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Manuscript accepted for publication

This is a preliminary PDF of the author-produced manuscript that has been peer-reviewed and accepted for publication. It has not been copy-edited, proofread, or finalized by Rad HAZU Production staff.

COMPUTABILITY OF CHAINABLE GRAPHS

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ABSTRACT. If every semicomputable set, in arbitrary computable topological space, which is homeomorphic to a topological space A is computable, then we say that A has computable type. A topological pair (A, B), $B \subseteq A$, has computable type if for all semicomputable sets S and T, in arbitrary computable topological space, such that S is homeomorphic to A by a homeomorphism which maps T to B, it holds that S is computable. It is known that (G, E) has computable type, where G is a certain kind of topological graph and E is the set of its endpoints. In this paper, we consider more general graphs $\widetilde{G}$ obtained by taking edges of $\widetilde{G}$ to be chainable continua (instead of arcs) and we prove that $(\widetilde{G}, E)$ has computable type, where E is the set of all endpoints of $\widetilde{G}$.

1. INTRODUCTION

A compact set $S \subseteq \mathbb{R}$ is semicomputable if its complement $\mathbb{R} \setminus S$ can be effectively exhausted by rational open intervals. A compact set $S \subseteq \mathbb{R}$ is computable if it is semicomputable and we can effectively enumerate all rational open intervals which intersect S. Semicomputable and computable sets can be defined in more general spaces - computable metric and topological spaces. Regardless of the ambient space, it is clear that each computable set is also semicomputable, but the converse does not hold. Namely, there exists $\gamma > 0$ such that $[0, \gamma]$ is a semicomputable set which is not computable [20]. Moreover, computable numbers are dense in every nonempty computable set in $\mathbb{R}$, while there exists a nonempty semicomputable set $S \subseteq \mathbb{R}$ which does not contain any computable number [22].

2020 *Mathematics Subject Classification.* 03D78, 03F60.

Key words and phrases. computable topological space, semicomputable set, computable set, computable type, chainable continuum, chainable graph.

Although the implication

$$S \text{ semicomputable} \Rightarrow S \text{ computable} \tag{1.1}$$

is not true in general, there are certain additional assumptions on S under which (1.1) holds. For instance, we know this: if S is a semicomputable circularly chainable continuum which is not chainable then S is computable. Also, the same is true if S is a continuum chainable from a to b, where a and b are computable points (i.e. $\{a, b\}$ is a semicomputable set). So, our main goal is to give an answer as good as possible to the following question: what topological conditions force a semicomputable set in a computable topological space to be a computable one? All the results will be stated in the form of *computable type*. We say that A has computable type if (1.1) holds in any computable topological space X whenever $S \subseteq X$ is homeomorphic to A. It is known that each sphere in Euclidean space has computable type, moreover each compact manifold has computable type [20, 13, 14, 18]. However, not only manifolds have computable type. For example, the Warsaw circle has computable type and it is not a manifold. As mentioned before, any circularly chainable continuum which is not chainable has computable type [12, 10, 16].

Likewise, we say that a topological pair (A, B) (i.e. a pair of topological spaces such that $B \subseteq A$) has computable type if (1.1) holds whenever there exists a homeomorphism $f : A \to S$ such that $f(B)$ is a semicomputable set in X. So, $[0, 1]$ does not have computable type (because of the example from the begining), but $([0, 1], \{0, 1\})$ has. Hence if S is a set in a computable topological space X and $f : [0, 1] \to S$ is a homeomorphism such that $f(0)$ and $f(1)$ are computable points (i.e. if S is an arc with computable endpoints), then implication (1.1) holds. Even more, as already mentioned, if S is a continuum chainable from a to b, then $(S, \{a, b\})$ has computable type [12, 10, 16].

Certain results regarding computable type and (in)computability of semicomputable sets can be found in [1, 2, 6, 19, 17, 11, 8, 9, 7, 23].

In this paper, we consider a space which resembles a graph whose edges are chainable continua. We call such a space a chainable graph. It will (easily) be shown that each graph defines a chainable graph. But since each chainable continuum, which is obviously not a graph (since it is not locally connected) defines a chainable graph, it is clear that we have a generalization of a notion of a graph. It is proved in [15] that (G, E) has computable type if G is a graph and E is the set of all its endpoints. Here, we prove a more general result: if G is a chainable graph and E is the set of all endpoints of G, then (G, E) has computable type

2. Preliminaries

In this section we give some basic facts about computable metric and topological spaces. See [21, 26, 24, 25, 4, 3, 12].

Let $k \in \mathbf{N}$, $k \geq 1$.

- A function $f : \mathbf{N}^k \to \mathbf{Q}$ is said to be **computable** if there are computable (i.e. recursive) functions $a, b, c : \mathbf{N}^k \to \mathbf{N}$ such that
$$f(x) = (-1)^{c(x)} \frac{a(x)}{b(x)+1},$$
for each $x \in \mathbf{N}^k$.
- A function $f : \mathbf{N}^k \to \mathbf{R}$ is said to be **computable** if there exists a computable function $F : \mathbf{N}^{k+1} \to \mathbf{Q}$ such that
$$|f(x) - F(x,i)| < 2^{-i},$$
for each $x \in \mathbf{N}^k$, $i \in \mathbf{N}$.
- For a set X, let $\mathcal{F}(X)$ denote the family of all finite subsets of X. A function $\Theta : \mathbf{N} \to \mathcal{F}(\mathbf{N})$ is called **computable** if the set
$$\{(x,y) \in \mathbf{N}^2 \mid y \in \Theta(x)\}$$
is computable and if there is a computable function $\varphi : \mathbf{N} \to \mathbf{N}$ such that
$$\Theta(x) \subseteq \{0, \dots, \varphi(x)\}$$
for each $x \in \mathbf{N}$.

From now on, let $\mathbf{N} \to \mathcal{F}(\mathbf{N})$, $j \to [j]$ be some fixed computable function whose range is the set of all nonempty finite subsets of $\mathbf{N}$.

2.1. *Computable metric space.* A triple (X, d, α) is said to be a **computable metric space** if (X, d) is a metric space, $\alpha = (\alpha_i)$ is a sequence in X such that $\alpha(\mathbf{N}) \subseteq X$ is dense in (X, d) and such that the function $\mathbf{N}^2 \to \mathbf{R}$, $(i, j) \mapsto d(\alpha_i, \alpha_j)$ is computable.

For example, if d is the Euclidean metric on $\mathbf{R}^n$, where $n \in \mathbf{N} \setminus \{0\}$, and $\alpha : \mathbf{N} \to \mathbf{Q}^n$ is some effective enumeration of $\mathbf{Q}^n$, then $(\mathbf{R}^n, d, \alpha)$ is a computable metric space.

Let (X, d, α) be a fixed computable metric space.

Let $i \in \mathbf{N}$ and $r \in \mathbf{Q}$, $r > 0$. We say that the set $B(\alpha_i, r) = \{x \in X \mid d(x, \alpha_i) < r\}$ is an **(open) rational ball** in a computable metric space (X, d, α). By $\overline{B}(\alpha_i, r)$ we will denote the corresponding **closed rational ball**.

Now, let $q : \mathbf{N} \to \mathbf{Q}$ be some fixed computable function whose image is the set of all positive rational numbers and let $\tau_1, \tau_2 : \mathbf{N} \to \mathbf{N}$ be some fixed computable functions such that $\{(\tau_1(i), \tau_2(i)) \mid i \in \mathbf{N}\} = \mathbf{N}^2$. For $i \in \mathbf{N}$ we define

$$I_i = B(\alpha_{\tau_1(i)}, q_{\tau_2(i)}). \tag{2.2}$$

Note that the sequence $(I_i)_{i \in \mathbf{N}}$ is an enumeration of all rational balls. Every finite union of rational balls will be called a **rational open set**. For $j \in \mathbf{N}$

we define

$$J_j = \bigcup_{i \in [j]} I_i.$$

Clearly, $\{J_j \mid j \in \mathbf{N}\}$ is the family of all rational open sets in (X, d, α).

Let $S \subseteq X$ be a closed set in (X, d). We say that S is a **computably enumerable** (c.e.) set in (X, d, α) if the set

$$\{i \in \mathbf{N} \mid I_i \cap S \neq \emptyset\}$$

is a c.e. subset of $\mathbf{N}$.

Let $S \subseteq X$ be a compact set in (X, d). We say that S is a **semicomputable** set in (X, d, α) if the set

$$\{j \in \mathbf{N} \mid S \subseteq J_j\}$$

is a c.e. subset of $\mathbf{N}$.

Finally, we say that S is a **computable set** in (X, d, α) if S is both c.e. and semicomputable in (X, d, α).

These definitions do not depend on the choice of functions q, τ_1, τ_2 and $([j])_{j \in \mathbf{N}}$.

It can be shown that a nonempty subset S of X is computable in (X, d, α) if and only if S can be effectively approximated by a finite subset of $\{\alpha_i \mid i \in \mathbf{N}\}$ with any given precision. More precisely, S is computable in (X, d, α) if and only if there exists a computable function $f : \mathbf{N} \to \mathbf{N}$ such that

$$d_H(S, \{\alpha_i \mid i \in [f(k)]\}) < 2^{-k},$$

for each $k \in \mathbf{N}$, where d_H is the Hausdorff metric (see Proposition 2.6 in [14]).

2.2. *Computable topological space.* A more general ambient space is a computable topological space. The notion of a computable topological space is not new, for example see [27, 28]. We will use the notion of a computable topological space which corresponds to the notion of a SCT_2 space from [27] (which is an effective second countable Hausdorff space).

Let $(X, \mathcal{T})$ be a topological space and let (I_i) be a sequence in $\mathcal{T}$ such that the set $\{I_i \mid i \in \mathbf{N}\}$ is a basis for $\mathcal{T}$. A triple $(X, \mathcal{T}, (I_i))$ is called a **computable topological space** if there exist c.e. subsets $C, D \subseteq \mathbf{N}^2$ such that:

1. if $i, j \in \mathbf{N}$ are such that $(i, j) \in C$, then $I_i \subseteq I_j$;
2. if $i, j \in \mathbf{N}$ are such that $(i, j) \in D$, then $I_i \cap I_j = \emptyset$;
3. if $x \in X$ and $i, j \in \mathbf{N}$ are such that $x \in I_i \cap I_j$, then there is $k \in \mathbf{N}$ such that $x \in I_k$ and $(k, i), (k, j) \in C$,
4. if $x, y \in X$ are such that $x \neq y$, then there are $i, j \in \mathbf{N}$ such that $x \in I_i$, $y \in I_j$ and $(i, j) \in D$.

Let $(X, \mathcal{T}, (I_i))$ be a fixed **computable topological space**. We define $J_j := \bigcup_{i \in [j]} I_i$.

We say that a closed set S in $(X, \mathcal{T})$ is **computably enumerable** in $(X, \mathcal{T}, (I_i))$ if $\{i \in \mathbf{N} \mid S \cap I_i \neq \emptyset\}$ is a c.e. subset od $\mathbf{N}$.

Furthermore, we say that S is **semicomputable** in $(X, \mathcal{T}, (I_i))$ if S is a compact set in $(X, \mathcal{T})$ and $\{j \in \mathbf{N} \mid S \subseteq J_j\}$ is a c.e. subset of $\mathbf{N}$.

We say that S is **computable** in $(X, \mathcal{T}, (I_i))$ if S is both c.e. and semicomputable in $(X, \mathcal{T}, (I_i))$.

The definition of a semicomputable set (and a computable set) does not depend on the choice of the sequence $([j])_{j\in\mathbf{N}}$.

If (X, d, α) is a computable metric space, then $(X, \mathcal{T}_d, (I_i))$ is a computable topological space where $\mathcal{T}_d$ is a topology induced by the metric d and (I_i) is the sequence defined by (2.2). Clearly, S is c.e./semicomputable/computable in (X, d, α) if and only if S is c.e./semicomputable/computable in $(X, \mathcal{T}_d, (I_i))$.

We say that $x \in X$ is a **computable point** in $(X, \mathcal{T}, (I_i))$ if $\{i \in \mathbf{N} \mid x \in I_i\}$ is c.e. subset of $\mathbf{N}$.

The proofs of the following facts, which will be used frequently in this paper, can be found in [18].

THEOREM 2.1. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space. There exist c.e. subsets $\mathcal{C}, \mathcal{D} \subseteq \mathbf{N}^2$ such that:*

1. *if $i, j \in \mathbf{N}$ are such that $(i, j) \in \mathcal{C}$, then $J_i \subseteq J_j$;*
2. *if $i, j \in \mathbf{N}$ are such that $(i, j) \in \mathcal{D}$, then $J_i \cap J_j = \emptyset$;*
3. *if $\mathcal{F}$ is a finite family of nonempty compact sets in $(X, \mathcal{T})$ and $A \subseteq \mathbf{N}$ is a finite subset of $\mathbf{N}$, then for each $K \in \mathcal{F}$ there is $i_K \in \mathbf{N}$ such that*
   (i) *$K \subseteq J_{i_k}$;*
   (ii) *if $K, L \in \mathcal{F}$ are such that $K \cap L = \emptyset$, then $(i_K, i_L) \in \mathcal{D}$;*
   (iii) *if $a \in A$ and $K \in \mathcal{F}$ are such that $K \subseteq J_a$, then $(i_k, a) \in \mathcal{C}$.*

PROPOSITION 2.2. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space, let $S \subseteq X$ be a semicomputable set in this space and let $m \in \mathbf{N}$. Then the set $S \setminus J_m$ is semicomputable in $(X, \mathcal{T}, (I_i))$.*

The proof of the following proposition can be found in [15].

PROPOSITION 2.3. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and let $x_0, \dots, x_n \in X$. Then the following holds:*

$$x_0, \dots, x_n \text{ are computable points} \iff \{x_0, \dots, x_n\} \text{ is a semicomputable set}$$
$$\iff \{x_0, \dots, x_n\} \text{ is a computable set.}$$

If $(X, \mathcal{T}, (I_i))$ is a computable topological space, then the topological space $(X, \mathcal{T})$ need not be metrizable (see Example 3.2 in [18]). However, if S is a compact set in $(X, \mathcal{T})$, then S, as a subspace of $(X, \mathcal{T})$, is a compact Hausdorff second countable space, which implies that S is a normal second countable space and therefore it is metrizable. This fact will be very important to us later and we will use it often.

Let A be a topological space. Suppose that the following holds: if $(X, \mathcal{T}, (I_i))$ is a computable topological space and S a semicomputable set in this space such that S and A are homeomorphic, then S is computable. Then we say that A has **computable type**.

Moreover, let A be a topological space and let B be a subspace of A. Suppose that the following holds: if $(X, \mathcal{T}, (I_i))$ is a computable topological space, S and T semicomputable sets in this space and $f : A \to S$ a homeomorphism such that $f(B) = T$, then S is computable. Then we say that (A, B) has **computable type**.

2.3. *Chainable and circularly chainable Hausdorff continua.* Let X be a set and $\mathcal{C} = (C_0, \dots, C_m)$ be a finite sequence of subsets of X. We say that $\mathcal{C}$ is a **chain** in X if the following holds:

$$C_i \cap C_j = \emptyset \iff 1 < |i - j|,$$

for all $i, j \in \{0, \dots, m\}$.

We say that $\mathcal{C}$ is a **circular chain** in X if the following holds:

$$C_i \cap C_j = \emptyset \iff 1 < |i - j| < m,$$

for all $i, j \in \{0, \dots, m\}$.

Let $A \subseteq X$ and $a, b \in A$. We say that $C_0, \dots, C_m$ **covers** A if $A \subseteq C_0 \cup \cdots \cup C_m$, and we say it **covers** A **from** a **to** b if it is also $a \in C_0$ and $b \in C_m$.

Let (X, d) be a metric space. A (circular) chain $C_0, \dots, C_m$ is said to be a ϵ-**(circular) chain**, for some $\epsilon > 0$, if $\operatorname{diam} C_i < \epsilon$, for each $i \in \{0, \dots, m\}$ and it is said to be an **open (circular) chain** if every C_i is open in (X, d). In the same way we define the notion of a **compact (circular) chain**.

Let (X, d) be a continuum, i.e. a connected and compact metric space. We say that (X, d) is a **(circularly) chainable continuum** if for every $\epsilon > 0$ there is an open ϵ-(circular) chain in (X, d) which covers X.

Suppose $a, b \in X$. We say that (X, d) is a **continuum chainable from** a **to** b if for every $\epsilon > 0$ there is an open ϵ-chain $C_0, \dots, C_m$ which covers X from a to b.

We similarly define the notions of an open and a compact (circular) chain in a topological space.

A **Hausdorff continuum** is a connected and compact Hausdorff topological space.

Let $\mathcal{A}$ and $\mathcal{B}$ be families of sets. We say that $\mathcal{A}$ **refines** $\mathcal{B}$ if for each $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $A \subseteq B$.

Let X be a Hausdorff continuum. We say that X is a **(circularly) chainable Hausdorff continuum** if for each open cover $\mathcal{U}$ of X there is an open (circular) chain $C_0, \dots, C_m$ in X which covers X and such that $\{C_0, \dots, C_m\}$ refines $\mathcal{U}$. We similarly define that a Hausdorff continuum is **chainable from** a **to** b.

It follows easily that a metric space (X, d) is a (circularly) chainable continuum if and only if topological space $(X, \mathcal{T}_d)$ is a (circularly) chainable Hausdorff continuum. Also, (X, d) is a continuum chainable from a to b if and only if $(X, \mathcal{T}_d)$ is a Hausdorff continuum chainable from a to b. See Section 3 in [10].

Remark 2.4. Let X and Y be topological spaces and let $f : X \to Y$ be a homeomorphism. Then is easy to see that If X is a (circularly) chainable Hausdorff continuum if and only if Y is a (circularly) chainable Hausdorff continuum. Furthermore, if $a, b \in X$, then X is a Hausdorff continuum chainable from a to b if and only if Y is a Hausdorff continuum chainable from $f(a)$ to $f(b)$.

The proofs of the following facts can be found in [16].

Proposition 2.5. *Let (X, d) be a continuum and $a, b \in X$. Then (X, d) is a chainable continuum from a to b if and only if for each $\epsilon > 0$ there is a compact ϵ-chain in (X, d) which covers X from a to b.*

Proposition 2.6. *Let (X, d) be a continuum. Then (X, d) is a (circularly) chainable continuum if and only if for each $\epsilon > 0$ there is a compact ϵ-(circular) chain in (X, d) which covers X.*

Example 2.7. We have that $[0, 1]$ (with the Euclidean metric) is a continuum chainable from 0 to 1. This can be easily concluded from Proposition 2.5. (Thus $[0, 1]$ with the Euclidean topology is a Hausdorff continuum chainable from 0 to 1.)

Similarly, the unit circle S^1 in $\mathbf{R}^2$ is a circularly chainable continuum. However, S^1 is not a chainable continuum (see [5]).

A topological space homeomorphic to $[0, 1]$ is called an **arc**. If A is an arc and $f : [0, 1] \to A$ a homeomorphism, then we say that $f(0)$ and $f(1)$ are **endpoints** of A (this definition does not depend on the choice of f).

If A is an arc with endpoints a and b, then by Example 2.7 and Remark 2.4 we have that A is a Hausdorff continuum chainable from a to b.

A topological space homeomorphic to S^1 is called a **topological circle**. By Example 2.7 and Remark 2.4 it is a circularly chainable Hausdorff continuum which is not chainable.

Example 2.8. Let

$$K = (\{0\} \times [-1, 1]) \cup \left\{ \left(x, \sin \frac{1}{x} \right) \mid 0 < x \leq 1 \right\}.$$

Let $a = (0, -1)$ and $b = (1, \sin 1)$. It is known that K is a continuum chainable from a to b. However, K is not an arc since K is not locally connected.

Furthermore, let

$$W = K \cup (\{0\} \times [-2, -1]) \cup ([0, 1] \times \{-2\}) \cup (\{1\} \times [-2, \sin 1]).$$

The space W is called the **Warsaw circle**. It is known that W is a circularly chainable continuum which is not chainable. Since W is not locally connected, W is not a topological circle.

THEOREM 2.9. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and let S be a semicomputable set in this space. Suppose S is, as a subspace of $(X, \mathcal{T})$, a Hausdorff continuum chainable from a to b, where a and b are computable points in $(X, \mathcal{T}, (I_i))$. Then S is computable.*

3. Chainable graphs

Let $n \in \mathbf{N}$ and let $\mathcal{I}$ be a nonempty finite family of (non-degenerate) line segments in $\mathbf{R}^n$ such that the following holds:

$$\text{if } I, J \in \mathcal{I} \text{ are such that } I \neq J \text{ and } I \cap J \neq \emptyset, \text{ then } I \cap J = \{a\}, \tag{3.3}$$

where a is an endpoint of both I and J. Then any topological space G homeomorphic to $\bigcup_{I \in \mathcal{I}} I$ is called a **graph**.

If G is a graph and $x \in G$, we say that x is an **endpoint** of G if there exists an open neighborhood N of x in G such that N is homeomorphic to $[0, \infty)$ by a homeomorphism which maps x to 0. If $\mathcal{I}$ is the family from the definition of G, then x is an endpoint of G if and only if there exists a unique $I \in \mathcal{I}$ such that x is an endpoint of I (see [15]).

The following result was proved in [15].

THEOREM 3.1. *Let G be a graph and let E be the set of all endpoints of G. Then (G, E) has computable type.*

In this section we consider spaces more general than graphs, so called chainable graphs, and we generalize Theorem 3.1 by showing that an analogue of this theorem for chainable graphs also holds.

Let A be a topological space. Suppose V is a finite subset of A and let $\mathcal{K}$ be a finite family of pairs $(K, \{a, b\})$, where $a, b \in V$, $a \neq b$, and K is a subspace of A such that K is a Hausdorff continuum chainable from a to b and $K \cap V = \{a, b\}$. Suppose

$$A = V \cup \bigcup_{(K, \{a,b\}) \in \mathcal{K}} K$$

and the following holds:

(i) for all $a, b \in V$, $a \neq b$, there exists at most one K such that $(K, \{a, b\}) \in \mathcal{K}$;

(ii) if $(K, \{a, b\}) \in \mathcal{K}$ and $(K, \{c, d\}) \in \mathcal{K}$, then $\{a, b\} = \{c, d\}$;

(iii) if $(K, \{a, b\}), (L, \{c, d\}) \in \mathcal{K}$ and $K \neq L$, then $K \cap L \subseteq V$.

Then we say that the triple $(A, \mathcal{K}, V)$ is a **chainable graph**.

Let $(A, \mathcal{K}, V)$ be a chainable graph and let $a \in V$. We say that a is an **endpoint** of $(A, \mathcal{K}, V)$ if there exists a unique $(K, \{c, d\}) \in \mathcal{K}$ such that $a \in \{c, d\}$.

EXAMPLE 3.2. Let G be a graph and let $\mathcal{I}$ be the family from the definition of G. Let V be the set of all endpoints of all $I \in \mathcal{I}$. Let $\mathcal{K}$ be the family of all $(I, \{a, b\})$ such that $I \in \mathcal{I}$ and a and b are endpoints of I. Then $(G, \mathcal{K}, V)$ is obviously a chainable graph. We have that x is an endpoint of G if and only if x is an endpoint of $(G, \mathcal{K}, V)$.

EXAMPLE 3.3. Suppose K is a Hausdorff continuum chainable from a to b.

If $a \neq b$, then $(K, \{(K, \{a, b\})\}, \{a, b\})$ is a chainable graph and a and b are all its endpoints.

If $a = b$, then it is easy to conclude that $K = \{a\}$ and we have that $(K, \emptyset, \{a\})$ is a chainable graph which has no endpoints.

If $(G, \mathcal{K}, V)$ is a chainable graph, then G need not be a graph, as the following example shows.

EXAMPLE 3.4. Let K, a and b be as in Example 2.8. Then $(K, \{(K, \{a, b\})\}, \{a, b\})$ is a chainable graph. But K is not a graph: K is not locally connected and each graph is easily seen to be locally connected.

REMARK 3.5. Let $(G, \mathcal{K}, V)$ be a chainable graph. Suppose $(K, \{a, b\})$, $(L, \{c, d\}) \in \mathcal{K}$ are such that $K \neq L$. Then $K \cap L \subseteq V$ and since $K \cap V = \{a, b\}$, we have $K \cap L \subseteq \{a, b\}$. If $K \cap L = \{a, b\}$, then $\{a, b\} \subseteq L \cap V = \{c, d\}$, hence $\{a, b\} = \{c, d\}$ and so $(L, \{a, b\}) \in \mathcal{K}$ which is impossible by property (i) from the definition of a chainable graph. We conclude that $\operatorname{card}(K \cap L) \leq 1$.

Also note that K and L can only intersect in one of the points a, b, c, d. So

$$(K \setminus \{a, b\}) \cap L = \emptyset = K \cap (L \setminus \{c, d\}).$$

In this section we are going to prove the following result:

THEOREM 3.6. *If $(A, \mathcal{K}, V)$ is a chainable graph and B is the set of all its endpoints, then (A, B) has computable type.*

By Examples 3.2 and 3.4, Theorem 3.6 is a generalization of Theorem 3.1. Furthermore, by Example 3.3, if $a \neq b$, Theorem 3.6 is a generalization of Theorem 2 from [10]: if K is a Hausdorff continuum chainable from a to b, then $(K, \{a, b\})$ has computable type.

Although Theorem 3.6 is a generalization of Theorem 3.1, the techniques of the proof of Theorem 3.1 in [15] cannot be easily applied in the case of chainable graphs. The main reason if that that proof essentially relies on the fact that for each $x \in [0, 1]$ such that $0 < x < 1$ and each open neighborhood U of x in $[0, 1]$ there exist subcontinua F, G and H of $[0, 1]$ which cover $[0, 1]$ and such that $0 \in F$, $1 \in H$, $F \cap H = \emptyset$ and $x \in G \subseteq U$ (it is obvious that we can find such F, G and H). The problem is that an analogous statement does not hold in general for chainable continua. For example, let K, a and b be as in Example 2.8, let $x = (0, 0)$ and $U = \{(x, y) \in K \mid 0 \leq x < \frac{1}{2},$

$-\frac{1}{2} < y < \frac{1}{2}\}$. Then U is an open neighborhood of x in K and it is not hard to see that there exist no subcontinua F, G and H of K which cover K and such that $a \in F$, $b \in H$, $F \cap H = \emptyset$ and $x \in G \subseteq U$.

Before we give a proof of Theorem 3.6 we need some further facts about chainable continua.

Let X be a set. Let $\mathcal{A} = (A_0, \dots, A_m)$ and $\mathcal{B} = (B_0, \dots, B_n)$ be finite sequences of subsets of X. We say that $\mathcal{A}$ **strongly refines** $\mathcal{B}$ if $\{A_0, \dots, A_m\}$ refines $\{B_0, \dots, B_n\}$, $A_0 \subseteq B_0$ and $A_m \subseteq B_n$.

The proof of the following lemma can be found in [12].

Lemma 3.7. *Let (X, d) be a metric space such in each closed ball is compact. Let $(\mathcal{C}^k)$, where $\mathcal{C}^k = (C_0^k, \dots, C_{m_k}^k)$, $k \in \mathbf{N}$, be a sequence of chains such that $\overline{C_0^{k+1}}, \dots, \overline{C_{m_{k+1}}^{k+1}}$ strongly refines $C_0^k, \dots, C_{m_k}^k$ and such that diam$(C_j^k) < 2^{-k}$, for each $k \in \mathbf{N}$ and for each $j \in \{0, \dots, m_k\}$. Let*

$$S = \bigcap_{k \in \mathbf{N}} (\overline{C_0^{k+1}} \cup \dots \cup \overline{C_{m_{k+1}}^{k+1}}).$$

Then S is a continuum chainable from a to b, where $a \in \bigcap_{k \in \mathbf{N}} C_0^k$, $b \in \bigcap_{k \in \mathbf{N}} C_{m_k}^k$.

Lemma 3.8. *Let (K, d) be a continuum chainable from a to b, $a, b \in K$. Let $\epsilon > 0$ be arbitrary. Then there exist $c \in K$ and $K' \subseteq K$ such that $c \neq a$, K' is a continuum chainable from a to c and $K' \subseteq B(a, \epsilon)$.*

Proof. Firstly, we will construct a sequence of chains $(\mathcal{C}^i)$, $\mathcal{C}^i = (C_0^i, \dots, C_{m_i}^i)$, $i \in \mathbf{N}$, such that for each $i \in \mathbf{N}$:

1. $\mathcal{C}^i$ covers K from a to b and $\operatorname{diam}(C_j^i) < 2^{-i}$, for each $j \in \{0, \dots, m_i\}$;
2. $\overline{C_0^{i+1}}, \dots, \overline{C_{m_{i+1}}^{i+1}}$ strongly refines $C_0^i, \dots, C_{m_i}^i$;
3. $\operatorname{diam}(C_j^0) < \min\{\frac{\epsilon}{3}, \frac{d(a,b)}{4}\}$, for each $j \in \{0, \dots, m_0\}$.

Because K is a continuum chainable from a to b, there is an open $\min\{\frac{\epsilon}{3}, \frac{d(a,b)}{4}, 1\}$-chain $\mathcal{C}^0 = (C_0^0, \dots, C_{m_0}^0)$ in (K, d) which covers K from a to b. Since $\operatorname{diam}(C_j^0) < \frac{d(a,b)}{4}$, for each $j \in \{0, \dots, m_0\}$, we easily conclude that $m_0 \geq 4$.

Let us assume that $\mathcal{C}^i = (C_0^i, \dots, C_{m_i}^i)$ is an open chain in (K, d) with property 1. Since $\mathcal{U} = \{C_0^i, \dots, C_{m_i}^i\}$ is an open cover of (K, d), there is a Lebesgue number λ of $\mathcal{U}$. Because $a \in C_0^i$ and C_0^i is an open set, there is $r_a > 0$ such that $B(a, r_a) \subseteq C_0^i$. Also, there is $r_b > 0$ such that $B(b, r_b) \subseteq C_{m_i}^i$.

Since K is a chainable continuum, there is a $\min\{r_a, r_b, \lambda, 2^{-(i+1)}\}$-open chain $\mathcal{C}^{i+1} = (C_0^{i+1}, \dots, C_{m_{i+1}}^{i+1})$ in (K, d) which covers K from a to b. Because $\operatorname{diam}(\overline{C_j^{i+1}}) = \operatorname{diam}(C_j^{i+1}) < \lambda$ for each $j \in \{0, \dots, m_{i+1}\}$, we have that $\overline{C_0^{i+1}}, \dots, \overline{C_{m_{i+1}}^{i+1}}$ refines $C_0^i, \dots, C_{m_i}^i$. Furthermore, $a \in C_0^{i+1}$ and $\operatorname{diam}(\overline{C_0^{i+1}}) = \operatorname{diam}(C_0^{i+1}) < r_a$, so $\overline{C_0^{i+1}} \subseteq B(a, r_a)$ and thus $\overline{C_0^{i+1}} \subseteq C_0^i$. Analogously,

$\overline{C^{i+1}_{m_{i+1}}} \subseteq C^i_{m_i}$. This concludes the recursive construction of the sequence $(\mathcal{C}^i)$ with properties 1-3.

Now, we want to choose, for each $i \in \mathbf{N}$,

$$n_i \in \{2, \dots, m_i - 2\} \tag{3.4}$$

so that

$$\overline{C^{i+1}_0}, \dots, \overline{C^{i+1}_{n_{i+1}}} \text{ strongly refines } C^i_0, \dots, C^i_{n_i}. \tag{3.5}$$

Let $n_0 = 2$. Since $m_0 \geq 4$, we have $n_0 \in \{2, \dots, m_0 - 2\}$. Let us assume that $i \in \mathbf{N}$ and $n_i \in \{2, \dots, m_i - 2\}$.

Let

$$k = \min\{j \in \{0, \dots, m_{i+1}\} \mid \overline{C^{i+1}_j} \subseteq C^i_{n_i}\}. \tag{3.6}$$

The number k is well defined. Namely, if we assume that there is no $j \in \{0, \dots, m_{i+1}\}$ such that $\overline{C^{i+1}_j} \subseteq C^i_{n_i}$ and we take into account that $\overline{C^{i+1}_0}, \dots, \overline{C^{i+1}_{m_{i+1}}}$ strongly refines $C^i_0, \dots, C^i_{m_i}$, we get that $K = U \cup V$, where

$$U = \bigcup \{C^{i+1}_j \mid \overline{C^{i+1}_j} \subseteq C^i_l \text{ for some } l < n_i\},$$

$$V = \bigcup \{C^{i+1}_j \mid \overline{C^{i+1}_j} \subseteq C^i_l \text{ for some } l > n_i\}.$$

The sets U and V are nonempty. Namely, $\overline{C^{i+1}_0} \subseteq C^i_0$ and $0 < n_i$ since $n_i \geq 2$, so $C^{i+1}_0 \subseteq U$, in particular $U \neq \emptyset$. Also, $\overline{C^{i+1}_{m_{i+1}}} \subseteq C^i_{m_i}$ and $m_i > n_i$ since $n_i \leq m_i - 2$, so $C^{i+1}_{m_{i+1}} \subseteq V$ and $V \neq \emptyset$.

If $l, l' \in \{0, \dots, m_i\}$ are such that $l < n_i < l'$, then $C^i_l \cap C^i_{l'} = \emptyset$ since $\mathcal{C}^i$ is a chain. This implies $U \cap V = \emptyset$.

Clearly U and V are open in (K, d). We conclude that (U, V) is a separation of (K, d), which is impossible since (K, d) is connected. So the number k is well defined.

Since $n_i \geq 2$, $C^i_{n_i} \cap C^i_0 = \emptyset$ holds. Because $\overline{C^{i+1}_k} \subseteq C^i_{n_i}$ and $\overline{C^{i+1}_0} \subseteq C^i_0$, $C^{i+1}_k \cap C^{i+1}_0 = \emptyset$ holds, also. Therefore, $k \geq 2$. Similary, $n_i \leq m_i - 2$, so $C^i_{n_i} \cap C^i_{m_i} = \emptyset$. Because $\overline{C^{i+1}_{m_{i+1}}} \subseteq C^i_{m_i}$ holds, it is $C^{i+1}_{m_{i+1}} \cap C^{i+1}_k = \emptyset$ and hence $k \leq m_{i+1} - 2$. So, $k \in \{2, \dots, m_{i+1} - 2\}$.

Now, let $n_{i+1} = k$.

We claim that $\overline{C^{i+1}_0}, \dots, \overline{C^{i+1}_{n_{i+1}}}$ strongly refines $C^i_0, \dots, C^i_{n_i}$. Certainly $\overline{C^{i+1}_0} \subseteq C^i_0$ and $\overline{C^{i+1}_{n_{i+1}}} \subseteq C^i_{n_i}$. We want to show that for each $j \in \{1, \dots, k-1\}$ there exists $j' \in \{0, \dots, n_i\}$ such that $\overline{C^{i+1}_j} \subseteq C^i_{j'}$. Suppose the opposite. Then the number

$$k' = \min\{j \in \{1, \dots, k-1\} \mid \exists j' \in \{n_i + 1, \dots, m_i\} \text{ such that } \overline{C^{i+1}_j} \subseteq C^i_{j'}\}$$

is well defined.

By the definition of k' we have $\overline{C^{i+1}_{k'-1}} \subseteq C^i_{j''}$ for some $j'' \leq n_i$. However, if $j'' = n_i$, then $\overline{C^{i+1}_{k'-1}} \subseteq C^i_{n_i}$, which, together with $k'-1 < k' \leq k-1 < k$, contradicts (3.6) (the choice of k). Hence $j'' < n_i$.

The inequalities $j'' < n_i < j'$ imply $C^i_{j'} \cap C^i_{j''} = \emptyset$ and consequently $C^{i+1}_{k'} \cap C^{i+1}_{k'-1} = \emptyset$ which is impossible.

We conclude that $\overline{C^{i+1}_0}, \dots, \overline{C^{i+1}_{n_{i+1}}}$ strongly refines $C^i_0, \dots, C^i_{n_i}$.

So there exists a sequence (n_i) such that (3.4) and (3.5) hold for each $i \in \mathbf{N}$. We know that $\operatorname{diam}(C^i_j) < 2^{-i}$ for each $i \in \mathbf{N}$ and each $j \in \{0, \dots, n_i\}$. Since each closed ball in a compact is also a compact, using Lemma 3.7 we conclude that

$$K' = \bigcap_{i \in \mathbf{N}} (\overline{C^{i+1}_0} \cup \dots \cup \overline{C^{i+1}_{n_{i+1}}}).$$

is a continuum chainable from a' to c, where $a' \in \bigcap_{i \in \mathbf{N}} C^i_0$ and $c \in \bigcap_{i \in \mathbf{N}} C^i_{n_i}$. However, by construction we have $a \in \bigcap_{i \in \mathbf{N}} C^i_0$ and $\operatorname{diam}(C^i_0) \to 0$, therefore $a = a'$.

On the other hand, $c \in C^0_{n_0} = C^0_2$ and so $a \neq c$. Furthermore,

$$K' \subseteq \overline{C^1_0} \cup \overline{C^1_1} \cup \dots \cup \overline{C^1_{n_1}} \subseteq C^0_0 \cup C^0_1 \cup C^0_2$$

and consequently

$$\operatorname{diam} K' \leq \operatorname{diam}(C^0_0 \cup C^0_1 \cup C^0_2) < \epsilon.$$

Now $a \in K'$ implies $K' \subseteq B(a, \epsilon)$. □

Remark 3.9. Using Remark 2.4, we conclude the following: if $(A, \mathcal{K}, V)$ is a chainable graph, B the set of all its endpoints, A' a topological space and $f : A \to A'$ a homeomorphism, then

$$(A', \{(f(K), \{f(a), f(b)\}) \mid (K, \{a, b\}) \in \mathcal{K}\}, f(V))$$

is a chainable graph and $f(B)$ is the set of all its endpoints.

Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and let S and T be subsets of X such that $S \subseteq T$. We say that S is **computably enumerable (c.e) up to** T if there exists a c.e. subset Ω of $\mathbf{N}$ such that for each $i \in \mathbf{N}$ the following holds:

$$\text{if } I_i \cap S \neq \emptyset, \text{ then } i \in \Omega;$$
$$\text{if } i \in \Omega, \text{ then } I_i \cap T \neq \emptyset.$$

It is obvious that if S is closed and S is c.e. up to S, then S is a c.e. set in $(X, \mathcal{T}, (I_i))$.

Remark 3.10. Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and let $S_0, \dots, S_k$ and T be subsets of X such that S_i is c.e. up to T for each $i \in \{0, \dots, k\}$. Then it readily follows that $S_0 \cup \dots \cup S_k$ is c.e. up to T.

Now we are ready to prove Theorem 3.6. In view of Remark 3.9 it is enough to prove the following theorem.

THEOREM 3.11. *Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and let S and T be semicomputable sets in this space. Suppose there exist $\mathcal{K}$ and V such that $(S, \mathcal{K}, V)$ is a chainable graph and T is the set of all its endpoints. Then S is computable.*

PROOF. The claim of the theorem is clear if $S = \emptyset$. Suppose $S \neq \emptyset$. Since S is compact in $(X, \mathcal{T})$, it is metrizable. Let d be the metric on S which induces the topology on S, i.e. the relative topology on S in $(X, \mathcal{T})$. So, for each $(K, \{a, b\}) \in \mathcal{K}$ the metric space $(K, d|_{K \times K})$ is a continuum chainable from a to b.

Let

$$\mathcal{K}' = \{K \mid \exists a, b \text{ such that } (K, \{a, b\}) \in \mathcal{K}\}.$$

Then, by the definition of a chainable graph,

$$S = \left(\bigcup_{K \in \mathcal{K}'} K \right) \cup \left(V \setminus \left(\bigcup_{K \in \mathcal{K}'} K \right) \right). \tag{3.7}$$

The set $V \setminus \left(\bigcup_{K \in \mathcal{K}'} K\right)$ is finite (since V is finite) and therefore compact in $(X, \mathcal{T})$. It is clearly disjoint with the compact set $\bigcup_{K \in \mathcal{K}'} K$ and so there exists $m \in \mathbf{N}$ such that $\bigcup_{K \in \mathcal{K}'} K \subseteq J_m$ and J_m is disjoint with $V \setminus \left(\bigcup_{K \in \mathcal{K}'} K\right)$. It follows from (3.7) that

$$V \setminus \left(\bigcup_{K \in \mathcal{K}'} K \right) = S \setminus J_m$$

and thus $V \setminus \left(\bigcup_{K \in \mathcal{K}'} K\right)$ is a semicomputable set (Proposition 2.2). Now Proposition 2.3 gives that $V \setminus \left(\bigcup_{K \in \mathcal{K}'} K\right)$ is a computable set in $(X, \mathcal{T}, (I_i))$, in particular it is c.e. So it is (trivially) c.e. up to S.

We want to prove that K is c.e. up to S for each $K \in \mathcal{K}'$. If we prove that, then (3.7) and Remark 3.10 will imply that S is c.e. up to S, hence S will be c.e. and thus computable.

Let $K \in \mathcal{K}'$. Then we have $(K, \{a, b\}) \in \mathcal{K}$ for some a and b. There are four cases: $a, b \notin T$; $a \in T$ and $b \notin T$; $a \notin T$ and $b \in T$; $a, b \in T$.

Suppose $a, b \notin T$. Let

$$\mathcal{M} = \{M \subseteq X \mid \exists m \in X \text{ such that } (M, \{m, a\}) \in \mathcal{K}\}.$$

Then $\mathcal{M} \setminus \{K\} \neq \emptyset$ (since a is not an endpoint of S). Let $M_0, \dots, M_k$ be all (mutually different) elements of $\mathcal{M} \setminus \{K\}$.

For each $i \in \{0, \dots, k\}$ let m_i be such that

$$(M_i, \{m_i, a\}) \in \mathcal{K}.$$

Then $m_i \neq m_{i'}$ for all $i, i' \in \{0, \dots, k\}$ such that $i \neq i'$ (by property (i) from the definition of a chainable graph since $M_i \neq M_{i'}$) and also $m_i \neq b$ for each $i \in \{0, \dots, k\}$. By Remark 3.5 we have

$$K \cap M_i = \{a\} \quad \text{for each } i \in \{0, \dots, k\}. \tag{3.8}$$

Similarly, let

$$\mathcal{L} = \{L \subseteq X \mid \exists l \in X \text{ such that } (L, \{l, b\}) \in \mathcal{K}\},$$

let $L_0, \dots, L_n$ be all (mutually different) elements of $\mathcal{L} \setminus \{K\}$ and let $l_0, \dots, l_n \in V \setminus \{a\}$ be mutually different such that

$$(L_j, \{l_j, b\}) \in \mathcal{K}$$

for each $j \in \{0, \dots, n\}$. It follows from Remark 3.5 that

$$K \cap L_j = \{b\} \quad \text{for each } j \in \{0, \dots, n\}. \tag{3.9}$$

Note also that by the same remark for all $i \in \{0, \dots, k\}$ and $j \in \{0, \dots, n\}$ we have

$$M_i \cap L_j \subseteq \{m_i\} \cap \{l_j\}. \tag{3.10}$$

Let $i \in \{0, \dots, k\}$. Then there exists a compact $\frac{d(a,m_i)}{4}$-chain $D_0, \dots, D_v$ in $(M_i, d|_{M_i \times M_i})$ which covers M_i from a to m_i. Since $a \in D_0$ and $m_i \in D_v$ and $D_0, \dots, D_v$ is a chain, the triangle inequality implies

$$d(a, m_i) \leq \operatorname{diam}(D_0) + \dots + \operatorname{diam}(D_v) \leq (v+1) \cdot \frac{d(a, m_i)}{4},$$

which yields $v \geq 3$. Let

$$M_i^1 = D_0, \quad M_i^2 = D_1 \cup \dots \cup D_{v-2} \quad \text{and} \quad M_i^3 = D_{v-1} \cup D_v.$$

Then M_i^1, M_i^2 and M_i^3 are nonempty compact sets in $(X, \mathcal{T})$ such that

$$M_i = M_i^1 \cup M_i^2 \cup M_i^3, \quad M_i^1 \cap M_i^3 = \emptyset, \quad a \in M_i^1 \quad \text{and} \quad m_i \in M_i^3 \tag{3.11}$$

and

$$m_i \notin M_i^1 \cup M_i^2. \tag{3.12}$$

Similarly, for each $j \in \{0, \dots, n\}$ there exist nonempty compact sets L_j^1, L_j^2 and L_j^3 in $(X, \mathcal{T})$ such that

$$L_j = L_j^1 \cup L_j^2 \cup L_j^3, \quad L_j^1 \cap L_j^3 = \emptyset, \quad b \in L_j^1 \quad \text{and} \quad l_j \in L_j^3 \tag{3.13}$$

and $l_j \notin L_j^1 \cup L_j^2$.

Let

$$\mathcal{F} = \{F \mid \exists c, d \text{ such that } (F, \{c, d\}) \in \mathcal{K} \text{ and } \{a, b\} \cap \{c, d\} = \emptyset\}.$$

Note that the families $\mathcal{F}$ and $\mathcal{M} \cup \mathcal{L} \cup \{K\}$ are disjoint and their union is $\mathcal{K}'$. Furthermore, by Remark 3.5 for all $F \in \mathcal{F}$, $i \in \{0, \dots, k\}$ and $j \in \{0, \dots, n\}$ the following holds

$$F \cap M_i \subseteq \{m_i\}, \quad F \cap L_j \subseteq \{l_j\} \quad \text{and} \quad F \cap K = \emptyset. \tag{3.14}$$

Let

$$W = V \setminus \{a, b, m_0, \dots, m_k, l_0, \dots, l_n\}.$$

The sets

$$K \cup \bigcup_{i=0}^{k} M_i^1 \cup \bigcup_{j=0}^{n} L_j^1 \quad \text{and} \quad \bigcup \mathcal{F} \cup W \cup \bigcup_{i=0}^{k} M_i^3 \cup \bigcup_{j=0}^{n} L_j^3 \tag{3.15}$$

are nonempty and compact in $(X, \mathcal{T})$ and they are also disjoint since it is obvious that the sets from the different unions are disjoint (in detail can be proven using (3.14), (3.11), (3.13), (3.10), (3.9) and (3.8)).

So there exist $\mu, \mu' \in \mathbf{N}$ such that $J_\mu \cap J_{\mu'} = \emptyset$,

$$\bigcup \mathcal{F} \cup W \cup \bigcup_{i=0}^{k} M_i^3 \cup \bigcup_{j=0}^{n} L_j^3 \subseteq J_\mu \text{ and } K \cup \bigcup_{i=0}^{k} M_i^1 \cup \bigcup_{j=0}^{n} L_j^1 \subseteq J_{\mu'}. \tag{3.16}$$

(this follows e.g. from Theorem 2.1). Therefore

$$\left(K \cup \bigcup_{i=0}^{k} M_i^1 \cup \bigcup_{j=0}^{n} L_j^1 \right) \cap J_\mu = \emptyset. \tag{3.17}$$

Note that the union of $\bigcup_{i=0}^{k} M_i^2 \cup \bigcup_{j=0}^{n} L_j^2$ and the sets in (3.15) is equal to S. From this and (3.16) it follows that

$$S \setminus J_\mu \subseteq K \cup \bigcup_{i=0}^{k} M_i^1 \cup \bigcup_{i=0}^{k} M_i^2 \cup \bigcup_{j=0}^{n} L_j^1 \cup \bigcup_{j=0}^{n} L_j^2. \tag{3.18}$$

Let us denote

$$A = \bigcup_{i=0}^{k} M_i^1 \cup \bigcup_{i=0}^{k} M_i^2 \quad \text{and} \quad B = \bigcup_{j=0}^{n} L_j^1 \cup \bigcup_{j=0}^{n} L_j^2.$$

By (3.17) and (3.18) we have

$$K \cup \bigcup_{i=0}^{k} M_i^1 \cup \bigcup_{j=0}^{n} L_j^1 \subseteq S \setminus J_\mu \subseteq K \cup A \cup B, \tag{3.19}$$

and $S \setminus J_\mu$ is a semicomputable set in $(X, \mathcal{T}, (I_i))$ (Proposition 2.2).

It follows from (3.10), (3.12), (3.8) and (3.9) that

$$A \cap B = \emptyset, \quad A \cap K = \{a\} \quad \text{and} \quad B \cap K = \{b\}. \tag{3.20}$$

By (3.16) we have $a, b \in J_{\mu'} \cap S$. Since $J_{\mu'} \cap S$ is open in (S, d), there is $\epsilon > 0$ such that $B(a, \epsilon) \subseteq J_{\mu'} \cap S \subseteq S \setminus J_\mu$, $B(b, \epsilon) \subseteq J_{\mu'} \cap S \subseteq S \setminus J_\mu$ and $B(a, \epsilon) \cap B(b, \epsilon) = \emptyset$.

Now, according to Lemma 3.8, there are $c \in M_0$, $c \neq a$, $\widetilde{M} \subseteq M_0$, $d \in L_0$, $d \neq b$, and $\widetilde{L} \subseteq L_0$ such that:

(3.21) $\widetilde{M}$ is a continuum chainable from c to a and $\widetilde{M} \subseteq B(a, \epsilon) \subseteq S \setminus J_\mu$;

(3.22) $\widetilde{L}$ is a continuum chainable from b to d and $\widetilde{L} \subseteq B(b, \epsilon) \subseteq S \setminus J_\mu$.

Therefore, $\widetilde{M} \cup K \cup \widetilde{L}$ is a connected set such that

$$\widetilde{M} \cup K \cup \widetilde{L} \subseteq S \setminus J_\mu. \tag{3.23}$$

Furthermore, $d \in B(b, \epsilon)$ and $B(b, \epsilon) \cap J_\mu = \emptyset$, so $d \notin J_\mu$. Since $l_0 \in L_0^3 \subseteq J_\mu$, we have $d \neq l_0$.

Because $d \neq b$ and $d \in L_0$, it follows from (3.9) that $d \notin K$. On the other hand, $d \neq l_0$, $d \in L_0$ and (3.10) imply $d \notin M_i$ for each $i \in \{0, \dots k\}$. Consequently, $d \notin A$. Hence

$$d \notin K \cup A.$$

Therefore we can choose $\alpha \in \mathbf{N}$ such that

$$K \cup A \subseteq J_\alpha \text{ and } d \notin J_\alpha. \tag{3.24}$$

Similary, we can choose $\beta \in \mathbf{N}$ such that

$$K \cup B \subseteq J_\beta \text{ and } c \notin J_\beta. \tag{3.25}$$

Let $\mathcal{C}$ and $\mathcal{D}$ be the subsets of $\mathbf{N}^2$ from Theorem 2.1 and let $f : \mathbf{N} \to \mathbf{N}$ be a computable function such that $I_i = J_{f(i)}$ for each $i \in \mathbf{N}$.

Suppose $i \in \mathbf{N}$ is such that $I_i \cap K \neq \emptyset$. Then there exists $x \in I_i \cap (K \setminus \{a, b\})$ and we can choose $0 < r < \min\{d(x, a), d(x, b)\}$ so that

$$B(x, r) \subseteq I_i \cap K \subseteq I_i = J_{f(i)}. \tag{3.26}$$

Furthermore, since K is a continuum chainable from a to b, there is a compact r-chain $K_0, \dots, K_t$ in $(K, d|_{K \times K})$ which covers K and such that $a \in K_0$ and $b \in K_t$. Let $p \in \{0, \dots, t\}$ be such that $x \in K_p$. Since $r < d(x, a)$ and $r < d(x, b)$, $p \neq 0$ and $p \neq t$ hold. Because of (3.26) and $\operatorname{diam}(K_p) < r$, we have that

$$K_p \subseteq I_i = J_{f(i)}. \tag{3.27}$$

Let us denote

$$F = A \cup K_0 \cup \cdots \cup K_{p-1} \text{ and } G = B \cup K_{p+1} \cup \cdots \cup K_t.$$

It is $K \cup A \cup B = F \cup K_p \cup G$, and because of (3.19) holds

$$S \setminus J_\mu \subseteq F \cup K_p \cup G. \tag{3.28}$$

It follows from (3.20) that $F \cap G = \emptyset$.

The sets F, K_p and G are compact in $(X, \mathcal{T})$, F and G are disjoint, by (3.24) and (3.25) we have $F \subseteq J_\alpha$ and $G \subseteq J_\beta$ and (3.27) holds, so according to Theorem 2.1, there are $u, v, w \in \mathbf{N}$ such that $F \subseteq J_u$, $K_p \subseteq J_v$, $G \subseteq J_w$, $(u, w) \in \mathcal{D}$, $(v, f(i)) \in \mathcal{C}$, $(u, \alpha) \in \mathcal{C}$ and $(w, \beta) \in \mathcal{C}$. Note that, by (3.28), $S \setminus J_\mu \subseteq J_u \cup J_v \cup J_w$ holds.

So, if $i \in \mathbf{N}$ is such that $I_i \cap K \neq \emptyset$, then there exist $u, v, w \in \mathbf{N}$ such that:

(1) $S \setminus J_\mu \subseteq J_u \cup J_v \cup J_w$;
(2) $(u, w) \in \mathcal{D}$;
(3) $(v, f(i)) \in \mathcal{C}$;
(4) $(u, \alpha) \in \mathcal{C}$;
(5) $(w, \beta) \in \mathcal{C}$.

Let Ω be the set of all $(i, u, v, w) \in \mathbf{N}^4$ for which the statements (1)-(5) hold. Since $S \setminus J_\mu$ is a semicomputable set, the set of all $(i, u, v, w) \in \mathbf{N}^4$ such that (1) holds is a c.e. set. It follows now easily that Ω is c.e. Let Γ be the set of all $i \in \mathbf{N}$ for which there exist $u, v, w \in \mathbf{N}$ such that $(i, u, v, w) \in \Omega$. Then Γ is c.e.

We have proved the following: if $I_i \cap K \neq \emptyset$, then $i \in \Gamma$.

Conversely, suppose $i \in \Gamma$. Then there exist $u, v, w \in \mathbf{N}$ such that $(i, u, v, w) \in \Omega$. So the statements (1)-(5) hold. We claim that $I_i \cap S \neq \emptyset$.

Suppose the opposite, i.e. $I_i \cap S = \emptyset$. Since $J_v \subseteq I_i$ by (3), we have $J_v \cap S = \emptyset$ and now (1) implies that

$$S \setminus J_\mu \subseteq J_u \cup J_w.$$

From this and (3.23) it follows

$$\widetilde{M} \cup K \cup \widetilde{L} \subseteq J_u \cup J_w.$$

Furthermore, $c \in \widetilde{M} \cup K \cup \widetilde{L}$ by (3.21), which implies that $c \in J_u \cup J_w$. If $c \in J_w$, then $c \in J_\beta$ by (5), which is a contradiction with (3.25). So $c \in J_u$, which means that J_u intersects $\widetilde{M} \cup K \cup \widetilde{L}$.

Similarly, using (3.22), (4) and (3.24) we conclude that $d \in J_w$, so J_w intersects $\widetilde{M} \cup K \cup \widetilde{L}$.

Hence the sets J_u and J_w are open in $(X, \mathcal{T})$ and disjoint, their union contains $\widetilde{M} \cup K \cup \widetilde{L}$ and each of them intersects $\widetilde{M} \cup K \cup \widetilde{L}$ which implies that $\widetilde{M} \cup K \cup \widetilde{L}$ is not connected, a contradiction. Therefore, $I_i \cap S \neq \emptyset$.

To summarize, for each $i \in \mathbf{N}$ the following two implications hold:

$$I_i \cap K \neq \emptyset \;\Rightarrow\; i \in \Gamma,$$

$$i \in \Gamma \;\Rightarrow\; I_i \cap S \neq \emptyset.$$

This means that K is c.e. up to S.

So K is c.e. up to S if $a, b \notin T$.

Let us now consider the case $a \in T$ and $b \notin T$. Then b is not an endpoint of $(S, \mathcal{K}, V)$ and let $\mathcal{L}$, $L_0, \dots, L_n$ and $l_0, \dots, l_n$ be as in the previous case.

Furthermore, let for each $j \in \{0, \dots, m\}$ the sets L_j^1, L_j^2 and L_j^3 be defined as before and let also $\mathcal{F}$ and B be defined as before. Let

$$W = V \setminus \{a, b, l_0, \dots, l_n\}.$$

The sets

$$K \cup \bigcup_{j=0}^{m} L_j^1 \quad \text{and} \quad \bigcup \mathcal{F} \cup W \cup \bigcup_{j=0}^{m} L_j^3$$

are nonempty, disjoint and compact in $(X, \mathcal{T})$, so there exist $\mu, \mu' \in \mathbf{N}$ such that

$$\bigcup \mathcal{F} \cup W \cup \bigcup_{j=0}^{m} L_j^3 \subseteq J_\mu \text{ and } K \cup \bigcup_{j=0}^{n} L_j^1 \subseteq J_{\mu'}$$

and

$$J_\mu \cap J_{\mu'} = \emptyset.$$

It follows

$$K \cup \bigcup_{j=0}^{m} L_j^1 \subseteq S \setminus J_\mu \subseteq K \cup B. \tag{3.29}$$

Let $\epsilon > 0$ be such that $B(b, \epsilon) \cap J_\mu = \emptyset$. By Lemma 3.8 there exist $d \in L_0$, $d \neq b$, and $\widetilde{L} \subseteq L_0$ such that $\widetilde{L}$ is a continuum chainable from b to d and $\widetilde{L} \subseteq B(b, \epsilon) \subseteq S \setminus J_\mu$.

It follows that $K \cup \widetilde{L}$ is a connected set such that

$$K \cup \widetilde{L} \subseteq S \setminus J_\mu. \tag{3.30}$$

Since $d \in L_0$ and $d \neq b$ we have $d \notin K$ (by (3.9)). So we can choose $\gamma \in \mathbf{N}$ such that

$$K \subseteq J_\gamma \text{ and } d \notin J_\gamma. \tag{3.31}$$

Suppose $i \in \mathbf{N}$ is such that $I_i \cap K \neq \emptyset$. In the same way as before we conclude that there exists a compact chain $K_0, \dots, K_t$ in $(K, d|_{K \times K})$ which covers K from a to b and $p \in \{1, \dots, t-1\}$ such that

$$K_p \subseteq I_i.$$

We define

$$F = K_0 \cup \cdots K_{p-1} \text{ and } G = K_{p+1} \cup \cdots \cup K_t \cup B.$$

It holds that $K \cup B = F \cup K_p \cup G$ and by (3.29)

$$S \setminus J_\mu \subseteq F \cup K_p \cup G.$$

As before, F and G are disjoint and thus there are $u, v, w \in \mathbf{N}$ such that $F \subseteq J_u$, $K_p \subseteq J_v$, $G \subseteq J_w$, $(u, w) \in \mathcal{D}$, $(v, f(i)) \in \mathcal{C}$ and $(u, \gamma) \in \mathcal{C}$. Also, $a \in F$, so $a \in J_u$.

So if $i \in \mathbf{N}$ is such that $I_i \cap K \neq \emptyset$, then there exist $u, v, w \in \mathbf{N}$ such that:

(1) $S \setminus J_\mu \subseteq J_u \cup J_v \cup J_w$;
(2) $(u, w) \in \mathcal{D}$;
(3) $(v, f(i)) \in \mathcal{C}$;
(4) $(u, \gamma) \in \mathcal{C}$;
(5) $a \in J_u$.

Let Ω be the set of all $(i, u, v, w) \in \mathbf{N}^4$ for which the statements (1)-(5) hold and let Γ be the set of all $i \in \mathbf{N}$ for which there exist u, v, w such that $(i, u, v, w) \in \Omega$.

The set T, as a subset of V, is finite and by Proposition 2.3 each point of T is computable. In particular a is a computable point and, again by Proposition 2.3, $\{a\}$ is a semicomputable set in $(X, \mathcal{T}, (I_i))$, which means that the set of all $u \in \mathbf{N}$ such that (5) holds is c.e. We conclude that Ω is a c.e. set and then Γ is also c.e.

We have seen that the following holds: if $I_i \cap K \neq \emptyset$, then $i \in \Gamma$.

On the other hand, suppose $i \in \Gamma$. Then there exist $u, v, w \in \mathbf{N}$ such that (1)-(5) hold. We want to prove that that $I_i \cap S \neq \emptyset$.

Suppose $I_i \cap S = \emptyset$. Then

$$S \setminus J_\mu \subseteq J_u \cup J_w,$$

and (3.30) implies

$$K \cup \widetilde{L} \subseteq J_u \cup J_w. \tag{3.32}$$

Clearly J_u and J_w are open and disjoint, so to get a contradiction with the connectedness of $K \cup \widetilde{L}$ it suffices to prove that both J_u and J_w intersect $K \cup \widetilde{L}$.

That J_u intersects $K \cup \widetilde{L}$ is obvious from (5). On the other hand we have $d \in \widetilde{L}$, so $d \in K \cup \widetilde{L}$ and, by (3.32), $d \in J_u$ or $d \in J_w$. If $d \in J_u$, then (4) implies $d \in J_\gamma$, which contradicts (3.31). Hence $d \in J_w$, which means that J_w intersects $K \cup \widetilde{L}$.

We conclude that $I_i \cap S \neq \emptyset$.

To summarize, if $I_i \cap K \neq \emptyset$, then $i \in \Gamma$, and if $i \in \Gamma$, then $I_i \cap S \neq \emptyset$. This means that K is c.e. up to S (in the case $a \in T$ and $b \notin T$).

Of course, the same conclusion holds if $a \notin T$ and $b \in T$.

Finally, let us assume $a, b \in T$. As in the previous case we see that a and b are computable points in $(X, \mathcal{T}, (I_i))$.

Let F be such that $(F, \{c, d\}) \in \mathcal{K}$ for some c and d and such that $F \neq K$. Then F and K are disjoint. Otherwise, since $F \cap K \subseteq \{a, b\} \cap \{c, d\}$ (Remark 3.5), we have $a \in \{c, d\}$ or $b \in \{c, d\}$, which is impossible since both a and b are endpoints of $(S, \mathcal{K}, V)$. We also know that $K \cap V = \{a, b\}$ and therefore we conclude that

$$S \setminus K = (V \setminus \{a, b\}) \cup \bigcup_{\substack{F \in \mathcal{K}' \\ F \neq K}} F.$$

This means that $S \setminus K$, as a finite union of compact sets, is a compact set.

So K and $S \setminus K$ are disjoint compact sets and there exists $\mu \in \mathbf{N}$ such that $S \setminus K \subseteq J_\mu$ and $K \cap J_\mu = \emptyset$. It follows that

$$K = S \setminus J_\mu$$

and thus K is a semicomputable set in $(X, \mathcal{T}, (I_i))$. It follows from Theorem 2.9 that K is a computable set in $(X, \mathcal{T}, (I_i))$. In particular K is a c.e. set and thus c.e. up to S.

We have proved that K is c.e. up to S for every $K \in \mathcal{K}'$. As noted before, this is enough to conclude that S is a computable set in $(X, \mathcal{T}, (I_i))$. □

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Izračunljivost grafova

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Sažetak. Ako je svaki poluizračunljiv skup, u bilo kojem izračunljivom topološkom prostoru, koji je homeomorfan topološkom prostoru A ujedno i izračunljiv, onda kažemo da A ima izračunljiv tip. Topološki par (A, B), $B \subseteq A$ ima izračunljiv tip ako za sve poluizračunljive skupove S i T, u bilo kojem izračunljivom topološkom prostoru, takve da je S homeomorfan prostoru A pri homeomorfizmu koji preslikava T u B, vrijedi da je S izračunljiv. Poznato je da uređeni par (G, E) ima izračunljiv tip, gdje je G određeni topološki graf, a E skup njegovih krajnjih točaka. U ovom radu promatramo općenitije objekte $\tilde{G}$ čiji su bridovi lančasti kontinuumi i pokazujemo da $(\tilde{G}, E)$ ima izračunljiv tip, gdje je E skup svih krajnjih točaka od $\tilde{G}$.