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A. J. Macleod

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SMALL-DEGREE PARAMETRIC SOLUTIONS FOR DEGREE 6 AND 7 IDEAL MULTIGRADES

Allan J. Macleod

Abstract. We derive parametric solutions for 7 and 8 term ideal multigrades. These are of significantly smaller degree than previous solutions, such as those of Chernick.

1. Introduction

A multigrade of degree N is an integer solution to

$$X_1^i + X_2^i + \ldots + X_M^i = Y_1^i + Y_2^i + \ldots + Y_M^i, \qquad i = 1, 2, \ldots, N\,, \tag{1.1}$$

where the sets $\{X_1, X_2, \ldots, X_M\} \neq \{Y_1, Y_2, \ldots, Y_M\}$. If they are just permutations, we call this a trivial solution. The book by Gloden [3] is the standard reference, though out-of-print for decades.

We write this as

$$X_1, X_2, \ldots, X_M \stackrel{N}{=} Y_1, Y_2, \ldots, Y_M\,.$$

An old theorem of Bastien states that a solution only exists when $M > N$. An "ideal" solution satisfies $M = N + 1$, and we will concentrate on this type of solution.

Numerical ideal solutions are known for degrees $N = 1-9$ and degree $N = 11$, see the web-site of Chen Shuwen [7]. Parametric solutions are only known

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for degrees $N = 1 - 7$, see Chernick [1]. In fact, for degree $N = 8$, only 2 numerical solutions are known! For degree $N = 9$, there are an infinite number of solutions parameterized by points on an elliptic curve, see Smyth [8].

The parametric solutions quoted by Chernick are small for degrees $1-5$, for example the following is a degree 5 solution

$$A_1, A_2, A_3, -A_1, -A_2, -A_3 \stackrel{5}{=} B_1, B_2, B_3, -B_1, -B_2, -B_3\,, \tag{1.2}$$

where

$$A_1 = -5t^2 + 4t - 3 \qquad A_2 = -3t^2 + 6t + 5 \qquad A_3 = -t^2 - 10t - 1\,,$$
$$B_1 = -5t^2 + 6t + 3 \qquad B_2 = -3t^2 - 4t - 5 \qquad B_3 = -t^2 + 10t - 1\,,$$

with $t \in \mathbb{Q}$.

For degree 6 and 7, the parametric solutions have much larger degree. In fact, he does not give these latter forms explicitly. These are the only parametric solutions quoted in Chen Shuwen's web-site [7]. Recently, Ajai Choudhry [2] presented a very nice method which produces simpler solutions.

The purpose of this note is to develop much simpler degree $6-7$ forms, by different methods, in the hope that they might suggest forms for degree 8 and higher.

2. Degree 6 Parametric Forms

We follow the basic method used by Chernick. Consider the relation
(2.1)

$$U_1, U_2, U_3, U_4, -V_1, -V_2, -V_3, -V_4 \stackrel{6}{=} -U_1, -U_2, -U_3, -U_4, V_1, V_2, V_3, V_4\,,$$

which automatically satisfies the degree $2, 4, 6$ relations. For odd degree, we have

$$U_1^n + U_2^n + U_3^n + U_4^n = V_1^n + V_2^n + V_3^n + V_4^n \qquad n = 1, 3, 5\,. \tag{2.2}$$

Define

$$U_1 = -X_1 + X_2 + X_3 \qquad U_2 = X_1 - X_2 + X_3\,,$$
$$U_3 = X_1 + X_2 - X_3 \qquad U_4 = -X_1 - X_2 - X_3\,,$$
$$V_1 = -Y_1 + Y_2 + Y_3 \qquad V_2 = Y_1 - Y_2 + Y_3\,,$$
$$V_3 = Y_1 + Y_2 - Y_3 \qquad V_4 = -Y_1 - Y_2 - Y_3\,,$$

Then the $n = 1$ identity of (2.2) is satisfied, and we have the following from the $n = 3$ and $n = 5$

$$X_1X_2X_3 = Y_1Y_2Y_3 \qquad X_1^2 + X_2^2 + X_3^2 = Y_1^2 + Y_2^2 + Y_3^2\,, \tag{2.3}$$

Chernick sets $U_1 = 0$ to give an ideal multigrade of degree 6, which corresponds to the constraint $X_1 = X_2 + X_3$. He also defines the variable $t = X_2/Y_1$, giving (2.3) as the two equations

$$(2t^2 - 1)Y_1^2 + 2X_3^2 + 2X_3Y_1t - Y_2^2 - Y_3^2 = 0\,, \tag{2.4}$$

and

$$X_3^2t + X_3Y_1t^2 - Y_2Y_3 = 0\,. \tag{2.5}$$

The latter equation gives $2X_3^2 + 2X_3Y_1t = 2Y_2Y_3/t$ so (2.4) is

$$Y_1^2(2t^2 - 1) - Y_2^2 + 2Y_2Y_3/t - Y_3^2 = 0\,,$$

and this quadric can be parameterized in the usual way. One possibility is

$$Y_1 = k^2(2t^2 - 1) + 2kt + 1 \tag{2.6}$$

$$Y_2 = k^2(t - 1)(2t^2 - 1) + 2k(2t^2 - 1) + t + 1 \tag{2.7}$$

$$Y_3 = t(1 - k^2(2t^2 - 1))\,, \tag{2.8}$$

where k is a rational parameter.

Substituting these in (2.5) gives a quadratic equation for X_3 which we want to have rational solutions. Thus, the discriminant must be a rational square, so

$$\begin{aligned}\square = (t-2)^2(2t^2-1)^2k^4 + 4(2t^2-1)(t^3-4t^2+2)k^3 + \\ 2(4t^4 - 9t^2 + 4)k^2 + 4(t^3 + 4t^2 - 2)k + (t + 2)^2\,.\end{aligned} \tag{2.9}$$

It is essentially at this point where we diverge from Chernick's method. He, basically, completes the square of the quartic with a method known since the time of Fermat. Straightforward algebra shows that the right hand side of (2.9) can be written $f(k, t)^2 + g(k, t)$ where

$$f(k,t) = (t-2)(2t^2-1)k^2 + \frac{2(t^3-4t^2+2)}{t-2}k + \frac{2t^6 - 25t^4 + 28t^3 - 16t + 8}{(t-2)^3(2t^2-1)}\,,$$

and

$$g(k,t) = \frac{16k(t^6 - 9t^4 + 12t^2 - 4)(2t-1)}{(t-2)^4(2t^2-1)} - \\ \frac{16(t^6 - 9t^4 + 12t^2 - 4)(2t^5 - 10t^4 + 15t^3 - 7t^2 - 4t + 3)}{(t-2)^6(2t^2-1)^2}z,\,.$$

Setting $g(k,t)=0$ gives a solution to (2.9), which is given by

$$k = \frac{(t^3-3t^2+1)(2t^2-4t+3)}{(t-2)^2(2t-1)(2t^2-1)} . \tag{2.10}$$

Putting this value of k into the above formulae results in a degree 6 ideal multigrade with the U_i and V_i terms being polynomials in t of degree 10 and 11.

The quartic (in k) clearly has a rational point $(0,(t+2))$, and so is birationally equivalent to an elliptic curve. Using the method described in Mordell [5] we find this curve (with $|t| \neq 1$) to be

$$v^2 = u(u+(t+1)^2(t^2+2t-2))(u+(t-1)^2(t^2-2t-2)) , \tag{2.11}$$

with

$$k = \frac{v(2-t)+u(t^3-4t^2+2)+t^7-9t^5+12t^3-4t}{u(t-2)^2+t^6-9t^4+12t^2-4} . \tag{2.12}$$

There are 3 clear finite points of order 2, and numerical experiments suggest that the torsion subgroup is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, but this is, of course, nowhere near a proof. This can be verified using some Magma code very kindly supplied by the referee.

These same numerical experiments also suggested that the rank was at least 2 except when $t=2$ which has rank 1. To find elements of the group of rational points, we first used the Pari-GP function *ellratpoints* to find smallish height rational points for specified t.

The most obvious result was that $u=9/4$ always gave such a point for each t, namely $v = \pm 3(2t^2+4t-1)(2t^2-4t)/8$. It is very unusual (in the author's experience) for u constant to always give a rational point. The positive v gives a fairly horrid value of k, but the negative value gives

$$k = \frac{2t^2+4t+3}{2(t+2)(1-2t^2)} ,$$

which leads to the following elements of the right-hand-side of (2.1) with $U_1 = 0$. The left-hand-side elements are just the negatives.

It might be thought that $u=9/4$ was bound to give a generator. It should be noted that elliptic curves with at least one torsion point of order 2 lead to a doubling formula resulting in a u-value which is a rational square. 9/4 is a

TABLE 1. Parametric Solution for Degree 6

i	Term
1	$12t^3 + 30t^2 + 6t - 3$
2	$4t^4 + 4t^3 - 18t^2 - 22t - 4$
3	$-4t^4 - 16t^3 - 12t^2 - 8t - 5$
4	$4t^5 + 12t^4 - 4t^3 - 22t^2 - 3t + 4$
5	$4t^5 + 16t^4 + 12t^3 - 16t^2 - 7t$
6	$-4t^5 - 12t^4 + 4t^3 + 28t^2 + 15t + 5$
7	$-4t^5 - 16t^4 - 12t^3 + 10t^2 + 19t + 3$

rational square and it is standard algebra to show that it is double a rational point and thus not a generator.

We find this rational point to be

$$(\,(t^2-1)^2, \pm(t^2-1)^2(2t^2-1)\,)\,,$$

which gives

$$k = \frac{3}{2t^3 - 4t^2 - 5t + 4} \qquad\qquad k = \frac{-1}{t}\,,$$

from the positive and negative values respectively.

The first leads to the same parametric form as before, whilst the second leads to a trivial solution $U_i = V_i$.

As we stated before the numerical solutions suggest the rank is at least 2. We found that the following point

$$(\,t^2 - t^4\,,\, 2t(t^2-1)(2t^2-1)\,)\,,$$

was often a second generator. Proving this would be difficult. The point gave

$$k = 0 \qquad\qquad k = \frac{2t - t^2}{t^3 - 3t^2 + 1}\,,$$

with Magma showing that this point and the previous one are linearly independent.

The second formula for k gives the following parametric ideal solution.

Table 2. Parametric Solution for Degree 6

i	Term
1	$t^4 - t^3 - 3t^2 + 2t$
2	$t^4 - 4t^3 + t^2 + 2t - 1$
3	$-2t^4 + 5t^3 - 2t^2 - 2t + 1$
4	$t^5 - 3t^4 + 3t^2 - t$
5	$t^5 - 4t^4 + 5t^3 - t$
6	$-t^5 + 3t^4 - t^3 - t^2 - t + 1$
7	$-t^5 + 4t^4 - 4t^3 + 2t^2 + t - 1$

3. Degree 7 Parametric Forms

We, again, follow Chernick by assuming the relationship

$$\{\pm X_1, \pm X_2, \pm X_3, \pm X_4\} \stackrel{7}{=} \{\pm Y_1, \pm Y_2, \pm Y_3, \pm Y_4\},$$

with $X_i \neq Y_j$.

Thus, we have

$$X_1^n + X_2^n + X_3^n + X_4^n = Y_1^n + Y_2^n + Y_3^n + Y_4^n \qquad n = 2, 4, 6\,. \tag{3.1}$$

In 1913 Crussol gave a method for this equations which the present author discussed in [4]. Included in that paper is the following table for a parametric solution.

Table 3. Parametric solution for X_i, Y_i

i	X_i	Y_i
1	$4j^5 - 4j^4 - 13j^3 + 15j^2 + 4j + 4$	$4j^5 - 8j^4 - 13j^3 - 32j^2 + 4j$
2	$4j^5 + 8j^4 - 13j^3 + 32j^2 + 4j$	$4j^5 + 4j^4 - 13j^3 - 15j^2 + 4j - 4$
3	$4j^4 - 32j^3 - 13j^2 - 8j + 4$	$4j^5 + 4j^4 + 15j^3 - 13j^2 - 4j + 4$
4	$4j^5 - 4j^4 + 15j^3 + 13j^2 - 4j - 4$	$4j^4 + 32j^3 - 13j^2 + 8j + 4$

In the current work, we use the form suggested by Piezas. Piezas uses $3b$ everywhere instead of b, but this only makes one condition slightly simpler.

With this form (3.1) is identically true for $n = 2$. For $n = 4$, we have

$$16xy(x + y)(x - y)(a + b)(a - b)(ab - 3c) = 0\,,$$

TABLE 4. Identities for X_i, Y_i

i	x_i	y_i
1	$xy+ax+by-c$	$xy+bx+ay-c$
2	$xy-ax-by-c$	$xy-bx-ay-c$
3	$xy+ay-bx+c$	$xy+ax-by+c$
4	$xy-ay+bx+c$	$xy-ax+by+c$

and we force a solution by setting $c = ab/3$.

For $n = 6$, we have

$$9x^2(10y^2 - a^2 - b^2) - 9y^2(a^2 + b^2) + 10a^2b^2 = 0\,,$$

Since we want rational solutions for x, y, we must have

$$(9y^2(a^2 + b^2) - 10a^2b^2)(90y^2 - 10a^2 - 10b^2)) = \square.$$

Piezas claims this is an elliptic curve, but such a quartic is only equivalent to an elliptic curve if there is at least one rational solution. We have (dividing by 9)

$$\square = 90(a^2 + b^2)y^4 - (9a^4 + 118a^2b^2 + 9b^4)y^2 + 10a^2b^2(a^2 + b^2)\,, \tag{3.2}$$

and it is not too hard to find $y = -a$ gives a right-hand-side of $a^2(3a+b)^2(3a-b)^2$.

Proceeding along standard lines [5], we eventually find the equivalent elliptic curve to be

$$v^2 = u\,(u + (a + 3b)^2(3a + b)^2)\,(u + (a - 3b)^2(3a - b)^2)\,, \tag{3.3}$$

with

$$y = \frac{a(v + u(a^2 + 11b^2) + (a + 3b)^2(a - 3b)^2(3a + b)(3a - b))}{-v - u(19a^2 + 9b^2) + (a + 3b)^2(a - 3b)^2(3a + b)(3a - b)}\,, \tag{3.4}$$

and, thus, $9a^2 - b^2 \neq 0$ and $a^2 - 9b^2 \neq 0$ must hold.

Numerical experiments suggested that the torsion subgroup was isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and that the rank was at least 1, though often exactly 1 for some a, b. We find the following points of order 4

$$(\,(a+3b)(a-3b)(3a+b)(3a-b)\,, \pm 6(a+3b)(a-3b)(a^2+b^2)(3a+b)(3a-b)\,)\,,$$

$$(\,-(a+3b)(a-3b)(3a+b)(3a-b)\,, \pm 20ab(a+3b)(a-3b)(3a+b)(3a-b)\,)\,,$$

These numerical experiments also suggest that the point

$$(-(3a+b)^2(a-3b)^2 \, , \, 12ab(3a+b)^2(a-3b)^2 \,) \, , \tag{3.5}$$

is a point of infinite order and often a generator. This gives

$$x = \frac{b(3a^2+5b^2)}{5a^2-13b^2} \qquad y = \frac{b(13a^2-5b^2)}{3(5a^2+3b^2)} \, , \tag{3.6}$$

which leads eventually to the parametric forms

Table 5. Parametric solution for X_i, Y_i

X_1	$2(5a^5+26a^4b+38a^3b^2-36a^2b^3+21ab^4+10b^5)$
X_2	$(b-a)(35a^4+48a^3b+74a^2b^2-48ab^3-45b^4)$
X_3	$(a+b)(45a^4-48a^3b-74a^2b^2+48ab^3-35b^4)$
X_4	$-2(10a^5-21a^4b-36a^3b^2-38a^2b^3+26ab^4-5b^5)$
Y_1	$2(10a^5+21a^4b-36a^3b^2+38a^2b^3+26ab^4+5b^5)$
Y_2	$(b-a)(45a^4+48a^3b-74a^2b^2-48ab^3-35b^4)$
Y_3	$(a+b)(35a^4-48a^3b+74a^2b^2+48ab^3-45b^4)$
Y_4	$-2(5a^5-26a^4b+38a^3b^2+36a^2b^3+21ab^4-10b^5)$

4. Piezas' Resultant Method

Later on, in the section on sixth powers with 8 terms, Piezas describes a simple-looking method. He sets, similar to the previous section

$$\begin{aligned} \{X_1,X_2,X_3,X_4\} &= \{a+bh, c+dh, e+fh, g+h\} \\ \{Y_1,Y_2,Y_3,Y_4\} &= \{a-bh, c-dh, e-fh, g-h\} \, , \end{aligned} \tag{4.1}$$

and forces

$$X_1^n+X_2^n+X_3^n+X_4^n = Y_1^n+Y_2^n+Y_3^n+Y_4^n \qquad n=1,2,4,6 \, . \tag{4.2}$$

For an ideal multigrade with 8 terms, he does not require the $n=1$ condition. Without it, however, we do not get as much simplification as we need to get an answer.

For $n=1,2$, we have the simple identities

$$f = -1-b-d \qquad g = -ab-cd-ef \, ,$$

reducing the number of parameters to 6.

The conditions for $n=4,6$ reduce to two equations for h

$$P_{22}h^2+P_{20}=0 \qquad P_{44}h^4+P_{42}h^2+P_{40}=0 \, ,$$

where the P_{ij} are complicated functions of a, b, c, d, e.

The resultant of these equations is of the form $F(a,b,c,d,e)^2 = 0$. It is very surprising that F factors into the product of 3 reasonable linear terms and a cubic term. The linear expressions are

$$(a+ab-c+cd-be-de) \quad (-a+ab+c+cd-be-de) \quad (a+ab+c+cd-2e-be-de)\,.$$

We consider the third of these factors, with the other two using the same methodology. We have

$$b = \frac{a + c(d+1) - e(d+2)}{e-a}\,, \tag{4.3}$$

which we substitute into the quadratic equation for h.

This has solutions

$$h = \frac{\pm(a-e)}{d+1} \qquad c = e \qquad e = \frac{(a+c)(c(d+1) - a(d-2))}{a(d+4) + c(2-d)}\,,$$

with the first 3 solutions leading to trivial multigrades. The final one does not.

Substituting the formula for e into the quartic, we find that it factorises into 2 linear terms in h and a quadratic of the form $Q_{22}h^2 - Q_{20}$, where Q_{22}, Q_{20} are functions of a, c, d. Solving for h in the linear terms just gives trivial solutions, so we concentrate on the quadratic.

For $h \in \mathbb{Q}$ we must have $\square = Q_{22}Q_{20}$. This latter expression is of degree 8 in a and c, but a quartic in d. The leading term is $9a^2c^2(a-c)^2(2a+c)^2$, so the quartic is birationally equivalent to an elliptic curve.

After some standard, but lengthy, calculations, we find the elliptic curve to be

$$v^2 = u(u^2+(9(a^4+c^4)-160ac(a^2+c^2)-418a^2c^2)u+1600a^2c^2(a+2c)^2(2a+c)^2)\,, \tag{4.4}$$

with

$$d = \frac{3v - (41a^2 + 98ac + 41c^2)u + 800ac(a+2c)^2(2a+c)^2}{2ac(c-a)(400(a+2c)^2(2a+c)^2 - 9u)}\,. \tag{4.5}$$

These curves are singular if $a = \pm c$, which we now assume does not happen. Numerical experiments on the curves, for simple integer a, c values, suggest that the torsion subgroup is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ with points of order 4 given by

$$(\,40ac(a+2c)(2a+c)\,,\ \pm 120ac(a+c)(a-c)(a+2c)(2a+c)\,)\,,$$

with none of the torsion points leading to non-trivial solutions.

These numerical experiments also suggested that the rank of the curves is at least 2, except when $a=2, c=1$, when the rank is only 1. We used the Pari-GP code ellratpoints to find rational points and then try to infer an algebraic form.

We found 2 simple points that seem to often give generators

$$(\,16ac(a+2c)(2a+c)\,,\ 48ac(a+2c)(2a+c)(a^2+4ac+c^2)\,)\,,$$

and

$$(\,64ac(a+2c)(2a+c)\,,\ 192ac(a+2c)(2a+c)(a^2+ac+c^2)\,)\,.$$

From the first point we find the following parametric form

Table 6. Parametric solution for X_i, Y_i

X_1	$-(30a^5+116a^4c+598a^3c^2+1179a^2c^3+823ac^4+170c^5)$
X_2	$170a^5+933a^4c+2080a^3c^2+2221a^2c^3+1017ac^4+140c^5$
X_3	$110a^5+763a^4c+1863a^3c^2+1756a^2c^3+581ac^4+30c^5$
X_4	$-140a^5-569a^4c-821a^3c^2-274a^2c^3+236ac^4+110c^5$
Y_1	$140a^5+1017a^4c+2221a^3c^2+2080a^2c^3+933ac^4+170c^5$
Y_2	$-(170a^5+823a^4c+1179a^3c^2+598a^2c^3+116ac^4+30c^5)$
Y_3	$30a^5+581a^4c+1756a^3c^2+1863a^2c^3+763ac^4+110c^5$
Y_4	$110a^5+236a^4c-274a^3c^2-821a^2c^3-569ac^4-140c^5$

whilst, from the second point

TABLE 7. Parametric solution for X_i, Y_i

X_1	$1040a^5 + 3732a^4c + 7438a^3c^2 + 8479a^2c^3 + 4266ac^4 + 560c^5$
X_2	$-(560a^5 + 2746a^4c + 5361a^3c^2 + 4321a^2c^3 + 614ac^4 + 480c^5)$
X_3	$1520a^5 + 4574a^4c + 3533a^3c^2 - 2069a^2c^3 - 3602ac^4 - 1040c^5$
X_4	$-480a^5 - 922a^4c + 625a^3c^2 + 4146a^2c^3 + 4588ac^4 + 1520c^5$
Y_1	$480a^5 - 614a^4c - 4321a^3c^2 - 5361a^2c^3 - 2746ac^4 - 560c^5$
Y_2	$560a^5 + 4266a^4c + 8479a^3c^2 + 7438a^2c^3 + 3732ac^4 + 1040c^5$
Y_3	$-1040a^5 - 3602a^4c - 2069a^3c^2 + 3533a^2c^3 + 4574ac^4 + 1520c^5$
Y_4	$1520a^5 + 4588a^4c + 4146a^3c^2 + 625a^2c^3 - 922ac^4 - 480c^5$

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Allan J. Macleod
Statistics, O.R. and Mathematics Group (Retired),
University of the West of Scotland,
High St., Paisley,
Scotland. PA1 2BE

E-mail: peediejenn@hotmail.com