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β-EXPANSION OF p-ADIC NUMBERS WITH PISOT-CHABAUTY BASES

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Abstract. The aim of this paper is to study some arithmetic properties about the finiteness and the periodicity of the β-expansion of p-adic numbers. We prove that if the β-expansion of unity or rationals verifies some conditions then β is a Pisot-Chabauty number.

1. Introduction

The β-expansion is a representation of real numbers in an arbitrary non integer base $\beta > 1$. The notion of the β-expansion was historically introduced in 1957 by A. Rényi [8]. Since then, several researchers have been interested in studying arithmetic, diophantine and ergodic properties of this β-expansion. Let β be a real number such that $\beta > 1$. Similarly to the case of integral bases, it is possible to define the β-expansion of a real number $x \in [0, 1]$ as the sequence $(x_i)_{i\geq 1}$ with values in $\{0, 1, \ldots, [\beta]\}$ produced by the β-transformation $T_\beta : x \longrightarrow \beta x \pmod 1$ as follows :

$$for\ all\ i \geq 1,\ x_i = [\beta T_\beta^{i-1}(x)],\ and\ so\ x = \sum_{i\geq 1} \frac{x_i}{\beta^i}.$$

Let's mention that an expansion of real number is finite if $(x_i)_{i\geq 1}$ is eventually 0. It's periodic if $p \geq 1$ and $m \geq 1$ exists and verifying $x_k = x_{k+p}$, for all $k \geq m$. The sets of real numbers in $\in [0, 1]$ with periodic β-expansions and finite β-expansions are
respectively denoted by $Per(\beta)$ and $Fin(\beta)$.

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These sets have been extensively investigated during the last fifty years especially with their relationship with Pisot and Salem numbers, so there are many important
results along these lines. For instance, C. Frougny and B. Solomyak have proved in [4] that if $\mathbb{N} \subset Fin(\beta)$, then β is a Pisot number(an algebraic integer > 1 whose conjugates have modulus strictly less than one) or a Salem number (an algebraic integer > 1 with conjugates having modulus ≤ 1 and at least one of them has a modulus equal to 1). Later, S. Akiyama has improved this previous result by showing in [1] that β can only be a Pisot number. In the same direction, C. Frougny and B. Solomyak [4] have determined the real having a finite β-expansion for a specific family of Pisot numbers in the following theorem:

THEOREM 1.1. *Let β be the positive root of the polynomial $P(X) = X^m - a_1X^{m-1} - a_2X^{m-2} - \cdots - a_m$, $a_i \in \mathbb{Z}$, and $a_1 \geq a_2 \geq \cdots \geq a_m > 0$. Then β is a Pisot number and $d_\beta(1) = a_1a_2 \ldots a_m$.*

Moreover, periodic β-expansions appeared in several works and afforded interesting results. As an example, it was shown by an easy argument that $Per(\beta) \subseteq \mathrm{Q}(\beta) \cap [0, 1)$ for every real number $\beta > 1$ (where $\mathrm{Q}(\beta)$ is the smallest field which contains Q and β). Later, in [9] K. Schmidt has enhanced this result by proving that if β is a Pisot number, then $Per(\beta)$=$\mathrm{Q}(\beta) \cap [0, 1)$. In particular, periodic β-expansion of rationals has also interested many researchers, among them K. Schmidt who has proved the following theorem in [9]:

THEOREM 1.2. *Let β be a real number > 1. If $\mathbb{Q} \cap [0, 1[\subset Per(\beta)$, then β is either a Pisot or a Salem number.*

Furthermore, the β-expansion of 1 plays an important role in the study of the
classification of algebraic numbers $\beta > 1$. Let's recall that numbers β such that their
β-expansion of 1 is ultimately periodic are called Parry numbers and those such that their β-expansion of 1 is finite are called simple Parry numbers. These families of numbers were introduced by W. Parry in [6], its elements were initially called β-numbers and it is easy to check that these elements are algebraic integers. Many works have been devoted to the study of these numbers such as F. Bassino who has studied the cubic simple beta-numbers and all the numbers having finite or periodic β-expansion in the Pisot cubic case. Particularly, she showed the following theorem [2].

THEOREM 1.3. *If β is a cubic simple beta-number, then β is a Pisot number.*

Besides, concerning the degree 4, D. Boyd has proved in [3] the following:

THEOREM 1.4. *If β is a Salem number of degree 4, then β is a beta-number.*

But, it is clear that there is not a full characterization of Parry numbers or simple Parry numbers. So, we conclude that the finiteness and the periodicity of the β-expansion of unity is a very important problem but still largely open untill now.

The objective of this paper is on the one hand to investigate the analogous of the notion of Parry numbers in the p-adic field in order to detect whether β is a Pisot Chabauty number and on the other hand to give a variant of K. Schmidt's Theorem (Theorem 1.2) in the field of p-adic numbers.

The present paper is organized as follows: In section 2, we define $\mathbb{Q}_p$, the field of p-adic numbers and we review some basic properties and notations necessary in our work. In section 3, we introduce the β-expansion algorithm for p-adic numbers and we give the suitable definition of β-p-adic numbers as well as the analogous to β-numbers in the real case. The last section is firstly devoted to characterize the β-expansion of p-adic numbers. Secondly, we prove that if β is a β-p-adic number, then β a Pisot-Chabauty number. Finally, we provide a variant of Theorem 1.2 about the periodicity of β-expansions of rationals.

2. FIELD OF p-ADIC NUMBERS

In order to introduce $\mathbb{Q}_p$ in an harmonious way, we start by presenting the following sets:
Let p be a prime and $\mathbb{A}_p = \{mp^n, m, n \in \mathbb{Z}\} = \mathbb{Z}[\frac{1}{p}]$.

Recall that $\begin{cases} \bullet\ \mathbb{A}_p \subset \mathbb{Q} \textit{ is a principal ring.} \\ \bullet\ \textit{The unit group of } \mathbb{A}_p \textit{ is } \{\pm p^k, k \in \mathbb{Z}\}. \\ \bullet\ \textit{The field of fractions of } \mathbb{A}_p \textit{ is } \mathbb{Q}. \end{cases}$

Particularly, we denote by $\mathbb{A'}_p = \mathbb{A}_p \cap [0, 1)$.
Now, let's define the p-adic valuation:

$$\begin{array}{rcl} v_p : \mathbb{A}_p & \longrightarrow & \mathbb{Z} \bigcup \{\infty\} \\ x & \longmapsto & \begin{cases} max\{n \in \mathbb{Z} : p^n \textit{ divides } x\} & \text{if } x \neq 0, \\ \infty & \text{if } x = 0, \end{cases} \end{array}$$

which fulfills the following properties:

- $v_p(0) = \infty$,
- $v_p(xy) = v_p(x) + v_p(y)$,
- $v_p(x+y) \geq min\{v_p(x), v_p(y)\}$ *with* $v_p(x+y) = min\{v_p(x), v_p(y)\}$, *if* $v_p(x) \neq v_p(y)$.

Then $v_p(.)$ is an exponential valuation on $\mathbb{A}_p$. Consequently, the p-adic absolute value $|.|_p$ is defined by

$$|x|_p = \begin{cases} p^{-v_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Thus $|.|_p$ is a non Archimedean absolute value on $\mathbb{A}_p$ which verifies the strict triangular inequality

$$|x+y|_p \leq max\{|x|_p,\ |y|_p\} \quad with$$

$$|x+y|_p = max\{|x|_p,\ |y|_p\} \quad if \quad |x|_p \neq |y|_p.$$

Let $|.|_\infty$ be the Archimedean absolute value. So $|x|_p$ and $|x|_\infty$ satisfy the following product formula

$$\prod_{p \in \mathbb{P} \cup \{\infty\}} |x|_p = 1 \quad for\ all \quad x \in \mathbb{Q} \setminus \{0\}$$

where $\mathbb{P}$ denote the set of primes.
Now, the completion of $\mathbb{A}_p$ with respect to $|.|_p$ is the field of p-adic numbers $\mathbb{Q}_p$, therefore we have

$$\mathbb{Z} \subset \mathbb{A}_p \subset \mathbb{Q} \subset \mathbb{Q}_p.$$

We mention that each element $x \in \mathbb{Q}_p$ $(x \neq 0)$ admits a unique p-adic expansion of the form

$$x = \sum_{n=n_0}^{\infty} x_n p^n, \quad such\ that \quad n_0 \in \mathbb{Z}, \quad x_{n_0} \neq 0 \quad and\ x_n \in \{0, \ldots, p-1\} \quad (\star)$$

From expansions of the form mentioned in $(\star)$, we will use the notation

$$x = \ldots p_2 p_1 p_0 . p_{-1} \ldots p_{n_0}.$$

Definition 2.1. *Each $x \in \mathbb{Q}_p$ of the form mentioned above in $(\star)$ has a unique Artin decomposition*

$$x = [x]_p + \{x\}_p$$

where

$$[x]_p = \sum_{n \geq 0} x_n p^n \quad and \quad \{x\}_p = \sum_{n<0} x_n p^n.$$

The number $[x]_p \in \mathbb{Z}_p$ is called $p-$adic integer part and $\{x\}_p \in \mathbb{A}_p \cap [0,1)$ is called $p-$adic fractional part of x.

Moreover, we can also extend v_p in $\mathbb{Q}_p$ as follows:
If $x = \sum\limits_{n=n_0}^{\infty} x_n p^n$, where $n_0 \in \mathbb{Z}$, $x_{n_0} \neq 0$, $x_n \in \{0, \ldots, p-1\}$, we define $v_p(x)$ by:

$$v_p(x) = \begin{cases} n_0 & if\ x \neq 0 \\ \infty & if\ x = 0. \end{cases}$$

Furthermore, $\mathbb{Q}_p$ is equivalent to the fraction field of the p-adic integers $\mathbb{Z}_p$ where

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p; |x|_p \leq 1\}.$$

Therefore, it easily follows that

$$\mathbb{Z} = \mathbb{A}_p \cap \mathbb{Z}_p = \{x \in \mathbb{A}_p; |x|_p \leq 1\} \text{ and } p\mathbb{Z}_p = \{x \in \mathbb{Q}_p; |x|_p < 1\}.$$

Now, we aim to define the Pisot-Chabauty numbers as the analogous to Pisot numbers in the real case. For this, we need some definitions.

DEFINITION 2.2. *An element α is called algebraic over $\mathbb{A}_p$, if there is a polynomial*

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{A}_p[x] \text{ with } a_n \neq 0 \text{ and } P(\alpha) = 0.$$

If P is irreducible over $\mathbb{A}_p$, then P is called a minimal polynomial of α. In addition, if $a_n = p^k$ for some $k \in \mathbb{Z}$, thus α is an algebraic integer. As p^k is a unit of $\mathbb{A}_p$, we can assume without loss of generality, that $a_n = 1$. If $a_0 = p^{k'}$ for some $k' \in \mathbb{Z}$, so α is called an algebraic unit.

It turns out that algebraic elements over $\mathbb{Q}$ are not necessarily contained in $\mathbb{Q}_p$. In our context, we will only need that $|.|_p$ can be extended uniquely from $\mathbb{Q}_p$ to all of its algebraic extensions. This follows from the next theorem, which holds generally in non-archimedean fields.

THEOREM 2.3 ([5], Chapter II, Theorem 4.8). *Let K be a field which is complete with respect to $|.|$ and L/K be an algebraic extension of degree m. Thus $|.|$ has a unique extension to L defined by : $|\alpha| = \sqrt[m]{|N_{L/K}(\alpha)|}$ and L is complete with respect to this extension.*

REMARK 2.4. In what follows, for algebraic elements β over $\mathbb{A}_p$ we will denote by $\beta_1, \ldots, \beta_n$ the non-Archimedean conjugates of β and by $\beta_{n+1}, \ldots, \beta_{2n}$ the Archimedean conjugates of β (the complex roots of the minimal polynomial of β).

Finally, we reach to give the definition of Pisot-Chabauty numbers.

DEFINITION 2.5. *A Pisot-Chabauty number (for short PC number) is a p-adic number $\beta \in \mathbb{Q}_p$, such that*

- *$\beta_1 = \beta$ is an algebraic integer over $\mathbb{A}_p$.*
- *$|\beta_1|_p > 1$ for one non-Archimedean conjugate of β.*
- *$|\beta_i|_p \leq 1$ for all non-Archimedean conjugates β_i, $i \in \{2, \ldots, n\}$ of β.*
- *$|\beta_i|_\infty < 1$ for all Archimedean conjugates β_i, $i \in \{n+1, \ldots, 2n\}$ of β.*

Let's define the following set:

DEFINITION 2.6.

$\xi_n = \{(r_1, ..., r_n) \in \mathbb{R}^n : x^n + r_n x^{n-1} + ... + r_1$ *has only complex roots* $\gamma \in \mathbb{C}$ *with* $|\gamma| < 1\}$

Definition 2.7. *Let* $\beta \in \mathbb{Q}_p$ *with* $|\beta|_p > 1$ *an algebraic number of degree* n
and $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0 \in \mathbb{A}_p[x]$ *is its minimal polynomial.*
If $(a_0, ..., a_n) \in \xi_n$*, we write* $\beta \in \mathbb{E}_n$*.*

3. $\beta-$expansion in the field $\mathbb{Q}_p$

Similarly to the classical β-expansions for the real numbers, we introduce the β-expansions for p-adic numbers. For this, let $\beta \in \mathbb{Q}_p$ with $|\beta|_p > 1$ and $x \in \mathbb{Z}_p$. A representation in base β (or β-representation) of x is a sequence $(d_i)_{i\geq 1}$, $d_i \in \mathbb{A}'_p = \mathbb{A}_p \cap [0, 1)$, such

$$x \quad = \quad \sum_{i\geq 1} \frac{d_i}{\beta^i}.$$

A particular β-representation of x is called the β-expansion of x and noted by
$d_\beta(x) = (d_i)_{i\geq 1}$ with values in $\mathbb{A}_{\beta,p} = [0, 1) \cap \{x \in \mathbb{A}_p : |x|_p \leq |\beta|_p\}$ produced by the β-transformation $T : \mathbb{Z}_p \to \mathbb{Z}_p$, which is given by the mapping $z \mapsto [\beta z]_p$. For $k \geq 0$, let's define

$$T^0(x) = x \text{ and } T^k(x) = T(T^{k-1}(x)).$$

So, $d_k = \{\beta T^{k-1}(x)\}_p$ for all $k \geq 1$.

An equivalent definition of the β-expansion can be obtained by using a greedy algorithm. This algorithm proceeds as follows :

$$r_0 = x;\ d_k = \{\beta r_{k-1}\}_p \text{ and } r_k = \lfloor \beta r_{k-1} \rfloor_p \text{ for all } k \geq 1.$$

The β-expansion of x will be noted as $d_\beta(x) = (d_k)_{k\geq 1}$.

Now, let $x \in \mathbb{Q}_p$ with $|x|_p > 1$. Thus there is a unique $k \in \mathbb{N}$ such that $|\beta|_p^k \leq |x|_p < |\beta|_p^{k+1}$. We can represent x by shifting $d_\beta(\beta^{-(k+1)}x)$ by k digits to the left. Therefore, if $d_\beta(x) = 0 \bullet d_1 d_2 d_3 \ldots$, then $d_\beta(\beta x) = d_1 \bullet d_2 d_3 \ldots$. Thereby, if $d_\beta(x) = d_1 d_2 d_3 \ldots d_k \bullet d_{k+1} d_{k+2} \ldots$. We denote x by

$$x = [x]_\beta + \{x\}_\beta$$

with

$$[x]_\beta = \sum_{1\leq i\leq k} d_i \beta^i \quad and \quad \{x\}_\beta = \sum_{i\geq k+1} \frac{d_i}{\beta^i}.$$

The number $[x]_\beta$ is called a p-adic β-integer part of x and the number $\{x\}_\beta$ is called a p-adic β-fractional part of x.
Moreover, we mention that $d_\beta(x)$ is finite if and only if there is a $k \geq 0$ with $T^k(x) = 0$, $d_\beta(x)$ is ultimately periodic if and only if there is some

smallest $n \geq 0$ (the pre-period length) and $s \geq 1$ (the period length) when $T^{n+s}(x) = T^n(x)$, namely the period length. In a special case, where $n = 0$, $d_\beta(x)$ is purely periodic.
Afterwards, we will use the following notations :

$$Fin(\beta) = \{x \in \mathbb{Z}_p : d_\beta(x) \ is \ finite\} \text{ and}$$
$$Per(\beta) = \{x \in \mathbb{Z}_p : d_\beta(x) \ is \ eventually \ periodic\}.$$

Hence, it is easy to check that

$$Fin(\beta) \subset Per(\beta).$$

Through the use of the previous sets and the PC numbers, K.Scheicher, V. F. Sirvent and P. Surer established the following theorem ([7]):

THEOREM 3.1. *Let β be a PC number. Then $Per(\beta) = \mathbb{Q}(\beta) \cap \mathbb{Z}_p$.*

Furthermore, analogously to the notion of β-numbers in the real case we define the β-p-adic numbers in $\mathbb{Q}_p$ as follows:

DEFINITION 3.2. *Let $\beta \in \mathbb{Q}_p$ where $|\beta|_p > 1$. β is called a β-p-adic number if*
$1 \in Per(\beta)$ and is called a simple β-p-adic number if $1 \in Fin(\beta)$.

4. Results

Before giving our main theorems, we begin by the following lemmas:

LEMMA 4.1. *Let $\beta \in \mathbb{Q}_p$ where $|\beta|_p > 1$ and $(a_i)_{i\geq 1}$ is a β-representation of x. Then $d_\beta(x) = (a_i)_{i\geq 1}$ if and only if $|a_i|_p \leq |\beta|_p$, for all $i \geq 1$.*

PROOF. The necessary condition is trivial. For the sufficient condition, by assumption we have $(a_i)_{i\geq 1}$ is a β-representation of x and $|a_i|_p \leq |\beta|_p$ *for all* $i \geq 1$, so

$$x = \sum_{i\geq 1} \frac{a_i}{\beta^i}.$$

If we multiply by β, we get

$$\beta x = a_1 + \sum_{i\geq 2} \frac{a_i}{\beta^{i-1}}.$$

As $|\sum_{i\geq 2} \frac{a_i}{\beta^{i-1}}|_p < 1$ and $a_1 \in \mathbb{A}'_p$, we obtain that $a_1 = \{\beta x\}_p$. Put now $r_0 = x$. We have

$$\beta x - a_1 = \sum_{i\geq 2} \frac{a_i}{\beta^{i-1}}$$

and if we multiply again by β, we get

$$\beta(\beta x - a_1) \quad = \quad a_2 + \sum_{i\geq 3} \frac{a_i}{\beta^{i-2}}.$$

Since $|\sum_{i\geq 3} \frac{a_i}{\beta^{i-2}}|_p < 1$ and $a_2 \in \mathbb{A}'_p$, we get $a_2 = \{\beta r_1\}_p$ where $r_1 = \beta x - a_1$. Therefore, it's clear that the sequence $(a_k)_{k\geq 1}$ verifies the recurrent condition

$$r_0 = x; a_k = \{\beta r_{k-1}\}_p \text{ and } r_k = \lfloor \beta r_{k-1} \rfloor_p,$$

which implies that $d_\beta(x) = (a_i)_{i\geq 1}$.

□

LEMMA 4.2. *Let $r \in \mathbb{Q}$. Let $\beta \in \mathbb{Q}_p$ with $|\beta|_p > 1$ and γ a conjugate of β such that $|\gamma|_p > 1$. If $r = \sum_{k\geq 1} a_k \beta^{-k}$ where $(a_k)_{k\geq 1}$ is a periodic sequence, then $r = \sum_{k\geq 1} a_k \gamma^{-k}$.*

PROOF. By assumption, we have $r \in \mathbb{Q}$ and $r = \sum_{k\geq 1} a_k \beta^{-k}$ where $(a_k)_{k\geq 1}$ is a periodic sequence. So $(a_k)_{k\geq 1} = a_1 ... a_p \overline{a_{p+1} ... a_{p+s}}$ with $a_p \neq a_{p+s}$. Hence, we get

$$r = \frac{a_1}{\beta} + \cdots + \frac{a_p}{\beta^p} + \frac{a_{p+1}}{\beta^{p+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}} + \frac{1}{\beta^s}(\frac{a_{p+1}}{\beta^{p+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}}) + \frac{1}{\beta^{2s}}(\frac{a_{p+1}}{\beta^{p+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}}) + \cdots .$$

Therefore

$$r = \frac{a_1}{\beta} + \cdots + \frac{a_p}{\beta^p} + (\frac{a_{p+1}}{\beta^{p+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}})(1 + \frac{1}{\beta^s} + \frac{1}{\beta^{2s}} + \frac{1}{\beta^{3s}} + \cdots),$$

which implies that

$$r = \frac{a_1}{\beta} + \cdots + \frac{a_p}{\beta^p} + (\frac{a_{p+1}}{\beta^{p+1}} + \cdots + \frac{a_{p+s}}{\beta^{p+s}})(\frac{1}{1 - \frac{1}{\beta^s}}).$$

Thus, for every conjugate γ of β, we have

$$r = \frac{a_1}{\gamma} + \cdots + \frac{a_p}{\gamma^p} + (\frac{a_{p+1}}{\gamma^{p+1}} + \cdots + \frac{a_{p+s}}{\gamma^{p+s}})(\frac{1}{1 - \frac{1}{\gamma^s}}).$$

Now, for every conjugate γ of β such that $|\gamma|_p > 1$,we obtain

$$r = \frac{a_1}{\gamma} + \cdots + \frac{a_p}{\gamma^p} + (\frac{a_{p+1}}{\gamma^{p+1}} + \cdots + \frac{a_{p+s}}{\gamma^{p+s}})(1 + \frac{1}{\gamma^s} + \frac{1}{\gamma^{2s}} + \frac{1}{\gamma^{3s}} + \cdots).$$

Consequently, we reach to our result by getting the following equality

$$r = \frac{a_1}{\gamma} + \cdots + \frac{a_p}{\gamma^p} + \frac{a_{p+1}}{\gamma^{p+1}} + \cdots + \frac{a_{p+s}}{\gamma^{p+s}} + \frac{1}{\gamma^s}(\frac{a_{p+1}}{\gamma^{p+1}} + \cdots + \frac{a_{p+s}}{\gamma^{p+s}}) + \frac{1}{\gamma^{2s}}(\frac{a_{p+1}}{\gamma^{p+1}} + \cdots + \frac{a_{p+s}}{\gamma^{p+s}}) + \cdots .$$

□

In the previous sections, we have seen that the β-expansion of 1 play a crucial role in the study of the algebraicity of real numbers and in the notion of beta numbers . This motivates as to study this notion in the field of p-adic numbers. For this, we begin by the following theorem:

Theorem 4.3. *Let $\beta \in \mathbb{Q}_p$ with $|\beta|_p > 1$ such that $\beta \in E_n$. If β is a β-p-adic number, then β is a PC number.*

Proof. We assume that β is a β-p-adic number and $d_\beta(1) = 0\bullet a_1 ... a_s \overline{a_{s+1} ... a_{s+r}}$. Therefore,

$$1 = \sum_{i \geq 1} \frac{a_i}{\beta^i}.$$

More precisely,

$$\begin{aligned}
1 &= \frac{a_1}{\beta} + \cdots + \frac{a_s}{\beta^s} + \frac{a_{s+1}}{\beta^{s+1}} + \cdots + \frac{a_{s+r}}{\beta^{s+r}} + \frac{a_{s+1}}{\beta^{s+r+1}} + \cdots + \frac{a_{s+r}}{\beta^{s+2r}} + \cdots \\
&= \frac{a_1}{\beta} + \cdots + \frac{a_s}{\beta^s} + \frac{a_{s+1}}{\beta^{s+1}} + \cdots + \frac{a_{s+r}}{\beta^{s+r}} + \frac{1}{\beta^r}(1 - \frac{a_1}{\beta} - \cdots - \frac{a_s}{\beta^s}) \\
&= \frac{a_1}{\beta} + \cdots + \frac{a_s}{\beta^s} + (\frac{a_{s+1}}{\beta^{s+1}} + \cdots + \frac{a_{s+r}}{\beta^{s+r}})(1 + \frac{1}{\beta^r} + \frac{1}{\beta^{2r}} + \cdots) \\
&= \frac{a_1}{\beta} + \cdots + \frac{a_s}{\beta^s} + (\frac{a_{s+1}}{\beta^{s+1}} + \cdots + \frac{a_{s+r}}{\beta^{s+r}})(\frac{1}{1 - \frac{1}{\beta^r}}),
\end{aligned}$$

which involves that β is an algebraic integer. (1)
Our purpose now is to prove that all the non-Archimedean conjugates have an absolute value less or equal to 1. For this, assume contrary and let α be a non-Archimedean conjugate of β such that $|\alpha|_p > 1$. Then, by Lemma 4.2 we get

$$1 = \sum_{i \geq 1} \frac{a_i}{\alpha^i}.$$

Thereby, we have

$$\begin{aligned}
0 &= a_1(\frac{1}{\beta} - \frac{1}{\alpha}) + a_2(\frac{1}{\beta^2} - \frac{1}{\alpha^2}) + \cdots \\
&= \sum_{i \geq 1} a_i(\frac{1}{\beta^i} - \frac{1}{\alpha^i}) \\
&= (\frac{1}{\beta} - \frac{1}{\alpha}) \left(a_1 + \sum_{i \geq 2} a_i F_i(\alpha, \beta) \right),
\end{aligned}$$

where $F_i(\alpha,\beta) = \dfrac{\dfrac{1}{\beta^i} - \dfrac{1}{\alpha^i}}{\dfrac{1}{\beta} - \dfrac{1}{\alpha}} = \dfrac{1}{\beta^{i-1}} + \dfrac{1}{\beta^{i-2}}\dfrac{1}{\alpha} + \cdots + \dfrac{1}{\alpha^{i-1}}, \quad for\ all\ i \geq 2.$

On the one hand, since $(\dfrac{1}{\beta} - \dfrac{1}{\alpha}) \neq 0$, then

$$a_1 + \sum_{i\geq 1} a_i F_i(\alpha,\beta) = 0. \qquad (\star\star)$$

On the other hand, as $|\dfrac{1}{\beta}|_p < 1$ and $|\dfrac{1}{\alpha}|_p < 1$, we obtain

$$|F_i(\alpha,\beta)|_p \leq \max_{1\leq k\leq i}\{|\frac{1}{\beta^{i-k}}\frac{1}{\alpha^{k-1}}|_p < 1, \quad for\ all\ i \geq 2.$$

Moreover, through Lemma 4.1, we get $|a_i|_p \leq |\beta|_p$ for all $i \geq 1$. Then

$$|a_i F_i(\alpha,\beta)|_p < |a_i|_p \leq |\beta|_p, \quad for\ all\ i \geq 2,$$

which yields that

$$|\sum_{i\geq 2} a_i F_i(\alpha,\beta)|_p < |\beta|_p.$$

Since $|a_1|_p = |\beta|_p$, we infer that

$$|a_1 + \sum_{i\geq 1} a_i F_i(\alpha,\beta)|_p = |a_1|_p = |\beta|_p,$$

which is in contradiction with $(\star\star)$. Therefore, we get that all the non-Archimedean conjugates have an absolute value less or equal to 1. (2)
Finally, by (1), (2) and under the assumption of the theorem that $\beta \in E_n$, we deduce that β is a PC number. □

From the previous Theorem, we display this immediate consequence:

COROLLARY 4.4. *Let $\beta \in \mathbb{Q}_p$ with $|\beta|_p > 1$ such that $\beta \in E_n$. If β is a simple*
β-p-adic-number, then β is a PC number.

It is natural now to give our final result which is a variant of Theorem 1.2.

THEOREM 4.5. *Let $\beta \in \mathbb{Q}_p$ with $|\beta|_p > 1$ and $\beta \in E_n$. If $\mathbb{Q} \cap (p\mathbb{Z}_p) \subset Per(\beta)$, then β is a PC number.*

PROOF. Since $\mathbb{Q} \cap (p\mathbb{Z}_p) \subset Per(\beta)$, then in particular we have

$$\begin{aligned} p &= \frac{a_1}{\beta} + \cdots + \frac{a_s}{\beta^s} + \frac{a_{s+1}}{\beta^{s+1}} + \cdots + \frac{a_{s+r}}{\beta^{s+r}} + \frac{a_{s+1}}{\beta^{s+r+1}} + \cdots + \frac{a_{s+r}}{\beta^{s+2r}} + \cdots \\ &= \frac{a_1}{\beta} + \cdots + \frac{a_s}{\beta^s} + \frac{a_{s+1}}{\beta^{s+1}} + \cdots + \frac{a_{s+r}}{\beta^{s+r}} + \frac{1}{\beta^r}(1 - \frac{a_1}{\beta} - \cdots - \frac{a_s}{\beta^s}) \\ &= \frac{a_1}{\beta} + \cdots + \frac{a_s}{\beta^s} + (\frac{a_{s+1}}{\beta^{s+1}} + \cdots + \frac{a_{s+r}}{\beta^{s+r}})(1 + \frac{1}{\beta^r} + \frac{1}{\beta^{2r}} + \cdots) \\ &= \frac{a_1}{\beta} + \cdots + \frac{a_s}{\beta^s} + (\frac{a_{s+1}}{\beta^{s+1}} + \cdots + \frac{a_{s+r}}{\beta^{s+r}})(\frac{1}{1 - \frac{1}{\beta^r}}), \end{aligned}$$

which involves that β is an algebraic integer. (1)
Our objective now is to prove that all the non-Archimedean conjugates have an absolute value less or equal to 1. For this, assume contrary and let γ be a non-Archimedean conjugate of β such that $|\gamma|_p > 1$. we aim at deriving a contradiction.
By the density of $\mathbb{Q}$ in $\mathbb{Q}_p$, there exists for all integer $m \geq 3$ a rational $r \in \mathbb{Q}$ such that
$|r - \frac{1}{\beta}|_p < \frac{1}{\beta^{m-1}}$, which implies that $r \in \mathbb{Q} \cap (p\mathbb{Z}_p)$ and $d_\beta(r) = (r_i)_{i\geq 1}$ with $r_1 = 1$, and $r_k = 0$ for $k = 2, \ldots, m-1$. Therefore, $d_\beta(r)$ is periodic and as $|\gamma|_p > 1$, we obtain according to Lemma 4.2,

$$\begin{aligned} r &= \beta^{-1} + \sum_{k\geq m} r_k \beta^{-k} \\ &= \gamma^{-1} + \sum_{k\geq m} r_k \gamma^{-k}, \end{aligned}$$

which involves that

$$|\beta^{-1} - \gamma^{-1}|_p = |\sum_{k\geq m} r_k(\gamma^{-k} - \beta^{-k})|_p.$$

Now, on the one hand we have $\lim\limits_{m\to+\infty}(\sum\limits_{k\geq m} r_k(\gamma^{-k} - \beta^{-k})) = 0$ (as the rest of a convergent series) and on the other hand we have $|\beta^{-1} - \gamma^{-1}|_p > 0$. Therefore, a contradiction is achieved, which means that that all the non-Archimedean conjugates have an absolute value less or equal to 1. (2)
Finally, From (1), (2) and under the assumption of the theorem $\beta \in E_n$, we conclude that β is a PC number and the proof of our theorem is reached.

$\square$

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References

[1] S. Akiyama, Cubic Pisot units with finite beta expansions. In F.Halter-Koch and R.F. Tichy, editors, Algebraic Number Theory and Diophantine Analysis, (2000),pp. 11-26.

[2] F. Bassino, β-expansions for cubic Pisot numbers, in LATIN02, LNCS, Vol.2286, (2002), pp. 141-152.

[3] D. Boyd, On beta-expansions for Pisot numbers, Math. Comp. 65(N214) (1996), pp. 841-860.

[4] C. Frougny and B. Solomyak, Finite beta-expansions. Ergod. Th. and Dynam. Sys, 12, no 4,(1992), pp. 713-723.

[5] J.Neukirch: Algebraic number theory, volume 322 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, (1999).

[6] W. Parry, On the beta-expansions of real numbers, Acta.Math. Acad. Sci. Hungar, 11, (1960), pp. 401-416.

[7] K.Scheicher, V.F.Sirvent, P.Surer, *beta*-expansions of p-adic numbers, Ergod. Th. Dynam. Sys, 36, (2016), pp. 924-943.

[8] A. Rényi, Representations for real numbers and their ergodic properties, Acta Math. Acad. Sci. Hung, 8, (1975), pp. 477-493,.

[9] K. Schmidt, On periodic expansions of Pisot numbers and Salem numbers. Bull. london Math . Soc, (1980) pp. 269-278.

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