RAD HRVATSKE AKADEMIJE ZNANOSTI I UMJETNOSTI
MATEMATIČKE ZNANOSTI

D. Pokaz

Generalized Hardy-type inequality via Lidstone interpolating polynomial and new Green functions

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GENERALIZED HARDY-TYPE INEQUALITY VIA LIDSTONE INTERPOLATING POLYNOMIAL AND NEW GREEN FUNCTIONS

Dora Pokaz

Abstract. For a given general setting, involving measure spaces with positive σ-finite measures, we present new results regarding Hardy-type inequality. We established a connection between the difference operator obtained from Hardy-type inequality and the expression that includes Lidstone interpolating polynomial and four new Green functions. We discuss about $2n$ convexity of the function and consider the main result depending on the parity of the part of exponent and index n. Applying Hölder inequality for conjugate exponents p and q we get some consequential results. Finally, we derived bounds for the identity using Čebyšev functional and Ostrowski-type bound for the generalized Hardy's inequality.

1. Preliminaries

The Lidstone series is a generalization of Taylor's series. It approximates a given function in the neighborhood of two points instead of one. In the beginning such series have been studied by G. J. Lidstone (1929), H. Poritsky (1932) and J. M. Wittaker (1934). It was continued by many others. It should be emphasized that Boas made a complete characterization, which was very important from a practical point of view. Apart from approximation theory, applications of this interpolating polynomial can be found in many branches of physical sciences. We start with the definition of the Lidstone series by Lidstone polynomial.

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Definition 1.1. *Let $f \in C^{\infty}([0,1])$, then the Lidstone series has the form*

$$\sum_{k=0}^{\infty}\left(f^{(2k)}(0)\Lambda_k(1-x)+f^{(2k)}(1)\Lambda_k(x)\right),$$

where Λ_n is a polynomial of degree $2n+1$ defined by the relations

$$\begin{aligned}&\Lambda_0(t)=t,\\ &\Lambda_n''(t)=\Lambda_{n-1}(t),\\ &\Lambda_n(0)=\Lambda_n(1)=0,\ n\geq 1.\end{aligned}\tag{1.1}$$

Other explicit representations of the Lidstone polynomial are given in [1] and [15],

$$\begin{aligned}\Lambda_n(t) &= (-1)^n\frac{2}{\pi^{2n+1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{2n+1}}\sin k\pi t,\\ \Lambda_n(t) &= \frac{1}{6}\left[\frac{6t^{2n+1}}{(2n+1)!}-\frac{t^{2n-1}}{(2n-1)!}\right]\\ &- \sum_{k=0}^{n-2}\frac{2(2^{2k+3}-1)}{(2k+4)!}B_{2k+4}\frac{t^{2n-2k-3}}{(2n-2k-3)!},\ n=1,2,\dots,\\ \Lambda_n(t) &= \frac{2^{2n+1}}{(2n+1)!}B_{2n+1}\left(\frac{1+t}{2}\right),\ n=1,2\dots,\end{aligned}$$

where B_{2k+4} is the $(2k+4)$-th Bernoulli number and $B_{2n+1}\left(\frac{1+t}{2}\right)$ is the Bernoulli polynomial.

In [16], Widder proved the fundamental lemma:

Lemma 1.2. *If $f \in C^{(2n)}([0,1])$, then*

$$f(t)=\sum_{k=0}^{n-1}\left[f^{(2k)}(0)\Lambda_k(1-t)+f^{(2k)}(1)\Lambda_k(t)\right]+\int_0^1 G_n(t,s)f^{(2n)}(s)ds,$$

where

$$G_1(t,s)=G(t,s)=\begin{cases}(t-1)s, & \text{if } s\leq t,\\ (s-1)t, & \text{if } t\leq s,\end{cases}\tag{1.2}$$

is homogeneous Green's function of the differential operator $\frac{d^2}{ds^2}$ on $[0,1]$, and with the successive iterates of $G(t,s)$

$$G_n(t,s)=\int_0^1 G_1(t,p)G_{n-1}(p,s)dp,\ n\geq 2.\tag{1.3}$$

The Lidstone polynomial can be expressed in terms of $G_n(t,s)$ as

$$\Lambda_n(t) = \int_0^1 G_n(t,s) s\, ds.$$

Based on the article [9] throughout the paper, $\tilde{G}_\gamma$, $\gamma = 1,2,3,4$, will denote the following Green functions defined on $[\alpha,\beta]\times[\alpha,\beta]$ with

$$\tilde{G}_1(t,s) = \begin{cases} \alpha - s\ , & \alpha \le s \le t; \\ \alpha - t, & t \le s \le \beta. \end{cases} \tag{1.4}$$

$$\tilde{G}_2(t,s) = \begin{cases} t - \beta\ , & \alpha \le s \le t; \\ s - \beta, & t \le s \le \beta. \end{cases} \tag{1.5}$$

$$\tilde{G}_3(t,s) = \begin{cases} t - \alpha\ , & \alpha \le s \le t; \\ s - \alpha, & t \le s \le \beta. \end{cases} \tag{1.6}$$

$$\tilde{G}_4(t,s) = \begin{cases} \beta - s\ , & \alpha \le s \le t; \\ \beta - t, & t \le s \le \beta. \end{cases} \tag{1.7}$$

Note that all these functions are continuous and convex with respect to both variables.

Lemma 1.3. *For $\phi \in C^2([\alpha,\beta])$, the following identities hold*

$$\phi(t) = \phi(\alpha) + (t-\alpha)\phi'(\beta) + \int_\alpha^\beta \tilde{G}_1(t,s)\phi''(s)ds, \tag{1.8}$$

$$\phi(t) = \phi(\beta) + (t-\beta)\phi'(\alpha) + \int_\alpha^\beta \tilde{G}_2(t,s)\phi''(s)ds, \tag{1.9}$$

$$\phi(t) = \phi(\beta) + (t-\alpha)\phi'(\alpha) - (\beta-\alpha)\phi'(\beta) + \int_\alpha^\beta \tilde{G}_3(t,s)\phi''(s)ds, \tag{1.10}$$

$$\phi(t) = \phi(\alpha) - (\beta-t)\phi'(\beta) + (\beta-\alpha)\phi'(\alpha) + \int_\alpha^\beta \tilde{G}_4(t,s)\phi''(s)ds, \tag{1.11}$$

where the functions $\tilde{G}_\gamma$, $\gamma = 1,\ldots,4$, are defined by (1.4)–(1.7).

2. Introduction

The aim of this article is to give a result related to the general Hardy-type inequality. The classical Hardy inequality from [6] is

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx,\ p>1, \tag{2.12}$$

where f is non-negative function such that $f \in L^p(\mathbb{R}_+)$ and $\mathbb{R}_+ = (0,\infty)$. The constant $\left(\frac{p}{p-1}\right)^p$ is sharp. Inequality (2.12) was generalized in many ways, see [7], [12] and [13]. Here we refer to settings and generalization from [8].

We begin by defining the settings that we continue to work with. Let $(\Sigma_1, \Omega_1, \mu_1)$ and $(\Sigma_2, \Omega_2, \mu_2)$ be measure spaces with positive σ-finite measures and A_k be an integral operator defined by

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,t) f(t) d\mu_2(t), \tag{2.13}$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative kernel, $f : \Omega_2 \to \mathbb{R}$ is measurable function and

$$0 < K(x) := \int_{\Omega_2} k(x,t) d\mu_2(t), \qquad x \in \Omega_1. \tag{2.14}$$

Throughout the article we will mark the open interval in $\mathbb{R}$ with I. The following result was given in [8].

Theorem 2.1. *Let u be a weight function, $k(x,y) \ge 0$. Assume that $\frac{k(x,y)}{K(x)} u(x)$ is locally integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v by*

$$v(y) := \int_{\Omega_1} \frac{k(x,y)}{K(x)} u(x) d\mu_1(x) < \infty. \tag{2.15}$$

If Φ is a convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\Omega_1} \Phi(A_k f(x)) u(x) d\mu_1(x) \le \int_{\Omega_2} \Phi(f(y)) v(y) d\mu_2(y) \tag{2.16}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $Im f \subseteq I$, where A_k is defined by (2.13) - (2.14).

Now, we start with the generalized Hardy-type inequality (2.16). In the settings where A_k is as in (2.13), a weight function u with v given by (2.15) and for $\gamma \in \{1,2,3,4\}$, we consider $\tilde{G}_\gamma$ to be as in (1.4)–(1.7). In addition, for $\phi \in C^2([\alpha,\beta])$, identities (1.8)–(1.11) and some simple calculations yield the following statements from [9]

$$\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x)$$

$$= \int_\alpha^\beta \left[\int_{\Omega_2} \tilde{G}_\gamma(f(y), s)v(y)d\mu_2(y) - \int_{\Omega_1} \tilde{G}_\gamma(A_k f(x), s)u(x)d\mu_1(x) \right] \phi''(s)ds. \tag{2.17}$$

Additionally, if ϕ is convex, then $\phi''(s) \geq 0$ for $s \in [\alpha, \beta]$.
In this article, we consider Lidstone series representation of $\Phi \in C^{(2n)}([\alpha, \beta])$ as:

$$\begin{aligned} \Phi(x) &= \sum_{m=0}^{n-1} (\beta-\alpha)^{2m} \left[\Phi^{(2m)}(\alpha)\Lambda_m\left(\frac{\beta-x}{\beta-\alpha}\right) + \Phi^{(2m)}(\beta)\Lambda_m\left(\frac{x-\alpha}{\beta-\alpha}\right) \right] \\ &\quad +(\beta-\alpha)^{2n-1} \int_\alpha^\beta G_n\left(\frac{x-\alpha}{\beta-\alpha}, \frac{s-\alpha}{\beta-\alpha}\right) \Phi^{(2n)}(s)ds. \end{aligned} \tag{2.18}$$

Further in the article, we will state our results for the class of n-convex functions, a more general class of functions that contains convex functions as a special case. We recall the basic definition and some properties of n-convex functions.

Definition 2.2. *The n-th order divided difference, $n \in \mathbb{N}_0$, of a function $\phi : [\alpha, \beta] \to \mathbb{R}$ at mutually distinct points $x_0, x_1, ..., x_n \in [\alpha, \beta]$ is defined recursively by*

$$[x_i; \phi] = \phi(x_i), \quad i = 0, ..., n$$

$$[x_0, ..., x_n; \phi] = \frac{[x_1, ..., x_n; \phi] - [x_0, ..., x_{n-1}; \phi]}{x_n - x_0}.$$

The value $[x_0, ..., x_n; \phi]$ is independent of the order of the points $x_0, ..., x_n$. A function $f : [\alpha, \beta] \to \mathbb{R}$ is n-convex if all its n-th order divided differences are non-negative, i. e. $[x_0, ..., x_n; f] \geq 0$ for all choices $x_i \in [\alpha, \beta]$. Thus, 0-convex functions are non-negative and 1-convex functions are non-decreasing, while 2-convex functions are convex in the classical sense. An n times differentiable is n-convex if and only if its n-derivative is non-negative (see [14]).

3. Main result

For settings given in the Introduction we start with our main result involving the Lidstone polynomial.

Theorem 3.1. *Let $n \in \mathbb{N}$, $n \geq 2$ and $\phi : I \to \mathbb{R}$ be such that $\phi^{(2(n-1))}$ is absolutely continuous and $\alpha, \beta \in I$, $\alpha < \beta$. Further, let A_k be as in* (2.13),

$\gamma \in \{1,2,3,4\}$, $\tilde{G}_\gamma$ as in (1.4)-(1.7), Λ_n as in (1.1) and u a weight function with v given by (2.15). Then

$$\begin{aligned}
&\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\
&= \int_\alpha^\beta \left[\int_{\Omega_2} \tilde{G}_\gamma(f(y),t)v(y)d\mu_2(y) - \int_{\Omega_1} \tilde{G}_\gamma(A_k f(x),t)u(x)d\mu_1(x) \right] \\
&\times \sum_{k=1}^{n-1} (\beta-\alpha)^{2k-2} \left[\phi^{(2k)}(\alpha)\Lambda_{k-1}\left(\frac{\beta-t}{\beta-\alpha}\right) + \phi^{(2k)}(\beta)\Lambda_{k-1}\left(\frac{t-\alpha}{\beta-\alpha}\right)\right] dt \\
&\quad + (\beta-\alpha)^{2n-3} \times \int_\alpha^\beta \phi^{(2n)}(s) \left[\int_\alpha^\beta \left(\int_{\Omega_2} \tilde{G}_\gamma(f(y),t)v(y)d\mu_2(y) \right.\right. \\
&\quad \left.\left. - \int_{\Omega_1} \tilde{G}_\gamma(A_k f(x),t)u(x)d\mu_1(x) \right) \times G_{n-1}\left(\frac{t-\alpha}{\beta-\alpha}, \frac{s-\alpha}{\beta-\alpha}\right) dt \right] ds. \qquad (3.19)
\end{aligned}$$

Proof. If we substitute Φ with Φ'' and n with $n-1$, then a function Φ defined by (2.18) becomes

$$\begin{aligned}
\phi''(t) \quad = \quad & \sum_{k=0}^{n-2} (\beta-\alpha)^{2k} \left[\phi^{(2k+2)}(\alpha)\Lambda_k\left(\frac{\beta-t}{\beta-\alpha}\right) + \phi^{(2k+2)}(\beta)\Lambda_k\left(\frac{t-\alpha}{\beta-\alpha}\right)\right] \\
+ \quad & (\beta-\alpha)^{2n-3} \int_\alpha^\beta G_{n-1}\left(\frac{t-\alpha}{\beta-\alpha}, \frac{s-\alpha}{\beta-\alpha}\right) \phi^{(2n)}(s)ds
\end{aligned}$$

which with a little calculation leads to

$$\begin{aligned}
\phi''(t) \quad = \quad & \sum_{k=1}^{n-1} (\beta-\alpha)^{2k-2} \left[\phi^{(2k)}(\alpha)\Lambda_{k-1}\left(\frac{\beta-t}{\beta-\alpha}\right) + \phi^{(2k)}(\beta)\Lambda_{k-1}\left(\frac{t-\alpha}{\beta-\alpha}\right)\right] \\
+ \quad & (\beta-\alpha)^{2n-3} \int_\alpha^\beta G_{n-1}\left(\frac{t-\alpha}{\beta-\alpha}, \frac{s-\alpha}{\beta-\alpha}\right) \phi^{(2n)}(s)ds. \qquad (3.20)
\end{aligned}$$

Finally, if we include Φ'' calculated in (3.20) into (2.17) we get the required result. $\square$

Starting from the result of the Theorem 3.1, depending on parity of n we have the following inequalities.

THEOREM 3.2. *Suppose that u, v A_k ,$\tilde{G}_\gamma$, $\gamma \in \{1,2,3,4\}$ and Λ_n be as in* Theorem 3.1. If $\phi : I \to \mathbb{R}$ is $2n$-convex, and

(i) n is odd number, then the inequality

$$\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x)$$
$$\geq \int_\alpha^\beta \left[\int_{\Omega_2} \tilde{G}_\gamma(f(y),t)v(y)d\mu_2(y) - \int_{\Omega_1} \tilde{G}_\gamma(A_k f(x),t)u(x)d\mu_1(x) \right]$$
$$\times \sum_{k=1}^{n-1} (\beta-\alpha)^{2k-2} \left[\phi^{(2k)}(\alpha)\Lambda_{k-1}\left(\frac{\beta-t}{\beta-\alpha}\right) + \phi^{(2k)}(\beta)\Lambda_{k-1}\left(\frac{t-\alpha}{\beta-\alpha}\right)\right] dt \tag{3.21}$$

holds.

(ii) n is even number, then the inequality

$$\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x)$$
$$\leq \int_\alpha^\beta \left[\int_{\Omega_2} \tilde{G}_\gamma(f(y),t)v(y)d\mu_2(y) - \int_{\Omega_1} \tilde{G}_\gamma(A_k f(x),t)u(x)d\mu_1(x) \right]$$
$$\times \sum_{k=1}^{n-1} (\beta-\alpha)^{2k-2} \left[\phi^{(2k)}(\alpha)\Lambda_{k-1}\left(\frac{\beta-t}{\beta-\alpha}\right) + \phi^{(2k)}(\beta)\Lambda_{k-1}\left(\frac{t-\alpha}{\beta-\alpha}\right)\right] dt \tag{3.22}$$

holds.

PROOF. (i) Since $\tilde{G}_\gamma(\cdot,t)$ is continuous and convex with respect to the first variable for each $\gamma \in \{1,2,3,4\}$ and $t \in [\alpha,\beta]$, according to the Theorem 3.1

$$\int_{\Omega_2} \tilde{G}_\gamma(f(y),t)v(y)d\mu_2(y) - \int_{\Omega_1} \tilde{G}_\gamma(A_k f(x),t)u(x)d\mu_1(x) \geq 0,$$

holds for every $t \in [\alpha,\beta]$. On the other hand $\tilde{G}_\gamma$ given by (1.4) - (1.7) are non-negative and so is ϕ^{2n} since ϕ is $2n$-convex. By definition (1.2) and (1.3) of the function G_n, we can see that if n is odd, then G_n is not positive. On the other hand, if n is positive, then G_n is non-negative. In (3.19) we have expression containing G_{n-1}. So, if $n-1$ is even, then n is odd and the required inequality (3.21) follows from the statement of Theorem 3.1.

(ii) Similarly, if $n-1$ is odd, then n is even so the inequality (3.22) holds. $\square$

Remark 3.3. If we consider the functional

$$\xi(\Phi) = \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x)$$

then for $2n$-convex function ϕ

$$\xi(\tilde{G}_\gamma(\cdot, t) \geq 0, t \in [\alpha, \beta]$$

holds.

In the next result we consider Hölder inequality for conjugate exponents p and q. As usual we suppose that $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The symbol $\| \cdot \|_p$ denotes the standard $L^p([\alpha, \beta])$ norm of a function, i. e.

$$\| g \|_p = \left(\int_\alpha^\beta |g(s)|^p ds \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$, while $\|g\|_\infty$ is the essential supremum of g.

Theorem 3.4. *Let $n \in \mathbb{N}$, $n \geq 2$ and $\phi : I \to \mathbb{R}$ be such that $\phi^{(2(n-1))}$ is absolutely continuous and $\phi^{(2n)} \in L^p[\alpha, \beta]$ for α, $\beta \in I$, $\alpha < \beta$. Further,let A_k be as in* (2.13), $\gamma \in \{1, 2, 3, 4\}$, $\tilde{G}_\gamma$ as in (1.4)-(1.7), Λ_n as in (1.1) and u a weight function with v given by (2.15). If (p, q) is a pair of conjugate exponents, then

$$\begin{aligned} |S_k(\phi)| \leq (\beta - \alpha)^{2n-3} \times \|\phi^{(2n)}\|_p \times \\ \left(\int_\alpha^\beta \left| \int_\alpha^\beta \left(\int_{\Omega_2} \tilde{G}_\gamma(f(y), t)v(y)d\mu_2(y) - \int_{\Omega_1} \tilde{G}_\gamma(A_k f(x), t)u(x)d\mu_1(x) \right) \right.\right. \\ \left.\left. \times\ G_{n-1}\left(\frac{t-\alpha}{\beta-\alpha}, \frac{s-\alpha}{\beta-\alpha} \right) dt \right|^q ds \right)^{\frac{1}{q}}, \end{aligned} \tag{3.23}$$

holds, where

$$
\begin{aligned}
S_k(\phi) &= \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\
&- \int_\alpha^\beta \left[\int_{\Omega_2} \tilde{G}_\gamma(f(y),t)v(y)d\mu_2(y) - \int_{\Omega_1} \tilde{G}_\gamma(A_k f(x),t)u(x)d\mu_1(x)\right] \\
&\times \sum_{k=1}^{n-1}(\beta-\alpha)^{2k-2}\left[\phi^{(2k)}(\alpha)\Lambda_{k-1}\left(\frac{\beta-t}{\beta-\alpha}\right) + \phi^{(2k)}(\beta)\Lambda_{k-1}\left(\frac{t-\alpha}{\beta-\alpha}\right)\right] dt.
\end{aligned} \tag{3.24}
$$

Proof. Applying the Hölder inequality on (3.19) considering the notation (3.24), we get

$$
\begin{aligned}
|S_k(\phi)| &= (\beta-\alpha)^{2n-3}\times \\
&\left|\int_\alpha^\beta \phi^{(2n)}\left(\int_\alpha^\beta\left(\int_{\Omega_2} \tilde{G}_\gamma(f(y),t)v(y)d\mu_2(y) - \int_{\Omega_1} \tilde{G}_\gamma(A_k f(x),t)u(x)d\mu_1(x)\right)\right.\right. \\
&\left.\left.\times\ G_{n-1}\left(\frac{t-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)dt\right)ds\right| \\
&\le (\beta-\alpha)^{2n-3}\times\|\phi^{(2n)}\|_p\times \\
&\left(\int_\alpha^\beta\left|\int_\alpha^\beta\left(\int_{\Omega_2} \tilde{G}_\gamma(f(y),t)v(y)d\mu_2(y) - \int_{\Omega_1} \tilde{G}_\gamma(A_k f(x),t)u(x)d\mu_1(x)\right)\right.\right. \\
&\left.\left.\times\ G_{n-1}\left(\frac{t-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)dt\right|^q ds\right)^{\frac{1}{q}}
\end{aligned}
$$

and obtain the required inequality. □

Remark 3.5. As special cases for boundary values p and q, from the inequality (3.23) we get the following inequalities:

$$
\begin{aligned}
|S_k(\phi)| &\le (\beta-\alpha)^{2n-3}\times\max|\phi^{(2n)}(s)|\times \\
&\left|\int_\alpha^\beta\int_\alpha^\beta\left(\int_{\Omega_2} \tilde{G}_\gamma(f(y),t)v(y)d\mu_2(y) - \int_{\Omega_1} \tilde{G}_\gamma(A_k f(x),t)u(x)d\mu_1(x)\right)\right. \\
&\left.\times\ G_{n-1}\left(\frac{t-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)dt\ ds\right|
\end{aligned}
$$

and

$$
\begin{aligned}
&|S_k(\phi)| \leq (\beta-\alpha)^{2n-3}\times \\
&\quad \max\left|\int_\alpha^\beta \int_\alpha^\beta \left(\int\limits_{\Omega_2} \tilde{G}_\gamma(f(y),t)v(y)d\mu_2(y) - \int\limits_{\Omega_1} \tilde{G}_\gamma(A_k f(x),t)u(x)d\mu_1(x)\right)\right. \\
&\qquad \left.\times\, G_{n-1}\left(\frac{t-\alpha}{\beta-\alpha}, \frac{s-\alpha}{\beta-\alpha}\right) dt\; ds\right| \times \left|\int_\alpha^\beta \phi^{(2n)}(s)ds\right|.
\end{aligned}
$$

4. Applications to the Čebyšev functional

Consider the Čebyšev functional,

$$
T(h,g) := \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)g(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)dt\cdot\frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)dt
$$

for Lebesgue integrable functions $h, g : [\alpha,\beta] \to \mathbb{R}$. Examples with upper bound obtained using Čebyšev functional can be found in [11]. The next two theorems from [5] provide Grüss and Ostrowski type inequalities involving the above functional.

Theorem 4.1. *Let $h, g : [\alpha,\beta] \to \mathbb{R}$ be two absolutely continuous functions with $(\cdot-\alpha)(\beta-\cdot)(h')^2, (\cdot-\alpha)(\beta-\cdot)(g')^2 \in L([\alpha,\beta])$. Then*

$$
|T(h,g)| \leq \frac{1}{\sqrt{2}}|T(h,h)|^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta (s-\alpha)(\beta-s)[g'(s)]^2 ds\right)^{\frac{1}{2}}. \tag{4.25}
$$

The constant $\frac{1}{\sqrt{2}}$ is the best possible in (4.25).

Theorem 4.2. *Assume that $g : [\alpha,\beta] \to \mathbb{R}$ is monotonic non-decreasing and $h : [\alpha,\beta] \to \mathbb{R}$ is absolutely continuous with $h' \in L^\infty([\alpha,\beta])$. Then*

$$
|T(h,g)| \leq \frac{1}{2(\beta-\alpha)}\|h'\|_\infty \int_\alpha^\beta (s-\alpha)(\beta-s)dg(s). \tag{4.26}
$$

The constant $\frac{1}{2}$ is the best possible in (4.26) .

To simplify notation, for $\gamma \in \{1,2,3,4\}$ we introduce the abbreviation $R_\gamma : [\alpha,\beta] \to \mathbb{R}$ in the form:

$$
\begin{aligned}
R_\gamma(s) &= \int_\alpha^\beta \left(\int\limits_{\Omega_2} \tilde{G}_\gamma(f(y),t)v(y)d\mu_2(y) - \int\limits_{\Omega_1} \tilde{G}_\gamma(A_k f(x),t)u(x)d\mu_1(x)\right) \\
&\times G_{n-1}\left(\frac{t-\alpha}{\beta-\alpha}, \frac{s-\alpha}{\beta-\alpha}\right) dt,
\end{aligned} \tag{4.27}
$$

where we assume that all the terms appearing in R_γ satisfy the assumptions of Theorem 3.1.

Theorem 4.3. *Let $n \in \mathbb{N}$, $n \geq 2$, R_γ be as in* (4.27) and $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi^{(2n)}$ is absolutely continuous with $(\cdot - \alpha)(\beta - \cdot)(\phi^{(2n+1)})^2 \in L([\alpha, \beta])$. If $(\cdot - \alpha)(\beta - \cdot)(R'_\gamma)^2 \in L([\alpha, \beta])$, then the remainder

$$\begin{aligned}
\varrho(\phi;\alpha,\beta) &= \int\limits_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int\limits_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\
&\quad - \int_\alpha^\beta \left[\int\limits_{\Omega_2} \tilde{G}_\gamma(f(y),t)v(y)d\mu_2(y) - \int\limits_{\Omega_1} \tilde{G}_\gamma(A_k f(x),t)u(x)d\mu_1(x) \right] \\
&\quad \times \sum_{k=1}^{n-1} (\beta-\alpha)^{2k-2} \left[\phi^{(2k)}(\alpha)\Lambda_{k-1}\left(\frac{\beta-t}{\beta-\alpha}\right) + \phi^{(2k)}(\beta)\Lambda_{k-1}\left(\frac{t-\alpha}{\beta-\alpha}\right) \right] dt \\
&\quad -(\beta-\alpha)^{2n-4}\left[\phi^{(2n-1)}(\beta) - \phi^{(2n-1)}(\alpha)\right] \int_\alpha^\beta R_\gamma(s)ds
\end{aligned} \tag{4.28}$$

is bounded by

$$|\varrho(\phi;\alpha,\beta)| \leq \frac{(\beta-\alpha)^{(2n-\frac{7}{2})}}{\sqrt{2}} |T(R_\gamma,R_\gamma)|^{\frac{1}{2}} \left(\int_\alpha^\beta (s-\alpha)(\beta-s)[\phi^{(2n+1)}(s)]^2 ds \right)^{\frac{1}{2}}. \tag{4.29}$$

Proof. From (3.19) and (4.28) we conclude

$$\begin{aligned}
\varrho(\phi;\alpha,\beta) &= (\beta-\alpha)^{2n-3} \int_\alpha^\beta R_\gamma(s)\phi^{(2n)}(s)ds \\
&\quad -(\beta-\alpha)^{(2n-4)}\left[\phi^{(2n-1)}(\beta) - \phi^{(2n-1)}(\alpha)\right] \int_\alpha^\beta R_\gamma(s)ds.
\end{aligned} \tag{4.30}$$

Assumptions of Theorem 4.1 are satisfied for $h = R_\gamma$ and $g = \phi^{(2n)}$, so

$$\begin{aligned}
&\left| \frac{1}{\beta-\alpha} \int_\alpha^\beta R_\gamma(s)\phi^{(2n)}(s)ds - \frac{1}{\beta-\alpha}\int_\alpha^\beta R_\gamma(s)ds \cdot \frac{1}{\beta-\alpha}\int_\alpha^\beta \phi^{(2n)}(s)ds \right| \\
&\leq \frac{1}{\sqrt{2}} |T(R_\gamma,R_\gamma)|^{\frac{1}{2}} \frac{1}{\sqrt{\beta-\alpha}} \left(\int_\alpha^\beta (s-\alpha)(\beta-s)[\phi^{(2n+1)}(s)]^2 ds \right)^{\frac{1}{2}}.
\end{aligned} \tag{4.31}$$

Therefore from (4.30) and (4.31) we get (4.29).

□

Theorem 4.4. *Let $n \in \mathbb{N}$, $n \geq 2$, R_γ be as in* (4.27) and $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi^{(2n)}$ is monotonic non-decreasing. If R_γ is absolutely continuous with $R'_\gamma \in L^\infty([\alpha, \beta])$, then the remainder $\varrho(\phi; \alpha, \beta)$ given by (4.28) is bounded by

$$\begin{aligned}
&|\varrho(\phi; \alpha, \beta)| \leq \\
&\quad (\beta-\alpha)^{(2n-3)} \|R'_\gamma\|_\infty \left[\frac{(\beta-\alpha)\left(\phi^{(2n-1)}(\beta) + \phi^{(2n-1)}(\alpha)\right)}{2} \right. \\
&\quad \left. - \left\{ \phi^{(2n-2)}(\beta) - \phi^{(2n-2)}(\alpha) \right\} \right]. \qquad (4.32)
\end{aligned}$$

Proof. Assumptions of Theorem 4.2 are satisfied for $h = R_\gamma$ and $g = \phi^{(2n)}$, so, taking into account (4.30), we have

$$\begin{aligned}
&\left| \frac{1}{\beta-\alpha} \int_\alpha^\beta R_\gamma(s) \phi^{(2n)}(s) ds - \frac{1}{\beta-\alpha} \int_\alpha^\beta R_\gamma(s) ds \cdot \frac{1}{\beta-\alpha} \int_\alpha^\beta \phi^{(2n)}(s) ds \right| \\
&\leq \frac{1}{2(\beta-\alpha)} \left\| R'_\gamma \right\|_\infty \int_\alpha^\beta (s-\alpha)(\beta-s) \phi^{(2n+1)}(s) ds. \qquad (4.33)
\end{aligned}$$

Simple calculation yields

$$\begin{aligned}
&\int_\alpha^\beta (s-\alpha)(\beta-s)\phi^{(2n+1)}(s) ds = \int_\alpha^\beta \left[2s - (\alpha+\beta)\right] \phi^{(2n)}(s) ds \\
&= (\beta-\alpha)\left[\phi^{(2n-1)}(\beta) + \phi^{(2n-1)}(\alpha)\right] - 2\left[\phi^{(2n-2)}(\beta) - \phi^{(2n-2)}(\alpha)\right].
\end{aligned}$$

Finally, inserting the last expression in (4.33) and taking into account (4.30) we get (4.32). ∎

The last theorem gives Ostrowski-type bound for the generalized Hardy's inequality. About Ostrowski-type inequalities can be found i.e. in [3] and [10].

Theorem 4.5. *Let $n \in \mathbb{N}$, $n \geq 2$, R_γ be as in* (4.27), $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that $\|\phi^{(2n)}\|_p < \infty$. Then

$$\Bigg| \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x)$$
$$- \int_\alpha^\beta \left[\int_{\Omega_2} \tilde{G}_\gamma(f(y), t)v(y)d\mu_2(y) - \int_{\Omega_1} \tilde{G}_\gamma(A_k f(x), t)u(x)d\mu_1(x) \right]$$
$$\times \sum_{k=1}^{n-1} (\beta - \alpha)^{2k-2} \left[\phi^{(2k)}(\alpha)\Lambda_{k-1}\left(\frac{\beta - t}{\beta - \alpha}\right) + \phi^{(2k)}(\beta)\Lambda_{k-1}\left(\frac{t - \alpha}{\beta - \alpha}\right)\right] dt \Bigg|$$
$$\leq (\beta - \alpha)^{2n-3} \left\| \phi^{(2n)} \right\|_p \|R_\gamma\|_q .$$

The constant $(\beta - \alpha)^{2n-3} \|R_\gamma\|_q$ is sharp when $1 < p \leq \infty$ and the best possible when $p = 1$.

In the future, we will try to further develop the idea and method using various known functionals.

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Poopćenje nejednakosti Hardyjevog tipa putem Lidstonovog interpolacijskog polinoma i novih Greenovih funkcija

Dora Pokaz

Sažetak. Za poopćeno okruženje, koje uključuje prostore mjera sa pozitivnim σ-konačnim mjerama, predstavili smo rezultat vezan uz Hardyjevu nejednakost. Uspostavili smo vezu izmedu operatora razlike dobivene pomoću Hardyjeve nejednakosti te izraza koji sadrži Lidstonov interpolacijski polinom i četiri nove Greenove funkcije. Raspravljali smo o $2n$ konveksnosti funkcije te dali rezultat u ovisnosti o parnosti dijela eksponenta i indeksa n. Primjenom Hölderove nejednakosti za konjugirane eksponente p i q dobili smo daljnje rezultate. Konačno, izveli smo gornje ograde za naš identitet uz pomoć Čebyšeljevog funkcionala te ogradu tipa Ostrowskog za generaliziranu Hardyjevu nejednakost.

First author
Faculty of Civil Engineering
University of Zagreb
10 000 Zagreb, Croatia
E-mail: dora.pokaz@grad.unizg.hr