a finite base for its topology, then $\mathfrak{P}(X)$ is finite, so $\dim X$ is finite. In any event, in this paper we are only going to study spaces that have infinite weight.

Lemma 3.5. *Let X be a space and $\mathcal{C} \in \mathfrak{P}(X)$ be an essential family in X. Then each nonempty subset $\mathcal{D} \subset \mathcal{C}$ is in $\mathfrak{P}(X)$ and is an essential family in X.*

Proof. By Lemma 2.6, $\mathcal{D} \in \mathfrak{P}(X)$. Suppose to the contrary that $\mathcal{D}$ is an inessential family in X; for each $C \in \mathcal{D}$, select a partition P_C of C having the property that $\bigcap\{P_C \,|\, C \in \mathcal{D}\} = \emptyset$. Now apply Definition 2.4. For each $C \in \mathcal{C} \setminus \mathcal{D}$, let P_C be a partition of C. Then, $\bigcap\{P_C \,|\, C \in \mathcal{C}\} \subset \bigcap\{P_C \,|\, C \in \mathcal{D}\} = \emptyset$. This contradicts the fact that $\mathcal{C}$ is an essential family in X. □

Lemma 3.6. *Let X be a regular or normal space, $\mathcal{C}$ be an essential family in X, and $\mathcal{B}$ be a base for the topology of X. Then*

(1) *for each $C \in \mathcal{C}$, there exist nonempty partitions P_C and S_C of C with $P_C \cap S_C = \emptyset$, and*
(2) *for each $C \in \mathcal{C}$ and $x \in C^0$, there exists $U_x \in \mathcal{B}$ with $x \in U_x \subset \operatorname{cl}_X U_x \subset C^0$, having the property that for some element $R_C \in \{P_C, S_C\}$, $\operatorname{cl}_X U_x \cap R_C = \emptyset$.*

Proof. In case X is regular, we get (1) from Lemma 3.3 and obtain (2) easily from regularity. If X is normal, then apply Lemma 2.7(4) to obtain P_C, S_C, and U_x as needed. □

We now present a technical lemma that shows how one might find conditions on an element $\mathcal{C}$ of $\mathfrak{P}(X)$ for a space X that will show that $\mathcal{C}$ is an inessential family in X.

Lemma 3.7. *Let X be a space and $\mathcal{C} \in \mathfrak{P}(X)$. Suppose that $\mathcal{D}$ is a nonempty subset of $\mathcal{C}$ and for each $C \in \mathcal{D}$, R_C is a partition of C and B_C is a subset of X with $\operatorname{cl}_X B_C \subset C^0 \setminus R_C$. If $\{\operatorname{cl}_X B_C \,|\, C \in \mathcal{D}\}$ is a cover of $\mathcal{D}^0$, then $\mathcal{C}$ is an inessential family in X.*

Proof. Suppose that $\mathcal{C}$ is an essential family in X. By Lemma 3.5, $\mathcal{D} \in \mathfrak{P}(X)$, and $\mathcal{D}$ is an essential family in X.

For each $C \in \mathcal{D}$, put $\widetilde{C} = (C^+ \cup \operatorname{cl}_X B_C, C^-)$. One sees that for each $C \in \mathcal{D}$, $\widetilde{C}$ is a disjoint closed pair, and hence, $\widetilde{\mathcal{C}} = \{\widetilde{C} \,|\, C \in \mathcal{D}\} \in \mathfrak{C}(X)$. Moreover, whenever $C \in \mathcal{D}$, $\widetilde{C}^0 \subset C^0$. It follows that $\widetilde{\mathcal{C}}^0 \subset \mathcal{D}^0$. So if $x \in \widetilde{\mathcal{C}}^0$, then $x \in \mathcal{D}^0$. By hypothesis, $\{\operatorname{cl}_X B_C \,|\, C \in \mathcal{D}\}$ is a cover of $\mathcal{D}^0$. So for some $C \in \mathcal{D}$, $x \in \operatorname{cl}_X B_C$. But then $x \notin \widetilde{C}^0$, which implies that $x \notin \widetilde{\mathcal{C}}^0$, a contradiction. This proves that $\widetilde{\mathcal{C}}^0 = \emptyset$.

Now notice that for each $C \in \mathcal{D}$, $R_C \subset \widetilde{C}^0$. Therefore, $\bigcap\{R_C \,|\, C \in \mathcal{D}\} \subset \bigcap\{\widetilde{C}^0 \,|\, C \in \mathcal{D}\} = \widetilde{\mathcal{C}}^0 = \emptyset$. Hence $\bigcap\{R_C \,|\, C \in \mathcal{D}\} = \emptyset$, and by assumption, for each $C \in \mathcal{D}$, R_C is a partition of C. So $\mathcal{D}$ is an inessential family in X, and we get a contradiction. □

4. Main Theorem

First we present our Main Lemma.

Lemma 4.1. *Let X be a space with infinite weight. Suppose that*

(1) *$\mathcal{C}$ is an essential family in X,*
(2) *for each $C \in \mathcal{C}$, there exist nonempty partitions P_C and S_C of C with $P_C \cap S_C = \emptyset$, and*
(3) *for each $C \in \mathcal{C}$ and $x \in C^0$, there are $R_C \in \{P_C, S_C\}$ and an open subset U of X having the property that $x \in U \subset \operatorname{cl}_X U \subset C^0$ and $\operatorname{cl}_X U \cap R_C = \emptyset$.*

Then $\operatorname{card} \mathcal{C} \leq \operatorname{wt} X$*.*

Proof. Let $\mathcal{B}$ be a base for the topology of X with $\operatorname{card} \mathcal{B} = \operatorname{wt} X$. To arrive at a contradiction to (1), suppose that $\operatorname{wt} X < \operatorname{card} \mathcal{C}$. Applying Lemma 3.5 we may assume that $\operatorname{card} \mathcal{C} = \aleph$ where $\aleph$ is the first cardinal with $\operatorname{card} \mathcal{B} < \aleph$.

Consider the following statement:

($*$) For each nonempty subset $\mathcal{D} \subset \mathcal{C}$ with $\operatorname{card} \mathcal{D} < \aleph$, and collections $\{R_C \,|\, C \in \mathcal{D}\}$ and $\{B_C \,|\, C \in \mathcal{D}\}$ such that for all $C \in \mathcal{D}$, $R_C \in \{P_C, S_C\}$, $B_C \in \mathcal{B}$, $B_C \subset \operatorname{cl}_X B_C \subset C^0$, and $\operatorname{cl}_X B_C \cap R_C = \emptyset$, it is true that $\{\operatorname{cl}_X B_C \,|\, C \in \mathcal{D}\}$ is not a cover of $\mathcal{D}^0$.

We are going to prove that ($*$) is false. This will show that

($**$) there exist a nonempty subset $\mathcal{D} \subset \mathcal{C}$ with $\operatorname{card} \mathcal{D} < \aleph$, and collections $\{R_C \,|\, C \in \mathcal{D}\}$ and $\{B_C \,|\, C \in \mathcal{D}\}$ such that for all $C \in \mathcal{D}$, $R_C \in \{P_C, S_C\}$, $B_C \in \mathcal{B}$, $B_C \subset \operatorname{cl}_X B_C \subset C^0$, $\operatorname{cl}_X B_C \cap R_C = \emptyset$, and $\{\operatorname{cl}_X B_C \,|\, C \in \mathcal{D}\}$ is a cover of $\mathcal{D}^0$.

Our reason for accomplishing this is the following. By (1), $\mathcal{C}$ is an essential family in X, so $\mathcal{C} \in \mathfrak{P}(X)$. We apply the latter, ($**$), and Lemma 3.7 to conclude that $\mathcal{C}$ is an inessential family in X, a contradiction. Hence our proof will be complete if we can show that ($*$) is false. We proceed by assuming the truth of ($*$) and in the end, reaching a contradiction.

Use $\preceq$ to denote a well-ordering of $\mathcal{C}$, let Q designate the first element of $\mathcal{C}$ under $\preceq$, and for each $C \in \mathcal{C}$, let $C+1$ denote the immediate successor of C. Put $\mathcal{D}_Q = [Q, Q+1)$; then $\mathcal{D}_Q = \{Q\}$, so $\operatorname{card} \mathcal{D}_Q = 1 < \aleph$. By (2), P_Q and S_Q are nonempty partitions of Q with $P_Q \cap S_Q = \emptyset$. Hence $\emptyset \neq P_Q \cup S_Q \subset Q^0$. Take $x \in Q^0$. Applying (3), we find $B_Q \in \mathcal{B}$ and $R_Q \in \{P_Q, S_Q\}$ having the property that $x \in B_Q \subset \operatorname{cl}_X B_Q \subset Q^0$, $\operatorname{cl}_X B_Q \cap R_Q = \emptyset$, and $\{B_Q\}$ is not a cover of $\mathcal{D}_Q^0 = Q^0$. Make the trivial observation that for all $\{C, C'\} \subset [Q, Q+1)$ with $C \neq C'$, $B_C \neq B_{C'}$.

Suppose inductively that $F \in \mathcal{C}$, $Q+1 \preceq F$, and for each $C \in [Q, F)$, we have chosen $R_C \in \{P_C, S_C\}$ and $B_C \in \mathcal{B}$ such that,

($\dagger_1$) $B_C \subset \operatorname{cl}_X B_C \subset C^0$, and
($\dagger_2$) $\operatorname{cl}_X B_C \cap R_C = \emptyset$,

in such a manner that

($\dagger_3$) for all $\{C, C'\} \subset [Q, F)$ with $C \neq C'$, $B_C \neq B_{C'}$.

Now $\emptyset \neq [Q, F) \subset \mathcal{C}$ and $\operatorname{card}[Q, F) < \aleph$. By ($*$) with $\mathcal{D} = [Q, F)$, $\{\operatorname{cl}_X B_C \,|\, C \in [Q, F)\}$ is not a cover of $\mathcal{D}^0$. Choose $x \in \mathcal{D}^0 \setminus \bigcup\{\operatorname{cl}_X B_C \,|\, C \in [Q, F)\}$. Apply (3) to find $R_F \in \{P_F, S_F\}$ and $B_F \in \mathcal{B}$ such that $x \in B_F \subset \operatorname{cl}_X B_F \subset F^0$ and $\operatorname{cl}_X B_F \cap R_F = \emptyset$. Then ($\dagger_1$) and ($\dagger_2$) are true for each $C \in [Q, F+1)$, and since $x \notin \bigcup\{\operatorname{cl}_X B_C \,|\, C \in [Q, F)\}$, ($\dagger_3$) also obtains when we replace $[Q, F)$ with $[Q, F+1)$.

We have just shown by the method of transfinite construction that for each $C \in \mathcal{C}$, there exist $B_C \in \mathcal{B}$ and that for all $\{C, C'\} \subset \mathcal{C}$ with $C \neq C'$, $B_C \neq B_{C'}$. Hence the function $C \mapsto B_C$ from $\mathcal{C}$ to $\mathcal{B}$ is injective, which is impossible since $\operatorname{card} \mathcal{B} < \aleph = \operatorname{card} \mathcal{C}$. □

Applying Lemmas 3.6 and 4.1, we arrive at our Main Theorem.

Theorem 4.2. *Let X be a regular or normal space with infinite weight, and $\mathcal{C}$ be an essential family in X. Then* $\operatorname{card} \mathcal{C} \leq \operatorname{wt} X$.

Applying Theorem 4.2 and Lemma 2.8, we may state the next two corollaries.

Corollary 4.3. *Let X be a normal space with infinite weight and $\mathcal{C}$ be an essential family in X in the classical sense. Then* $\operatorname{card} \mathcal{C} \leq \operatorname{wt} X$.

Corollary 4.4. (J. van Mill) *Let X be a separable metrizable space of infinite weight and $\mathcal{C}$ be an essential family in X in the classical sense. Then $\mathcal{C}$ is a countable family.* □

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