RAD HRVATSKE AKADEMIJE ZNANOSTI I UMJETNOSTI
MATEMATIČKE ZNANOSTI

M. Jukić Bokun and I. Soldo

On triangular $D(-1)$-pairs with odd elements

Manuscript accepted for publication

ON TRIANGULAR $D(-1)$-PAIRS WITH ODD ELEMENTS

MIRELA JUKIĆ BOKUN AND IVAN SOLDO

ABSTRACT. In this paper, we study the extendibility of the triangular $D(-1)$-pair $\{1,7\}$, the smallest such pair consisting entirely of odd elements. We consider the third element c in the form $c = p^n q$, where n is a non-negative integer and p, q are distinct primes. In the case of $p = 2$ or $q = 2$ we show that the only admissible values are $c \in \{22, 106\}$, while in the case of odd integers $p^n, q \leq 10^{24000}$ the admissible values are $c \in \{5461, 26797, 1728749401\}$.

Moreover, we prove the non-extendibility of the triples corresponding to the values of $c \in \{22, 106, 5461\}$. These results support the conjecture that triangular $D(-1)$-quadruples are extremely rare.

1. INTRODUCTION

The study of sets of integers possessing special multiplicative-additive properties originates from Diophantus of Alexandria and remains an ongoing research topic in number theory. Let n be a nonzero integer and m a positive integer. A Diophantine m-tuple with the property $D(n)$ is defined as a set of m distinct non-zero integers $\{a_1, a_2, \ldots, a_m\}$ such that

$$a_i a_j + n$$

is a perfect square for all $1 \leq i < j \leq m$. Numerous results concerning such sets have been established for various values of n, as well as in different algebraic settings (see [6, 7] for a comprehensive survey).

A natural generalization of this classical concept involves replacing perfect squares with triangular numbers, i.e. numbers of the form

$$\Delta_r = \frac{r(r+1)}{2}, \quad r \in \mathbb{N},$$

2020 *Mathematics Subject Classification.* 11D09, 11D59, 11Y50, 11D61 .

Key words and phrases. Diophantine m-tuple, Pell-type equation, triangular number, generalized Ramanujan-Nagell equation, Thue equation .

which correspond to sequence A000217 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [17].

Definition 1.1. *Let n be an integer and m a positive integer. A set of m distinct positive integers $\{a_1, \dots, a_m\}$ satisfying that $a_i a_j + n$ is a triangular number for all $1 \leq i < j \leq m$ is called a triangular Diophantine m-tuple, or briefly triangular $D(n)$-m-tuple.*

This problem was first systematically studied by Filipin and Szalay [10], who introduced the notion of triangular Diophantine sets and examined the extendibility of certain triangular $D(1)$-pairs. They proved that the triangular $D(1)$-pair $\{1, 2\}$ cannot be extended to a quadruple, establishing one of the first non-existence results in this setting. Later, Filipin, Jurasić, and Szalay [8] investigated the analogous problem for triangular $D(3)$-pair $\{3, 6\}$ and obtained a similar result, i.e., there is no triangular $D(3)$-quadruple of the form $\{3, 6, c, d\}$. Recently, Filipin, Jurasić, and Togbe [9] studied the problem of extending the triangular $D(k)$-pair $\{k, 2k\}$ for $k = 10$ and $k = 28$. They showed that, in each of these cases, the pair cannot be extended to a triangular $D(k)$-quadruple. In a related recent work, Jukić Bokun and Soldo [11] considered the case $n = -1$. They completely characterized the triangular $D(-1)$-triples of the form $\{1, 2, c\}$, where $c = 2^l p$, l is a non-negative integer, and p is a prime. They also showed that there are only finitely many such triples and that none extends to a quadruple. These studies rely on the transformation of the defining conditions into systems of Pell-type equations. Additionally, they involve the analysis of integer points on certain elliptic curves associated with the problem.

The investigation of triangular Diophantine sets reveals a rich interplay between Diophantine equations, recurrence relations, and number-theoretic methods. Although there does not exist an infinite set of triangular $D(n)$-tuples for any integer n (see [11]), many open questions remain, particularly regarding the maximal possible size of such sets. Current results, together with numerical and theoretical considerations, suggest that the maximum possible size of such sets does not exceed 4. Ongoing research continues to explore these structures both theoretically and computationally, providing deeper insight into their arithmetic properties.

In this paper, we consider the extendibility of the triangular $D(-1)$-pair $\{1, 7\}$, which is the smallest such pair consisting entirely of odd elements. Motivated by previous results on the pair $\{1, 2\}$, where only finitely many triples and no quadruples exist [11], we investigate whether $\{1, 7\}$ can be extended to a triangular $D(-1)$-quadruple of the form $\{1, 7, c, d\}$. We focus on the case in which the third element c is of the form $c = p^n q$, where n is a non-negative integer and $p \neq q$ are primes satisfying one of the following conditions: either $p = 2$ or $q = 2$, or both p and q are odd and $p^n, q \leq 10^{24000}$. The multiplicative structure of c is particularly useful, as it allows the defining conditions either to be handled directly or to be reduced to generalized Ramanujan–Nagell or Pell-type equations. Among the resulting Pell-type equations, one is a classical Pell equation. Although this equation admits infinitely many integer solutions (see the end of Section 2.1), the relevant solutions are those for which one component is either a prime or a prime power. Identifying such solutions is a highly nontrivial problem. A well-known example illustrating the difficulty of such questions is the longstanding open problem of whether there exist infinitely many Fibonacci primes; see [5, Chapter 4.5]. We therefore impose the explicit bound $p^n, q \leq 10^{24000}$. The numerical value 10^{24000} was chosen to make an exhaustive computational search feasible while still covering a substantial search range. Within this range, we determine all admissible values of c.

Furthermore, the motivation for studying this pair also arises from the general expectation that triangular Diophantine quadruples are exceedingly rare. Numerical experiments provide strong evidence in support of this expectation. Nevertheless, triangular $D(1)$-quadruples do exist. For instance, $\{14, 31, 135, 3510\}$ is such a quadruple. By contrast, no example of a triangular $D(-1)$-quadruple is currently known. Thus, the study of the pair $\{1, 7\}$ provides further insight into the limitations of triangular $D(-1)$-sets and contributes to understanding whether any non-trivial triangular $D(-1)$-quadruples can exist.

The main results of this paper are presented below:

THEOREM 1.2. *Let n be a non-negative integer, and let $p \neq q$ be primes. Set $c = p^n q$ and assume that $\{1, 7, c\}$ is a triangular $D(-1)$-triple. Then the following holds:*

(i) If p and q are odd primes and $p^n, q \leq 10^{24000}$, then

$$c \in \{5\,461, 26\,797, 1\,728\,749\,401\}.$$

(ii) If $p = 2$ or $q = 2$, then $c \in \{22, 106\}$.

Since Theorem 1.2 remains valid for $n = 0$, it also applies when c is a prime. Consequently, we obtain the following corollary:

COROLLARY 1.3. *Triangular $D(-1)$-pair $\{1, 7\}$ cannot be extended to a triangular $D(-1)$-triple by adding a prime number.*

Furthermore, taking into account triangular $D(-1)$-quadruples of the form $\{1, 7, c, d\}$, where the element c is as in Theorem 1.2, we were able to prove the following result:

THEOREM 1.4. *If $c \in \{22, 106, 5\,461\}$, then there does not exist a triangular $D(-1)$-quadruple of the form $\{1, 7, c, d\}$.*

A remark concerning the computations in the remaining cases, namely $c = 26\,797$ and $c = 1\,728\,749\,401$, is given in Remark 3.2.

The paper is organized as follows. In Section 2, we consider extensions of the triangular $D(-1)$-pair $\{1,7\}$ to triples, while in Section 3, we study extensions to quadruples. Symbolic algebraic manipulations were carried out by using Wolfram Mathematica 14.3 [18], while all remaining computations were performed by using PARI/GP 2.15.4 [16].

2. Triangular $D(-1)$-triples $\{1,7,c\}$

Let $\{1,7,c\}$ be a triangular $D(-1)$-triple. Then there exist integers x_c, y_c such that

$$\begin{aligned} c-1 &= \frac{x_c(x_c+1)}{2}, \\ 7c-1 &= \frac{y_c(y_c+1)}{2}. \end{aligned} \tag{2.1}$$

Setting $x = 2x_c+1, y = 2y_c+1$, from (2.1) we get the following equivalent system of equations

$$\begin{aligned} 8c-7 &= x^2, \\ 56c-7 &= y^2, \end{aligned} \tag{2.2}$$

which yields the Pell-type equation $y^2 - 7x^2 = 42$ with infinitely many solutions in odd integers. Thus, there are infinitely many triangular $D(-1)$-triples of the form $\{1,7,c\}$. Note that (2.2) implies that c is not a multiple of 7. Further, from (2.2) we get the equation

$$y^2 - x^2 = 48c. \tag{2.3}$$

In what follows, we study the existence of such triangular $D(-1)$-triples for particular forms of c.

2.1. *Case 1: $c = p^n q$, where n is a non-negative integer and p and q are distinct primes, with p odd.* Let p and q be distinct primes, with p an odd prime, and let $c = p^n q$, where n is a non-negative integer. From (2.3) we get the equation

$$y^2 - x^2 = 2^4 3\, p^n q. \tag{2.4}$$

Noting that x, y are odd integers, we have to consider the following possible cases:

$$\begin{aligned} (y-x, y+x) \in \{&(2^i p^k, 2^{4-i} 3 p^{n-k} q), (2^i 3 p^k, 2^{4-i} p^{n-k} q), \\ &(2^i p^k q, 2^{4-i} 3 p^{n-k}), (2^i 3 p^k q, 2^{4-i} p^{n-k}) : \\ &k, i \in \mathbb{Z}, 1 \le i \le 3, 0 \le k \le n\}. \end{aligned}$$

In each case, we get the equation of one of the following forms:

$$\begin{aligned} &a_1 p^{2n-2k} q^2 + a_2 p^n q + a_3 p^{2k} + 7 = 0, \\ &a_1 p^{2n-2k} + a_2 p^n q + a_3 p^{2k} q^2 + 7 = 0, \quad a_i \in \mathbb{Z}, 1 \le i \le 3, 0 \le k \le n. \end{aligned} \tag{2.5}$$

For $n = 0$, it is easily checked that the resulting quadratic equations in q have no prime solutions. In what follows, let n be a positive integer. Since c is not divisible by 7, equations (2.5) imply that the cases $0 < k < n$ cannot occur.

Furthermore, for $q = 2$, solving the remaining equations for p^n shows that the only possibilities admitting integer solutions are $(y-x, y+x) \in \{(2p^n, 2^4 3), (2^4 3, 2p^n)\}$, which lead to the equation

$$p^{2n} - 64p^n + 583 = 0.$$

These cases yield $p^n \in \{11, 53\}$, corresponding to $c = 22$ and $c = 106$, respectively. This proves the case $q = 2$ of Theorem 1.2 (ii).

In what follows, we assume that q is an odd prime. It follows from parity considerations, or from solving the above equations for $p^n q$, that the following cases are impossible:

$$\begin{aligned}
(y-x, y+x) = &(2^2 q, 2^2 3p^n), (2^2 3q, 2^2 p^n), \\
&(2^i, 2^{4-i} 3p^n q), (2^i 3, 2^{4-i} p^n q), i \in \{1, 2, 3\}, \\
(y-x, y+x) = &(2^2 p^n, 2^2 3q), (2^2 3p^n, 2^2 q), \\
&(2^i p^n q, 2^{4-i} 3), (2^i 3p^n q, 2^{4-i}), i \in \{1, 2, 3\}.
\end{aligned}$$

The cases $(y-x, y+x) = (2p^n, 2^3 3q), (2^3 3q, 2p^n)$, both lead to the equation

$$144q^2 - 32p^n q + p^{2n} + 7 = 0,$$

and hence to

$$p^n = 16q \pm \sqrt{112q^2 - 7}. \tag{2.6}$$

Thus we conclude that $112q^2 - 7$ has to be a square and we obtain Pell-type equation $u^2 - 112q^2 = -7$. Using Nagell's theorem [15, Theorem 108a], all solutions in positive integers are given by

$$u_m + q_m\sqrt{112} = (21 + 2\sqrt{112})(127 + 12\sqrt{112})^m, \quad m \in \mathbb{N}_0.$$

In other words, the solutions $u_m + q_m\sqrt{112}$ can be written in the form of the following recurrence sequence:

$$\begin{aligned}
&u_0 = 21, q_0 = 2; \quad u_1 = 5355, q_1 = 506; \\
&u_{m+1} = 254u_m - u_{m-1},\ q_{m+1} = 254q_m - q_{m-1}, \quad m \geq 1.
\end{aligned}$$

Since q is an odd prime, whereas q_m is even for every $m \in \mathbb{N}_0$, we conclude that this case is impossible.

The cases $(y-x, y+x) = (2^3 3p^n, 2q), (2q, 2^3 3p^n)$ both lead to the equation

$$144p^{2n} - 32p^n q + q^2 + 7 = 0$$

and

$$q = 16p^n \pm \sqrt{112p^{2n} - 7}. \tag{2.7}$$

Thus we conclude that $112p^{2n}-7$ has to be a square and we obtain Pell-type equation $u^2-112Q^2=-7$, $Q=p^n$. This equation is the same as in the previous case, and therefore its second component Q is even. On the other hand, $Q=p^n$ is odd, since p is an odd prime. This contradiction shows that this case is impossible.

Suppose that $(y-x, y+x)=(2^3p^n, 2\cdot 3q), (2\cdot 3q, 2^3p^n)$. In both cases, we obtain

$$16p^{2n}-32p^nq+9q^2+7=0,$$

and hence

$$p^n=q\pm\frac{\sqrt{7q^2-7}}{4}.$$

This leads to the Pell equation $q^2-112r^2=1$. All positive integer solutions are given by $q_m+r_m\sqrt{112}=(127+12\sqrt{112})^m$, $m\in\mathbb{N}$, or by the recurrence sequence

$$\begin{aligned}&q_1=127, r_1=12;\quad q_2=32257, r_2=3048;\\&q_{m+2}=2q_1q_{m+1}-q_m,\ r_{m+2}=2q_1r_{m+1}-r_m,\quad m\in\mathbb{N}.\end{aligned}$$

Under the assumptions $q, p^n\leq 10^{24000}$, a computation in PARI/GP [16] shows that the only admissible pairs are $q=127, p^n\in\{43, 211\}$ and $q=32257, p^n=53593$. This gives $c\in\{5\,461, 26\,797, 1\,728\,749\,401\}$.

The remaining possible cases are $(y-x, y+x)=(2\cdot 3p^n, 2^3q), (2^3q, 2\cdot 3p^n)$. In both cases, we obtain the equation

$$9p^{2n}-32p^nq+16q^2+7=0,$$

which leads to

$$q=p^n\pm\frac{\sqrt{7p^{2n}-7}}{4}.$$

This yields the Pell equation $Q^2-112r^2=1$, with $Q=p^n$, which is the same as in the previous case. By assumption $q, p^n\leq 10^{24000}$, a computer search yields $p^n=127$, $q\in\{43, 211\}$ and $p^n=32257$, $q=53593$. Again, this gives $c\in\{5\,461, 26\,797, 1\,728\,749\,401\}$.

This completes the proof of Theorem 1.2 (i).

Remark 2.1. Considering the Pell equation $q^2-112r^2=1$, which occurs in the last two cases above, a computer search for solutions with prime $q\leq 10^{24000}$ include the first 9980 terms of the corresponding Pell sequence. Among the solutions obtained in this search, we found exactly one additional solution for which q is prime, namely the solution given by $q+r\sqrt{112}$, where

$$\begin{aligned}q&=150038171394905030432003281854339710977,\\r&=14177274595635847752648111737532309312.\end{aligned}$$

But in that case for the corresponding p^n we get

$$p^n \in \{249279093564355964700540064017065876161, \\ 507972492254540961634664996916135457 93\}$$

and they are composite numbers with more than two distinct prime factors.

As seen, we obtained only three solutions for which q is prime. We conjecture that no further prime values of q occur. As mentioned at the beginning of the paper, all solutions of this Pell equation can be described explicitly. However, determining which of them yield prime values of q is a substantially more difficult problem. At present, we are not aware of any available method for studying Pell equations and the associated recurrence sequences that would allow us to prove this conjecture.

Also, let us mention that if ChebyshevT$[2^n, x]$ is the 2^nth Chebyshev polynomial of the first kind evaluated at x, then $q = 32257 =$ ChebyshevT$[2^2, 8]$, while $q = 150038171394905030432003281854339710977 =$ ChebyshevT$[2^5, 8]$ is the smallest prime of the form ChebyshevT$[2^5, x]$ (*see* [17, *A*219280]).

2.2. *Case 2: $c = 2^n q$, where n is a non-negative integer and q is an odd prime.* We will use the results on the number of solutions of the generalized Ramanujan-Nagell equation. To facilitate the reader's understanding of the content presented in this section, we begin it by outlining the methodology employed in [4]. One can also see [2, 12].

Let D_1 and D_2 be coprime positive integers and let $k \geq 2$ be an integer coprime with D_1D_2. Let $\lambda \in \{1, 2^{1/2}, 2\}$ be such that $\lambda = 2$ if k is even and we consider the equation

$$D_1x^2 + D_2 = \lambda^2 k^n, \text{ in integers } x \geq 1, n \geq 1. \tag{2.8}$$

For $\lambda \in \{2^{1/2}, 2\}$ assume that D_1 and D_2 are odd positive integers. Let $N(\lambda, D_1, D_2, k)$ be the number of solutions (x, n) of equation (2.8). Denote by (F_k) the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and satisfying $F_k = F_{k-1} + F_{k-2}$, for all $k \geq 2$. Further, denote by (L_k) the Lucas sequence defined by $L_0 = 2$, $L_1 = 1$ and satisfying $L_k = L_{k-1} + L_{k-2}$, for all $k \geq 2$. Then set

$$\begin{aligned}
&\mathcal{F} = \{(F_{k-2e}, L_{k+e}, F_k) : k \geq 2, e \in \{-1, 1\}\},\\
&\mathcal{G} = \{(1, 4k^r - 1, k) : k \geq 2, r \geq 1\},\\
&\mathcal{H} = \{(D_1, D_2, k) : \text{there exist positive integers } r \text{ and } s \text{ such that}\\
&\qquad D_1s^2 + D_2 = \lambda^2 k^r \text{ and } 3D_1s^2 - D_2 = \pm\lambda^2\}.
\end{aligned}$$

The following is a result due to [4], with two exceptional cases subsequently resolved in [13] and [14]:

THEOREM 2.2 (see [4, Theorem 1], [13, Theorem 2.3] and [14]). *Let p be a prime number. Then, we have $N(\lambda, D_1, D_2, p) \leq 1$ except for*

$$\begin{aligned} N(2,13,3,2) = N(\sqrt{2},7,11,3) &= N(2,7,1,2) = N(\sqrt{2},1,1,5) \\ &= N(\sqrt{2},1,1,13) = N(2,1,3,7) = N(1,6,1,7) = 2, \\ N(1,2,1,3) = 3, \end{aligned}$$

and when (D_1, D_2, p) belongs to one of the infinite families $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$. In the latter case, we always have $N(\lambda, D_1, D_2, p) = 2$, except for

$$N(2,1,7,2) = 5 \text{ and } N(2,3,5,2) = N(2,1,11,3) = N(2,1,19,5) = 3.$$

Let $c = 2^n q$, where n is a non-negative integer and q is an odd prime. It follows from the discussion below equations (2.5) that the case $n = 0$ is impossible. Hence, throughout the remainder of the proof, we assume that n is a positive integer. We consider the equation

$$y^2 - x^2 = 3 \cdot 2^{n+4} q, \tag{2.9}$$

and get the following possible cases:

$$\begin{aligned} (y-x, y+x) \in \{&(2^{k+1}, 2^{n+3-k}3q), (2^{k+1}3, 2^{n+3-k}q), \\ &(2^{k+1}q, 2^{n+3-k}3), (2^{k+1}3q, 2^{n+3-k}) : \\ &k \in \mathbb{Z}, 0 \leq k \leq n+2\}. \end{aligned}$$

Combining this with (2.2) leads to the following possible solutions for q:

$$\begin{aligned} q_{1,2} =& \frac{1}{9}\left(2^{2k-n} \pm 2^{k-n-2}\sqrt{7(2^{2k}-9)}\right), \\ q_{3,4} =& 2^{2k-n} \pm 2^{k-n-2}\sqrt{7(2^{2k}-1)}, \\ q_{5,6} =& 2^{n-2k+4} \pm 2^{-k}\sqrt{7(2^{2n-2k+4}-1)}, \\ q_{7,8} =& \frac{1}{9}\left(2^{n-2k+4} \pm 2^{-k}\sqrt{7(2^{2n-2k+4}-9)}\right). \end{aligned}$$

From the assumptions that q is an odd prime and $0 \leq k \leq n+2$, it can be concluded that $k = n+2$ in the cases of $q_{1,2}$ and $q_{3,4}$, while $k = 0$ holds in the cases of $q_{5,6}$ and $q_{7,8}$. This implies

$$q = \frac{1}{9}\left(2^{n+4} \pm \sqrt{7(2^{2n+4}-9)}\right), \tag{2.10}$$

$$q = 2^{n+4} \pm \sqrt{7(2^{2n+4}-1)}. \tag{2.11}$$

In that way, we derive the following Diophantine equations

$$7m^2 + 9 = 2^{2n+4} = 4 \cdot 2^l, \tag{2.12}$$

$$7m^2 + 1 = 2^{2n+4} = 4 \cdot 2^l, \tag{2.13}$$

where $l = 2n + 2$, m is a positive integer, and n is a positive integer. Comparing (2.12) and (2.13) with (2.8) and applying Theorem 2.2, we have that $N(2, 7, 9, 2) \leq 1$ and $N(2, 7, 1, 2) = 2$.

It is easy to conclude that $(l, m) = (2, 1)$ solves equation (2.12), whereas $(l, m) = (1, 1), (4, 3)$ are solutions of equation (2.13). Since $l = 2n + 2$, the solution $(l, m) = (1, 1)$ does not correspond to an integer n. The solution $(l, m) = (2, 1)$ yields $n = 0$, which is not possible. Finally, the solution $(l, m) = (4, 3)$ yields $n = 1$. Substitution into (2.11) gives $q = 11$ or $q = 53$, corresponding to $c = 22$ and $c = 106$, respectively.

The preceding analysis completes the proof of Theorem 1.2 (ii).

3. Triangular $D(-1)$-quadruples $\{1, 7, c, d\}$

This section is devoted to studying whether some of the triangular $D(-1)$-triples presented in Section 2 can be extended to quadruples. The analysis presented here leads directly to the proof of Theorem 1.4.

Let $c \in \{22, 106, 5\,461\}$ and assume that $\{1, 7, c, d\}$ is a triangular $D(-1)$-quadruple with integer d. Then there exist integers x_d, y_d, and z_d satisfying the following conditions:

$$\begin{aligned} d - 1 &= \frac{x_d(x_d+1)}{2}, \\ 7d - 1 &= \frac{y_d(y_d+1)}{2}, \\ cd - 1 &= \frac{z_d(z_d+1)}{2}. \end{aligned}$$

Setting $X = 2x_d + 1, Y = 2y_d + 1, Z = 2z_d + 1$, we obtain

$$\begin{aligned} 8d - 7 &= X^2, \\ 56d - 7 &= Y^2, \\ 8cd - 7 &= Z^2. \end{aligned}$$

As a consequence of eliminating d, the problem reduces to a system of Pell-type equations

$$\begin{aligned} Y^2 - 7X^2 &= 42, \\ Z^2 - cX^2 &= 7\,(c-1)\,, \end{aligned} \tag{3.14}$$

equivalent to

$$Y^2 - 7X^2 = 42, \tag{3.15}$$

$$(c-7)X^2 - (c-1)Y^2 + 6Z^2 = 0. \tag{3.16}$$

To proceed, we analyze the system of Pell-type equations outlined above. We solve equation (3.16) in integers and combine the obtained solution with equation (3.15). For this purpose, we apply the following theorem due to Alekseyev [1].

THEOREM 3.1 (see [1, Theorem 5]). *Let A, B, C be non-zero integers and let (x_0, y_0, z_0) be a particular non-trivial integer solution of the Diophantine equation $Ax^2 + By^2 + Cz^2 = 0$ with $z_0 \neq 0$. The general integer solution to the above equation is given by*

$$(x, y, z) = \frac{p}{q}(P_x(m,n), P_y(m,n), P_z(m,n)),$$

where m, n as well as p, q are coprime integers with $q > 0$ dividing $2\operatorname{lcm}(A,B)Cz_0^2$, and

$$\begin{aligned}
P_x(m,n) &= x_0Am^2 + 2y_0Bmn - x_0Bn^2,\\
P_y(m,n) &= -y_0Am^2 + 2x_0Amn + y_0Bn^2,\\
P_z(m,n) &= z_0Am^2 + z_0Bn^2.
\end{aligned}$$

Equation (3.16) has a particular solution $(X_0, Y_0, Z_0) = (1, 1, 1)$. Using the result stated above, we conclude that its general integer solution is given by

$$(X, Y, Z) = \frac{p}{q}\left(P_X(m,n), P_Y(m,n), P_Z(m,n)\right),$$

where m, n as well as p, q are coprime integers with $q > 0$, $q \mid 12\operatorname{lcm}(c-7, c-1)$, and

$$\begin{aligned}
P_X(m,n) &= (c-7)m^2 - 2(c-1)mn + (c-1)n^2,\\
P_Y(m,n) &= -(c-7)m^2 + 2(c-7)mn - (c-1)n^2,\\
P_Z(m,n) &= (c-7)m^2 - (c-1)n^2.
\end{aligned}$$

Inserting these expressions into equation (3.15) yields:

$$\begin{aligned}
-\frac{6p^2}{q^2} \cdot ((c-7)^2m^4 - 4c(c-7)m^3n + (6c^2 - 16c - 14)m^2n^2&\\
- 4c(c-1)mn^3 + (c-1)^2n^4) = 42&,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&(c-7)^2m^4 - 4c(c-7)m^3n + (6c^2 - 16c - 14)m^2n^2\\
&- 4c(c-1)mn^3 + (c-1)^2n^4 = -7\frac{q^2}{p^2}.
\end{aligned} \tag{3.17}$$

Since p and q are coprime and $p^2 \mid 7q^2$, it follows that $p^2 \mid 7$, and hence $p = \pm 1$. Therefore, for each positive divisor q of $12 \operatorname{lcm}(c-7, c-1)$, equation (3.17) reduces to the following Thue equation:

$$\begin{aligned}
&(c-7)^2m^4 - 4c(c-7)m^3n + (6c^2 - 16c - 14)m^2n^2\\
&- 4c(c-1)mn^3 + (c-1)^2n^4 = -7q^2.
\end{aligned} \tag{3.18}$$

Using PARI/GP [16], we determined all coprime integer pairs (m, n) and the corresponding values of q, X, Y, Z, and d. The results are summarized in Table 1. In the column headed $(\pm m, \pm n)$, the signs are taken simultaneously.

c	q	$(\pm m, \pm n)$	X	Y	Z	d
22	3	$(2,1)$	-1	-7	13	1
	30	$(3,5)$	1	-7	-13	
	42	$(7,3)$	1	-7	13	
	105	$(7,10)$	-1	-7	-13	
106	45	$(5,6)$	-1	-7	-29	1
	126	$(7,9)$	1	-7	-29	
	231	$(14,11)$	-1	-7	29	
	330	$(15,11)$	1	-7	29	
	54	$(11,9)$	$-\frac{17}{3}$	$-\frac{49}{3}$	$\frac{193}{3}$	$\frac{44}{9}$
	99	$(8,11)$	$\frac{17}{3}$	$-\frac{49}{3}$	$-\frac{193}{3}$	
	945	$(35,24)$	$\frac{17}{3}$	$-\frac{49}{3}$	$\frac{193}{3}$	
	6930	$(105,121)$	$-\frac{17}{3}$	$-\frac{49}{3}$	$-\frac{193}{3}$	
5461	1404	$(26,27)$	1	-7	-209	1
	1890	$(35,36)$	-1	-7	-209	
	15756	$(104,101)$	-1	-7	209	
	21210	$(105,101)$	1	-7	209	
	1620	$(30,29)$	$\frac{1}{27}$	$-\frac{175}{27}$	$\frac{5279}{27}$	$\frac{638}{729}$
	14742	$(91,88)$	$-\frac{1}{27}$	$-\frac{175}{27}$	$\frac{5279}{27}$	

TABLE 1. Solutions of (3.18) for $c \in \{22, 106, 5461\}$, together with the corresponding candidate values of X, Y, Z, and d.

As can be seen from Table 1, none of the cases yields a new integer value of d. This completes the proof of Theorem 1.4.

REMARK 3.2. In the case of $c = 26\,797$, we have $q \mid 1\,435\,729\,680$. In this case, we obtain 960 values of q. We were able to perform calculations in PARI/GP for the smallest 537 of these (smallest 952 under the assumption of the Generalized Riemann Hypothesis (GRH)) and found: $(q, \pm m, \pm n) \in \{(6\,612, 58, 57), (8\,778, 77, 76), (81\,780, 232, 235), (108\,570, 231, 235)\}$ and $d = 1$, yielding a contradiction. The conjecture is that there does not exist a triangular $D(-1)$-quadruple of the form $\{1, 7, 26\,797, d\}$.

In the case of $c = 1\,728\,749\,401$, we have $q \mid 5\,977\,148\,955\,255\,727\,200$. In this case, we were unable to perform calculations in PARI/GP, with or without the assumption of the GRH. Moreover, in both cases, the calculations in Magma 2.29-4 [3] were also not successful, even in the simpler cases.

Furthermore, substituting the above values of c into (3.14), we obtain an explicit system of Pell-type equations. At present, we are unable to solve this system by means of the hypergeometric method and the reduction method, which are standard tools in the study of such systems. The main problem is connected with the fundamental solutions of the resulting system of Pell-type equations, primarily because it is difficult to obtain sufficiently sharp estimates for these fundamental solutions which should be used later on. Consequently, solving systems of this type arising from triangular Diophantine m-tuples appears to be a challenging problem with the currently available methods.

Acknowledgements.

The authors were supported by the Croatian-Austrian scientific and technological cooperation project *Diophantine m-tuples and related topics.* The authors are grateful to their colleagues at Paris Lodron University of Salzburg for several valuable discussions related to the topic of this paper during the authors' visit to the university. The authors were supported by the Croatian Science Foundation under the project No. IP-2022-10-5008.

The authors would like to express their gratitude to the referee for the careful reading of the manuscript and for the valuable and detailed suggestions that significantly improved presentation of this paper.

References

[1] M. A. Alekseyev, *On the Intersections of Fibonacci, Pell, and Lucas Numbers*, Integers **11** (2011), 239–259.

[2] A. Bérczes, I. Pink, *On generalized Lebesgue-Ramanujan-Nagell equations*, An. St. Univ. Ovidius Constanta, Ser. Mat. **22** (2014), 51–71.

[3] W. Bosma, J. Cannon, C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235–265.

[4] Y. Bugeaud, T. N. Shorey, *On the number of solutions of the generalized Ramanujan-Nagell equation*, J. Reine Angew. Math. **539** (2001), 55–74.

[5] A. Dujella, *Number Theory*, Školska knjiga, Zagreb, 2021.

[6] A. Dujella, *Diophantine m-tuples and elliptic curves*, Springer Cham, Cham, 2024.

[7] A. Dujella, *Diophantine m-tuples page*, `http://web.math.pmf.unizg.hr/~duje/dtuples.html`.

[8] A. Filipin, A. Jurasić, L. Szalay, *Extensions of the Triangular $D(3)$-pair $\{3, 6\}$*, Math. Slovaca **75** (2025), 775–790.

[9] A. Filipin, A. Jurasić, A. Togbé, *A note on the extension of some triangular Diophantine sets*, An. St. Univ. Ovidius Constanta, Ser. Mat. **34** (2026), 29–43.

[10] A. Filipin, L. Szalay, *Triangular Diophantine tuples from $\{1, 2\}$*, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan. **27** (2023), 55–70.

[11] M. Jukić Bokun, I. Soldo, *Triangular $D(-1)$-tuples*, Bull. Math. Soc. Sci. Math. Roum. **69** (2026), 233–239.

[12] M. Le, G. Soydan, *A brief survey on the generalized Lebesgue-Ramanujan-Nagell equation*, Surv. Math. Appl. **15** (2020), 473–523.
[13] M.-G. Leu, G.-W. Li, *The diophantine equation* $2x^2+1=3^n$, Proc. Amer. Math. Soc. **131** (2003), 3643–3645.
[14] R. A. Mollin, *A Note on the Diophantine Equation* $D_1x^2+D_2=ak^n$, Acta Math. Acad. Paedagog. Nyíregyházi. **21** (2005), 21–24.
[15] T. Nagell, *Introduction to Number Theory*, Wiley, New York, 1951.
[16] The PARI Group, *PARI/GP, version 2.15.4*, Univ. Bordeaux, 2023, `http://pari.math.u-bordeaux.fr`.
[17] N. J. A. Sloane, *The on-line encyclopedia of integer sequences*, available at `https://oeis.org/`.
[18] Wolfram Research, Inc., *Mathematica, version 14.3*, Champaign, IL, 2025, `https://www.wolfram.com/mathematica`.

O trokutnim $D(-1)$-parovima s neparnim elementima

Mirela Jukić Bokun i Ivan Soldo

Sažetak. U ovom radu proučavamo proširivost trokutnog $D(-1)$-para $\{1,7\}$, najmanjeg takvog para koji se u cijelosti sastoji od neparnih elemenata.
Treći element c promatramo u obliku $c=p^nq$, gdje je n nenegativan cijeli broj, a p i q različiti prosti brojevi. U slučaju $p=2$ ili $q=2$ pokazujemo da su jedine moguće vrijednosti $c\in\{22,106\}$, dok su u slučaju kada su p^n i q neparni te $p^n,q\le 10^{24000}$ jedine moguće vrijednosti

$$c\in\{5\,461, 26\,797, 1\,728\,749\,401\}.$$

Osim toga, dokazujemo da se trojke koje odgovaraju vrijednostima $c\in\{22,106,5\,461\}$ ne mogu proširiti na četvorke. Dobiveni rezultati podupiru slutnju da su trokutne $D(-1)$-četvorke iznimno rijetke.

Mirela Jukić Bokun
School of Applied Mathematics and Informatics
University of Osijek
31 000 Osijek, Croatia
E-mail: `mirela@mathos.hr`

Ivan Soldo
School of Applied Mathematics and Informatics
University of Osijek
31 000 Osijek, Croatia
E-mail: `isoldo@mathos.hr`