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Character recurrences and level one formulas for $B_2^{(1)}$

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CHARACTER RECURRENCES AND LEVEL ONE FORMULAS FOR $B_2^{(1)}$

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Abstract. We obtain recurrence relations for formal characters of Feigin–Stoyanovsky type subspaces of standard $B_2^{(1)}$-modules at fixed level. The recurrences are derived directly from the combinatorial bases described by difference and initial conditions. Unlike in type $A_\ell^{(1)}$, the right side in general involves several shifted characters. We give explicit polynomial coefficients and prove that the resulting system, together with the constant term normalization, determines all formal characters at fixed level uniquely. At level 1 we solve the recurrence system explicitly and obtain fermionic formulas for all three characters. We also relate these characters to the multigraded generating functions for exponent triples parametrizing the Meurman–Primc monomial basis of standard $A_1^{(1)}$-modules.

1. Introduction

The study of combinatorial bases of standard modules for affine Lie algebras originates in the work of Lepowsky and Wilson on Rogers–Ramanujan type identities (cf. [5, 6]). By using Lepowsky–Wilson's approach with untwisted vertex operators, Lepowsky and Primc obtained in [4] a combinatorial basis of standard $A_1^{(1)}$-modules in the homogeneous picture.

Feigin and Stoyanovsky later introduced principal subspaces of standard modules as a tool for studying characters through functional and geometric models [2]. A similar class of subspaces, called Feigin–Stoyanovsky type subspaces, has been studied through bases parametrized by colored partitions satisfying difference and initial conditions. Such bases were obtained for affine Lie algebras of type $A_\ell^{(1)}$ in [10], for type $C_\ell^{(1)}$ in [1], and for type $B_2^{(1)}$ in [8].

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Formal characters of Feigin–Stoyanovsky type subspaces are generating functions of the colored partitions that parametrize the bases. Relations among such generating functions at fixed level give recurrence relations for the characters. For affine Lie algebras of type $A_\ell^{(1)}$, a system of such recurrences was obtained in [3] from exact sequences constructed by means of intertwining operators and a simple current operator.

In this paper we derive analogous recurrence relations at fixed level for the affine Lie algebra $B_2^{(1)}$ directly from the combinatorial basis of [8]. In contrast with the $A_\ell^{(1)}$ case, the right side of the recurrence in Theorem 4.1 in general involves several shifted characters. This reflects the fact that, after lowering the relevant initial conditions, several degree -1 triples have to be removed. Together with the relation for the case $k_0+k_2=0$ and the constant term normalization, these recurrences determine all formal characters at fixed level uniquely.

At level 1, the recurrence system can be solved explicitly. Theorem 5.1 gives fermionic formulas for the three characters, written in terms of a single basic series and its shifts. Finally, we compare the inequalities for the $B_2^{(1)}$ basis of [8] with those for the exponent triples parametrizing the Meurman–Primc monomial basis of standard $A_1^{(1)}$-modules [7]. This gives a decomposition of the multigraded generating function of these exponent triples into formal characters.

The paper is organized as follows. Section 2 fixes the notation, recalls the basis theorem of [8], and introduces the formal characters. Section 3 describes the difference $\Pi_K \setminus \Pi_{K^-}$ and establishes the deletion bijection. Section 4 derives the recurrence relations and proves uniqueness of the resulting system. Section 5 solves the level 1 system explicitly. Section 6 relates the exponent triples parametrizing the Meurman–Primc monomial basis of standard $A_1^{(1)}$-modules [7] to the formal characters introduced in Section 2. It then uses the level 1 formulas of Section 5 to write the level 1 Meurman–Primc generating functions explicitly.

2. Preliminaries

2.1. *Affine Lie algebra and standard modules.* Let $\mathfrak{g}$ be a simple Lie algebra of type B_2 and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We realize the root system in $\mathbb{R}^2$, with canonical basis $\varepsilon_1, \varepsilon_2$, as $R = \{\pm(\varepsilon_1-\varepsilon_2), \pm(\varepsilon_1+\varepsilon_2)\} \cup \{\pm\varepsilon_1, \pm\varepsilon_2\}$. We fix simple roots $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2$. The maximal root is $\theta = \varepsilon_1 + \varepsilon_2$.

For the coweight $\omega = \omega_1 = \varepsilon_1$ we have $\Gamma = \{\alpha \in R \mid \alpha(\omega) = 1\} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1, \varepsilon_1 + \varepsilon_2\}$. This gives the $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where

$$\mathfrak{g}_{\pm 1} = \sum_{\alpha \in \pm\Gamma} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_0 = \mathfrak{h} + \sum_{\alpha(\omega)=0} \mathfrak{g}_\alpha.$$

We write $\underline{2} = \varepsilon_1 - \varepsilon_2$, $0 = \varepsilon_1$, and $2 = \varepsilon_1 + \varepsilon_2$, so that $\Gamma = \{\underline{2}, 0, 2\}$, and denote the corresponding root vectors by $x_{\underline{2}}, x_0, x_2$. The elements of Γ are the colors. We use the same symbols $\underline{2}, 0, 2$ for these colors.

Let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the affine Lie algebra of type $B_2^{(1)}$, and write $x(n) = x \otimes t^n$ for $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$. The grading on $\mathfrak{g}$ induces $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$, where $\tilde{\mathfrak{g}}_1 = \mathfrak{g}_1 \otimes \mathbb{C}[t, t^{-1}]$ is a commutative Lie subalgebra spanned by $x_{\underline{2}}(n), x_0(n), x_2(n)$, $n \in \mathbb{Z}$.

Let $L(\Lambda)$ be a standard $\tilde{\mathfrak{g}}$-module with dominant integral highest weight $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$ and level $k = \Lambda(c) = k_0 + k_1 + k_2$, where $k_0, k_1, k_2 \in \mathbb{Z}_{\geq 0}$. We fix a highest weight vector $v_\Lambda \in L(\Lambda)$. We use $K = (k_0, k_1, k_2)$ to record the coefficients of Λ.

2.2. *Feigin–Stoyanovsky type subspaces and combinatorial bases.* For an integral dominant weight Λ, define the Feigin–Stoyanovsky type subspace of $L(\Lambda)$ by $W(\Lambda) = U(\tilde{\mathfrak{g}}_1)v_\Lambda \subset L(\Lambda)$.

Let $\pi\colon \{x_\gamma(-j) \mid \gamma \in \Gamma,\ j \geq 1\} \to \mathbb{Z}_{\geq 0}$ be a colored partition in these colors with finite support, and let $x(\pi) = \prod x_\gamma(-j)^{\pi(x_\gamma(-j))}$ be the corresponding monomial in $U(\tilde{\mathfrak{g}}_1) = S(\tilde{\mathfrak{g}}_1)$. We identify π with the sequence $a_1, b_1, c_1, a_2, b_2, c_2, \ldots$, where $a_j = \pi(x_2(-j))$, $b_j = \pi(x_0(-j))$, and $c_j = \pi(x_{\underline{2}}(-j))$. Then

$$x(\pi) = \cdots x_{\underline{2}}(-2)^{c_2} x_0(-2)^{b_2} x_2(-2)^{a_2} x_{\underline{2}}(-1)^{c_1} x_0(-1)^{b_1} x_2(-1)^{a_1}.$$

For $K = (k_0, k_1, k_2)$, let Π_K be the set of colored partitions satisfying the difference conditions

$$\text{(DC)}\qquad \begin{aligned} c_{j+1} + b_{j+1} + c_j &\leq k, \\ b_{j+1} + a_{j+1} + c_j &\leq k, \\ a_{j+1} + c_j + b_j &\leq k, \\ a_{j+1} + b_j + a_j &\leq k \end{aligned}$$

for all $j \geq 1$, and the initial conditions

$$\text{(IC1)}\qquad a_1 \leq k_0,$$

$$\text{(IC2)}\qquad b_1 + a_1 \leq k_0 + k_2,$$

$$\text{(IC3)}\qquad c_1 + b_1 \leq k_0 + k_2.$$

The labels DC, IC1, IC2, and IC3 denote the difference conditions and the three initial inequalities. The colored partitions in Π_K are the admissible partitions for K. By Theorem 3.1 of [8], the vectors

$$x(\pi)v_\Lambda, \qquad \pi \in \Pi_K,$$

form a basis of the level k Feigin–Stoyanovsky type subspace $W(\Lambda)$. We denote this basis by

$$\mathcal{B}_K = \{x(\pi)v_\Lambda \mid \pi \in \Pi_K\}.$$

When no confusion can arise, $\mathcal{B}_K$ is identified with the set Π_K of colored partitions parametrizing these basis vectors.

2.3. *Formal characters.* For $K = (k_0, k_1, k_2)$ and $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$, we define the formal character of $W(\Lambda)$ by

$$\chi_K(z_1, z_2, z_3, q) = \sum_{\pi\in\Pi_K} z_1^{|\pi|_2} z_2^{|\pi|_0} z_3^{|\pi|_{\underline{2}}} q^{\deg(\pi)},$$

where

$$|\pi|_\gamma = \sum_{j\geq 1} \pi(x_\gamma(-j)), \qquad \deg(\pi) = \sum_{\gamma\in\Gamma}\sum_{j\geq 1} j\,\pi(x_\gamma(-j)).$$

We write $\mathbf{z} = (z_1, z_2, z_3)$, so that z_1, z_2, z_3 track the colors 2, 0, $\underline{2}$, respectively. The character $\chi_K(\mathbf{z}, q)$ is the multigraded generating function of the colored partitions in Π_K.

3. The set difference $\Pi_K \setminus \Pi_{K^-}$

3.1. *Definition of K^-.*

Definition 3.1. *For $K = (k_0, k_1, k_2)$ with $k_0 + k_2 > 0$, define*

$$K^- = \begin{cases} (k_0,\ k_1 + 1,\ k_2 - 1) & \text{if } k_2 > 0,\\ (k_0 - 1,\ k_1 + 1,\ 0) & \text{if } k_2 = 0,\ k_0 > 0.\end{cases}$$

In both cases K^- is again at level k, and the initial conditions for K^- have the same shape as the initial conditions for K, with the right sides of IC2 and IC3 reduced from $k_0 + k_2$ to $k_0 + k_2 - 1$. When $k_2 = 0$, the right side of IC1 is also reduced from k_0 to $k_0 - 1$.

Since the difference conditions are the same for K and K^-, and the initial conditions for K^- are at least as restrictive as those for K, we have $\Pi_{K^-} \subseteq \Pi_K$. Hence the difference of the two formal characters is the generating function of $\Pi_K \setminus \Pi_{K^-}$:

$$\chi_K(\mathbf{z}, q) - \chi_{K^-}(\mathbf{z}, q) = \sum_{\pi\in\Pi_K\setminus\Pi_{K^-}} z_1^{|\pi|_2}\, z_2^{|\pi|_0}\, z_3^{|\pi|_{\underline{2}}}\, q^{\deg(\pi)}. \tag{3.1}$$

We partition $\Pi_K \setminus \Pi_{K^-}$ by the degree -1 triple (a_1, b_1, c_1) of π and analyze each part.

3.2. *The set $\mathcal{S}(K)$.*

Proposition 3.2. *Assume $k_0 + k_2 > 0$. The set of degree -1 triples (a_1, b_1, c_1) of partitions in $\Pi_K \setminus \Pi_{K^-}$ is*

$$\mathcal{S}(K) = \left\{ (a_1, b_1, c_1) \in \mathbb{Z}^3_{\geq 0} \;\middle|\; \begin{array}{c} a_1 \leq k_0,\ b_1 + a_1 \leq k_0 + k_2,\ c_1 + b_1 \leq k_0 + k_2, \\ b_1 + a_1 = k_0 + k_2 \text{ or } c_1 + b_1 = k_0 + k_2 \end{array} \right\}.$$

Proof. The difference conditions are the same for K and K^-. Hence a partition in Π_K fails to lie in Π_{K^-} precisely when it violates at least one initial condition for K^-. We analyze the two cases in the definition of K^- separately.

Case $k_2 > 0$: The initial conditions for $K^- = (k_0, k_1 + 1, k_2 - 1)$ are

$$\begin{aligned} a_1 &\le k_0, \\ b_1 + a_1 &\le k_0 + k_2 - 1, \\ c_1 + b_1 &\le k_0 + k_2 - 1. \end{aligned}$$

Since IC1 is unchanged, violation requires

$$b_1 + a_1 \ge k_0 + k_2 \quad \text{or} \quad c_1 + b_1 \ge k_0 + k_2.$$

Together with the initial conditions for K, this implies $b_1 + a_1 = k_0 + k_2$ or $c_1 + b_1 = k_0 + k_2$.

Case $k_2 = 0$: The initial conditions for $K^- = (k_0 - 1, k_1 + 1, 0)$ are

$$\begin{aligned} a_1 &\le k_0 - 1, \\ b_1 + a_1 &\le k_0 - 1, \\ c_1 + b_1 &\le k_0 - 1. \end{aligned}$$

If $a_1 \ge k_0$, then $b_1 + a_1 \ge k_0$. Hence violation is equivalent to

$$b_1 + a_1 \ge k_0 \quad \text{or} \quad c_1 + b_1 \ge k_0.$$

Together with the initial conditions for K, this implies

$$b_1 + a_1 = k_0 \quad \text{or} \quad c_1 + b_1 = k_0.$$

Since $k_2 = 0$, this is equivalent to $b_1 + a_1 = k_0 + k_2$ or $c_1 + b_1 = k_0 + k_2$.

Conversely, every triple satisfying the defining inequalities of $\mathcal{S}(K)$ is the degree -1 triple of some partition in $\Pi_K \setminus \Pi_{K^-}$. Namely, set $a_j = b_j = c_j = 0$ for all $j \ge 2$. The resulting partition satisfies (DC) at every j, satisfies the initial conditions for K by assumption, and violates an initial condition for K^- by the equality $b_1 + a_1 = k_0 + k_2$ or $c_1 + b_1 = k_0 + k_2$. ∎

3.3. *Constraints at degree -2.*

Proposition 3.3. *Assume $k_0 + k_2 > 0$ and fix $(a_1, b_1, c_1) \in \mathcal{S}(K)$. For $\pi \in \Pi_K \setminus \Pi_{K^-}$ with degree -1 triple (a_1, b_1, c_1), the conditions* (DC) *with $j = 1$ are equivalent to*

$$\begin{aligned} a_2 &\le k_1, \\ b_2 + a_2 &\le k - c_1, \\ c_2 + b_2 &\le k - c_1. \end{aligned}$$

These are exactly the initial conditions for $\Pi_{(k_1, c_1, k_0 + k_2 - c_1)}$.

PROOF. Evaluating the four inequalities in (DC) at $j = 1$ gives

$$\begin{aligned} a_2 &\le \min(k - c_1 - b_1,\ k - b_1 - a_1), \\ b_2 + a_2 &\le k - c_1, \\ c_2 + b_2 &\le k - c_1. \end{aligned} \tag{3.2}$$

The initial conditions for $\Pi_{(k_1,c_1,k_0+k_2-c_1)}$ are

$$\begin{aligned} a_2 &\le k_1, \\ b_2 + a_2 &\le k - c_1, \\ c_2 + b_2 &\le k - c_1. \end{aligned}$$

The last two inequalities already agree with (3.2). For the first, the defining conditions of $\mathcal{S}(K)$ give $b_1 + a_1 \le k_0 + k_2$ and $c_1 + b_1 \le k_0 + k_2$ with at least one equality, so $\max(b_1 + a_1,\ c_1 + b_1) = k_0 + k_2$. Hence

$$\begin{aligned} \min(k - c_1 - b_1,\ k - b_1 - a_1) &= k - \max(c_1 + b_1,\ b_1 + a_1) \\ &= k - (k_0 + k_2) = k_1. \end{aligned}$$

□

The values of c_1 on $\mathcal{S}(K)$ are exactly $0, \ldots, k_0+k_2$, since $(0, k_0+k_2-i, i) \in \mathcal{S}(K)$ for each such i. The weights $(k_1, c_1, k_0 + k_2 - c_1)$ in Proposition 3.3 therefore run over $(k_1, i, k_0 + k_2 - i)$, $i = 0, \ldots, k_0 + k_2$.

LEMMA 3.4. *Fix $(a_1, b_1, c_1) \in \mathcal{S}(K)$. For a partition $\pi = (a_j, b_j, c_j)_{j\ge1}$ with degree -1 triple (a_1, b_1, c_1), define $\pi' = (a'_j, b'_j, c'_j)_{j\ge1}$ by*

$$\begin{aligned} a'_j &= a_{j+1}, \\ b'_j &= b_{j+1}, \\ c'_j &= c_{j+1}. \end{aligned}$$

Then $\pi \mapsto \pi'$ is a bijection from the set of all $\pi \in \Pi_K \setminus \Pi_{K^-}$ with degree -1 triple (a_1, b_1, c_1) onto $\Pi_{(k_1,c_1,k_0+k_2-c_1)}$.

If

$$d = \deg(\pi'), \qquad n_1 = |\pi'|_2, \quad n_2 = |\pi'|_0, \quad n_3 = |\pi'|_{\underline{2}},$$

then

$$\deg(\pi) = (a_1 + b_1 + c_1) + d + n_1 + n_2 + n_3.$$

PROOF. We verify that the map $\pi \mapsto \pi'$ is well-defined, injective, and surjective, and then compute the degree.

Well-defined: By Proposition 3.2, the fixed triple (a_1, b_1, c_1) satisfies the initial conditions for K and fails the initial conditions for K^-. The initial conditions for $\Pi_{(k_1,c_1,k_0+k_2-c_1)}$ are

$$\begin{aligned} a_1' &\le k_1, \\ b_1' + a_1' &\le k - c_1, \\ c_1' + b_1' &\le k - c_1. \end{aligned}$$

Since $a_1' = a_2$, $b_1' = b_2$, and $c_1' = c_2$, these are exactly the conditions obtained in Proposition 3.3. For the conditions (DC) with $j \ge 2$, reindexing by $i = j-1$ gives the difference conditions for π' with $i \ge 1$.

Injective: The degree -1 triple (a_1, b_1, c_1) is fixed, so distinct partitions with degree -1 triple (a_1, b_1, c_1) have distinct images.

Surjective: Given $\pi' \in \Pi_{(k_1,c_1,k_0+k_2-c_1)}$, write $\pi' = (a_j', b_j', c_j')_{j\ge1}$. Let π be the partition whose degree -1 triple is (a_1, b_1, c_1) and whose remaining entries are

$$(a_j, b_j, c_j) = (a_{j-1}', b_{j-1}', c_{j-1}') \quad \text{for } j \ge 2.$$

The initial conditions for π' are exactly the inequalities obtained from (DC) at $j = 1$ by the calculation in Proposition 3.3. Therefore, the difference conditions for π hold at $j = 1$. For $j \ge 2$, the difference conditions for π follow from the difference conditions for π' by reindexing. Because $(a_1, b_1, c_1) \in \mathcal{S}(K)$, the initial conditions for K hold and the initial conditions for K^- fail. Hence the constructed partition lies in $\Pi_K \setminus \Pi_{K^-}$ and has image π'.

Degree statement: The partition π has degree

$$\begin{aligned} \sum_{j\ge1} j(a_j + b_j + c_j) &= (a_1 + b_1 + c_1) + \sum_{j\ge2} j(a_{j-1}' + b_{j-1}' + c_{j-1}') \\ &= (a_1 + b_1 + c_1) + \sum_{i\ge1} (i+1)(a_i' + b_i' + c_i') \\ &= (a_1 + b_1 + c_1) + \sum_{i\ge1} i(a_i' + b_i' + c_i') + \sum_{i\ge1} (a_i' + b_i' + c_i') \\ &= (a_1 + b_1 + c_1) + d + (n_1 + n_2 + n_3). \end{aligned}$$

□

4. The recurrence relations

THEOREM 4.1. *Let $K = (k_0, k_1, k_2)$ be a level k weight.*

If $k_0 + k_2 > 0$, let K^- and $\mathcal{S}(K)$ be as in Definition 3.1 and Proposition 3.2. Then

$$\chi_K(\mathbf{z}, q) - \chi_{K^-}(\mathbf{z}, q) = \sum_{i=0}^{k_0+k_2} \mathcal{P}_i(\mathbf{z}, q)\, \chi_{(k_1,i,k_0+k_2-i)}(\mathbf{z}q, q), \tag{4.1}$$

where $\mathbf{z}q = (z_1q, z_2q, z_3q)$ *and*

$$\mathcal{P}_i(\mathbf{z}, q) = \sum_{\substack{(a_1,b_1,c_1)\in\mathcal{S}(K)\\ c_1=i}} z_1^{a_1}\, z_2^{b_1}\, z_3^{c_1}\, q^{a_1+b_1+c_1}. \tag{4.2}$$

If $k_0 + k_2 = 0$*, then* $K = (0, k, 0)$ *and*

$$\chi_{(0,k,0)}(\mathbf{z}, q) = \chi_{(k,0,0)}(\mathbf{z}q, q). \tag{4.3}$$

Proof. Assume first that $k_0 + k_2 > 0$. By (3.1), the left side of (4.1) is the generating function of $\Pi_K \setminus \Pi_{K^-}$. By Lemma 3.4, the set $\Pi_K \setminus \Pi_{K^-}$ is partitioned by the degree -1 triple $(a_1, b_1, c_1) \in \mathcal{S}(K)$, and for each fixed triple the deletion bijection maps the subset with degree -1 triple (a_1, b_1, c_1) onto $\Pi_{(k_1,c_1,k_0+k_2-c_1)}$.

For a partition with degree -1 triple (a_1, b_1, c_1) whose image π' satisfies $|\pi'|_2 = n_1$, $|\pi'|_0 = n_2$, $|\pi'|_{\underline{2}} = n_3$, and $\deg(\pi') = d$, the contribution to the generating function is

$$z_1^{a_1+n_1}\, z_2^{b_1+n_2}\, z_3^{c_1+n_3}\, q^{(a_1+b_1+c_1)+(d+n_1+n_2+n_3)} = z_1^{a_1} z_2^{b_1} z_3^{c_1} q^{a_1+b_1+c_1} \cdot z_1^{n_1} z_2^{n_2} z_3^{n_3} q^{d+n_1+n_2+n_3}.$$

Summing over all images gives

$$z_1^{a_1} z_2^{b_1} z_3^{c_1} q^{a_1+b_1+c_1} \cdot \chi_{(k_1,c_1,k_0+k_2-c_1)}(\mathbf{z}q, q).$$

Summing over all $(a_1, b_1, c_1) \in \mathcal{S}(K)$ and grouping by $c_1 = i$ yields (4.1).

It remains to consider the case $k_0 + k_2 = 0$. Then $K = (0, k, 0)$, and the initial conditions

$$\begin{aligned} a_1 &\le 0,\\ b_1 + a_1 &\le 0,\\ c_1 + b_1 &\le 0 \end{aligned}$$

force $a_1 = b_1 = c_1 = 0$. Evaluating (DC) at $j = 1$ gives

$$\begin{aligned} a_2 &\le k,\\ b_2 + a_2 &\le k,\\ c_2 + b_2 &\le k, \end{aligned}$$

which are the initial conditions for $\Pi_{(k,0,0)}$. Thus deleting the zero degree -1 triple and reindexing gives a bijection from $\Pi_{(0,k,0)}$ to $\Pi_{(k,0,0)}$. After the zero degree -1 triple is deleted and the remaining entries are reindexed, each remaining factor contributes one extra power of q, and hence

$$\chi_{(0,k,0)}(\mathbf{z}, q) = \chi_{(k,0,0)}(\mathbf{z}q, q),$$

which is (4.3). ∎

4.1. *Explicit form of $\mathcal{P}_i$.*

COROLLARY 4.2. *For $i = 0, \ldots, k_0 + k_2$,*

(4.4)
$$\mathcal{P}_i(\mathbf{z}, q) = z_3^i \left[q^{k_0+k_2+i} \sum_{a_1=i}^{k_0} z_2^{k_0+k_2-a_1} z_1^{a_1} + z_2^{k_0+k_2-i} q^{k_0+k_2} \sum_{a_1=0}^{\min(i-1,k_0)} (z_1 q)^{a_1} \right],$$

with the convention that the first sum is empty when $i > k_0$ and the second is empty when $i = 0$.

PROOF. The triples in $\mathcal{S}(K)$ with $c_1 = i$ split into two disjoint cases. If $b_1 + a_1 = k_0 + k_2$, then $b_1 = k_0 + k_2 - a_1$. Since $c_1 + b_1 \le k_0 + k_2$ and $b_1 = k_0 + k_2 - a_1$, we have $c_1 \le a_1$, hence $i \le a_1 \le k_0$. The contribution from triples satisfying $b_1 + a_1 = k_0 + k_2$ to (4.2) is

$$z_3^i q^{k_0+k_2+i} \sum_{a_1=i}^{k_0} z_2^{k_0+k_2-a_1} z_1^{a_1}.$$

If $c_1 + b_1 = k_0 + k_2$ and $b_1 + a_1 < k_0 + k_2$, then $b_1 = k_0 + k_2 - i$ and $0 \le a_1 \le \min(i-1, k_0)$. The contribution from triples satisfying $c_1 + b_1 = k_0 + k_2$ and $b_1 + a_1 < k_0 + k_2$ is

$$z_3^i z_2^{k_0+k_2-i} q^{k_0+k_2} \sum_{a_1=0}^{\min(i-1,k_0)} (z_1 q)^{a_1}.$$

Summing the two contributions gives (4.4). □

4.2. *Uniqueness.* We regard the formal characters as elements of

$$\mathbb{Z}[z_1, z_2, z_3][[q]].$$

For each d there are only finitely many colored partitions of degree d, and therefore the coefficient of q^d in every character is a polynomial in z_1, z_2, z_3.

THEOREM 4.3. *The system* (4.1)–(4.3)*, together with the condition that the constant term of each series $\chi_K(\mathbf{z}, q)$ is equal to* 1*, uniquely determines $\chi_K(\mathbf{z}, q)$ for all K at level k.*

PROOF. Write
$$\chi_K(\mathbf{z}, q) = \sum_{d \ge 0} f_K^{(d)}(\mathbf{z}) q^d.$$

The condition on the constant terms gives $f_K^{(0)}(\mathbf{z}) = 1$ for all weights K at level k. We prove by induction on d that $f_K^{(d)}$ is uniquely determined for all such weights K.

The recurrence (4.1) expresses $\chi_K - \chi_{K^-}$ in terms of products $\mathcal{P}_i(\mathbf{z}, q) \cdot \chi_{K'}(\mathbf{z}q, q)$ for various K'. Since $k_0 + k_2 > 0$, every triple in $\mathcal{S}(K)$ has $a_1 + b_1 + c_1 \ge 1$. Hence, by (4.2), each $\mathcal{P}_i(\mathbf{z}, q)$ has minimum q-degree at least 1.

Under the substitution $\mathbf{z} \mapsto \mathbf{z}q$, a monomial $\mathbf{z}^{\mathbf{n}}q^{d'}$ in $\chi_{K'}$ becomes $\mathbf{z}^{\mathbf{n}}q^{d'+|\mathbf{n}|}$, where $|\mathbf{n}| = n_1 + n_2 + n_3 \geq 0$. Hence the coefficient of q^d in the product $\mathcal{P}_i \cdot \chi_{K'}(\mathbf{z}q, q)$ depends only on coefficients $f_{K'}^{(d')}$ with $d' < d$.

Fix $d \geq 1$ and assume that all coefficients of degree $< d$ are known. We first determine $f_{(0,k,0)}^{(d)}$. By (4.3),

$$\chi_{(0,k,0)}(\mathbf{z}, q) = \chi_{(k,0,0)}(\mathbf{z}q, q).$$

The coefficient of q^d on the right depends only on coefficients $f_{(k,0,0)}^{(d')}$ with $d' < d$: the empty monomial contributes only to degree 0, and every nonempty monomial gains a positive power of q under the substitution $\mathbf{z} \mapsto \mathbf{z}q$. Hence $f_{(0,k,0)}^{(d)}$ is determined.

Now let K be any level k weight with $k_0 + k_2 > 0$. Since each step $K \mapsto K^-$ decreases $k_0 + k_2$ by one, iterating $K \mapsto K^-$ reaches $(0, k, 0)$ after finitely many steps. Along this chain, (4.1) determines

$$f_K^{(d)} - f_{K^-}^{(d)}$$

from coefficients of degree $< d$, which are already known. Starting from the already determined coefficient for $(0, k, 0)$, we determine $f_K^{(d)}$ successively along the chain. Thus all coefficients $f_K^{(d)}$ are uniquely determined. □

5. Level one character formulas

Write

$$\begin{aligned} \chi_{i,j,l} &= \chi_{(i,j,l)}(\mathbf{z}, q), \\ \chi_{i,j,l}^{+} &= \chi_{(i,j,l)}(\mathbf{z}q, q). \end{aligned} \tag{5.1}$$

At level 1, Theorem 4.1 and Corollary 4.2 give

$$\chi_{1,0,0} - \chi_{0,1,0} = q(z_2 + z_1)\chi_{0,0,1}^{+} + qz_3(1 + z_1 q)\chi_{0,1,0}^{+}, \tag{5.2}$$

$$\chi_{0,0,1} - \chi_{0,1,0} = qz_2\chi_{0,0,1}^{+} + qz_3\chi_{0,1,0}^{+}, \tag{5.3}$$

$$\chi_{0,1,0} = \chi_{1,0,0}^{+}. \tag{5.4}$$

Although Theorem 4.3 determines the characters recursively at every level, the level 1 system can be solved explicitly. For nonnegative integers m, set

$$(q)_0 = 1, \qquad (q)_m = \prod_{j=1}^{m}(1 - q^j).$$

For integers r and s with $0 \leq s \leq r$, define the Gaussian binomial coefficient with base q^2 by

$$\binom{r}{s}_{q^2} = \prod_{j=1}^{s} \frac{1 - q^{2(r-s+j)}}{1 - q^{2j}}. \tag{5.5}$$

We set $\binom{r}{s}_{q^2} = 0$ if $s < 0$ or $s > r$.

THEOREM 5.1. *Define*

$$F(\mathbf{z}, q) = \sum_{n_1,n_2,n_3 \geq 0} \frac{q^{n_1^2+n_3^2+\binom{n_2+1}{2}+n_1n_2+n_2n_3}}{(q)_{n_2}(q)_{n_1+n_3}} \binom{n_1+n_3}{n_1}_{q^2} z_1^{n_1} z_2^{n_2} z_3^{n_3}. \tag{5.6}$$

Here n_1, n_2, n_3 record the multiplicities of the colors $2, 0, \underline{2}$, corresponding to the variables z_1, z_2, z_3, respectively. Then the level 1 characters are given by

$$\begin{aligned} \chi_{1,0,0} &= F(\mathbf{z}, q), \\ \chi_{0,1,0} &= F(\mathbf{z}q, q), \\ \chi_{0,0,1} &= F\big((z_1 q,\, z_2,\, z_3 q),\, q\big) + q z_3\, F\big((z_1 q^2,\, z_2 q,\, z_3 q^2),\, q\big). \end{aligned} \tag{5.7}$$

PROOF. By Theorem 4.3, it is enough to show that the three series in (5.7) satisfy (5.2)–(5.4) and have constant term 1. The first two series have constant term 1. For the third series, the first summand has constant term 1, while the second summand is divisible by q. Hence the third series also has constant term 1.

Let f_{n_1,n_2,n_3} be the coefficient of $z_1^{n_1} z_2^{n_2} z_3^{n_3}$ in $F(\mathbf{z}, q)$, and set $f_{n_1,n_2,n_3} = 0$ if one of the indices is negative. By (5.6),

$$f_{n_1,n_2,n_3} = \frac{q^{n_1^2+n_3^2+\binom{n_2+1}{2}+n_1n_2+n_2n_3}}{(q)_{n_2}\,(q)_{n_1+n_3}} \binom{n_1+n_3}{n_1}_{q^2}. \tag{5.8}$$

In each of (5.2)–(5.4), the expression after the equality sign is a combination of F evaluated at shifted arguments. Hence extracting the coefficient of $z_1^{n_1} z_2^{n_2} z_3^{n_3}$ produces values of f with one or two indices lowered by 1. By using (5.8) and (5.5), the coefficients with lowered indices are related to f_{n_1,n_2,n_3} by the following ratios:

$$\begin{aligned} \frac{f_{n_1,n_2-1,n_3}}{f_{n_1,n_2,n_3}} &= q^{-n_1-n_2-n_3}(1-q^{n_2}), \\ \frac{f_{n_1-1,n_2,n_3}}{f_{n_1,n_2,n_3}} &= q^{1-2n_1-n_2}\, \frac{1-q^{2n_1}}{1+q^{n_1+n_3}}, \\ \frac{f_{n_1,n_2,n_3-1}}{f_{n_1,n_2,n_3}} &= q^{1-n_2-2n_3}\, \frac{1-q^{2n_3}}{1+q^{n_1+n_3}}, \\ \frac{f_{n_1,n_2-1,n_3-1}}{f_{n_1,n_2,n_3}} &= q^{2-n_1-2n_2-3n_3}(1-q^{n_2})\, \frac{1-q^{2n_3}}{1+q^{n_1+n_3}}, \\ \frac{f_{n_1-1,n_2,n_3-1}}{f_{n_1,n_2,n_3}} &= q^{2-2n_1-2n_2-2n_3}\, \frac{(1-q^{2n_1})(1-q^{2n_3})}{(1+q^{n_1+n_3})(1+q^{n_1+n_3-1})}. \end{aligned} \tag{5.9}$$

With the convention $f_{a,b,c} = 0$ for negative indices, both sides of each identity in (5.9) vanish whenever a lowered index is negative, so these identities may be applied for all $n_1, n_2, n_3 \geq 0$.

Relation (5.2). We compare coefficients of $z_1^{n_1} z_2^{n_2} z_3^{n_3}$ in (5.2). From (5.7) and the definition of f_{n_1,n_2,n_3}, the coefficient of $z_1^{n_1} z_2^{n_2} z_3^{n_3}$ in $\chi_{1,0,0} - \chi_{0,1,0}$ is

$$(1 - q^{n_1+n_2+n_3})\, f_{n_1,n_2,n_3}. \tag{5.10}$$

It remains to compute the coefficient of $z_1^{n_1} z_2^{n_2} z_3^{n_3}$ in

$$q(z_2 + z_1)\chi^+_{0,0,1} + qz_3(1 + z_1 q)\chi^+_{0,1,0}.$$

By (5.1) and (5.7), this expression is

$$\begin{aligned} &q(z_2 + z_1)\Big(F\big((z_1 q^2,\, z_2 q,\, z_3 q^2),\, q\big) + q^2 z_3 F\big((z_1 q^3,\, z_2 q^2,\, z_3 q^3),\, q\big)\Big) \\ &\quad + qz_3(1 + z_1 q)F\big((z_1 q^2,\, z_2 q^2,\, z_3 q^2),\, q\big). \end{aligned} \tag{5.11}$$

To extract the coefficient of $z_1^{n_1} z_2^{n_2} z_3^{n_3}$ in (5.11), the monomial prefactor first determines which coefficient of F is needed. The powers of q are then read from the shifted arguments of F. For example, the term

$$qz_2 F\big((z_1 q^2,\, z_2 q,\, z_3 q^2),\, q\big)$$

uses the coefficient f_{n_1,n_2-1,n_3} of F, and contributes

$$q \cdot q^{2n_1+(n_2-1)+2n_3} f_{n_1,n_2-1,n_3} = q^{2n_1+n_2+2n_3} f_{n_1,n_2-1,n_3}.$$

Thus the coefficient of $z_1^{n_1} z_2^{n_2} z_3^{n_3}$ in (5.11) is

$$\begin{aligned} &q^{2n_1+n_2+2n_3}\, f_{n_1,n_2-1,n_3} + q^{3n_1+2n_2+3n_3-2}\, f_{n_1,n_2-1,n_3-1} \\ &+ q^{2n_1+n_2+2n_3-1}\, f_{n_1-1,n_2,n_3} + q^{2n_1+2n_2+2n_3-1}\, f_{n_1,n_2,n_3-1} \\ &+ q^{2n_1+2n_2+2n_3-2}(1 + q^{n_1+n_3-1})\, f_{n_1-1,n_2,n_3-1}. \end{aligned} \tag{5.12}$$

By (5.9), the coefficient in (5.12) is equal to

$$\begin{aligned} &q^{n_1+n_3}(1 - q^{n_2}) f_{n_1,n_2,n_3} \\ &\quad + q^{2n_1}(1 - q^{n_2})\frac{1 - q^{2n_3}}{1 + q^{n_1+n_3}} f_{n_1,n_2,n_3} \\ &\quad + q^{2n_3}\frac{1 - q^{2n_1}}{1 + q^{n_1+n_3}} f_{n_1,n_2,n_3} \\ &\quad + q^{2n_1+n_2}\frac{1 - q^{2n_3}}{1 + q^{n_1+n_3}} f_{n_1,n_2,n_3} \\ &\quad + \frac{(1 - q^{2n_1})(1 - q^{2n_3})}{1 + q^{n_1+n_3}} f_{n_1,n_2,n_3}. \end{aligned} \tag{5.13}$$

In (5.13), the second and fourth summands combine through $(1 - q^{n_2}) + q^{n_2} = 1$. Hence the coefficient in (5.11) is

$$f_{n_1,n_2,n_3}\Bigg(q^{n_1+n_3}(1 - q^{n_2}) + \frac{1}{1 + q^{n_1+n_3}}\Big(q^{2n_1}(1 - q^{2n_3})$$

$$+ q^{2n_3}(1-q^{2n_1}) + (1-q^{2n_1})(1-q^{2n_3})\Big)\Bigg)$$

$$= f_{n_1,n_2,n_3}\Big(q^{n_1+n_3}(1-q^{n_2}) + \frac{1-q^{2n_1+2n_3}}{1+q^{n_1+n_3}}\Big)$$

$$= (1-q^{n_1+n_2+n_3})f_{n_1,n_2,n_3}. \tag{5.14}$$

Together with (5.10), equation (5.14) proves (5.2).

Relation (5.3). We compare coefficients of $z_1^{n_1}z_2^{n_2}z_3^{n_3}$ in (5.3). From (5.7), the definition of f_{n_1,n_2,n_3}, and (5.9), the coefficient of $z_1^{n_1}z_2^{n_2}z_3^{n_3}$ in $\chi_{0,0,1} - \chi_{0,1,0}$ is

$$q^{n_1+n_3}f_{n_1,n_2,n_3} + q^{2n_1+n_2+2n_3-1}f_{n_1,n_2,n_3-1} - q^{n_1+n_2+n_3}f_{n_1,n_2,n_3}$$

$$= q^{n_1+n_3}(1-q^{n_2})f_{n_1,n_2,n_3} + q^{2n_1}\frac{1-q^{2n_3}}{1+q^{n_1+n_3}}f_{n_1,n_2,n_3}. \tag{5.15}$$

It remains to compute the coefficient of $z_1^{n_1}z_2^{n_2}z_3^{n_3}$ in

$$qz_2\chi^+_{0,0,1} + qz_3\chi^+_{0,1,0}.$$

By (5.1) and (5.7), this expression is

$$\begin{aligned} &qz_2\Big(F\big((z_1q^2,\, z_2q,\, z_3q^2),\, q\big) + q^2z_3F\big((z_1q^3,\, z_2q^2,\, z_3q^3),\, q\big)\Big)\\ &\quad + qz_3F\big((z_1q^2,\, z_2q^2,\, z_3q^2),\, q\big). \end{aligned} \tag{5.16}$$

The coefficient of $z_1^{n_1}z_2^{n_2}z_3^{n_3}$ in (5.16) is

$$\begin{aligned} &q^{2n_1+n_2+2n_3}\,f_{n_1,n_2-1,n_3} + q^{3n_1+2n_2+3n_3-2}\,f_{n_1,n_2-1,n_3-1}\\ &+ q^{2n_1+2n_2+2n_3-1}\,f_{n_1,n_2,n_3-1}. \end{aligned} \tag{5.17}$$

By (5.9), the coefficient in (5.17) is equal to

$$\begin{aligned} &q^{n_1+n_3}(1-q^{n_2})f_{n_1,n_2,n_3} + q^{2n_1}(1-q^{n_2})\frac{1-q^{2n_3}}{1+q^{n_1+n_3}}f_{n_1,n_2,n_3}\\ &+ q^{2n_1+n_2}\frac{1-q^{2n_3}}{1+q^{n_1+n_3}}f_{n_1,n_2,n_3}. \end{aligned} \tag{5.18}$$

The last two summands in (5.18) combine through $(1-q^{n_2})+q^{n_2}=1$, so the coefficient in (5.16) is

$$q^{n_1+n_3}(1-q^{n_2})f_{n_1,n_2,n_3} + q^{2n_1}\frac{1-q^{2n_3}}{1+q^{n_1+n_3}}f_{n_1,n_2,n_3}.$$

This is the same coefficient as in (5.15), and hence (5.3) follows.

Relation (5.4). By (5.1) and (5.7) we immediately have

$$\chi^+_{1,0,0} = \chi_{1,0,0}(\mathbf{z}q, q) = F(\mathbf{z}q, q) = \chi_{0,1,0}.$$

$\square$

6. Connection to the Meurman–Primc basis of standard $A_1^{(1)}$-modules

For $K = (k, 0, 0)$, it was observed in [8] that the admissible partitions for the $B_2^{(1)}$ basis of $W(k\Lambda_0)$ satisfy the same inequalities as the exponent triples for the Meurman–Primc monomial basis of the standard $A_1^{(1)}$-module $L(k\Lambda_0)$. The parametrization for all standard $A_1^{(1)}$-modules is recalled in [9]. The comparison decomposes according to the value of c_0. Combined with the recurrences of Section 4, this decomposition expresses the multi-graded Meurman–Primc character of every standard $A_1^{(1)}$-module in terms of the characters χ_K, which are determined by Theorems 4.1 and 4.3. In this sense the $B_2^{(1)}$ recurrences at level k also determine the corresponding $A_1^{(1)}$ characters.

We use the parametrization of the Meurman–Primc monomial basis recalled in [9]. Fix $\bar{k}_0, \bar{k}_1 \in \mathbb{Z}_{\geq 0}$ with $k = \bar{k}_0 + \bar{k}_1$, and let $\widetilde{\mathcal{B}}_{(\bar{k}_0, \bar{k}_1)}$ denote the set of exponent triples parametrizing this basis for the standard $A_1^{(1)}$-module of highest weight $\bar{k}_0\Lambda_0 + \bar{k}_1\Lambda_1$. It consists of triples $(a_j, b_j, c_j)_{j\geq 0}$ of nonnegative integers satisfying

$$a_0 = b_0 = 0, \qquad c_0 \leq \bar{k}_1, \qquad a_1 \leq \bar{k}_0,$$

together with the difference conditions

$$\begin{aligned} a_j + b_j + a_{j+1} &\leq k, \\ c_j + b_j + a_{j+1} &\leq k, \\ c_j + b_{j+1} + a_{j+1} &\leq k, \\ c_j + b_{j+1} + c_{j+1} &\leq k \end{aligned} \tag{6.1}$$

for all $j \geq 0$.

We identify the three exponent sequences with the colors 2, 0, $\underline{2}$ as follows:

$$a_j \leftrightarrow 2, \qquad b_j \leftrightarrow 0, \qquad c_j \leftrightarrow \underline{2}.$$

After reordering the summands in each line, the inequalities (6.1) at $j \geq 1$ are exactly (DC). For fixed $c_0 = m \in \{0, \ldots, \bar{k}_1\}$, the inequalities at $j = 0$ become the initial conditions for the $B_2^{(1)}$ weight $(\bar{k}_0, m, \bar{k}_1 - m)$.

Proposition 6.1. *For each $m \in \{0, \ldots, \bar{k}_1\}$, the map*

$$(a_j, b_j, c_j)_{j\geq 0} \longmapsto (a_j, b_j, c_j)_{j\geq 1}$$

gives a bijection from the subset of $\widetilde{\mathcal{B}}_{(\bar{k}_0, \bar{k}_1)}$ consisting of exponent triples with $c_0 = m$ onto $\Pi_{(\bar{k}_0,\, m,\, \bar{k}_1 - m)}$.

Proof. Fix $c_0 = m$. The inequalities (6.1) at $j = 0$ with $a_0 = b_0 = 0$ give

$$a_1 \leq k,$$

$$\begin{aligned} a_1 &\le k-m,\\ b_1+a_1 &\le k-m,\\ c_1+b_1 &\le k-m. \end{aligned}$$

Since $m \le \bar{k}_1$ implies $\bar{k}_0 \le k-m$, both $a_1 \le k$ and $a_1 \le k-m$ follow from $a_1 \le \bar{k}_0$. The inequality $a_1 \le \bar{k}_0$, together with

$$\begin{aligned} b_1+a_1 &\le k-m,\\ c_1+b_1 &\le k-m \end{aligned}$$

is exactly the set of initial conditions for the $B_2^{(1)}$ weight $(\bar{k}_0, m, \bar{k}_1 - m)$, because

$$k-m = \bar{k}_0 + \bar{k}_1 - m = \bar{k}_0 + (\bar{k}_1 - m).$$

The difference conditions (DC) at $j \ge 1$ coincide with (6.1) at $j \ge 1$ after reordering the summands in each line. Hence the map sends the subset with $c_0 = m$ into $\Pi_{(\bar{k}_0, m, \bar{k}_1 - m)}$.

Conversely, given $(a'_j, b'_j, c'_j)_{j\ge1} \in \Pi_{(\bar{k}_0, m, \bar{k}_1 - m)}$, define $(a_j, b_j, c_j)_{j\ge0}$ by

$$a_0 = b_0 = 0, \qquad c_0 = m, \qquad (a_j, b_j, c_j) = (a'_j, b'_j, c'_j) \quad (j \ge 1).$$

Then the resulting triple satisfies (6.1) at every $j \ge 0$ together with the Meurman–Primc initial conditions, so it lies in $\widetilde{\mathcal{B}}_{(\bar{k}_0,\bar{k}_1)}$ with $c_0 = m$. □

For $\pi = (a_j, b_j, c_j)_{j\ge0} \in \widetilde{\mathcal{B}}_{(\bar{k}_0,\bar{k}_1)}$, define

$$|\pi|_2 = \sum_{j\ge0} a_j, \qquad |\pi|_0 = \sum_{j\ge0} b_j, \qquad |\pi|_{\underline{2}} = \sum_{j\ge0} c_j$$

and

$$\deg(\pi) = \sum_{j\ge0} j(a_j + b_j + c_j).$$

Define now the multigraded formal character of the standard $A_1^{(1)}$-module $L(\bar{k}_0\Lambda_0 + \bar{k}_1\Lambda_1)$, with respect to the Meurman–Primc monomial basis, by

$$\widetilde{\chi}_{(\bar{k}_0,\bar{k}_1)}(\mathbf{z}, q) = \sum_{\pi \in \widetilde{\mathcal{B}}_{(\bar{k}_0,\bar{k}_1)}} z_1^{|\pi|_2}\, z_2^{|\pi|_0}\, z_3^{|\pi|_{\underline{2}}}\, q^{\deg(\pi)}.$$

Corollary 6.2. *The character $\widetilde{\chi}_{(\bar{k}_0,\bar{k}_1)}$ satisfies*

$$\widetilde{\chi}_{(\bar{k}_0,\bar{k}_1)}(\mathbf{z}, q) = \sum_{m=0}^{\bar{k}_1} z_3^m\, \chi_{(\bar{k}_0,\, m,\, \bar{k}_1 - m)}(\mathbf{z}, q). \tag{6.2}$$

Proof. By Proposition 6.1, the map sending $\pi = (a_j, b_j, c_j)_{j\ge0}$ to $\pi' = (a_j, b_j, c_j)_{j\ge1}$ is a bijection from the subset of $\widetilde{\mathcal{B}}_{(\bar{k}_0,\bar{k}_1)}$ with $c_0 = m$ onto $\Pi_{(\bar{k}_0,m,\bar{k}_1-m)}$. Since $a_0 = b_0 = 0$ and the terms with $j = 0$ do not contribute to $\deg(\pi)$, deleting the $j = 0$ entries preserves $|\pi|_2$, $|\pi|_0$, and $\deg(\pi)$. It changes only the $\underline{2}$-count:

$$|\pi|_{\underline{2}} = c_0 + |\pi'|_{\underline{2}} = m + |\pi'|_{\underline{2}}.$$

Therefore the contribution of the triples with $c_0 = m$ to $\widetilde{\chi}_{(\bar{k}_0,\bar{k}_1)}$ is

$$z_3^m \, \chi_{(\bar{k}_0,m,\bar{k}_1-m)}(\mathbf{z}, q).$$

Summing over $m = 0, \dots, \bar{k}_1$ gives (6.2). ∎

Corollary 6.3. *The multigraded Meurman–Primc characters of the two level* 1 *standard* $A_1^{(1)}$*-modules are*

$$\widetilde{\chi}_{(1,0)}(\mathbf{z}, q) = F(\mathbf{z}, q),$$

$$\widetilde{\chi}_{(0,1)}(\mathbf{z}, q) = F\big((z_1q,\, z_2,\, z_3q),\, q\big) + qz_3F\big((z_1q^2,\, z_2q,\, z_3q^2),\, q\big) + z_3F(\mathbf{z}q, q),$$

where F *is the series of Theorem 5.1.*

Proof. Specialize Corollary 6.2 to $k = 1$: $\widetilde{\chi}_{(1,0)} = \chi_{(1,0,0)}$ and $\widetilde{\chi}_{(0,1)} = \chi_{(0,0,1)} + z_3\chi_{(0,1,0)}$. Substituting the formulas of Theorem 5.1 gives the result. ∎

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Rekurzivne relacije za karaktere i formule nivoa jedan za $B_2^{(1)}$

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SAŽETAK. U radu se izvode rekurzivne relacije za formalne karaktere potprostora tipa Feigin–Stoyanovskoga za standardne $B_2^{(1)}$-module općeg nivoa. Rekurzivne relacije izvode se izravno iz kombinatornih baza opisanih uvjetima razlike i početnim uvjetima. Za razliku od tipa $A_\ell^{(1)}$, desna strana relacija općenito uključuje nekoliko pomaknutih karaktera, za koje su eksplicitno određeni polinomni koeficijenti. Potom se dokazuje da dobiveni sustav, zajedno s normalizacijom konstantnog člana, jednoznačno određuje sve formalne karaktere na zadanom nivou. Za nivo 1 eksplicitno se rješava sustav rekurzija, čime se dolazi do fermionskih formula za sva tri karaktera. Također, pokazuje se veza između karaktera za $B_2^{(1)}$ i funkcija izvodnica za trojke eksponenata koje parametriziraju Meurman–Primčevu monomijalnu bazu standardnih $A_1^{(1)}$-modula.

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