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CHARACTERIZING STRONG INFINITE-DIMENSION, WEAK INFINITE-DIMENSION, AND DIMENSION IN INVERSE SYSTEMS

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ABSTRACT. We present internal characterizations for an inverse system of compact Hausdorff spaces that show when its limit will be strongly infinite-dimensional, weakly infinite-dimensional, or have its dimension $\dim \in \mathbb{N}_{\geq 0}$. The technique involves essential families.

1. Introduction

Essential families provide a useful tool for the study of dimension theory ([RSW], [Ru1], [Ru2], [Ru3], [Ru4], [Wa], [Sa]). On the other hand, inverse systems are fundamental for the study of compact Hausdorff spaces, and in the case of dimension theory, sometimes a condition on an inverse system can be used to place an upper bound on the dimension dim of its limit. But as for strong and weak infinite-dimension, there are no such results.

In Section 2 we will review the definitions of an essential family, weak and strong infinite-dimensionality, and inverse systems. Our new concepts of n-essential and ω-essential families in an inverse system will appear in Section 3. The objective is to lift the notions of essential families in spaces into parallel concepts for inverse systems. Then in Theorems 4.1-4.4, which are the main results of this paper, we shall use these and ideas spawned from them to provide characterizations of strong infinite-dimension, weak infinite-dimension, and dim of the limit of an inverse system of nonempty compact Hausdorff spaces strictly in terms of the system itself.

We have structured this paper so that the new definitions and the four main theorems can be stated in Sections 2-4. Examples that illustrate applications of our main theorems will be provided in the final section, Section 11, which is accessible to the reader if desired immediately after Section 4. Sections 5-9 are relegated to developing the theory necessary to obtain short proofs of the four main theorems in Section 10.

2. Basic Definitions

We review here the concept of an essential family (see p. 261 of [Sa]) in a space, avoiding the classical but cumbersome notation involving indexing. We shall recall (Theorem 2.4) that essential families are useful in characterizing finite dimension dim, and are necessary for defining both strong and weak infinite-dimension (Definition 2.6). Although our definitions generally apply to arbitrary classes of spaces, our main results will be only about the

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class of compact Hausdorff spaces. The term "map" will always refer to "continuous function."

Let $C = (A, B)$ be an ordered pair of sets. We call C a *disjoint pair* if $A \cap B = \emptyset$. If X is a set with $A \cup B \subset X$, then we say that C is an ordered pair in X. Whenever X and Y are sets, $f : X \to Y$ is a function, and C is an ordered pair in X, then $f(C)$ is defined to be the ordered pair $(f(A), f(B))$ in Y. In case $\mathcal{C}$ is a collection of ordered pairs in X, then $f(\mathcal{C})$ will denote $\{f(C) \,|\, C \in \mathcal{C}\}$. Similarly if C is an ordered pair in Y, then $f^{-1}(C)$ is defined to be the ordered pair $(f^{-1}(A), f^{-1}(B))$ in X. If $\mathcal{C}$ is a collection of ordered pairs in Y, then $f^{-1}(\mathcal{C}) = \{f^{-1}(C) \,|\, C \in \mathcal{C}\}$.

Definition 2.1. *Let X be a space and $C = (A, B)$ an ordered pair in X. We say that C is a closed (respectively open) pair in X if each of A and B is closed (respectively open) in X.*

Definition 2.2. *If X is a space and $C = (A, B)$ is a closed pair in X, then a closed subset P of X is called a* ***partition***[1] *of C in X if there exists an open disjoint pair (U, V) in X such that $X \setminus P = U \cup V$, $A \subset U$, and $B \subset V$.*

Let us remark that partitions can be empty.

Definition 2.3. *An* ***essential*** *family in a space X is a nonempty collection $\mathcal{C}$ of closed disjoint pairs in X such that for each collection $\{P_C \,|\, C \in \mathcal{C}\}$ of respective partitions P_C of C in X, $\bigcap\{P_C \,|\, C \in \mathcal{C}\} \neq \emptyset$. An* ***inessential*** *family in a space X is a nonempty collection $\mathcal{C}$ of closed disjoint pairs in X that is not an essential family in X, that is, there exists a collection $\{P_C \,|\, C \in \mathcal{C}\}$ of respective partitions P_C of C such that $\bigcap\{P_C \,|\, C \in \mathcal{C}\} = \emptyset$.*

The following is a rewording of Theorem 5.2.17 of [Sa]:

Theorem 2.4. *Let X be a nonempty normal* [2] T_2*-space and $n \in \mathbb{N}$. Then* $\dim X = n$ *if and only if there exists an essential family $\mathcal{C}$ in X with* $\mathrm{card}(\mathcal{C}) = n$ *and for every essential family $\mathcal{C}_0$ in X,* $\mathrm{card}(\mathcal{C}_0) \leq n$. □

For $\dim = -1$ or $\dim = 0$, we make the following definition.

Definition 2.5. *Let X be a space. Then* $\dim X = -1$ *if and only if $X = \emptyset$. If $X \neq \emptyset$, then we say that* $\dim X = 0$ *if there is no essential family in X.*

Definition 2.6. *Let X be a space. We say that X is*

(1) ***infinite-dimensional*** *if for all $n \in \mathbb{N}$ there exists an essential family $\mathcal{C}$ in X with* $\mathrm{card}(\mathcal{C}) = n$;
(2) ***strongly infinite-dimensional*** (SID) *if there exists a countably infinite essential family in X; and*
(3) ***weakly infinite-dimensional*** (WID) *if it is infinite-dimensional and not strongly infinite-dimensional.*

In order to prepare for stating the main theorems in Section 4, we now refresh the reader with the definition of inverse system and delineate the conventions that we shall use concerning its projections and bonding maps.

Definition 2.7. *An* ***inverse system*** $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *consists of the following: a pre-ordered, directed set* $(\Gamma, \preceq)$; *for each $\gamma \in \Gamma$, a space X_γ; for*

[1] also called a separator

[2] We do not assume that normal spaces are T_2.

each pair $\gamma \preceq \gamma'$ from Γ, a ***bonding map*** $p_{\gamma\gamma'} : X_{\gamma'} \to X_\gamma$. *The bonding maps must satisfy the following two conditions*:

(1) $p_{\gamma\gamma} = \mathrm{id}_{X_\gamma}$, *and*
(2) *if* $\gamma \preceq \gamma' \preceq \gamma''$, *then* $p_{\gamma\gamma''} = p_{\gamma\gamma'} p_{\gamma'\gamma''}$.

A ***thread*** *of* $\mathbf{X}$ *is an element* $x \in \prod\{X_\gamma \mid \gamma \in \Gamma\}$ *such that whenever* $\gamma \preceq \gamma'$, *then* $p_\gamma(x) = p_{\gamma\gamma'} p_{\gamma'}(x)$. *Here, for each* $\gamma_0 \in \Gamma$, $p_{\gamma_0} : \prod\{X_\gamma \mid \gamma \in \Gamma\} \to X_{\gamma_0}$ *is the coordinate projection. The inverse limit of* $\mathbf{X}$, *denoted* $\lim \mathbf{X}$, *is the set of threads of* $\mathbf{X}$ *endowed with the topology it inherits from* $\prod\{X_\gamma \mid \gamma \in \Gamma\}$.

Conventions. We shall use $p_\gamma : X = \lim \mathbf{X} \to p_\gamma(X)$ to denote the (surjective) restriction of the coordinate projection. Similarly, if $\gamma \preceq \gamma'$, then the surjective map $p_{\gamma\gamma'} : p_{\gamma'}(X) \to p_\gamma(X)$ will denote the restriction of the bonding map.

3. Essentiality in a System

The purpose of this section is to provide the definitions of n-essential and ω-essential families in an inverse system and some lemmas and definitions related to them. The definitions are precisely the concepts that will be needed in order to state our characterization theorems in Section 4.

Definition 3.1. *Let* $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *be an inverse system of nonempty compact Hausdorff spaces*, $X = \lim \mathbf{X}$, *and* $(\mu, \mathcal{D})$ *a pair such that* $\mu \in \Gamma$ *and* $\mathcal{D}$ *is a nonempty family of closed disjoint pairs in* $p_\mu(X)$. *We shall say that* $(\mu, \mathcal{D})$ *is an* ***essential pair*** *in* $\mathbf{X}$, *or more briefly, is* ***essential*** *in* $\mathbf{X}$, *if for all* $\gamma \in \Gamma$ *with* $\mu \preceq \gamma$, $p_{\mu\gamma}^{-1}(\mathcal{D})$ *is an essential family in* $p_\gamma(X)$.

Lemma 3.2. *Let* $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *be an inverse system of nonempty compact Hausdorff spaces*, $X = \lim \mathbf{X}$, *and* $(\mu, \mathcal{D})$ *be essential in* $\mathbf{X}$. *Then for all* $\gamma \in \Gamma$ *with* $\mu \preceq \gamma$, $p_{\mu\gamma}^{-1}(\mathcal{D})$ *is a nonempty family of closed disjoint pairs in* $p_\gamma(X)$. *In particular*, $\mathcal{D}$ *is a nonempty family of closed disjoint pairs in* $p_\mu(X)$.

Proof. This follows from Definitions 3.1 and 2.3. □

Definition 3.3. *Let* A *be a set with a relation* $\preceq$ *and* B *be a set with a relation* $\preceq_1$. *We shall say that* $f : (A, \preceq) \to (B, \preceq_1)$ *is* ***relation preserving*** *if* $f : A \to B$ *is a function having the property that* $a \preceq b \Rightarrow f(a) \preceq_1 f(b)$.

Definition 3.4. *Let* $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *be an inverse system of nonempty compact Hausdorff spaces*, $X = \lim \mathbf{X}$, *and* $n \in \mathbb{N}$. *An* n-***essential sequence*** *in* $\mathbf{X}$ *consists of a pair* $(\mu, (\mathcal{D}_k))$ *where* $\mu : (\{1, \ldots, n\}, \leq) \to (\Gamma, \preceq)$ *is a relation preserving function, and* $(\mathcal{D}_k)$, $k = 1, \ldots, n$, *is a finite sequence such that for each* $1 \leq k \leq n$,

(1) $\mathcal{D}_k$ *is a nonempty family of closed disjoint pairs in* $p_{\mu(k)}(X)$,
(2) $\mathrm{card}(\mathcal{D}_k) = k$,
(3) $(\mu(k), \mathcal{D}_k)$ *is essential in* $\mathbf{X}$, *and*
(4) *if* $k < n$, *then* $p_{\mu(k)\mu(k+1)}^{-1}(\mathcal{D}_k) \subset \mathcal{D}_{k+1}$.

Definition 3.5. *Let* $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *be an inverse system of nonempty compact Hausdorff spaces and* $X = \lim \mathbf{X}$. *An* ω-***essential sequence*** *in* $\mathbf{X}$ *consists of a pair* $(\mu, (\mathcal{D}_n))$ *where* $\mu : (\mathbb{N}, \leq) \to (\Gamma, \preceq)$ *is a relation preserving function, and* $(\mathcal{D}_n)$ *is a sequence such that for each* $n \in \mathbb{N}$,

(1) $\mathcal{D}_n$ *is a nonempty family of closed disjoint pairs in* $p_{\mu(n)}(X)$,
(2) $\mathrm{card}(\mathcal{D}_n) = n$,
(3) $(\mu(n), \mathcal{D}_n)$ *is essential in* $\mathbf{X}$, *and*
(4) $p^{-1}_{\mu(n)\mu(n+1)}(\mathcal{D}_n) \subset \mathcal{D}_{n+1}$.

Lemma 3.6. *Let* $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *be an inverse system of nonempty compact Hausdorff spaces.*

(1) *If* $n \in \mathbb{N}$ *and there exists an* n*-essential sequence in* $\mathbf{X}$, *then for all* $n_0 \in \mathbb{N}$ *with* $n_0 < n$, *there exists an* n_0*-essential sequence in* $\mathbf{X}$.
(2) *If there exists an* ω*-essential sequence in* $\mathbf{X}$, *then for all* $n \in \mathbb{N}$, *there exists an* n*-essential sequence in* $\mathbf{X}$. □

4. Main Theorems

We now state our main results; their proofs will be given in Section 10. Theorem 4.1 provides a characterization of SID for the limit of an inverse system of nonempty compact Hausdorff spaces strictly in terms internal to it; Theorem 4.2 does this for WID, and Theorems 4.3 and 4.4 accomplish this for $\dim = n \in \mathbb{N}$, and $\dim = 0$ respectively.

Theorem 4.1. *Let* $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *be an inverse system of nonempty compact Hausdorff spaces and* $X = \lim \mathbf{X}$. *Then the following are equivalent.*

(1) X *is* SID.
(2) *There exists an* ω*-essential sequence in* $\mathbf{X}$.

Theorem 4.2. *Let* $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *be an inverse system of nonempty compact Hausdorff spaces and* $X = \lim \mathbf{X}$. *Then the following are equivalent.*

(1) X *is* WID.
(2) *There is no* ω*-essential sequence in* $\mathbf{X}$, *but for each* $n \in \mathbb{N}$, *there exists an* n*-essential sequence in* $\mathbf{X}$.

Theorem 4.3. *Let* $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *be an inverse system of nonempty compact Hausdorff spaces,* $X = \lim \mathbf{X}$, *and* $n \in \mathbb{N}$. *Then the following are equivalent.*

(1) $\dim X = n$.
(2) *There exists an* n*-essential sequence in* $\mathbf{X}$, *but for all* $n_0 \in \mathbb{N}$ *with* $n_0 > n$, *there is no* n_0*-essential sequence in* $\mathbf{X}$.

Theorem 4.4. *Let* $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *be an inverse system of nonempty compact Hausdorff spaces and* $X = \lim \mathbf{X}$. *Then the following are equivalent.*

(1) $\dim X = 0$.
(2) *There exists no* 1*-essential sequence in* $\mathbf{X}$.

5. Basics of Essential Families

We reviewed the idea of an essential family in Section 2. In the current section we are going to provide the reader with the most fundamental "point-set" notions needed to facilitate our use of essential families.

Lemma 5.1. *Let* X *and* Y *be sets,* $f : X \to Y$ *a function, and* C *an ordered pair in* $f(X)$. *If* $f^{-1}(C)$ *is a disjoint pair in* X, *then* C *is a disjoint pair in* $f(X)$. □

Lemma 5.2. *If a closed pair C in a space X has a partition in X, then C is a closed disjoint pair in X.* $\square$

Lemma 5.3. *Let $C = (A, B)$ be a closed pair in a space X. If either $A = \emptyset$ or $B = \emptyset$, then $\emptyset$ is a partition of C in X, and C is a closed disjoint pair in X.* $\square$

Lemma 5.4. *Let X and Y be spaces, $f : X \to Y$ a map, and $\mathcal{C}$ a collection of closed disjoint pairs in $f(X)$. Then the following statements are true.*

(1) *$f^{-1}(\mathcal{C})$ is a collection of closed disjoint pairs in X, and* $\mathrm{card}(\mathcal{C}) = \mathrm{card}(f^{-1}(\mathcal{C}))$.
(2) *If $C \in \mathcal{C}$ and P is a partition of C in $f(X)$, then $f^{-1}(P)$ is a partition of $f^{-1}(C)$ in X.*
(3) *If $\mathcal{C}$ is an inessential family in $f(X)$, then $f^{-1}(\mathcal{C})$ is an inessential family in X. More strongly, if for each $C \in \mathcal{C}$, P_C is a partition of C in $f(X)$ and $\bigcap\{P_C \mid C \in \mathcal{C}\} = \emptyset$, then $\bigcap\{f^{-1}(P_C) \mid C \in \mathcal{C}\} = \emptyset$.*
(4) *If $f^{-1}(\mathcal{C})$ is an essential family in X, then $\mathcal{C}$ is an essential family in $f(X)$.* $\square$

In Lemma 5.7 we get a type of "squeezing" phenomenon. To introduce it, we need a definition and a lemma.

Definition 5.5. *If $C = (A, B)$ is an ordered pair of sets, then a* ***thickening*** *of C is an ordered pair $C^* = (A^*, B^*)$ of sets such that $A \subset A^*$ and $B \subset B^*$, and we call C^* a* ***thickening into disjoint pairs*** *if C^* is a disjoint pair. If X is a set and $A^* \cup B^* \subset X$, then we call C^* a thickening of C in X. If $\mathcal{C}$ is a collection of ordered pairs of sets, then a collection $\mathcal{C}^* = \{C^* \mid C \in \mathcal{C}\}$ is called a thickening of $\mathcal{C}$ (into disjoint pairs) if for each $C \in \mathcal{C}$, C^* is a thickening of C (into disjoint pairs). It is a thickening in a set X if for each $C \in \mathcal{C}$, C^* is a thickening of C in X. In case X is a space, then we say that a thickening C^* of C in X is closed (respectively open) if C^* is closed (respectively open) in X. We shall apply the same language to a collection $\mathcal{C}$ of ordered pairs of sets in a set X or a space X, speaking of a thickening $\mathcal{C}^*$ of $\mathcal{C}$ in X, and open or closed thickenings in X where appropriate. In case it is true that for each $C \in \mathcal{C}$, C^* is a disjoint pair in X, then we shall say that $\mathcal{C}^*$ is a thickening of $\mathcal{C}$ into disjoint pairs in X.*

Lemma 5.6. *Let C be a closed pair in a space X and C^* a closed thickening of C in X. If P is a partition of C^* in X, then P is also a partition of C in X.* $\square$

Lemma 5.7. *Let X be a space and $\mathcal{C}$, $\mathcal{C}^*$ be collections of closed disjoint pairs in X such that $\mathcal{C}^*$ is a closed thickening of $\mathcal{C}$. If $\mathcal{C}^*$ is an inessential family in X, then so is $\mathcal{C}$.* $\square$

An application of Urysohn's Lemma shows the following.

Lemma 5.8. *Let $C = (A, B)$ be a closed disjoint pair in a normal space X. Then either there exists a map $f : X \to [0, 1]$ such that $f(A) = \{0\}$, $f(B) = \{1\}$, and $f(X) \subset \{0, 1\}$ or there exists a surjective map $f : X \to [0, 1]$ such that $f(A) = \{0\}$, and $f(B) = \{1\}$. In the former case, $\emptyset$ is a partition of C in X; in the latter case there exist two partitions P, P' of C in X such that $P \cap P' = \emptyset$. In either case, there exists a partition of C in X.* $\square$

Lemma 5.9. *Let $\mathcal{C}$ be an essential family in a normal space X. Then for each nonempty subset $\mathcal{C}_0 \subset \mathcal{C}$, $\mathcal{C}_0$ is an essential family in X.*

Proof. Suppose that $\mathcal{C}_0 \subset \mathcal{C}$ is nonempty and $\mathcal{C}_0$ is an inessential family in X. Then there exist a collection $\{P_C \mid C \in \mathcal{C}_0\}$, each P_C being a partition of C in X, such that $\bigcap\{P_C \mid C \in \mathcal{C}_0\} = \emptyset$. For each $C \in \mathcal{C} \setminus \mathcal{C}_0$, use the last part of Lemma 5.8 to find a partition P_C of C in X. It follows that $\bigcap\{P_C \mid C \in \mathcal{C}_0\} = \emptyset$, which contradicts the fact that $\mathcal{C}$ is an essential family in X. □

Lemmas 5.10 and 5.11 will provide convenient ways of organizing essential families in spaces. We ask the reader to provide proofs of them, making use of Lemma 5.9.

Lemma 5.10. *Let $n \in \mathbb{N}$, X be a normal space, and $\mathcal{C}$ be an essential family in X with $\operatorname{card}(\mathcal{C}) = n$. Then there exists a finite sequence $(\mathcal{C}_k)$, $k = 1, \ldots, n$, of essential families in X such that $\mathcal{C}_n = \mathcal{C}$, and for all $1 \leq k \leq n$,*

(1) $\operatorname{card}(\mathcal{C}_k) = k$, *and*
(2) *if $k < n$, then $\mathcal{C}_k \subset \mathcal{C}_{k+1}$.* □

Lemma 5.11. *Let X be a normal space and $\mathcal{C}$ be a countably infinite essential family in X. Then there exists a sequence $(\mathcal{C}_n)$ of essential families in X such that $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\} = \mathcal{C}$, and for all $n \in \mathbb{N}$,*

(1) $\operatorname{card}(\mathcal{C}_n) = n$, *and*
(2) $\mathcal{C}_n \subset \mathcal{C}_{n+1}$. □

Lemma 5.12 shows that thickening a finite essential family in a normal space into closed disjoint pairs does not change the cardinality.

Lemma 5.12. *Let $n \in \mathbb{N}$ and $\mathcal{C}$ be an essential family in a normal space X with $\operatorname{card}(\mathcal{C}) = n$. Suppose that $\mathcal{C}^* = \{C^* \mid C \in \mathcal{C}\}$ is a closed thickening of $\mathcal{C}$ in X into disjoint pairs. Then $\operatorname{card}(\mathcal{C}^*) = n$.*

Proof. The lemma is surely true for every normal space X and $n = 1$. So let us suppose that $n \in \mathbb{N}$ and it is true that for every normal space X and every essential family $\mathcal{C}$ in X with $\operatorname{card}(\mathcal{C}) = n$, and every closed thickening $\mathcal{C}^* = \{C^* \mid C \in \mathcal{C}\}$ of $\mathcal{C}$ into disjoint pairs in X, $\operatorname{card}(\mathcal{C}^*) = n$. Let X be a normal space and $\mathcal{C}$ an essential family in X with $\operatorname{card}(\mathcal{C}) = n+1$. Suppose that $\mathcal{C}^* = \{C^* \mid C \in \mathcal{C}\}$ is a closed thickening of $\mathcal{C}$ in X into disjoint pairs. We must prove that $\operatorname{card}(\mathcal{C}^*) = n+1$.

Of course $\operatorname{card}(\mathcal{C}^*) \leq n+1$. Fix $C_0 \in \mathcal{C}$, let $\mathcal{C}_0 = \mathcal{C} \setminus \{C_0\}$, and put $\mathcal{C}_0^* = \{C^* \mid C \in \mathcal{C}_0\}$. Then $\mathcal{C}_0^*$ is a closed thickening of $\mathcal{C}_0$ in X into disjoint pairs. By Lemma 5.9, $\mathcal{C}_0$ is an essential family in X, and surely $\operatorname{card}(\mathcal{C}_0) = n$; so the inductive assumption yields that $\operatorname{card}(\mathcal{C}_0^*) = n$. Hence if $\operatorname{card}(\mathcal{C}^*) < n+1$, it would be true that for some $C \in \mathcal{C}_0$, $C^* = C_0^*$. Using Lemma 5.8, we have two possibilities for C^*. One is that $\emptyset$ is a partition of C^* in X. But then by Lemma 5.6, $\emptyset$ is a partition of C in X, so $\{C\}$ is an inessential family in X. This contradicts Lemma 5.9. In the other case we find two partitions P_C and P_0 of $C^* = C_0^*$ in X so that $P_C \cap P_0 = \emptyset$. Then by Lemma 5.6, P_C is a partition of C in X and P_0 is a partition of C_0 in X. Using this and Lemma 5.6, one sees that $\{C, C_0\}$ is an inessential family in X, again contradicting Lemma 5.9. □

Lemma 5.13. *Let X be a compact Hausdorff space and $\mathcal{C}$ a nonempty collection of closed disjoint pairs in X. If $\{P_C \,|\, C \in \mathcal{C}\}$ is a collection of respective partitions P_C of C in X, and $\bigcap\{P_C \,|\, C \in \mathcal{C}\} = \emptyset$, then there exists a nonempty finite subset $\mathcal{C}_0 \subset \mathcal{C}$ such that $\bigcap\{P_C \,|\, C \in \mathcal{C}_0\} = \emptyset$. Therefore if $\mathcal{C}$ is an inessential family in X, then it has a nonempty finite subset that is an inessential family in X.* □

6. Inverse Systems, Essential Families

In this section we are going to gather some facts about inverse systems of compact Hausdorff spaces that rely on the commutativity of diagrams in them or the fact that the coordinate projections are closed maps. Before that we state a well-known proposition.

Proposition 6.1. *Let $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ be an inverse system of compact Hausdorff spaces. Then $\lim \mathbf{X}$ is compact and Hausdorff; moreover, $\lim \mathbf{X} \neq \emptyset$ if and only if for each $\gamma \in \Gamma$, $X_\gamma \neq \emptyset$.* □

Lemma 6.2. *Let $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ be an inverse system. Then whenever $\gamma \preceq \gamma'$, $A \subset \lim \mathbf{X}$, and $B \subset p_\gamma(X)$,*

1. *$p_{\gamma'}(A) \subset p^{-1}_{\gamma\gamma'}(p_\gamma(A))$, and*
2. *$p^{-1}_\gamma(B) = p^{-1}_{\gamma'}(p^{-1}_{\gamma\gamma'}(B))$.*

Proof. For (1), let $x \in A$. We need to show that $p_{\gamma'}(x) \in p^{-1}_{\gamma\gamma'}(p_\gamma(A))$, that is, that $p_{\gamma\gamma'}p_{\gamma'}(x) \in p_\gamma(A)$. But, $p_{\gamma\gamma'}p_{\gamma'}(x) = p_\gamma(x) \in p_\gamma(A)$. Item (2) follows from the fact that $p_\gamma = p_{\gamma\gamma'}p_{\gamma'}$. □

Lemma 6.3. *Let $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ be an inverse system of compact Hausdorff spaces and $X = \lim \mathbf{X}$. Then the following are true.*

1. *If A is a closed subspace of X and U an open neighborhood of A in X, then there exists $\beta \in \Gamma$ such that for all $\gamma \in \Gamma$ with $\beta \preceq \gamma$, $p^{-1}_\gamma(p_\gamma(A)) \subset U$.*
2. *If C is a closed disjoint pair in X, then there exists $\beta \in \Gamma$ such that for all $\gamma \in \Gamma$ with $\beta \preceq \gamma$, $p_\gamma(C)$ is a closed disjoint pair in $p_\gamma(X)$, and hence $p^{-1}_\gamma(p_\gamma(C))$ is a disjoint pair which is a closed thickening of C in X.*
3. *If $\mathcal{C}$ is a finite and non-empty family of closed disjoint pairs in X, then there exists $\beta \in \Gamma$ such that for all $\gamma \in \Gamma$ with $\beta \preceq \gamma$, $p_\gamma(\mathcal{C})$ is a family of closed disjoint pairs in $p_\gamma(X)$, and hence $\mathcal{C}^* = p^{-1}_\gamma(p_\gamma(\mathcal{C}))$ is a closed thickening of $\mathcal{C}$ in X into disjoint pairs.*
4. *If C is a closed disjoint pair in X, $\beta \in \Gamma$, and $p_\beta(C)$ is a disjoint pair in $p_\beta(X)$, then for all $\gamma \in \Gamma$ with $\beta \preceq \gamma$, $p^{-1}_{\beta\gamma}(p_\beta(C))$ is a disjoint pair in $p_\gamma(X)$ which is a closed thickening of $p_\gamma(C)$ in $p_\gamma(X)$.*
5. *If $\mathcal{C}$ is a finite essential family in X, then there exists $\beta \in \Gamma$ such that for all $\gamma \in \Gamma$ with $\beta \preceq \gamma$, $p_\gamma(\mathcal{C})$ is a family of closed disjoint pairs in $p_\gamma(X)$, $\mathrm{card}(p_\gamma(\mathcal{C})) = \mathrm{card}(\mathcal{C})$, and $p_\gamma(\mathcal{C})$ is an essential family in $p_\gamma(X)$.*

Proof. Let us prove (1). In case $A = \emptyset$, then any choice of β will suffice, so assume that $A \neq \emptyset$. Using Proposition 2.5.5 of [En], for each $x \in A$, there exists $\gamma_x \in \Gamma$ and an open neighborhood V_x of $p_{\gamma_x}(x)$ in X_{γ_x} with $x \in p^{-1}_{\gamma_x}(V_x) \subset U$. Since A is compact there exists a finite subset $B \subset A$

such that $\{p_{\gamma_x}^{-1}(V_x) \,|\, x \in B\}$ covers A. Now $(\Gamma, \preceq)$ is a directed set and B is finite, so there exists $\beta \in \Gamma$ having the property that for all $x \in B$, $\gamma_x \preceq \beta$.

Suppose that $\gamma \in \Gamma$, $\beta \preceq \gamma$, and $a \in A$. It will be sufficient to prove that $p_\gamma^{-1}(p_\gamma(a)) \subset U$. Choose $x \in B$ with $a \in p_{\gamma_x}^{-1}(V_x)$. Now $\gamma_x \preceq \beta \preceq \gamma$, so by Lemma 6.2(2), $p_{\gamma_x}^{-1}(V_x) = p_\gamma^{-1}(p_{\gamma_x\gamma}^{-1}(V_x))$. Hence $a \in p_\gamma^{-1}(p_{\gamma_x\gamma}^{-1}(V_x))$, so $p_\gamma(a) \in p_{\gamma_x\gamma}^{-1}(V_x)$, and it follows from this and Lemma 6.2(2) that $p_\gamma^{-1}(p_\gamma(a)) \subset p_\gamma^{-1}(p_{\gamma_x\gamma}^{-1}(V_x)) = p_{\gamma_x}^{-1}(V_x) \subset U$. This completes our proof of (1).

Let $C = (A, B)$ be a closed disjoint pair in X. Since X and all our spaces X_γ are compact and Hausdorff, one has that for all $\gamma \in \Gamma$, $p_\gamma(C)$ is a closed pair in $p_\gamma(X)$. Select a disjoint pair (U_A, U_B) in X such that U_A is an open neighborhood of A and U_B is an open neighborhood of B. Using (1), find $\beta_A \in \Gamma$ such that for all $\gamma \in \Gamma$ with $\beta_A \preceq \gamma$, $p_\gamma^{-1}(p_\gamma(A)) \subset U_A$. Similarly, find $\beta_B \in \Gamma$ such that for all $\gamma \in \Gamma$ with $\beta_B \preceq \gamma$, $p_\gamma^{-1}(p_\gamma(B)) \subset U_B$. Now choose $\beta \in \Gamma$ with $\beta_A \preceq \beta$ and $\beta_B \preceq \beta$ and suppose that $\gamma \in \Gamma$ with $\beta \preceq \gamma$. Then both $\beta_A \preceq \gamma$ and $\beta_B \preceq \gamma$. Hence $p_\gamma^{-1}(p_\gamma(A)) \subset U_A$, $p_\gamma^{-1}(p_\gamma(B)) \subset U_B$, and $U_A \cap U_B = \emptyset$. Moreover, $A \subset p_\gamma^{-1}(p_\gamma(A))$ and $B \subset p_\gamma^{-1}(p_\gamma(B))$. Since both $p_\gamma^{-1}(p_\gamma(A))$ and $p_\gamma^{-1}(p_\gamma(B))$ are closed in X, our proof of (2) is complete.

Item (3) follows from (2). One proves (4) as follows. Surely $p_\beta(C)$ is a closed disjoint pair in $p_\beta(X)$. Suppose that $\gamma \in \Gamma$ and $\beta \preceq \gamma$. Applying Lemma 5.4(1), one sees that $p_{\beta\gamma}^{-1}(p_\beta(C))$ is a closed disjoint pair in $p_\gamma(X)$. Lemma 6.2(1) can now be applied to each coordinate of the pair C.

To prove (5), first get β that satisfies (3); let $\gamma \in \Gamma$ with $\beta \preceq \gamma$. By (3), $\mathcal{C}^* = p_\gamma^{-1}(p_\gamma(\mathcal{C}))$ is a closed thickening of $\mathcal{C}$ in X into disjoint pairs. From Lemma 5.12, $\operatorname{card}(\mathcal{C}^*) = \operatorname{card}(\mathcal{C})$. Of course, $\operatorname{card}(\mathcal{C}^*) \le \operatorname{card}(p_\gamma(\mathcal{C})) \le \operatorname{card}(\mathcal{C})$. Hence $\operatorname{card}(p_\gamma(\mathcal{C})) = \operatorname{card}(\mathcal{C})$. From (3) again, one sees that $p_\gamma(\mathcal{C})$ is a family of closed disjoint pairs in $p_\gamma(X)$. Suppose that $p_\gamma(\mathcal{C})$ is an inessential family in $p_\gamma(X)$. Lemma 5.4(3) then yields that $\mathcal{C}^*$ is an inessential family in X. In turn, Lemma 5.7 shows that $\mathcal{C}$ is an inessential family in X, a contradiction. This proves (5). □

Definition 6.4. *Let* $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *be an inverse system of nonempty compact Hausdorff spaces and* $X = \lim \mathbf{X}$. *If* $\mathcal{C}$ *is a nonempty family of closed pairs in* X *and* $\beta \in \Gamma$, *then we say that* $(\mathcal{C}, \beta)$ *satisfies the* ***projection criterion*** *in* $\mathbf{X}$ *if for all* $\gamma \in \Gamma$ *with* $\beta \preceq \gamma$,

(1) $p_\gamma(\mathcal{C})$ *is an essential family in* $p_\gamma(X)$,
(2) $\operatorname{card}(p_\gamma(\mathcal{C})) = \operatorname{card}(\mathcal{C})$,
(3) $p_{\beta\gamma}^{-1}(p_\beta(\mathcal{C}))$ *is a closed thickening of* $p_\gamma(\mathcal{C})$ *in* $p_\gamma(X)$ *into disjoint pairs*,
(4) $p_{\beta\gamma}^{-1}(p_\beta(\mathcal{C}))$ *is an essential family in* $p_\gamma(X)$, *and*
(5) $\operatorname{card}(p_{\beta\gamma}^{-1}(p_\beta(\mathcal{C}))) = \operatorname{card}(p_\gamma(\mathcal{C}))$.

Lemma 6.5. *Let* $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *be an inverse system of nonempty compact Hausdorff spaces and* $X = \lim \mathbf{X}$. *If* $\mathcal{C}$ *is a nonempty finite essential family in* X, *then there exists* $\beta \in \Gamma$ *such that for all* $\beta_0 \in \Gamma$ *with* $\beta \preceq \beta_0$, $(\mathcal{C}, \beta_0)$ *satisfies the projection criterion in* $\mathbf{X}$.

Proof. Take β as in Lemma 6.3(5), and let $\beta_0 \in \Gamma$ be chosen so that $\beta \preceq \beta_0$. Suppose that $\gamma \in \Gamma$ with $\beta_0 \preceq \gamma$. Then $\beta \preceq \gamma$, so from Lemma 6.3(5), we get (1) and (2) of Definition 6.4. Item (3) results from Lemma 6.3(4). We get (4) using (1), (3), and Lemma 5.7. Finally apply Lemma 5.12 and (3) to see that (5) obtains. $\square$

Lemma 6.6. *Let* $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *be an inverse system of nonempty compact Hausdorff spaces,* $X = \lim \mathbf{X}$, $\mathcal{C}$ *be a finite essential family in* X, $\mu \in \Gamma$, *and* $(\mathcal{C}, \mu)$ *satisfy the projection criterion in* $\mathbf{X}$. *Let* $\mathcal{C}^*$ *be a finite essential family in* X *with* $\mathcal{C} \subset \mathcal{C}^*$. *Then there exists* $\mu^* \in \Gamma$ *with* $\mu \preceq \mu^*$ *such that* $(\mathcal{C}^*, \mu^*)$ *satisfies the projection criterion in* $\mathbf{X}$.

Proof. Apply Lemma 6.5 to the finite essential family $\mathcal{C}^*$ in X to obtain β as indicated there. Select $\mu^* \in \Gamma$ so that both $\beta \preceq \mu^*$ and $\mu \preceq \mu^*$. According to Lemma 6.5, $(\mathcal{C}^*, \mu^*)$ satisfies the projection criterion in $\mathbf{X}$. $\square$

Corollary 6.7. *Let* $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *be an inverse system of nonempty compact Hausdorff spaces and* $X = \lim \mathbf{X}$. *Suppose that* $(\mathcal{C}_n)$ *is a sequence of nonempty finite essential families in* X *such that for each* $n \in \mathbb{N}$, $\mathcal{C}_n \subset \mathcal{C}_{n+1}$. *Then there exists a relation preserving function* $\mu : (\mathbb{N}, \leq) \to (\Gamma, \preceq)$ *such that for each* $k \in \mathbb{N}$, $(\mathcal{C}_k, \mu(k))$ *satisfies the projection criterion in* $\mathbf{X}$.

Proof. Apply Lemma 6.5 to find $\mu(1) \in \Gamma$ such that $(\mathcal{C}_1, \mu(1))$ satisfies the projection criterion in $\mathbf{X}$. Then apply Lemma 6.6 recursively to find the function μ as requested. $\square$

Corollary 6.8. *Let* $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *be an inverse system of nonempty compact Hausdorff spaces,* $X = \lim \mathbf{X}$, *and* $n \in \mathbb{N}$. *Suppose that* $(\mathcal{C}_k)$, $k = 1, \ldots, n$, *is a finite sequence of nonempty finite essential families in* X *such that for each* $1 \leq k < n$, $\mathcal{C}_k \subset \mathcal{C}_{k+1}$. *Then there exists a relation preserving function* $\mu : (\{1, \ldots, n\}, \leq) \to (\Gamma, \preceq)$ *such that for each* $1 \leq k \leq n$, $(\mathcal{C}_k, \mu(k))$ *satisfies the projection criterion in* $\mathbf{X}$.

Proof. For each $k \in \mathbb{N}$ with $n < k$, put $\mathcal{C}_k = \mathcal{C}_n$, apply Corollary 6.7 to the sequence $(\mathcal{C}_k)$, let μ^* be the relation preserving function obtained therefrom, and define the function $\mu = \mu^* | \{1, \ldots, n\}$. $\square$

Lemma 6.9 will provide the first step in an inductive argument in the proof of Lemma 9.2.

Lemma 6.9. *Let* $\mathcal{C}_1$ *be a family of closed pairs in* X *with* $\mathrm{card}(\mathcal{C}_1) = 1$. *Suppose that* $\beta \in \Gamma$ *and that* $(\mathcal{C}_1, \beta)$ *satisfies the projection criterion in* $\mathbf{X}$. *Define* $\mu : \{1\} \to \Gamma$ *by* $\mu(1) = \beta$ *and put* $\mathcal{D}_1 = p_{\mu(1)}(\mathcal{C}_1)$. *Then* $(\mu, (\mathcal{D}_1))$ *is a* 1*-essential sequence in* $\mathbf{X}$.

Proof. We have to show that the four conditions in Definition 3.4 are satisfied by the pair $(\mu, (\mathcal{D}_1))$. By Definition 6.4(1), $\mathcal{D}_1$ is an essential family in $p_{\mu(1)}(X) = p_\beta(X)$, so by Definition 2.3, we get (1) of Definition 3.4. Plainly $\mathrm{card}(\mathcal{D}_1) = 1$, so we also get (2). To show that Definition 3.4(3) is operational, i.e., that $(\mu(1), \mathcal{D}_1))$ is essential in X, suppose that $\gamma \in \Gamma$ and $\mu(1) \preceq \gamma$. According to Definition 3.1, we are required to show that $p^{-1}_{\mu(1)\gamma}(\mathcal{D}_1)$ is an essential family in $p_\gamma(X)$. By Definition 6.4(1), $p_\gamma(\mathcal{C}_1)$ is an essential family in $p_\gamma(X)$, and by Definition 6.4(3), $p^{-1}_{\mu(1)\gamma}(\mathcal{D}_1) = p^{-1}_{\mu(1)\gamma}(p_{\mu(1)}(\mathcal{C}_1))$ is a closed thickening of $p_\gamma(\mathcal{C}_1)$ in $p_\gamma(X)$ into disjoint closed pairs. Apply Lemma 5.7 to conclude that $p^{-1}_{\mu(1)\gamma}(\mathcal{D}_1)$ is an essential family in $p_\gamma(X)$. Finally, Definition 3.4(4) is true vacuously. $\square$

7. Finite Inessential Families in Inverse Limits

Lemma 7.3 might be thought of as providing the "fundamental" step in the process of relating a nonempty finite inessential family in the limit of an inverse system of nonempty Hausdorff compacta to the system itself. It will be critical in our proof of Lemma 8.1. For use in our proof of this lemma, we provide the definition of a *swelling* (p. 472 of [En]) and cite the "swelling" Theorem, Theorem 7.1.4 of [En], in the form needed here.

Definition 7.1. *A **swelling** of a family $\{A_s \mid s \in S\}$ of subsets of a space X is a family $\{B_s \mid s \in S\}$ such that $A_s \subset B_s$ for all $s \in S$ and for every finite subset $S_0 \subset S$, $\bigcap\{A_s \mid s \in S_0\} = \emptyset$ if and only if $\bigcap\{B_s \mid s \in S_0\} = \emptyset$.*

Theorem 7.2. *Let Γ_0 be a nonempty finite set and $\{F_\gamma \mid \gamma \in \Gamma_0\}$ be a family of closed subsets of a normal space X. Then $\{F_\gamma \mid \gamma \in \Gamma_0\}$ has a swelling $\{U_\gamma \mid \gamma \in \Gamma_0\}$ into open subsets U_γ of X.* □

Lemma 7.3. *Let $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ be an inverse system of compact Hausdorff spaces and $X = \lim \mathbf{X}$. If $\Gamma_0 \subset \Gamma$ is finite and nonempty and $\{C_\gamma \mid \gamma \in \Gamma_0\}$ is an inessential family in X, then there exists a common successor γ_0 of Γ_0 in Γ such that for all $\beta \in \Gamma$ with $\gamma_0 \preceq \beta$, $\{p_\beta(C_\gamma) \mid \gamma \in \Gamma_0\}$ is an inessential family in $p_\beta(X)$.*

Proof. Since $\{C_\gamma \mid \gamma \in \Gamma_0\}$ is an inessential family in X, then there exists a collection $\{P_\gamma \mid \gamma \in \Gamma_0\}$ such that for each $\gamma \in \Gamma_0$, P_γ is a partition of C_γ in X, and $\bigcap\{P_\gamma \mid \gamma \in \Gamma_0\} = \emptyset$. For each $\gamma \in \Gamma_0$, denote $C_\gamma = (A_\gamma, B_\gamma)$. The definition of partition gives us a disjoint open pair (U_γ, V_γ) in X such that,

$$A_\gamma \subset U_\gamma \text{ and } B_\gamma \subset V_\gamma, \tag{1}$$

$$X \setminus P_\gamma = U_\gamma \cup V_\gamma. \tag{2}$$

Making use of Theorem 7.2, choose a collection $\{Q_\gamma \mid \gamma \in \Gamma_0\}$ of open subsets of X with the following properties:

$$\text{for each } \gamma \in \Gamma_0, P_\gamma \subset Q_\gamma, \tag{3}$$

$$\bigcap\{Q_\gamma \mid \gamma \in \Gamma_0\} = \emptyset, \text{ and} \tag{4}$$

$$\text{for each } \gamma \in \Gamma_0, Q_\gamma \subseteq X \setminus (A_\gamma \cup B_\gamma). \tag{5}$$

Then set

$$G_\gamma = \overline{U}_\gamma \cap (X \setminus Q_\gamma) \text{ and } H_\gamma = \overline{V}_\gamma \cap (X \setminus Q_\gamma).$$

Thus we have that for all $\gamma \in \Gamma_0$,

$$G_\gamma \text{ and } H_\gamma \text{ are both closed in } X,$$
$$X \setminus Q_\gamma = G_\gamma \cup H_\gamma, \text{ and}$$
$$A_\gamma \subset G_\gamma \subset U_\gamma \text{ and } B_\gamma \subset H_\gamma \subset V_\gamma.$$

Employing Lemma 6.3(1), for each $\gamma \in \Gamma_0$, choose $\beta_\gamma \in \Gamma$ so that $\gamma \preceq \beta_\gamma$ and for all $\beta \in \Gamma$ with $\beta_\gamma \preceq \beta$, the following inclusions hold:

$$(\dagger)\ p_\beta^{-1}(p_\beta(G_\gamma)) \subset U_\gamma \text{ and } p_\beta^{-1}(p_\beta(H_\gamma)) \subset V_\gamma.$$

Let γ_0 be a common successor to $\{\beta_\gamma \mid \gamma \in \Gamma_0\}$; then of course, γ_0 is also a common successor to Γ_0. Moreover, for each $\beta \in \Gamma$ with $\gamma_0 \preceq \beta$, and for all $\gamma \in \Gamma_0$, $(\dagger)$ holds true. Applying Lemma 6.3(2) if necessary, we may assume in addition that for each $\beta \in \Gamma$ with $\gamma_0 \preceq \beta$ and each $\gamma \in \Gamma_0$, $p_\beta(C_\gamma)$ is a closed disjoint pair in $p_\beta(X)$.

To see that γ_0 satisfies the hypothesis, fix $\beta \in \Gamma$ with $\gamma_0 \preceq \beta$. We must demonstrate that $\{p_\beta(C_\gamma) \mid \gamma \in \Gamma_0\}$ is an inessential family in $p_\beta(X)$. Since each p_β is a closed map, then for each $\gamma \in \Gamma_0$, we can choose open neighborhoods K_γ of $p_\beta(G_\gamma)$ and L_γ of $p_\beta(H_\gamma)$ in $p_\beta(X)$ such that

$$p_\beta^{-1}(p_\beta(G_\gamma)) \subset p_\beta^{-1}(K_\gamma) \subset U_\gamma \text{ and } p_\beta^{-1}(p_\beta(H_\gamma)) \subset p_\beta^{-1}(L_\gamma) \subset V_\gamma.$$

Note that since (U_γ, V_γ) is a disjoint pair in X, then we have that $p_\beta^{-1}(K_\gamma, L_\gamma)$ is a disjoint pair in X as well.

For each $\gamma \in \Gamma_0$ we now set

$$(*) \quad R_\gamma = p_\beta(X) \setminus (K_\gamma \cup L_\gamma).$$

We claim that R_γ is a partition of $p_\beta(A_\gamma, B_\gamma)$ in $p_\beta(X)$. Indeed, we have that $p_\beta(A_\gamma) \subset p_\beta(G_\gamma) \subset K_\gamma$, $p_\beta(B_\gamma) \subset p_\beta(H_\gamma) \subset L_\gamma$, $p_\beta(X) \setminus R_\gamma = K_\gamma \cup L_\gamma$. But $K_\gamma \cap L_\gamma = \emptyset$ for the following reasons. If $x \in K_\gamma \cap L_\gamma \subset p_\beta(X)$, then the surjectivity of $p_\beta : X \to p_\beta(X)$ shows that $\emptyset \neq p_\beta^{-1}(x) \subset p_\beta^{-1}(K_\gamma) \cap p_\beta^{-1}(L_\gamma)$. However, we showed above that $p_\beta^{-1}(K_\gamma, L_\gamma)$ is a disjoint pair in X.

We will now demonstrate that $\bigcap\{R_\gamma \mid \gamma \in \Gamma_0\} = \emptyset$. Of course $\bigcap\{R_\gamma \mid \gamma \in \Gamma_0\} \subset p_\beta(X)$. Note that for each $\gamma \in \Gamma_0$,

$$\begin{aligned} X \setminus Q_\gamma &= G_\gamma \cup H_\gamma \\ &\subset p_\beta^{-1}(p_\beta(G_\gamma)) \cup p_\beta^{-1}(p_\beta(H_\gamma)) \\ &\subset p_\beta^{-1}(K_\gamma) \cup p_\beta^{-1}(L_\gamma). \end{aligned}$$

It follows from this and $(*)$ that

$$\begin{aligned} p_\beta^{-1}(R_\gamma) &= p_\beta^{-1}(p_\beta(X) \setminus (K_\gamma \cup L_\gamma)) \\ &= X \setminus (p_\beta^{-1}(K_\gamma) \cup p_\beta^{-1}(L_\gamma)) \\ &\subset Q_\gamma. \end{aligned}$$

From (4) we have that $\bigcap\{Q_\gamma \mid \gamma \in \Gamma_0\} = \emptyset$, and so $\bigcap\{p_\beta^{-1}(R_\gamma) \mid \gamma \in \Gamma_0\} = p_\beta^{-1}(\bigcap\{R_\gamma \mid \gamma \in \Gamma_0\}) = \emptyset$. Since $p_\beta : X \to p_\beta(X)$ is surjective, then

$$\bigcap\{R_\gamma \mid \gamma \in \Gamma_0\} = \emptyset.$$

And so $\{p_\beta(C_\gamma) \mid \gamma \in \Gamma_0\}$ is an inessential family in $p_\beta(X)$. $\square$

8. Induced Essential Families

Lemma 8.1 will show that if $(\mu, \mathcal{D})$ is essential in an inverse system $\mathbf{X}$ (see Definition 3.1), then it will induce an essential family in $\lim \mathbf{X}$ of precisely the same cardinality as that of $\mathcal{D}$. Lemma 8.2 will show how, under certain conditions, a nonempty collection of closed disjoint pairs in a coordinate space X_{μ_1} can be "pulled up" to a coordinate space X_{μ_2}, $\mu_1 \preceq \mu_2$, so that it is a subset of another collection of closed disjoint pairs in X_{μ_2}.

Lemma 8.1. *Let $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ be an inverse system of nonempty compact Hausdorff spaces and $X = \lim \mathbf{X}$. Suppose that $(\mu, \mathcal{D})$ is essential in $\mathbf{X}$. Put $\mathcal{C} = p_\mu^{-1}(\mathcal{D})$. Then $\mathcal{C}$ is an essential family in X with* $\operatorname{card}(\mathcal{C}) = \operatorname{card}(\mathcal{D})$.

Proof. Using Definition 3.1 with $\gamma = \mu$, we have that $\mathcal{D}$ is a family of closed disjoint pairs in $p_\mu(X)$. An application of Lemma 5.4(1) yields that $\mathcal{C}$ is a collection of closed disjoint pairs in X with $\mathrm{card}(\mathcal{C}) = \mathrm{card}(\mathcal{D})$. Suppose, however, that $\mathcal{C}$ is not an essential family in X. Applying Lemma 5.13, we find a nonempty finite subset $\mathcal{D}_0 \subset \mathcal{D}$ such that $\mathcal{C}_0 = p_\mu^{-1}(\mathcal{D}_0)$ is an inessential family in X.

Apply Lemma 7.3 to find $\gamma \in \Gamma$ with $\mu \preceq \gamma$ such that $p_\gamma(\mathcal{C}_0)$ is an inessential family in $p_\gamma(X)$. From Lemma 6.2(2) one sees that $\mathcal{C}_0 = p_\gamma^{-1}(p_{\mu\gamma}^{-1}(\mathcal{D}_0))$. It follows that $p_\gamma(\mathcal{C}_0) = p_{\mu\gamma}^{-1}(\mathcal{D}_0) \subset p_{\mu\gamma}^{-1}(\mathcal{D})$ which is an essential family in $p_\gamma(X)$ by Definition 3.1. By virtue of Lemma 5.9, $p_\gamma(\mathcal{C}_0)$ is also an essential family in $p_\gamma(X)$. Thus we arrive at a contradiction. $\square$

Lemma 8.2. *Let* $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *be an inverse system of nonempty compact Hausdorff spaces and* $X = \lim \mathbf{X}$*. Suppose that*

(1) $\{\mu_1, \mu_2\} \subset \Gamma$ *with* $\mu_1 \preceq \mu_2$,
(2) $\mathcal{D}_1$ *is a nonempty collection of closed disjoint pairs in* $p_{\mu_1}(X)$,
(3) $\mathcal{D}_2$ *is a nonempty collection of closed disjoint pairs in* $p_{\mu_2}(X)$, *and*
(4) $p_{\mu_1\mu_2}^{-1}(\mathcal{D}_1) \subset \mathcal{D}_2$.

Define $\mathcal{C}_1 = p_{\mu_1}^{-1}(\mathcal{D}_1)$ *and* $\mathcal{C}_2 = p_{\mu_2}^{-1}(\mathcal{D}_2)$*. Then* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are collections of closed disjoint pairs in* X *with* $\mathcal{C}_1 \subset \mathcal{C}_2$*. If in addition, both* $(\mu_1, \mathcal{D}_1)$ *and* $(\mu_2, \mathcal{D}_2)$ *are essential in* $\mathbf{X}$, *then both* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are essential families in* X, $\mathrm{card}(\mathcal{C}_1) = \mathrm{card}(\mathcal{D}_1)$, *and* $\mathrm{card}(\mathcal{C}_2) = \mathrm{card}(\mathcal{D}_2)$.

Proof. An application of (2), (3), and the first part of Lemma 5.4(1) yields that $\mathcal{C}_1$ and $\mathcal{C}_2$ are collections of closed disjoint pairs in X. Use the inclusion in (4) and the definitions of $\mathcal{C}_1$ and $\mathcal{C}_2$ in conjunction with Lemma 6.2(2) to get that $\mathcal{C}_1 = p_{\mu_1}^{-1}(\mathcal{D}_1) = p_{\mu_2}^{-1}(p_{\mu_1\mu_2}^{-1}(\mathcal{D}_1)) \subset p_{\mu_2}^{-1}(\mathcal{D}_2) = \mathcal{C}_2$. Finally, apply Lemma 8.1 to see that both $\mathcal{C}_1$ and $\mathcal{C}_2$ are essential families in X and to obtain the cardinality facts. $\square$

9. Finite Essential Families in Inverse Limits

Important outputs of this section are Lemmas 9.2 and Lemma 9.3. The former shows that an essential family $\mathcal{C}$ in the limit of an inverse system $\mathbf{X}$ of nonempty compact Hausdorff spaces will provide us with an n-essential or ω-essential family in $\mathbf{X}$, depending on the cardinality of $\mathcal{C}$. The latter does this in reverse. We need Lemma 9.1 to support our proof of Lemma 9.2.

Lemma 9.1. *Let* $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *be an inverse system of nonempty compact Hausdorff spaces,* $X = \lim \mathbf{X}$, $n \in \mathbb{N}$, *and* $(\mathcal{C}_k)$, $k = 1, \dots, n+1$, *be a finite sequence of essential families in* X *such that for each* $1 \le k \le n+1$, $\mathrm{card}(\mathcal{C}_k) = k$*. Suppose in addition that* $\mu : (\{1, \dots, n+1\}, \le) \to (\Gamma, \preceq)$ *is a relation preserving function, for all* $1 \le k \le n+1$, $(\mathcal{C}_k, \mu(k))$ *satisfies the projection criterion in* $\mathbf{X}$, *and for all* $1 \le k \le n$, $\mathcal{C}_k \subset \mathcal{C}_{k+1}$.

Assume that there exists a finite sequence $(\mathcal{D}_k)$, $k = 1, \dots, n$, *such that* $\mathcal{D}_1 = p_{\mu(1)}(\mathcal{C}_1)$, $(\mu|\{1, \dots, n\}, (\mathcal{D}_k))$ *is an* n*-essential sequence in* $\mathbf{X}$, *for each* $1 \le k < n$, $p_{\mu(k)\mu(k+1)}^{-1}(\mathcal{D}_k) \subset \mathcal{D}_{k+1}$, *and for each* $1 \le k \le n$ *and* $\gamma \in \Gamma$ *with* $\mu(k) \preceq \gamma$, $p_{\mu(k)\gamma}^{-1}(\mathcal{D}_k)$ *is a closed thickening of* $p_\gamma(\mathcal{C}_k)$ *in* $p_\gamma(X)$*. Put*

$$(\dagger)\ \ \mathcal{D}_{n+1} = p_{\mu(n)\mu(n+1)}^{-1}(\mathcal{D}_n) \cup \{p_{\mu(n+1)}(C_{n+1})\},$$

where C_{n+1} is the unique element of $\mathcal{C}_{n+1} \setminus \mathcal{C}_n$. Then:

(1) $p^{-1}_{\mu(n)\mu(n+1)}(\mathcal{D}_n) \subset \mathcal{D}_{n+1}$,
(2) *for all $\gamma \in \Gamma$ with $\mu(n+1) \preceq \gamma$, $p^{-1}_{\mu(n+1)\gamma}(\mathcal{D}_{n+1})$ is a closed thickening of $p_\gamma(\mathcal{C}_{n+1})$ in $p_\gamma(X)$ into disjoint pairs, and*
(3) *$(\mu, (\mathcal{D}_k))$ is an $(n+1)$-essential sequence in $\mathbf{X}$.*

Proof. We get (1) from (†); proceed as follows for (2). Fix $\gamma \in \Gamma$ with $\mu(n+1) \preceq \gamma$. Of course, $\mu(n) \preceq \mu(n+1) \preceq \gamma$. By hypothesis, $p^{-1}_{\mu(n)\mu(n)}(\mathcal{D}_n) = \mathcal{D}_n$ is a closed thickening of $p_{\mu(n)}(\mathcal{C}_n)$ in $p_{\mu(n)}(X)$. This and Lemma 6.3(4) yield that,

($\dagger_1$) $p^{-1}_{\mu(n)\gamma}(\mathcal{D}_n) = (p_{\mu(n)\mu(n+1)}p_{\mu(n+1)\gamma})^{-1}(\mathcal{D}_n) = p^{-1}_{\mu(n+1)\gamma}(p^{-1}_{\mu(n)\mu(n+1)}(\mathcal{D}_n))$ is a closed thickening of $p_\gamma(\mathcal{C}_n)$ in $p_\gamma(X)$ into disjoint pairs.

Also by hypothesis, $(\mathcal{C}_{n+1}, \mu(n+1))$ satisfies the projection criterion in $\mathbf{X}$. So by Definition 6.4(1),

($\dagger_2$) $p_\gamma(\mathcal{C}_{n+1})$ is an essential family in $p_\gamma(X)$.

Applying ($\dagger_2$) and Definition 2.3, one sees that $p_\gamma(\mathcal{C}_{n+1})$ is a collection of closed disjoint pairs in $p_\gamma(X)$; as a special case,

($\dagger_3$) $p_\gamma(C_{n+1})$ is a closed disjoint pair in $p_\gamma(X)$ which is trivially a closed thickening of $p_\gamma(C_{n+1})$ in $p_\gamma(X)$.

From ($\dagger_3$), one has,

($\dagger_4$) $p_{\mu(n+1)}(C_{n+1})$ is a closed disjoint pair in $p_{\mu(n+1)}(X)$.

Moreover, the fact from ($\dagger_4$) that $p_{\mu(n+1)}(C_{n+1})$ is a closed disjoint pair in $p_{\mu(n+1)}(X)$ along with Lemma 5.4(1), show that $p^{-1}_{\mu(n+1)\gamma}(p_{\mu(n+1)}(C_{n+1}))$ is a closed disjoint pair in $p_\gamma(X)$. Apply Lemma 6.2(1) to detect that

($\dagger_5$) $p^{-1}_{\mu(n+1)\gamma}(p_{\mu(n+1)}(C_{n+1}))$ is a closed thickening of $p_\gamma(C_{n+1})$ in $p_\gamma(X)$ into a disjoint pair.

Use (†), ($\dagger_1$), and ($\dagger_5$) to conclude that (2) holds true.

To prove (3) we have to show that (1)-(4) of Definition 3.4 hold true for μ and the finite sequence $(\mu(k), \mathcal{D}_k)$, $k = 1, \ldots, n+1$. By assumption, (1)-(4) of Definition 3.4 hold true for $\mu|\{1, \ldots, n\}$ and the finite sequence $(\mu(k), \mathcal{D}_k)$, $k = 1, \ldots, n$. So it remains to demonstrate that,

(A1) $\mathcal{D}_{n+1}$ is a nonempty family of closed disjoint pairs in $p_{\mu(n+1)}(X)$,
(B1) $\mathrm{card}(\mathcal{D}_{n+1}) = n+1$,
(C1) $(\mu(n+1), \mathcal{D}_{n+1})$ is essential in $\mathbf{X}$, and
(D1) $p^{-1}_{\mu(n)\mu(n+1)}(\mathcal{D}_n) \subset \mathcal{D}_{n+1}$.

Since $p_{\mu(n+1)}(C_{n+1}) \in \mathcal{D}_{n+1}$, then $\mathcal{D}_{n+1} \neq \emptyset$. Apply (2) with $\gamma = \mu(n+1)$ to see that $\mathcal{D}_{n+1}$ is a family of closed disjoint pairs in $p_{\mu(n+1)}(X)$. So (A1) is established. By hypothesis $\mathrm{card}(\mathcal{C}_{n+1}) = n+1$, and Definition 6.4(2) obtains here. Hence we have,

($\dagger_6$) $\mathrm{card}(p_{\mu(n+1)}(\mathcal{C}_{n+1})) = \mathrm{card}(\mathcal{C}_{n+1}) = n+1$.

Apply (2) and Lemma 5.12 to obtain (B1).

To prove (C1) we must show that $p^{-1}_{\mu(n+1)\gamma}(\mathcal{D}_{n+1})$ is an essential family in $p_\gamma(X)$. From (2), $p^{-1}_{\mu(n+1)\gamma}(\mathcal{D}_{n+1})$ is a closed thickening of $p_\gamma(\mathcal{C}_{n+1})$ in $p_\gamma(X)$ into disjoint pairs. Apply this and Lemma 5.7 to get (C1). Of course, (D1) is true because of (†). □

Lemma 9.2. *Let $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ be an inverse system of nonempty compact Hausdorff spaces and $X = \lim \mathbf{X}$.*

(1) *Suppose that* $n \in \mathbb{N}$, $\mathcal{C}_1, \dots, \mathcal{C}_n$ *is a finite sequence of essential families in* X, *for each* $1 \leq k \leq n$, $\operatorname{card}(\mathcal{C}_k) = k$, *and for each* $1 \leq k < n$, $\mathcal{C}_k \subset \mathcal{C}_{k+1}$. *Then there is an* n*-essential sequence* $(\mu, (\mathcal{D}_k))$ *in* $\mathbf{X}$ *such that* $\mathcal{D}_1 = p_{\mu(1)}(\mathcal{C}_1)$, *and for each* $1 \leq k < n$, $p^{-1}_{\mu(k)\mu(k+1)}(\mathcal{D}_k) \subset \mathcal{D}_{k+1}$.

(2) *Suppose that* $(\mathcal{C}_n)$ *is a sequence of essential families in* X *such that for each* $n \in \mathbb{N}$, $\operatorname{card}(\mathcal{C}_n) = n$ *and* $\mathcal{C}_n \subset \mathcal{C}_{n+1}$. *Then there is an* ω*-essential sequence* $(\mu, (\mathcal{D}_n))$ *in* $\mathbf{X}$ *such that* $\mathcal{D}_1 = p_{\mu(1)}(\mathcal{C}_1)$ *and for each* $n \in \mathbb{N}$, $p^{-1}_{\mu(n)\mu(n+1)}(\mathcal{D}_n) \subset \mathcal{D}_{n+1}$.

Proof. The lemma is true in (1) for $n = 1$ via $\mu : (\{1\}, \leq) \to (\Gamma, \preceq)$ from Lemma 6.9. Use this and and Lemma 9.1(1,3) recursively to complete the proof for each of (1) and (2). $\square$

Lemma 9.3. *Let* $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, (\Gamma, \preceq)\}$ *be an inverse system of nonempty compact Hausdorff spaces and* $X = \lim \mathbf{X}$.

(1) *Suppose that* $n \in \mathbb{N}$ *and there exists an* n*-essential sequence* $(\mu, (\mathcal{D}_k))$ *in* $\mathbf{X}$. *For each* $1 \leq k \leq n$, *define* $\mathcal{C}_k = p^{-1}_{\mu(k)}(\mathcal{D}_k)$. *Then for each* $1 \leq k \leq n$, $\mathcal{C}_k$ *is an essential family in* X, $\operatorname{card}(\mathcal{C}_k) = k$, *and if* $k < n$, *then* $\mathcal{C}_k \subset \mathcal{C}_{k+1}$.

(2) *Suppose that there exists an* ω*-essential sequence* $(\mu, (\mathcal{D}_n))$ *in* $\mathbf{X}$. *For each* $n \in \mathbb{N}$ *define* $\mathcal{C}_n = p^{-1}_{\mu(n)}(\mathcal{D}_n)$. *Then for each* $n \in \mathbb{N}$, $\mathcal{C}_n$ *is an essential family in* X, $\operatorname{card}(\mathcal{C}_n) = n$, *and* $\mathcal{C}_n \subset \mathcal{C}_{n+1}$

Proof. Applying Definitions 3.4(3) and 3.5(3) in both (1) and (2), one sees that for each $k \in \operatorname{dom}(\mu)$, $(\mu(k), \mathcal{D}_k)$ is essential in $\mathbf{X}$. Therefore the proof is completed by applying Lemmas 8.1 and 8.2 respectively. $\square$

10. Proofs of the Main Results

Proof of Theorem 4.1.

(1) $\Rightarrow$ (2). Since X is SID, by Definition 2.6(2), there is a countably infinite essential family $\mathcal{C}$ in X. Using Lemma 5.11, write $\mathcal{C} = \bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ where for each $n \in \mathbb{N}$, $\mathcal{C}_n$ is an essential family in X, $\operatorname{card}(\mathcal{C}_n) = n$, and $\mathcal{C}_n \subset \mathcal{C}_{n+1}$. Then apply Lemma 9.2(2) to see that there is an ω-essential sequence in $\mathbf{X}$.

(2) $\Rightarrow$ (1). We are given that there is an ω-essential sequence in $\mathbf{X}$. By Lemma 9.3(2), there is a sequence $(\mathcal{C}_n)$ of essential families in X such that for each $n \in \mathbb{N}$, $\operatorname{card}(\mathcal{C}_n) = n$ and $\mathcal{C}_n \subset \mathcal{C}_{n+1}$. An application of Lemma 5.13 shows that $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ is a countably infinite essential family in X, so by Definition 2.6(2), X is SID. $\square$

Proof of Theorem 4.2.

(1) $\Rightarrow$ (2). Using (2) $\Rightarrow$ (1) of Theorem 4.1 and the fact that X is not SID, we see that there does not exist an ω-essential sequence in $\mathbf{X}$. Let $n \in \mathbb{N}$. Since X is WID, then by Definition 2.6, there exists an essential family $\mathcal{C}$ in X with $\operatorname{card}(\mathcal{C}) = n$. Use Lemma 5.10 for $\mathcal{C}$ to find a finite sequence $(\mathcal{C}_k)$, $k = 1, \dots, n$, of essential families in X such that for each $1 \leq k \leq n$, $\operatorname{card}(\mathcal{C}_k) = k$, and if $1 \leq k < n$, then $\mathcal{C}_k \subset \mathcal{C}_{k+1}$. An application of Lemma 9.2(1) yields an n-essential sequence in $\mathbf{X}$.

(2) $\Rightarrow$ (1). Since there is no ω-essential sequence in $\mathbf{X}$, then Theorem 4.1 (1) $\Rightarrow$ (2) shows that X is not SID. Let $n \in \mathbb{N}$; by hypothesis there exists an n-essential sequence in $\mathbf{X}$. By Lemma 9.3(1), we see that there exists an essential family in X, given there as $\mathcal{C}_n$, with $\operatorname{card}(\mathcal{C}_n) = n$. $\square$

Proof of Theorem 4.3.

(1) $\Rightarrow$ (2). By Theorem 2.4, $\dim X = n$ implies that there exists an essential family $\mathcal{C}$ in X with $\mathrm{card}(\mathcal{C}) = n$. Use Lemma 5.10 for $\mathcal{C}$ to find a finite sequence $(\mathcal{C}_k)$, $k = 1, \ldots, n$, of essential families in X such that for each $1 \leq k \leq n$, $\mathrm{card}(\mathcal{C}_k) = k$, and if $1 \leq k < n$, then $\mathcal{C}_k \subset \mathcal{C}_{k+1}$. An application of Lemma 9.2(1) yields an n-essential sequence in $\mathbf{X}$.

To arrive at a contradiction, suppose that $n_0 > n$ and there exists an n_0-essential sequence $(\mu, (\mathcal{D}_k))$ in $\mathbf{X}$. Then $\mathrm{card}(\mathcal{D}_{n_0}) = n_0$ and $(\mu(n_0), \mathcal{D}_{n_0})$ is essential in $\mathbf{X}$. By Lemma 8.1, X has an essential family $\mathcal{C}_0$ with $\mathrm{card}(\mathcal{C}_0) = n_0$. But $\dim X = n$, so this contradicts Theorem 2.4.

(2) $\Rightarrow$ (1). Since there exists an n-essential sequence in $\mathbf{X}$, then Lemma 8.1 implies there exists an essential family in X with cardinality n. To prove that $\dim X = n$, let $n_0 \in \mathbb{N}$ with $n_0 > n$. According to Theorem 2.4, we have to show that there is no essential family in X whose cardinality is n_0. Suppose that there is an essential family $\mathcal{C}$ in X with $\mathrm{card}(\mathcal{C}) = n_0$. Use Lemma 5.10 for $\mathcal{C}$ to find a finite sequence $(\mathcal{C}_k)$, $k = 1, \ldots, n_0$, of essential families in X such that for each $1 \leq k \leq n_0$, $\mathrm{card}(\mathcal{C}_k) = k$, and if $1 \leq k < n_0$, then $\mathcal{C}_k \subset \mathcal{C}_{k+1}$. An application of Lemma 9.2(1) yields an n_0-essential sequence in $\mathbf{X}$. This gives us a contradiction. $\square$

We shall leave the proof of Theorem 4.4 to the reader.

11. Examples

Our examples will involve inverse sequences $\mathbf{X} = (X_k, p_{kk+1}, (\mathbb{N}, \leq))$ of nonempty compact metrizable spaces X_k. Their limits are nonempty metrizable compacta. Let $X = \lim \mathbf{X}$ and $n \geq 0$. A typical limit theorem states that if for each $k \in \mathbb{N}$, $\dim X_k \leq n$, then $\dim X \leq n$. Such a theorem does not provide us conditions that would show that $\dim X = n$. In the event that $\dim X_k \leq n$ is not necessarily true for each k, then it could still be true that $\dim X \leq n$. Once again, those typical limit theorems are not helpful in detecting this.

As usual, for each $k \in \mathbb{N}$, let $I^k = [0,1]^k \subset \mathbb{R}^k$, and if $j \in \mathbb{N}$ with $k < j$, let $q^j_k : I^j \to I^k$ be the coordinate projection. Put $I^0 = \{0\}$ and for each $k \in \mathbb{N}$, let $q^k_0 : I^k \to I^0$ be the constant map. For each $k \geq 0$, $\dim I^k = k$. Denote

$$(\dagger) \quad Y_k = \bigsqcup \{I^j \mid 0 \leq j \leq k\}.$$

Thus Y_k is the topological sum of $\{I^0, \ldots, I^k\}$.

For each $k \geq 1$, the *standard* essential family $\mathcal{D}_k$ in I^k with $\mathrm{card}(\mathcal{D}_k) = k$ consists of the pairs $D^k_i = (L^k_i, R^k_i)$, $1 \leq i \leq k$, of *opposite faces* of I^k (Corollary 5.2.16 of [Sa]). Here $L^k_i = \{(x_1, \ldots, x_k) \in I^k \mid x_i = 0\}$ and $R^k_i = \{(x_1, \ldots, x_k) \in I^k \mid x_i = 1\}$.

Lemma 11.1. *Let $k \in \mathbb{N}$. Then for each $1 \leq i \leq k$, $(q^{k+1}_k)^{-1}(D^k_i) = D^{k+1}_i$. Hence $(q^{k+1}_k)^{-1}(\mathcal{D}_k) \subset \mathcal{D}_{k+1}$.* $\square$

From Lemma 11.1 one sees that all but one element of the standard essential family for I^{k+1} comes from "pulling up" the standard essential family from I^k.

The Hilbert cube I^∞ is $\prod\{I_j \mid j \in \mathbb{N}\}$ where for each j, $I_j = [0,1]$. One can define the countably infinite collection of opposite face pairs in I^∞ by analogy with what we just did for finite-dimensional cubes; it is known that this family is essential in I^∞ (Theorem 5.6.1 of [Sa]). So the Hilbert cube is SID. Alternatively consider the next example.

Example 1. Let $\mathbf{X} = (X_k, p_{kk+1}, (\mathbb{N}, \leq))$ be the inverse sequence where for each $k \in \mathbb{N}$, $X_k = I^k$ and $p_{kk+1} = q_k^{k+1}$. Put $X = \lim \mathbf{X}$.

It can be proved that $X \cong I^\infty$. Now define the relation preserving function $\mu : (\mathbb{N}, \leq) \to (\mathbb{N}, \leq)$ by $\mu = \mathrm{id}_{\mathbb{N}}$. Using Lemma 11.1, one can see that $p_{kk+1}^{-1}(\mathcal{D}_k) \subset \mathcal{D}_{k+1}$. It is not difficult to see that for each $k \in \mathbb{N}$, $(\mu(k), \mathcal{D}_{\mu(k)}) = (k, \mathcal{D}_k)$ is essential in $\mathbf{X}$. Hence $(\mu, (\mathcal{D}_k))$ is an ω-essential sequence in $\mathbf{X}$. By Theorem 4.1, X is SID. So Example 1 provides an alternate way of proving that I^∞ is SID.

Although it is possible to define a WID compactum inside I^∞, our next example will produce a WID metrizable compactum indirectly. This compactum of course could be embedded in I^∞.

Example 2. Let $\mathbf{Y} = (Y_k, r_{kk+1}, (\mathbb{N}, \leq))$ (see (†)) be the inverse sequence where for each $k \in \mathbb{N}$, $r_{kk+1}(x) = x$ for $x \in \bigsqcup\{I^j \mid 1 \leq j \leq k\}$, and $r_{kk+1}|I^{k+1} = q_0^{k+1} : I^{k+1} \to I^0$. Put $Y = \lim \mathbf{Y}$.

We leave it to the reader to see that there is no ω-essential sequence in $\mathbf{Y}$ but that for each $n \in \mathbb{N}$, there exists an n-essential sequence in $\mathbf{Y}$. Hence Theorem 4.2 shows that Y is WID.

Example 3. For each $n \in \mathbb{N}$, there exists an inverse sequence $\mathbf{Y}^n = (Y_k, s_{kk+1}, (\mathbb{N}, \leq))$ such that with $Y^n = \lim \mathbf{Y}^n$, $\dim Y^n = n$.

We leave it to the reader to define the bonding maps s_{kk+1} by adjusting the maps r_{kk+1} in Example 2 so that Theorem 4.3 can be applied. In this example there is no upper bound on the dimension of the coordinate spaces Y_k.

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