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On degrees in family of maps constructed via modular forms

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ON DEGREES IN FAMILY OF MAPS CONSTRUCTED VIA MODULAR FORMS

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Abstract. This paper is a continuation of our previous works (see Muić in Monatsh. Math. 180, no. 3, 607–629, (2016)) and (Muić, Kodrnja in Ramanujan J. 55, no. 2, 393–420, (2021)) where we have studied maps from $X_0(N)$ into $\mathbb{P}^2$ (and more general) constructed via modular forms of the same weight. In this short note we study how degrees of the maps and degrees of the resulting curve change when we let modular forms vary.

1. Introduction

In our earlier paper [13] we gave fairly general study of complex holomorphic maps $X_0(N) \longrightarrow \mathbb{P}^2$ (and more general) and proved a formula for the degrees [13, Theorem 1-4] described below in Theorem 1.1. Based on [13, Theorem 1-4], we developed the test for birationality of the maps (see the introduction in [13]). Using these results, the problems of constructing birational maps into $\mathbb{P}^2$ has been studied in [14] with emphasis on the explicit computations in SAGE. The paper [14] constructs various models over $\mathbb{C}$ of $X_0(N)$ complementing previous works such as [2], [3], [4], [5], [7], [10], [12], [17] and [20]. We also mention interesting new direction regarding smooth models [1]. The purpose of the present short note is to study how degrees of the maps and degrees of the resulting curve change when we let modular forms vary. The main result is Theorem 1.2 (see below).

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We continue by recalling some standard facts from [8]. Let $\mathbb{H}$ be the upper half–plane. Then the group $SL_2(\mathbb{R})$ acts on $\mathbb{H}$ as follows:

$$g.z = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).$$

We let $j(g,z) = cz+d$. The function j satisfies the cocycle identity:

$$j(gg', z) = j(g, g'.z)j(g', z). \tag{1.1}$$

Next, $SL_2(\mathbb{R})$–invariant measure on $\mathbb{H}$ is defined by $dxdy/y^2$, where the coordinates on $\mathbb{H}$ are written in a usual way $z = x + \sqrt{-1}y$, $y > 0$. A discrete subgroup $\Gamma \subset SL_2(\mathbb{R})$ is called a Fuchsian group of the first kind if

$$\iint_{\Gamma\backslash\mathbb{H}} \frac{dxdy}{y^2} < \infty.$$

Then, adding a finite number of points in $\mathbb{R} \cup \{\infty\}$ called cusps, $\mathcal{F}_\Gamma$ can be compactified. In this way we obtain a compact Riemann surface $\mathfrak{R}_\Gamma$. Let $g(\Gamma)$ be the genus of $\mathfrak{R}_\Gamma$. One of the most important examples are the groups

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); \quad c \equiv 0 \pmod{N} \right\}, \quad N \geq 1.$$

We write $X_0(N)$ for $\mathfrak{R}_{\Gamma_0(N)}$. As usual we consider $\mathfrak{R}_\Gamma$ as a smooth irreducible projective curve over $\mathbb{C}$ with the field of rational functions $\mathbb{C}(\mathfrak{R}_\Gamma)$ to be the field of meromorphic functions on $\mathfrak{R}_\Gamma$.

Let Γ be a Fuchsian group of the first kind. Let χ be a character $\Gamma \to \mathbb{C}^\times$ of finite order. Let $m \geq 1$. We consider the space $M_m(\Gamma, \chi)$ (resp., $S_m(\Gamma, \chi)$) of all modular (resp., cuspidal) forms of weight m; this is a space of all holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ such that $f(\gamma.z) = \chi(\gamma) j(\gamma, z)^m f(z)$ ($z \in \mathbb{H}$, $\gamma \in \Gamma$) which are holomorphic (resp., holomorphic and vanish) at every cusp for Γ. When χ is trivial, we write $M_m(\Gamma)$ and $S_m(\Gamma)$ instead of $M_m(\Gamma, \chi)$ and $S_m(\Gamma, \chi)$, respectively.

Assume that $\dim M_m(\Gamma, \chi) \geq 3$. We select three linearly independent modular forms f, g, and h in $M_m(\Gamma, \chi)$, and construct the holomorphic map $\mathfrak{R}_\Gamma \longrightarrow \mathbb{P}^2$ which is uniquely determined by being initially defined by

$$z \longmapsto (f(z) : g(z) : h(z)) \tag{1.2}$$

on the complement of a finite set of Γ–orbits in $\mathfrak{R}_\Gamma$ of common zeroes of f, g and h. The image is an irreducible projective plane curve, which we denote by $\mathcal{C}(f, g, h)$ (see [13, Lemma 3-1]). We denote by

$$\mathbb{C}(\mathcal{C}(f, g, h))$$

the field of rational functions on $\mathcal{C}(f, g, h)$. It can be realized as a subfield of the field $\mathbb{C}(\mathfrak{R}_\Gamma)$ of rational functions on $\mathfrak{R}_\Gamma$ generated over $\mathbb{C}$ by g/f and h/f. By the usual definition, the degree of the map (1.2), denoted by

$$d(f, g, h),$$

is the degree of the field extension

$$\mathbb{C}\left(\mathcal{C}(f,g,h)\right) \subset \mathbb{C}\left(\mathfrak{R}_\Gamma\right).$$

Let l be a line in $\mathbb{P}^2$ in general position with respect to $\mathcal{C}(f,g,h)$. Then, it intersects $\mathcal{C}(f,g,h)$ in different points the number of which is the degree of $\mathcal{C}(f,g,h)$. We denote the degree of $\mathcal{C}(f,g,h)$ by $\deg \mathcal{C}(f,g,h)$. The main result of [13] (see [13, Theorem 1-4]) proves the following:

THEOREM 1.1. *Assume that* $\dim M_m(\Gamma,\chi) \geq 3$. *Assume that* $f,g,h \in M_m(\Gamma,\chi)$ *are linearly independent. Then, we have the following:*

$$d(f,g,h)\cdot \deg \mathcal{C}(f,g,h) = \eta_m(f,g,h),$$

where

$$\eta_m(f,g,h) \overset{def}{=} \frac{m}{4\pi}\iint_{\Gamma\backslash\mathbb{H}} \frac{dxdy}{y^2} - \sum_{\mathfrak{a}\in\mathfrak{R}_\Gamma} \min\left(\nu_{\mathfrak{a}}(f), \nu_{\mathfrak{a}}(g), \nu_{\mathfrak{a}}(h)\right).$$

Now, we discuss the main result of the present paper. We introduce some notation. Let $\Xi \subset M_m(\Gamma,\chi)$ be a subspace such that $\dim \Xi \geq 3$. Assume that $f,g \in \Xi$ are linearly independent. For each $l \geq 1$ (resp., $k \geq 1$), we let X_l (resp., Z_k) be the set of all $h \in \Xi - (\mathbb{C}f + \mathbb{C}g)$ such that $\deg \mathcal{C}(f,g,h) = l$ (resp., $d(f,g,h) = k$). By Theorem 1.1, we have

$$X_l \text{ and } Z_k \text{ are empty for } k,l > \frac{m}{4\pi}\iint_{\Gamma\backslash\mathbb{H}} \frac{dxdy}{y^2}.$$

Obviously, we have

$$\Xi - (\mathbb{C}f + \mathbb{C}g) = \cup_{l\geq 1}\, X_l = \cup_{k\geq 1}\, Z_k.$$

We remark that $X_1 = \emptyset$ since f, g, and h are linearly independent. The main result of the present paper is the following theorem:

THEOREM 1.2. *Let* $\Xi \subset M_m(\Gamma,\chi)$ *be a subspace such that* $\dim \Xi \geq 3$. *We equip* Ξ *with Zariski topology. Assume that* $f,g \in \Xi$ *are linearly independent. Then, we have the following:*

(i) *The sets* X_l *are locally closed. The set of all* $h \in \Xi - (\mathbb{C}f + \mathbb{C}g)$ *such that* $\deg \mathcal{C}(f,g,h)$ *is largest possible is open (i.e., the largest* L *such that* $X_L \neq \emptyset$*).*

(ii) *For each* l *such that* $X_l \neq \emptyset$*, the set of all* $h \in X_l$ *such that* $d(f,g,h)$ *is smallest possible in* X_l *is an open set in* X_l.

(iii) *The sets* Z_k *are constructible. The set* Z_k *is empty set unless* k *divides* $[\mathbb{C}(\mathfrak{R}_\Gamma) : \mathbb{C}(g/f)]$.

Recall that we say that Ξ **determines the field of rational functions** $\mathbb{C}(\mathfrak{R}_\Gamma)$ if there exists a basis $f_0,\ldots,f_{s-1}$ of W, such that $\mathbb{C}(\mathfrak{R}_\Gamma)$ is generated over $\mathbb{C}$ by the quotients f_i/f_0, $1 \leq i \leq s-1$ (see [14, Definition 1.3]). The notion is independent of the basis. The introduction of [14] contains many examples of such spaces Ξ (called W there). For example, we may take $\Xi = S_2(\Gamma)$ if Γ is not hyperelliptic ([15] has determined all $\Gamma_0(N)$ such that $X_0(N)$ is not hyperelliptic (implies $g(\Gamma_0(N)) \geq 3$)). Also, for $m \geq 4$ is even, if $\dim S_m(\Gamma) \geq \max(g(\Gamma)+2, 3)$, then we can take $\Xi = S_m(\Gamma)$ by general theory of algebraic curves [11, Corollary 3.4].

Corollary 1.3. *Maintaining assumptions of Theorem 1.2, we assume also that Ξ determines the field of rational functions $\mathbb{C}(\mathfrak{R}_\Gamma)$. Let L be the largest possible such that $X_L \neq 0$. Then, $Z_1 \cap X_L$ contains a non–empty open set. In other words, the set of all $h \in \Xi - (\mathbb{C}f + \mathbb{C}g)$ such that $\deg \mathcal{C}(f,g,h)$ is largest possible and $d(f,g,h) = 1$ is non–empty open set.*

Proof. By [14, Theorem 1.4], Z_1 contains a non–empty open set. On the other hand, by Theorem 1.2, X_L is open. The corollary follows. $\square$

The corollary improves on [14, Theorem 1.4] since in the language of the present paper there we could just construct open set in $\Xi - (\mathbb{C}f + \mathbb{C}g)$ such that $d(f,g,h) = 1$ without control of $\deg \mathcal{C}(f,g,h)$. The corollary also generalizes [14, Corollary 3.7] with a similar conclusion but with more restrictive assumptions.

2. The Proof of Theorem 1.2

Let $h \in \Xi - (\mathbb{C}f + \mathbb{C}g)$. Let $P_{f,g,h}$ be an irreducible homogeneous polynomial which locus is $C(f,g,h)$. Equivalently, $P_{f,g,h}(f(z), g(z), h(z)) = 0$, for all $z \in \mathbb{H}$. The polynomial $P_{f,g,h}$ is unique up to a multiplication by a non–zero constant. The dehomogenization $Q_{f,g,h}$ of $P_{f,g,h}$ with respect to the last variable satisfies $Q_{f,g,h}(g/f, h/f) = 0$ in the field of rational functions $\mathbb{C}(\mathfrak{R}_\Gamma)$. It is very easy to check that $Q_{f,g,h}(g/f, \cdot)$ is irreducible as a polynomial with coefficients in the field $\mathbb{C}(g/f)$. Thus, it is a minimal polynomial of h/f over $\mathbb{C}(g/f)$. Hence, it is equal to the degree of the field extension $\mathbb{C}(g/f) \subset \mathbb{C}(\mathcal{C}(f,g,h))$

$$[\mathcal{C}(f,g,h) : \mathbb{C}(g/f)] = \deg Q_{f,g,h}(g/f, \cdot).$$

If we consider the field extensions

$$\mathbb{C}(g/f) \subset \mathbb{C}(\mathcal{C}(f,g,h)) \subset \mathbb{C}(\mathfrak{R}_\Gamma),$$

and compute their degrees, then we obtain the next lemma.

Lemma 2.1. *For $h \in \Xi - (\mathbb{C}f + \mathbb{C}g)$, the product $\deg Q_{f,g,h}(g/f, \cdot) \cdot d(f,g,h)$ does not depend on h. It is equal to the degree $[\mathbb{C}(\mathfrak{R}_\Gamma) : \mathbb{C}(g/f)]$ (i.e, to the degree of divisor of poles of g/f). In particular, Z_k is empty set unless k divides $[\mathbb{C}(\mathfrak{R}_\Gamma) : \mathbb{C}(g/f)]$.*

We continue with the following two lemmas:

LEMMA 2.2. *For each $k \geq 1$, let Y_k be the set of all $h \in \Xi-(\mathbb{C}f+\mathbb{C}g)$ such that $\deg Q_{f,g,h}(g/f,\cdot)=k$. Then, Y_k is empty unless k divides $[\mathbb{C}(\mathfrak{R}_\Gamma):\mathbb{C}(g/f)]$. If this is so, we have $Z_k = Y_{[\mathbb{C}(\mathfrak{R}_\Gamma):\mathbb{C}(g/f)]/k}$.*

PROOF. This follows immediately from Lemma 2.1. □

LEMMA 2.3. *For each $l \geq 1$, X_l is the set of all $h \in \Xi-(\mathbb{C}f+\mathbb{C}g)$ such that $\deg P_{f,g,h}=l$*

PROOF. It is well–known and easy to check directly that $\deg \mathcal{C}(f,g,h) = \deg P_{f,g,h}$. □

LEMMA 2.4. *For all $h \in \Xi-(\mathbb{C}f+\mathbb{C}g)$, we have $\deg Q_{f,g,h}(g/f,\cdot) \leq \deg P_{f,g,h} \leq \eta_m(f,g,h) \leq \frac{m}{4\pi}\iint_{\Gamma\backslash\mathbb{H}}\frac{dxdy}{y^2}$.*

PROOF. This follows from Theorem 1.1 and the fact that $\deg \mathcal{C}(f,g,h) = \deg P_{f,g,h}$. □

Now, we prove the key lemma.

LEMMA 2.5. *We have the following:*

- (i) *The sets X_l are locally closed, $X_1=\emptyset$, and the set of all $h \in \Xi-(\mathbb{C}f+\mathbb{C}g)$ such that $\deg P_{f,g,h}$ is largest possible is open (well–defined because of Lemma 2.4).*
- (ii) *For each l such that $X_l \neq \emptyset$, the set of all $h \in X_l$ such that $\deg Q_{f,g,h}$ is largest possible in X_l is an open set in X_l.*
- (iii) *The sets $Y_k \cap X_l$ are locally closed, and Y_k are constructible sets for all k,l.*

PROOF. We let $h=\lambda_0 f+\lambda_1 g+\lambda_2 f_2+\cdots+\lambda_{s-1}f_{s-1} \in \Xi-(\mathbb{C}f+\mathbb{C}g)$. Let $l \geq 1$ be an integer. For $\alpha=(\alpha_0,\alpha_1,\alpha_2) \in \mathbb{Z}^3_{\geq 0}$ such that $|\alpha| \overset{def}{=} \alpha_0+\alpha_1+\alpha_2=l$, and $(i_0,i_1,\ldots,i_{s-1}) \in \mathbb{Z}^s_{\geq 0}$ such that $\sum_{j=0}^{s-1} i_j=\alpha_2$, we consider a cusp form

$$f^{\alpha_0+i_0}g^{\alpha_1+i_1}f_2^{i_2}\cdots f_{s-1}^{i_{s-1}}=\sum_{n=1}^{\infty} b_n(\alpha,i_0,i_1,\ldots,i_{s-1})q^n \in S_{lm}(\Gamma).$$

We define homogeneous polynomials for all $n \geq 1$ as follows:

$$B_{\alpha,n}(\lambda_0,\ldots,\lambda_{s-1}) = \tag{2.3}$$
$$=\sum_{i_0+i_1+\cdots+i_{s-1}=\alpha_2}\binom{\alpha_2}{i_0,i_1,\ldots,i_{s-1}}b_n(\alpha,i_0,i_1,\ldots,i_{s-1})\lambda_0^{i_0}\cdots\lambda_{s-1}^{i_{s-1}}.$$

We order all $\binom{l+2}{2}$ α's in the lexicographical order: $(0,0,l) < (0,1,l-1) < \cdots < (l,0,0)$. Then, we can consider vectors:

$$
\begin{aligned}
(2.4)\quad & C_n\left(\lambda_0, \ldots, \lambda_{s-1}\right) \stackrel{def}{=} \\
& \stackrel{def}{=} \left(B_{(0,0,l),n}\left(\lambda_0, \ldots, \lambda_{s-1}\right), \ldots, B_{(l,0,0),n}\left(\lambda_0, \ldots, \lambda_{s-1}\right)\right) \in \mathbb{C}^{\binom{l+2}{2}}.
\end{aligned}
$$

Let $C\left(\lambda_0, \ldots, \lambda_{s-1}\right)$ be an infinite matrix with rows $C_n\left(\lambda_0, \ldots, \lambda_{s-1}\right)$. Next, the reader can easily check that $(a_{(0,0,l)}, \ldots, a_{(l,0,0)}) \in \mathbb{C}^{\binom{l+2}{2}}$ is the solution of the system of homogeneous equations: [1]

$$
\sum_{\alpha} B_{\alpha,n}\left(\lambda_0, \ldots, \lambda_{s-1}\right) a_\alpha = 0, \quad n \geq 1, \tag{2.5}
$$

if and only if

$$
\sum_{\alpha} a_\alpha f(z)^{\alpha_0} g(z)^{\alpha_1} h(z)^{\alpha_2} = 0, \quad z \in \mathbb{H}.
$$

Equivalently, the corresponding homogeneous polynomial belongs to the ideal of the curve $\mathcal{C}(f,g,h)$. This implies that the system (2.5) has only a trivial solution if $l < \deg P_{f,g,h}$ while for $l \geq \deg P_{f,g,h}$ there always exist a non–trivial solution. The solution is unique up to a multiplication by a non–zero element in $\mathbb{C}$ if $l = \deg P_{f,g,h}$, and determines $P_{f,g,h}$. On the other hand, by Linear algebra, the system (2.5) has a zero solution if and only if $\mathbb{C}$–span of vectors $C_n\left(\lambda_0, \ldots, \lambda_{s-1}\right)$, $n \geq 1$, is whole $\mathbb{C}^{\binom{l+2}{2}}$. Equivalently, there is a minor of $C\left(\lambda_0, \ldots, \lambda_{s-1}\right)$ of size $\binom{l+2}{2} \times \binom{l+2}{2}$ which is non–zero. Similarly, when $l = \deg P_{f,g,h}$, $\mathbb{C}$–span of vectors $C_n\left(\lambda_0, \ldots, \lambda_{s-1}\right)$, $n \geq 1$, must be of codimension one in $\mathbb{C}^{\binom{l+2}{2}}$. Equivalently, there must exists a non–zero minor of $C\left(\lambda_0, \ldots, \lambda_{s-1}\right)$ of size $\binom{l+2}{2} - 1 \times \binom{l+2}{2} - 1$ while all minors of size size $\binom{l+2}{2} \times \binom{l+2}{2}$ must be equal to zero.

Above discussion implies the first claim in (i) i.e., the sets X_l are locally closed. $X_1 = \emptyset$ since f, g, and h are linearly independent. Let L be the largest possible degree of $\deg P_{f,g,h}$ when h ranges over $\Xi - (\mathbb{C}f + \mathbb{C}g)$. We prove that X_L is open. First we note that all $\binom{L+2}{2} \times \binom{L+2}{2}$ minors are identically equal to zero on $\mathbb{C}^s$. Indeed, if there would be a non–identically zero minor, say M, of that size, then the system (2.5) has a trivial solution when $M(\lambda_0, \ldots, \lambda_{s-1}) \neq 0$. Hence, $\deg P_{f,g,h} > L$ for h such that $M(\lambda_0, \ldots, \lambda_{s-1}) \neq 0$ which is a contradiction. Now, X_L is open since $h \in X_L$ if and only if there exists a minor M of size $\binom{L+2}{2} - 1 \times \binom{L+2}{2} - 1$ such that $M(\lambda_0, \ldots, \lambda_{s-1}) \neq 0$. This completes the proof of (i).

Now, we prove (ii). Assume that l satisfies $X_l \neq \emptyset$. Let L be the largest possible degree of $\deg Q_{f,g,h}$ when h ranges over X_l. Let $h \in X_l$. Then, the solution of the corresponding system (2.5) satisfies $a_\beta = 0$ for all β of the

[1] By using Sturm bound for $S_{lm}(\Gamma)$ we can bound the number of equation, but this is not important here.

form $\beta = (\beta_0, \beta_1, L')$ with $L' > L$. Of course, for $h \in X_l$ such that $a_\beta \neq 0$, for some β of the form (β_0, β_1, L), we have $\deg Q_{f,g,h} = L$.

Open sets $M(\lambda_0, \ldots, \lambda_{s-1}) \neq 0$, where M ranges over all minors $\binom{l+2}{2} - 1 \times \binom{l+2}{2} - 1$ cover X_l. Let us fix such minor M. Then, M is obtained by removing some column (and using $\binom{l+2}{2} - 1$ rows but they are not important here), say γ-th. Then, rewriting the system (2.5) in the form

$$\sum_{\alpha \neq \gamma} B_{\alpha,n}(\lambda_0, \ldots, \lambda_{s-1})\, a_\alpha = -B_{\gamma,n}(\lambda_0, \ldots, \lambda_{s-1})\, a_\gamma, \quad n \geq 1,$$

we see that the solution is given by

$$a_\alpha = -a_\gamma \frac{M^\alpha(\lambda_0, \ldots, \lambda_{s-1})}{M(\lambda_0, \ldots, \lambda_{s-1})}, \quad \text{for all} \quad \alpha \neq a_\gamma, \tag{2.6}$$

where the determinant M^α is obtained from M by replacing α-th column with the column of the corresponding $B_{\gamma,n}$'s (i.e., using n's that determine rows of M). Using (2.6), we see that $a_\gamma \neq 0$. Also, we see that $a_\alpha \neq 0$ if and only if $M^\alpha(\lambda_0, \ldots, \lambda_{s-1}) \neq 0$. Let us fix now arbitrary α. Then, letting M vary, we see that the set of all $h \in X_l$ such that $a_\alpha \neq 0$ is open in X_l. This immediately implies that the set of all h such that one of the coefficients a_β, for β of the form (β_0, β_1, L), is non–zero is open. This completes the proof of (ii).

For the claim (iii), we recall that a constructible set is a finite union of locally closed sets. Then, it is enough to prove that $Y_k \cap X_l$ is locally closed for each l. As in the previous part of the proof, this intersection corresponds to the solution of the system (2.5) such that $a_\alpha = 0$, for all α of the form $\alpha = (\alpha_0, \alpha_1, k')$ with $k' > k$, and there exists β such that $a_\beta \neq 0$ where $\beta = (\beta_0, \beta_1, k)$. As before, using (2.6) this set is intersection of one closed set, one open set and X_l. But since X_l is itself intersection of a closed and an open set, see the same holds for $Y_k \cap X_l$. This means that $Y_k \cap X_l$ is locally closed. This completes the proof of the lemma. □

Finally, we complete the proof of Theorem 1.2. (i) follows from Lemma 2.5 (i) and Lemma 2.3. (ii) follows from Lemma 2.5 (ii) and Lemma 2.2. The first claim in (iii) follows also from Lemma 2.2 and Lemma 2.5 (iii). The second claim of (iii) is contained in Lemma 2.1.

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O stupnjevima u familiji preslikavanja konstruiranih preko modularnih formi

Goran Muić

Sažetak. Ovaj rad je nastavak naših ranijih radova (vidi Muić, Monatsh. Math. 180, no. 3, 607–629, (2016)) i (Muić, Kodrnja, Ramanujan J. 55, no. 2, 393–420, (2021)) u kojima proučavamo preslikavanja s $X_0(N)$ u $\mathbb{P}^2$ (i općenitije) konstruiranih preko modularnih formi iste težine. U ovom kratkom priopćenju proučavamo kako se stupnjevi preslikavanja i stupnjevi odgovarajućih krivulja mijenjanju kada dopustimo da se modularne forme mijenjaju.

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